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On the Green's function of Laplace's tidal equation, an application to the northern Adriatic Sea

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ABSTRACT

To evaluate the effect of external forces on the northern water motion of Adriatic Sea, we calculate the Green's Function of Laplace's tidal equation for a simple geometry. As an application, we use known data on sea level variations at various stations, in order to determine the external forces. We then compute the "theoretical" sea level at Venice. Its comparison with "experimental" data taken at Venice gives encouraging results.

1. Introduction

The Laplace Tidal Equations are widely used to study motion in oceans, seas and lakes, particularly tides. Many methods have been used to study the problem. In practice, however, the most natural method for solving inhomogeneous linear equations, i.e. by using their Green's Function, was only recently introduced by Webb (1974, 1976). Also Miles (1974) and Garrett and Greenberg (1976) have applied this method to harbours. Important previous research on similar aspects of tides was carried out by Fairbairn (1954), Cartwright and Munk (1966) and Proudman (1925). In this work we use Webb's formalism to obtain the sea level variation as the integral of the external forces applied to the fluid. Owing to the mathematical form of the Green's Function, our approach to the real solution is like Webb's. Because our experimental data came from comparatively few oceanic stations, we were obliged to consider only the lowest eigenmodes of the basin. This is not a bad approximation, as also Garrett and Greenberg (1976) remarked. A simple algebraic relation between sea level variation and external forces is then obtained. We use precisely this relation to compute the external forces from sea level data at various stations of the northern Adriatic Sea. This basin was chosen because of its very simple form (Fig. 1), but the method is rather general. We then used the computed external forces to obtain the sea level variation at Venice from sea level variations of other stations. The computed results are in reasonable agreement with the sea level data.

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2. The equations and their Green Functions

The Laplace Tidal equations read

$$\frac{\partial}{\partial t} u - fv = -g \frac{\partial}{\partial x} \xi + X$$

$$\frac{\partial}{\partial t} v + fu = -g \frac{\partial}{\partial y} \xi + Y$$

$$\frac{\partial}{\partial t} \xi + h_0 \left(\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v \right) = 0 \qquad (1)$$

where f is the Coriolis parameter, g the gravity, $\mathbf{v} = (u, v)$ the depth-averaged velocities, $\xi(x, y, t)$ the sea level variation, h_0 the bottom depth. We assume

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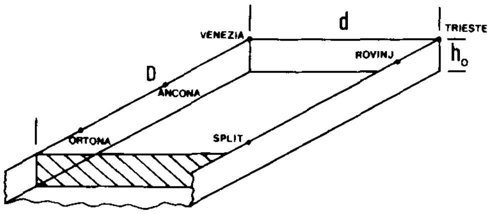

Fig. 1. Schematic representation of the northern Adriatic Sea and location of the stations.

h_0 to be constant, a fairly realistic hypothesis for the northern Adriatic Sea, schematized roughly as a box of dimensions h_0, d, D (Fig. 1); X, Y are external forces.

The boundary conditions are: $\mathbf{v}\cdot\mathbf{n} = 0$, where $\mathbf{n}$ is the normal to the coast, and $\xi = Z(x, t)$ on the open boundary.

Introducing the Fourier transform

$\tilde g(\omega) = \int_{-\infty}^{\infty} e^{i\omega t}\, g(t)\, dt$

the preceding equations can be written

$$(\omega^2 - f^2)\,\tilde\xi + gh_0\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\tilde\xi = h_0\left\{\frac{\partial}{\partial x}\tilde X + \frac{\partial}{\partial y}\tilde Y + \frac{f}{i\omega}\left[\frac{\partial}{\partial y}\tilde X - \frac{\partial}{\partial x}\tilde Y\right]\right\} \equiv \tilde F \qquad (2)$$

with $\hat\xi = Z(x, \omega)$ on the open boundary.

The mathematics of the problem become simpler if, instead of the above boundary conditions, at the coast we assume

$$\frac{\partial}{\partial n}\,\xi = 0$$

This is tantamount to disregarding the effect of the Coriolis force on the boundaries; it is allowed if $|\omega| > f$. To deal with the open boundary conditions, without introducing derivatives of the Green's Function, we rewrite the equations in terms of

$$\varphi(x, y, t) = \xi(x, y, t) - Z(x, t)$$

In order to use a more compact formalism, we introduce the $\hat{\mathscr{L}}$ operator as

$$\hat{\mathscr{L}}\Psi = (\omega^2 - f^2)\,\Psi + gh_0\,\nabla^2\,\Psi$$

and the "forces" Φ as

$$\tilde\Phi(x, y, \omega) = \tilde F(x, y, \omega) - (\omega^2 - f^2)\,\tilde Z - gh_0\,\frac{\partial}{\partial x^2}\,\tilde Z$$

Equation (2) is now

$$\hat{\mathscr{L}}\tilde\varphi = \tilde\Phi \qquad (3)$$

with

$$\frac{\partial}{\partial n}\,\tilde\varphi = 0 \quad \text{along the coast}$$

$$\tilde\varphi = 0 \quad \text{along the open boundary}$$

We introduce the Green Function $\mathscr{G}(x, x_0, y, y_0, t, t_0)$ as the "inverse" of the operator $\hat{\mathscr{L}}$.

$$\hat{\mathscr{L}}\mathscr{G} = \delta(x - x_0)\cdot\delta(y - y_0)\cdot\delta(t - t_0)$$

where $\delta(x)$ is the Dirac function. This definition allows us to write in a comparatively formal way

$$\varphi(x, y, t) \equiv \xi(x, y, t) - Z(x, t) = \int\int_{\text{basin}} dx_0\, dy_0\, dt_0$$

$$x\mathscr{G}(x, x_0, y, y_0, t, t_0)\;\; \Phi(x_0, y_0, t_0) \qquad (4)$$

The sea level variation $\xi(x, y, t)$ can be said to be the result of the effect of some unknown "forces" obtained through the operator $\mathscr{G}$. These forces are schematized in Φ and can be divided into atmospheric "forces" $F(x, y, t)$ and the effect of the input of water from the open southern boundary—the terms in Φ being proportional to $Z(x, t)$. These formulae can be explicitly computed once the Green's function is known.

For the northern Adriatic Sea the Green's function has a very simple form

$$\tilde{\mathscr{G}}(x, x_0, y, y_0, \omega) = \sum_{l=0}^{\infty}\sum_{j=0}^{\infty} \frac{A_l(x)\, A_l(x_0)\, B_j(y)\, B_j(y_0)}{\omega^2 - f^2 - h_0\, g(\alpha_l^2 + \beta_j^2)} \qquad (5)$$

where $A_l(x)$, $B_j(y)$ are the eigenmodes

$$A_l = \sqrt{\frac{2}{d}}\cos(\alpha_l x); \quad \alpha_l = l\frac{\pi}{d}; \quad l = 0, 1, 2 \ldots, \infty$$

$$B_j = \sqrt{\frac{2}{D}}\sin(\beta_j y); \quad \beta_j = \left(j + \frac{1}{2}\right)\frac{\pi}{D}$$

$$j = 0, 1, 2 \ldots, \infty$$

(Morse and Feshbach, 1953). Then eq. (4) becomes

$$\tilde{\xi}(x,y,\omega) = \tilde{Z}(x,\omega) + \iint_{\text{basin}} \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{A_l(x)\,B_j(y)\cdot A_l(x_0)\cdot B_j(y_0)}{\omega^2 - f^2 - h_0 g(\alpha_l^2 + \beta_j^2)} \cdot\tilde{\Phi}(x_0,y_0,\omega)\,dx_0\,dy_0 = \tilde{Z}(x,\omega) + \sum_{l=0}^{\infty}\sum_{j=0}^{\infty} A_l(x)\,B_j(y) \cdot\iint_{\text{basin}} \frac{A_l(x_0)\,B_j(y_0)}{\omega^2 - f^2 - h_0 g(\alpha_l^2 + \beta_j^2)} \times \tilde{\Phi}(x_0,y_0,\omega)\,dx_0\,dy_0 \tag{6}$$

3. Application to the northern Adriatic Sea

Equation (6) relates water levels to the forcing functions themselves, $\tilde{Z}(x,\omega)$ and $\tilde{\Phi}(x,y,\omega)$. Let us now assume that the $\xi(x, y, t)$ is known at the oceanographic stations, (x_i, y_i). We thus know the values

$$\xi_i(t) \equiv \xi(x_i, y_i, t) \quad i = 1, \dots N$$

Table 1 gives the positions (x_i, y_i) of the stations; the values are normalized with respect to the basin dimensions. As we have only a finite quantity N of informations, we must take a finite number of terms in the sums inside the Green's Function (6).

Moreover, these stations are approximately on the $x = 0$ and $x = d$ bank of the northern Adriatic Sea. So we are not obliged to deal with an unknown function $\tilde{Z}(x,\omega)$ of two variables but we can limit ourselves to only two unknown functions of one variable, $\tilde{Z}_l(\omega)$ and $\tilde{Z}_r(\omega)$. As sea level data are available from only five stations plus the Venice station (see Fig. 1 and Table 1), in order to have as many linear equations as unknowns, only three eigenmodes can be considered. Then eq. (6) gives:

Table 1. *Position of the stations in the northern Adriatic Sea*

	Station	$\frac{x_i}{d}$	$\frac{y_i}{D}$
1	Ortona	0	0.2
2	Ancona	0	0.6
3	Trieste	1.0	1.0
4	Rovinj	1.0	0.8
5	Split	1.0	0.1
6	Venezia	0	1.0

$$\tilde{\xi}(x_i,y_i,\omega) = \tilde{Z}(x_i,\omega) + \sum_{K=1}^{3} A_l(x_i)\cdot B_j(y_i) \cdot\iint_{\text{basin}} \frac{A_l(x_0)\,B_j(y_0)}{\omega^2 - f^2 - h_0 g(\alpha_l^2 + \beta_j^2)} \cdot\tilde{\Phi}(x_0,y_0,\omega)\,dx_0\,dy_0$$

where

$K = 1$ means $l = 0,\ j = 0$
$K = 2$ means $l = 0,\ j = 1$
$K = 3$ means $l = 1,\ j = 0$

We note that this system of linear equations allows a matrix description

$$\tilde{\xi}_i(w) = \sum_{m=0}^{5} M_{im}\,\tilde{\Psi}_m(\omega)$$

if we call Ψ_m the "forces", that is

$$\Psi_1(\omega) = \tilde{Z}_l(\omega) \equiv \tilde{Z}(o,\omega); \quad \Psi_2 = \tilde{Z}_r(\omega) \equiv \tilde{Z}(d,\omega)$$

$$\Psi_3(\omega) = \iint_{\text{basin}} \frac{A_0(x_0)\,B_0(y_0)}{\omega^2 - f^2 - h_0 g(\alpha_0^2 + \beta_0^2)} \cdot\tilde{\Phi}\,dx_0\,dy_0$$

$$\Psi_4(\omega) = \iint_{\text{basin}} \frac{A_0(x_0)\,B_1(y_0)}{\omega^2 - f^2 - h_0 g(\alpha_0^2 + \beta_1^2)} \times\tilde{\Phi}\,dx_0\,dy_0$$

$$\Psi_5(\omega) = \iint_{\text{basin}} \frac{A_1(x_0)\,B_0(y_0)}{\omega^2 - f^2 - h_0 g(\alpha_1^2 + \beta_0^2)} \times\tilde{\Phi}\,dx_0\,dy_0$$

and M_{im} the matrix elements.

$M_{i1} = \delta_{i1} + \delta_{i2}$
$M_{i2} = \delta_{i3} + \delta_{i4} + \delta_{i5} \quad i = 1, \dots, 5$
$M_{i3} = A_0(x_i)\,B_0(x_i)$
$M_{i4} = A_0(x_i)\,B_1(y_i)$
$M_{i5} = A_1(x_i)\,B_0(y_i)$.

We have in this way defined the values $\Psi_m(\omega)$, $m = 1, \dots, 5$ as the unknown forcing factors, and M_{im}, the "response" matrix.

We can now use the preceding relations for concrete purposes. From the knowledge of $\zeta_l(t)$ we can, for instance, infer the forcing on the basin by inverting the matrix by the Gaussian method. Only the three components Ψ_3, Ψ_4, Ψ_5 and the $\Psi_1 = Z_l$, $\Psi_2 = Z_2$ can obviously be detected. We thus have

$$\tilde{\Psi}_m(\omega) = \sum_{m=1}^{5} (M^{-1})_{ml}\, \tilde{\zeta}_l(\omega) \tag{7}$$

Once the "forces" $\tilde{\Psi}_m(\omega)$ are known as linear combinations of $\tilde{\xi}_l(\omega)$ $(i = 1, \dots 5)$ it is possible to check the method. Let us compute the theoretical effect of the forcing terms $\tilde{\Psi}_n(\omega)$ on the sea level $\tilde{\xi}_v(\omega)$ at a specific point x_v, y_v (i.e. Venice). We can now say

$$\tilde{\xi}_v(\omega) = \tilde{Z}_l(\omega) + A_0(x_v)\, B_0(y_v)\, \tilde{\Psi}_3(\omega) + A_0(x_v)\, B_1(y_v)\, \tilde{\Psi}_4(\omega) + A_1(x_v)\, B_0(y_v)\, \tilde{\Psi}_5(\omega) \tag{8}$$

since the M matrix is not time dependent, we can make a Fourier antitransformation and obtain exactly the same equation for time dependent quantities $\zeta_l(t)$

$$\zeta_v(t) = Z_l(t) + A_0(x_v)\, B_0(y_v) \sum_{p=1}^{5} (M^{-1})_{3p}\, \zeta_p(t) + A_0(x_v)\, B_1(y_v) \cdot \sum_{p=1}^{5} (M^{-1})_{4p}\, \zeta_p(t) + A_1(x_v)\, B_0(y_v) \cdot \sum_{p=1}^{5} (M^{-1})_{5p}\, \zeta_p(t)$$

We thus obtain the sea level at Venice as a linear combination of sea levels of adjacent stations (Fig. 1). We must add that the experimental data are usual sea level data, taken every hour. They were supplied by Osservatorio Geofisico Sperimentale, Triests.

There is another possible approximation. The $f > |\omega|$ approximation in the rigid boundary condition could be useful in studying long-term (1 week or more) motions.

This implies $\xi = \text{cost} = 0$ on the rigid boundary. The A_l, B_j are now

$$A_l = \sqrt{\frac{2}{d}} \sin\left(\frac{\pi}{d} lx\right); \quad B_j = \sqrt{\frac{2}{D}} \sin\left(\frac{\pi}{D} jy\right)$$

The formalism remains unchanged.

4. Discussion

In order to check the model, the period 4–18 October 1966 was chosen as a period of normal tides. The numerical results are shown in Fig. 2. A remarkable agreement between theoretical and experimental data can be observed. This is a rather surprising result, because the hypotheses, in particular $(\partial/\partial n)\xi = 0$ on the rigid boundary, are rather strained. One possible explanation is that the motion of the Adriatic Sea was first studied in one dimension with reasonable results (Palmieri and

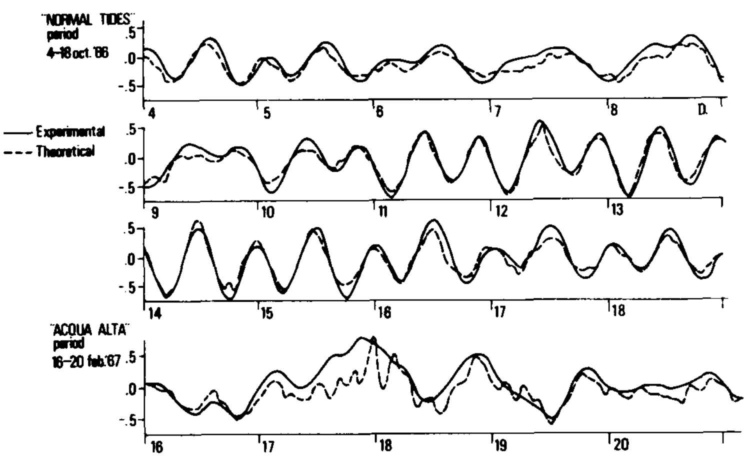

Fig. 2. Theoretical and experimental values for the period 4–18 October 1966— a normal period without particularly strong storms and 1967 16–20 February—a period with an "acqua alta". The full line refers to experimental data; the dotted line refers to theoretical values. The heights are in meters.

Finizio, 1970). The present work shows up the smaller effect of the transverse modes and of the Coriolis parameter—in this kind of problem.

We have also checked the capacity for predicting the "acqua alta" at Venice, a combined effect of wind, atmospheric pressure and seiches that can increase the sea level by as much as 1.5 meters above normal tide level. We feel that once the ξ becomes rather large ($\xi/h_0 \simeq 0, 1$) our hypothesis of linearized Euler equation is no longer valid. This could explain why in the periods of "acqua alta" as from 16–20 February 1967 (see Fig. 2) the amplitudes are in poor agreement with the experimental data, even if the phase is rather correct.

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О ФУНКЦИИ ГРИНА ПРИЛИВНОГО УРАВНЕНИЯ ЛАПЛАСА; ПРИЛОЖЕНИЕ К СЕВЕРНОЙ АДРИАТИКЕ

Для оценки эффекта внешнего возбуждения на движение воды в северной части Адриатического моря в случае простой геометрии вычисляется функция Грина приливного уравнения Лапласа. В начале приложения используются известные данные по изменению уровня моря на разных станциях с целью определения внешнего возбуждения. Далее вычисляется теоретический уровень моря в Венеции. Его сравнение с"экспериментальными" данными приводит к обнадеживающим результатам.