J. Math. Soc. Japan
Vol. 49, No. 4, 1997

# A functional decomposition theorem for the conformal representation

By Massimo LANZA DE CRISTOFORIS

(Received Jun. 20, 1995)
(Revised Sept. 25, 1995)

## 1. Introduction.

This paper concerns the nonlinear operator which maps an injective function $\phi$ defined on the unit disk in $\boldsymbol{R}^2$ with values in $\boldsymbol{R}^2$, of class $C^{m,\alpha}$, and with nonvanishing jacobian, into the unique holomorphic, one-to-one map $g[\phi]$ of the unit disk $\mathscr{D}$ onto the Jordan domain $\phi(\mathscr{D})$, normalized by $g[\phi](0)=\phi(0)$, $g'[\phi](0)\in]0, +\infty[$. By virtue of Warschawski's work, it is easy to see that under our assumptions $g[\phi]\in C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$ if $\alpha\in]0, 1[$. We have chosen to represent the Jordan domain as the image of $\phi$, rather than as the more customary Jordan curve which parametrizes the boundary of the Jordan domain to allow the application of the methods of this paper, but as we show in section 2, there is no loss of generality.

The main finding of this paper is that the nonlinear operator $\phi\mapsto g[\phi]$ can be decomposed as $\phi\mapsto\phi\circ S[\phi]^{(-1)}$, where $S[\phi]^{(-1)}$ denotes the inverse function of $S[\phi]$, and the operator $\phi\mapsto S[\phi]$ is analytic from a set of 'admissible' $\phi$'s in $C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$ to $C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$. In other words, $\phi\mapsto g[\phi]^{(-1)}\circ\phi$ is analytic. As we shall explain, this result easily allows to give precise information on the regularity of $\phi\mapsto g[\phi]$.

The analyticity statement for $S[\cdot]$ may sound surprizing. Indeed, in section 5 we show that $\phi\mapsto g[\phi]$ is not even differentiable from the set of admissible $\phi$'s in $C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$ to $C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$, and we show that in order to have differentiability of $g[\cdot]$, we must increase the regularity of the elements in the domain of $g[\cdot]$. We note that the problem of expanding $g$ in a power series of a parameter $\varepsilon$ ranging on some interval $I$ of the real line, when the Jordan domain depends on $\varepsilon$ and is for each $\varepsilon$ parametrized by the elements of $\partial\mathscr{D}$, has been solved by Kantorovich in the thirties (cf. Kantorovich & Krylov

1991 Mathematics Subject Classification. 30C20, 47H30.
Key words and phrases. Conformal representation, Nonlinear operators.
The author wishes to express his gratitude to the referee for a number of useful suggestions, some of which have enabled the author to simplify the proofs of Lemma 2.2 and of Theorem 3.5.

(1964, ch. V, §6) and references therein), who hypothesized that the curve binding the parameter dependent Jordan domain is analytic both in $\varepsilon$ and in the variable in $\partial\mathcal{D}$. The present paper shows in particular that a stronger expansion Theorem for $S$ (not for $g$) holds when no analyticity assumption on the dependence of $\phi$ on the variable in $\partial\mathcal{D}$ is formulated, thus confirming the remarkable difference between $\phi\mapsto g[\phi]$ and $\phi\mapsto g[\phi]^{(-1)}\circ\phi$. Several authors have investigated the dependence of the conformal representation upon the domain. Among them we mention Radó (1923), who proved a well-known result on the continuity of $\phi\mapsto g[\phi]$, Rosenblatt (1936), Zeitlin (1957), and Yoshikawa (1960), whose work is reported in the extensive monograph of Gaier (1964). Their methods are different from those of this paper, and this paper complements their work. In particular the author has never seen the nonlinear operator $S$ treated in the literature.

The proof is based on a simple but key idea. We observe that $S[\phi]$ is, for each $\phi$ the unique solution of a nonlinear elliptic system (cf. Theorem 3.5). Then we study such system locally by means of the Implicit Function Theorem.

By using the regularity Theorem for $\phi\mapsto S[\phi]$, together with two 'higher order' differentiability results for the superposition and for the inversion operator, we can prove that $\phi\mapsto g[\phi]$ is of class $C^k$, $k\in\boldsymbol{N}$ if the derivative loss between domain and range of $g$ is $k$, and if the domain of $g[\cdot]$ is restricted to the closure of the smooth functions in the norm of the domain (cf. Theorem 4.7). The sharpness of such statement, at least in case $k=1$ has been shown by means of the two 'inverse results' of section 5. In Remark 4.11, we briefly explain how such differentiability statements can be used to study the dependence of the spectrum of the Laplacian upon perturbation of the domain. Finally, we note that 'functional regularity' theorems for $g[\cdot]$ and for $S[\cdot]$ find application in fluid-solid interaction problems where the conformal representation can be used to handle domain-dependent exterior boundary-value problems (cf. Lanza & Antman (1991), Lanza (1993)). For example, assume as in Lanza & Antman (1991), that an elastic body is subject to an external pressure due to a steady-state incompressible, inviscid, irrotational flow. Then, as well-known, the pressure field on the boundary of the body can be given in terms of the boundary values of the conformal representation of the domain exterior to the body, in terms of certain parameters such as the pressure $P$ and the velocity $U$ of the external flow at infinity, and in terms of the constant density of the flow (cf. Lanza & Antman (1991, p. 1214)). Since the pressure field applied to the body enters crucially in the system of the governing equations for the position field of the body, which is the unknown of the problem, it is clear that if we wish to study the smoothness of the solution branches of the system of the governing equations, as the parameters $U$, $P$ are varied, or if we wish to justify a formal expansion of the solutions in terms of $U$, $P$, then we must have precise information on how smoothly the pressure field depends on $U$, $P$ and on the exterior domain of the flow, which is identified by the position field of the body. This can be done by using the results contained in this paper.

NOTE ADDED IN PROOF. By reading volume 818 (1995) of Zentralblatt für Mathematik, the author has been informed that Wu (1993), with the advice of R. Coifman, and by exploiting ideas of Coifman & Meyer (1983), has proved two analyticity statements for an operator, which can be easily related to the composition of $S$ with the restriction to $\partial\mathcal{D}$, for Jordan domains bounded by an arc-length parametrized Jordan curve sufficiently close to a circle, with prescribed length $2\pi$ and with direction of the tangent vector prescribed by a periodic function of class BMO with period $2\pi/n$, $\boldsymbol{N}\ni n\geqq 2$ (cf. Wu (1993, Th. 3.4)) or with a $2\pi$-periodic function of class $C^{0,\alpha}$ (cf. Wu (1993, Th. 6.4 in the case indicated in Remark 1, p. 1325)). The validity of a variant of the latter result follows from Theorem 3.10 of this paper, no matter if the curve is parametrized by arclength or if the curve is close to a circle, and also when the norm of $C^{0,\alpha}$ is replaced with that of $C^{m,\alpha}$, $m\geqq 1$. Wu (1993) exploits integral equation and operator theory methods and his approach is completely different from that of this paper.

## 2. Technical Preliminaries and Notation.

Let $\mathcal{X}$, $\mathcal{Y}$ be real normed spaces. We say that $\mathcal{X}$ is imbedded in $\mathcal{Y}$ provided that $\mathcal{X}\subseteq\mathcal{Y}$ and that the inclusion map is continuous. $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ denotes the normed space of the continuous linear maps of $\mathcal{X}$ into $\mathcal{Y}$ and is equipped with the topology of the uniform convergence on the unit sphere of $\mathcal{X}$. For standard definitions of Calculus in normed spaces, we refer *e.g.* to Berger (1977), Prodi & Ambrosetti (1973). The inverse function of a function $f$ is denoted $f^{(-1)}$ as opposed to the reciprocal of a complex-valued function $g$ or the inverse of a matrix $A$, which are denoted $g^{-1}$ and $A^{-1}$ respectively. A dot '$\cdot$' denotes the inner product in $\boldsymbol{R}^2$, or the matrix product between matrices with real entries. $M_r(\boldsymbol{R})$, with $r\in\boldsymbol{N}\setminus\{0\}$ denotes the set of $r\times r$ matrices with real entries. Let $A$ be a matrix. Then ${}^tA$ denotes the transpose matrix of $A$. Throughout the paper, we make no formal distinction between complex numbers and pairs of real numbers, so for example $\mathcal{D}$ denotes the open unit disk both in $\boldsymbol{C}$ and in $\boldsymbol{R}^2$. Similarly, if $f=(f_1, f_2)$ is a map of $\boldsymbol{R}^2$ to $\boldsymbol{R}^2$, and if $f_1+if_2$ is holomorphic in the complex variable $x_1+ix_2$, then $f'$ denotes the complex derivative of $f_1+if_2$. Let $B\subseteq\boldsymbol{R}^n$. Then $\mathrm{cl}\, B$ denotes the closure of $B$ and $\mathrm{int}\, B$ denotes the interior of $B$. Let $\Omega$ be an open subset of $\boldsymbol{R}^n$. The space of $m$-times continuously differentiable real-valued functions on $\Omega$, is denoted by $C^m(\Omega)$. Let

$f \in (C^m(\Omega))^n$. The $j$-th component of $f$ is denoted $f_j$, and $Df$ denotes the gradient matrix $(\partial f_j/\partial x_l)_{j,l=1,\cdots,n}$. Let $\boldsymbol{N}$ be the set of nonnegative integers including 0, and let $\eta \equiv (\eta_1, \cdots, \eta_n) \in \boldsymbol{N}^n$, $|\eta| \equiv \eta_1 + \cdots + \eta_n$. Then $D^\eta$ denotes $\partial^{|\eta|}/\partial x_1^{\eta_1} \cdots \partial x_n^{\eta_n}$. The subspace of $C^m(\Omega)$ of those functions which can be extended with continuity to $\mathrm{cl}\,\Omega$ together with their derivatives $D^\eta f$ of order $|\eta| \leqq m$ is denoted $C^m(\mathrm{cl}\,\Omega)$. Let $f \in C^m(\mathrm{cl}\,\Omega)$. The unique continuous extension of $D^\eta f$, $|\eta| \leqq m$ to $\mathrm{cl}\,\Omega$ is still denoted by the same symbol. Let $\Omega$ be a bounded open subset of $\boldsymbol{R}^n$. $C^m(\mathrm{cl}\,\Omega)$ equipped with the norm $\|f\|_m \equiv \sum_{|\eta| \leq m} \sup_{\mathrm{cl}\,\Omega} |D^\eta f|$ is a Banach space. The subspace of $C^m(\mathrm{cl}\,\Omega)$ whose functions have $m$-th order derivatives that are Hölder continuous with exponent $\alpha \in (0, 1]$ is denoted $C^{m,\alpha}(\mathrm{cl}\,\Omega)$, (cf. Kufner, John & Fučik (1977)). Let $B \subseteqq \boldsymbol{R}^n$. Then $C^{m,\alpha}(\mathrm{cl}\,\Omega, B)$ denotes $\{f \in (C^{m,\alpha}(\mathrm{cl}\,\Omega))^n : f(\mathrm{cl}\,\Omega) \subseteqq B\}$, and the elements of $C^{m,\alpha}(\mathrm{cl}\,\Omega, B)$ are always thought as row vectors. $C^{m,\alpha}(\mathrm{cl}\,\Omega, M_r(\boldsymbol{R}))$ denotes the space of functions of $\mathrm{cl}\,\Omega$ to $M_r(\boldsymbol{R})$, whose components are of class $C^{m,\alpha}$. If $f \in C^{0,\alpha}(\mathrm{cl}\,\Omega)$, then its Hölder quotient is $|f|_\alpha \equiv \sup\{|f(x)-f(y)|/|x-y|^\alpha : x, y \in \mathrm{cl}\,\Omega, x \neq y\}$. The space $C^{m,\alpha}(\mathrm{cl}\,\Omega)$ is equipped with its usual norm $\|f\|_{m,\alpha} = \|f\|_m + \sum_{|\eta|=m} |D^\eta f|_\alpha$. It is well-known that $\mathrm{cl}\,\mathscr{D}$ is a compact $C^\infty$ manifold with boundary imbedded in $\boldsymbol{R}^2$. Namely, $\mathscr{D}$ is an open subset of $\boldsymbol{R}^2$, and for all points $P \in \partial\mathscr{D}$, there exists an open neighborhood $W_P$ of $P$ in $\boldsymbol{R}^2$ and a homeomorphism $\psi_P$ of $\mathrm{cl}\,\mathscr{D}$ onto $\mathrm{cl}\,W_P$ such that $\psi_P\,(\{(x_1, x_2) \in \mathscr{D} : x_2 \geqq 0\}) = W_P \cap \mathrm{cl}\,\mathscr{D}$ and that $\psi_P \in C^\infty(\mathrm{cl}\,\mathscr{D})$, $\psi_P^{(-1)} \in C^\infty(\mathrm{cl}\,W_P)$. $C^{m,\alpha}(\partial\mathscr{D})$ denotes the set of functions $f$ of $\partial\mathscr{D}$ to $\boldsymbol{R}$ such that $f \circ \psi_P(\cdot, 0) \in C^{m,\alpha}([-1, 1])$, $\forall P \in \partial\mathscr{D}$. It is well-known that such definition of $C^{m,\alpha}(\partial\mathscr{D})$ does not depend on the chosen family $\{\psi_P\}$ with the properties above. Now let $\{P_1, \cdots, P_r\}$ be a finite collection of points of $\partial\mathscr{D}$ such that $\bigcup_{j=1}^r (W_{P_j} \cap \partial\mathscr{D}) = \partial\mathscr{D}$. Then $f \in C^{m,\alpha}(\partial\mathscr{D})$ if and only if $f \circ \psi_{P_j}(\cdot, 0) \in C^{m,\alpha}([-1, 1])$ $\forall j \in \{1, \cdots, r\}$, and $\sup_{j=1,\cdots,r} \|f \circ \psi_{P_j}(\cdot, 0)\|_{m,\alpha}$ defines a norm on $C^{m,\alpha}(\partial\mathscr{D})$. Finally, we obtain an equivalent norm by choosing another finite family of maps with the above properties. It is also well-known that all the elements of $C^{m,\alpha}(\partial\mathscr{D})$ are restrictions of some element of $C^{m,\alpha}(\mathrm{cl}\,\mathscr{D})$ (cf. *e.g.* Troianiello (1987, p. 16)).

The following lemma has been shown in Lanza (1991, Cor. 4.24, Prop. 4.29).

2.1. LEMMA. *Let $\phi \in C^1(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$, and $l[\phi] \equiv \inf\{|\phi(x)-\phi(y)|/|x-y| : x, y \in \mathrm{cl}\,\mathscr{D}, x \neq y\}$. Then $l[\phi] > 0$ holds if and only if $\phi$ is injective and* $\det D\phi(x) \neq 0$, *$\forall x \in \mathrm{cl}\,\mathscr{D}$. The map $\phi \mapsto l[\phi]$ is continuous from $C^1(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$ to $\boldsymbol{R}$, and for all $\delta \geqq 0$, the set $Y_\delta \equiv \{\phi \in C^1(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2) : l[\phi] > \delta\}$ is open in $C^1(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$.*

Then we have the following.

2.2. LEMMA. *Let $\phi \in Y_0$. Then $\partial\phi(\mathrm{cl}\,\mathscr{D}) = \phi(\partial\mathscr{D})$, $\phi(\partial\mathscr{D}) = \partial\phi(\mathscr{D})$, $\phi(\mathscr{D})$ equals the interior of $\phi(\mathrm{cl}\,\mathscr{D})$, and $\phi$ is a homeomorphism of* $\mathrm{cl}\,\mathscr{D}$ *onto* $\phi(\mathrm{cl}\,\mathscr{D})$.

PROOF. Since by Lemma 2.1, $\phi$ is continuous and one-to-one on the compact set $\mathrm{cl}\,\mathcal{D}$, then $\phi$ is a homeomorphism of $\mathrm{cl}\,\mathcal{D}$ onto $\phi(\mathrm{cl}\,\mathcal{D})$. By applying Brouwer's theorem on the invariance of domain (cf. *e.g.* Hurewicz & Wallman (1948, p. 95)) to $\phi$ and to $\phi^{(-1)}$, we deduce that $\phi(\mathcal{D})$ equals the interior of $\phi(\mathrm{cl}\,\mathcal{D})$. Since $\phi(\mathrm{cl}\,\mathcal{D})$ is closed and $\phi$ is injective, we have $\partial\phi(\mathrm{cl}\,\mathcal{D})=\mathrm{cl}\,\phi(\mathrm{cl}\,\mathcal{D})\backslash\mathrm{int}\,\phi(\mathrm{cl}\,\mathcal{D})=\phi(\mathrm{cl}\,\mathcal{D})\backslash\phi(\mathcal{D})=\phi(\partial\mathcal{D})$. We now show that $\partial\phi(\mathcal{D})=\phi(\partial\mathcal{D})$. Since $\phi(\mathcal{D})$ is open, we have $\partial\phi(\mathcal{D})=\mathrm{cl}\,\phi(\mathcal{D})\backslash\phi(\mathcal{D})$. Then, by the injectivity of $\phi$, it suffices to show that $\mathrm{cl}\,\phi(\mathcal{D})=\phi(\mathrm{cl}\,\mathcal{D})$. Since $\phi(\mathcal{D})\subseteq\phi(\mathrm{cl}\,\mathcal{D})$ and $\phi(\mathrm{cl}\,\mathcal{D})$ is closed, we have $\mathrm{cl}\,\phi(\mathcal{D})\subseteq\phi(\mathrm{cl}\,\mathcal{D})$, while the inclusion $\mathrm{cl}\,\phi(\mathcal{D})\supseteq\phi(\mathrm{cl}\,\mathcal{D})$ is an immediate consequence of the continuity of $\phi$. □

Let $\phi\in Y_0$. By Lemma 2.2, $\phi(\mathcal{D})$ is a Jordan domain bounded by a curve of class $C^1$ with nonvanishing tangent vector. Then by the Riemann Mapping Theorem, there exists a unique holomorphic homeomorphism

$$g[\phi]:\mathcal{D}\longrightarrow\phi(\mathcal{D}) \tag{2.3}$$

satisfying

$$g[\phi](0)=\phi(0),\quad g[\phi]'(0)\in\,]0,+\infty[\,. \tag{2.4}$$

By a Theorem of Caratheodory (cf. Pommerenke (1975, Th. 9.10)) the function $g[\phi]$ can be extended to the closure of $\mathcal{D}$, and the extension

$$g[\phi]:\mathrm{cl}\,\mathcal{D}\longrightarrow\mathrm{cl}\,\phi(\mathcal{D})\quad\text{is a homeomorphism.} \tag{2.5}$$

We now introduce the following notation. Let $L>0$, $m\in\boldsymbol{N}\backslash\{0\}$, $\alpha\in\,]0,1[$. Let

$$C_P^{m,\alpha}([0,L],\boldsymbol{R}^2)\equiv\{\zeta\in C^{m,\alpha}([0,L],\boldsymbol{R}^2):\zeta^{(j)}(0)=\zeta^{(j)}(L),\ j=0,\cdots,m\},$$

$$\sigma(s_1,s_2)\equiv\min\{|t_1-t_2|:t_j\in\boldsymbol{R},\ e^{i2\pi t_j/L}=e^{i2\pi s_j/L},\ j=1,2\}, \tag{2.6}$$

$$l_*[\zeta]\equiv\inf\left\{\left|\frac{\zeta(s)-\zeta(t)}{\sigma(s,t)}\right|:s,t\in[0,L],\ \sigma(s,t)\neq 0\right\}.$$

Then we have the following (cf. Lanza (1992, p. 124)).

2.7. LEMMA. *Let $\zeta\in C_P^{m,\alpha}([0,L],\boldsymbol{R}^2)$. Then $l_*[\zeta]>0$ if and only if the curve $\zeta$ is simple and $|\zeta'(s)|>0$, $\forall s\in[0,L]$. The set $\{\zeta\in C_P^{m,\alpha}([0,L],\boldsymbol{R}^2):l_*[\zeta]>0\}$ is open in the space $C_P^{m,\alpha}([0,L],\boldsymbol{R}^2)$.*

By $\mathcal{J}[\zeta]$ we denote the bounded connected component of $\boldsymbol{C}\backslash\{\zeta([0,L])\}$. Now we have the following, which is basically a restatement of Warschawski (1935, Th. IIIc, III*, Remark 1.a, p. 319).

2.8. THEOREM. *Let $\alpha\in\,]0,1[$, $m\in\boldsymbol{N}$, $\delta>0$. If $\phi\in C^{m+1,\alpha}(\mathrm{cl}\,\mathcal{D},\boldsymbol{R}^2)$ and $l[\phi]>0$, then $g[\phi]\in C^{m+1,\alpha}(\mathrm{cl}\,\mathcal{D},\boldsymbol{R}^2)$. If $\{\phi_n\}$ is a bounded sequence of $C^{m+1,\alpha}(\mathrm{cl}\,\mathcal{D},\boldsymbol{R}^2)$ such that*

$$l[\phi_n] > \delta, \tag{2.9}$$

*then there exists a positive constant* $c>0$ *such that*

$$\frac{1}{c} < |g'[\phi_n](z)| < c, \quad \forall z \in \mathrm{cl}\,\mathcal{D}, \tag{2.10}$$

$$\|g[\phi_n]\|_{m+1,\alpha} \leqq c. \tag{2.11}$$

*Let* $\phi$, $\phi_n \in C^{m+1,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2) \cap Y_0$. *If* $\{\phi_n\}$ *converges to* $\phi$ *in* $C^{m+1,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)$, *then* $\{g[\phi_n]\}$ *converges to* $g[\phi]$ *in* $C^{m+1,\beta}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)$, $\forall \beta \in ]0, \alpha[$.

PROOF. If $\phi \in C^{m+1,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)$ and $l[\phi]>0$, then by Lemmas 2.1 and 2.2, $\phi(\mathcal{D})$ is a Jordan domain bounded by the simple curve $\phi(e^{it})$, $t\in[0, 2\pi]$, which is of class $C^{m+1,\alpha}$ and has nonvanishing tangent vector. Then, with the aid of Lemma 3.18 of Lanza (1992), we can apply the results of Warschawski (1935) cited above to deduce that $g[\phi] \in C^{m+1,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)$. Assume now that (2.11) does not hold. Then for some subsequence $\{\phi_n\}$, we have $\lim_n \|g[\phi_n]\|_{m+1,\alpha} = +\infty$. Now let $\beta \in ]0, \alpha[$. Since $\{\phi_n\}$ is bounded and since the imbedding of $C^{m+1,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)$ into $C^{m+1,\beta}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)$ is compact, there exists a subsequence $\{\phi_{n_k}\}$ converging to some $\phi$ in the space $C^{m+1,\beta}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)$. By the continuity of $\phi \mapsto l[\phi]$ on $C^{m+1,\beta}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)$, we have $l[\phi] \geqq \delta$, and accordingly $\phi$ is injective and $\phi(0)$ belongs to the interior of $\mathrm{cl}\,\phi(\mathcal{D})$. If we take $h_1>0$ to be less than the distance of $\phi(0)$ from the boundary of $\phi(\mathcal{D})$, and $h_2 > 2\sup\{\|\phi_n\|_{m+1,\alpha}\}$, then clearly there exists $k_0 \in \boldsymbol{N}$ such that

$$h_1 < |\phi_{n_k}(e^{it}) - \phi(0)| < h_2, \quad \forall t \in [0, 2\pi], \tag{2.12a}$$

for $k \geqq k_0$. Furthermore, it is easy to see that

$$l_*[\phi_{n_k}(e^{it}) - \phi(0)] \geqq l_*[e^{it}]\, l[\phi_{n_k}] > \delta l_*[e^{it}]. \tag{2.12b}$$

Conditions (2.12) and Lanza (1992, Th. 3.22), which is basically a restatement of Warschawski (1935), implies that $\sup_k \|g[\phi_{n_k} - \phi(0)]\|_{m+1,\alpha} < \infty$. Then clearly $\sup_k \|g[\phi_{n_k}]\|_{m+1,\alpha} < \infty$, a contradiction. Similarly, we can deduce (2.10) by the work of Warschawski. We now consider the last statement. By Lemma 2.1, we can assume that $l[\phi_n] > l[\phi]/2$. Then (2.11), the pointwise convergence of $\{g[\phi_n]\}$ to $g[\phi]$, which holds by Radó Theorem (cf. Radó (1923)) and the well-known compactness of the imbedding of $C^{m+1,\alpha}(\mathrm{cl}\,\mathcal{D})$ in $C^{m+1,\beta}(\mathrm{cl}\,\mathcal{D})$ imply the validity of the last statement (cf. Lanza (1992, Lemma 2.3 (iv))). □

Next we show that to each sufficiently regular simple closed curve $\zeta$, we can associate a diffeomorphism $\phi$ of $\mathrm{cl}\,\mathcal{D}$ onto $\phi(\mathrm{cl}\,\mathcal{D}) \subseteq \boldsymbol{R}^2$, so that $\phi_{|\partial D}$ parametrizes the given curve. We also show that we can locally choose such correspondence $\zeta \mapsto \phi[\zeta]$ to be affine, so from every statement of (high order) differentiability or analyticity for a nonlinear map say $\mathcal{F}$ depending on $\phi$, we can deduce a corresponding result for $\zeta\mapsto\mathcal{G}[\phi[\zeta]]$. So for example, if we define $g_\zeta$ to be the conformal representation of the domain enclosed by $\zeta$, a differentiability theorem for $\zeta\mapsto g_\zeta$ can be deduced by a corresponding Theorem for $\phi\mapsto g[\phi]$. Indeed, by properly choosing the normalizing conditions, we have $g[\phi[\zeta]]=g_\zeta$. Now we have the following.

2.13. LEMMA. *Let* $m\in\boldsymbol{N}\setminus\{0\}$, $\alpha\in]0, 1[$, $\zeta_0\in C_P^{m,\alpha}([0, L], \boldsymbol{R}^2)$, $l_*[\zeta_0]>0$, $z_0\in\mathcal{I}[\zeta_0]$. *Then the following hold.*

(i) *There exists at least an element* $\phi_0\in C^{m,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)\cap Y_0$ *such that*

$$\phi_0(e^{i\theta}) = \zeta_0(\theta L/2\pi), \quad \forall\theta\in[0, 2\pi], \qquad \phi_0(0)=z_0. \tag{2.14}$$

(ii) *There exists a continuous linear operator* $F$ *of* $C_P^{m,\alpha}([0, L], \boldsymbol{R}^2)$ *to* $C^{m,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)$, *such that the affine map* $\zeta\mapsto\phi_0+F[\zeta-\zeta_0]\equiv\phi[\zeta]$ *maps an open neighborhood of* $\zeta_0$ *in* $\{\zeta\in C_P^{m,\alpha}([0, L], \boldsymbol{R}^2) : l_*[\zeta]>0\}$ *into* $\{\phi\in C^{m,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)\cap Y_0 : \phi(0)=z_0\}$, *and satisfies*

$$\phi[\zeta](e^{i\theta}) = \zeta(\theta L/2\pi), \quad \forall\theta\in[0, 2\pi]. \tag{2.15}$$

PROOF. Assume for example that the winding number of $\zeta_0$ with respect to $z_0$ is $+1$. By the Riemann Mapping Theorem and by Warschawski (1935), there exists a homeomorphism $G_0\in C^{m,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)$ of $\mathrm{cl}\,\mathcal{D}$ onto $\mathrm{cl}\,\mathcal{I}[\zeta_0]$, such that $G_0$ is holomorphic in $\mathcal{D}$, $G_0(0)=z_0$, and $\det DG_0\neq 0$ in $\mathrm{cl}\,\mathcal{D}$. Then clearly, $G_0^{(-1)}\circ\zeta_0\in C_P^{m,\alpha}([0, L], \partial\mathcal{D})$, $|(G_0^{(-1)}\circ\zeta_0(s))'|\neq 0$, $\forall s\in[0, L]$, so that there exists a function $\psi_0\in C^{m,\alpha}([0, 2\pi])$, such that $G_0^{(-1)}\circ\zeta_0(\theta L/2\pi)=e^{i\psi_0(\theta)}$, $\psi_0'\neq 0$, $\psi_0(2\pi)=2\pi+\psi_0(0)$, $\psi_0^{(j)}(0)=\psi_0^{(j)}(2\pi)$, $j=1, \cdots, m$. Now let $k\in C^\infty([0, 1], [0, 1])$ be such that $k([0, (1/3)])=\{0\}$, $k([(2/3), 1])=\{1\}$, $k'>0$ in $](1/3), (2/3)[$. Let $\arg(x)\in[0, 2\pi[$ be the argument of $x\in\mathrm{cl}\,\mathcal{D}\setminus\{0\}$. Then, by use of polar coordinates, it is not hard to prove that the map $\Phi_0$ defined by

$$\begin{aligned}\Phi_0(x_1, x_2) \equiv \sqrt{x_1^2+x_2^2}(&\cos\{\arg(x_1, x_2)+k((x_1^2+x_2^2)^{1/2})[\psi_0(\arg(x_1, x_2))-\arg(x_1, x_2)]\},\\ &\sin\{\arg(x_1, x_2)+k((x_1^2+x_2^2)^{1/2})[\psi_0(\arg(x_1, x_2))-\arg(x_1, x_2)]\}),\\ &\qquad\forall(x_1, x_2)\in\mathrm{cl}\,\mathcal{D}\setminus\{0\},\\ \Phi_0(0, 0) \equiv (0, 0),&\end{aligned} \tag{2.16}$$

which coincides with the identity if $\sqrt{x_1^2+x_2^2}\leq 1/3$, is an injection of class $C^{m,\alpha}$ of $\mathrm{cl}\,\mathcal{D}$ onto $\mathrm{cl}\,\mathcal{D}$ and satisfies $\det(D\Phi_0)\neq 0$ in $\mathrm{cl}\,\mathcal{D}$, and $\Phi_0(e^{i\theta})=e^{i\psi_0(\theta)}$. Then $\phi_0\equiv G_0\circ\Phi_0$ satisfies (i). We now prove (ii). We first observe that the map $A$ defined by $A[w]\equiv w(\arg(\cdot)L/2\pi)$, is linear and continuous from $C_P^{m,\alpha}([0, L], \boldsymbol{R}^2)$ to $C^{m,\alpha}(\partial\mathcal{D}, \boldsymbol{R}^2)$. We also note that it is well-known that there exists a continuous and linear extension operator $E$ of $C^{m,\alpha}(\partial\mathcal{D}, \boldsymbol{R}^2)$ to $C^{m,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)$ such that $(Ef)_{|\partial\mathcal{D}}=f$, $\forall f\in C^{m,\alpha}(\partial\mathcal{D}, \boldsymbol{R}^2)$. By possibly multiplying the components of $E$ by a $C^\infty$ function equal to 1 on $\partial\mathcal{D}$, and to 0 at 0, there is no loss of generality in assuming that $Ef(0, 0)=(0, 0)$. Then we can choose $F\equiv E\circ A$. Indeed, (2.15) is clearly satisfied, and we can choose $\|\zeta-\zeta_0\|_{m,\alpha}$ sufficiently small, so that $\|F[\zeta-\zeta_0]\|_{m,\alpha}$ is small and $l_*[\zeta]>0$, $l[\phi_0+F[\zeta-\zeta_0]]>0$. □

We now close the present section by collecting some known properties of elementary operators in the following Lemma. We note that, throughout the paper, 'analytic' means 'real analytic'. For the definition of analytic operator, we refer the reader to Prodi & Ambrosetti (1973, p. 89).

2.17. LEMMA. *Let* $r, m\in\boldsymbol{N}$, $r>0$, $\alpha\in]0, 1]$. *Let* $F^{-1}$ *be the inverse matrix of (an invertible)* $F\in C^{m,\alpha}(\mathrm{cl}\,\mathcal{D}, M_r(\boldsymbol{R}))$. *Then we have the following.*

(i) *The pointwise matrix product, which reduces to the pointwise product of functions when* $r=1$, *is bilinear and continuous and henceforth analytic from* $C^{m,\alpha}(\mathrm{cl}\,\mathcal{D}, M_r(\boldsymbol{R}))\times C^{m,\alpha}(\mathrm{cl}\,\mathcal{D}, M_r(\boldsymbol{R}))$ *to* $C^{m,\alpha}(\mathrm{cl}\,\mathcal{D}, M_r(\boldsymbol{R}))$.

(ii) *If* $m>0$, *the map* $F\mapsto F^{-1}$ *is analytic from* $\{F\in C^{m,\alpha}(\mathrm{cl}\,\mathcal{D}, M_r(\boldsymbol{R})):\det F>0 \text{ on } \mathrm{cl}\,\mathcal{D}\}$ *to itself, and its differential at the element* $F_0$ *is given by the map* $M\mapsto -F_0^{-1}\cdot M\cdot F_0^{-1}$.

PROOF. To prove (i), it suffices to remember that the pointwise product is bilinear and continuous in $C^{m,\alpha}(\mathrm{cl}\,\mathcal{D})$ (cf. *e.g.* Kufner, John & Fučik (1977)). Then (ii) easily follows because to each invertible matrix of $C^{m,\alpha}(\mathrm{cl}\,\mathcal{D}, M_r(\boldsymbol{R}))$ we can associate, in a linear and therefore analytic way, a linear and invertible element of $\mathcal{L}(C^{m,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^r), C^{m,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^r))$, and the inversion of the invertible linear operators is analytic (cf. *e.g.* Prodi & Ambrosetti (1973, p. 109)). □

We also mention that continuous (multi)linear operators between normed spaces are analytic (cf. *e.g.* Prodi & Ambrosetti (1973)). Finally, we remark that the analyticity of some operator, say $\phi\mapsto A[\phi]$ of $C^{m,\alpha}(\mathrm{cl}\,\mathcal{D})$ to itself, should not be confused with the analyticity of $\mathcal{D}\ni x\mapsto A[\phi](x)\in\boldsymbol{R}$, which in general does not follow from the analyticity of $A$.

## 3. The functional decomposition for the conformal representation.

In this section we show that the conformal representation operator $\phi\mapsto g[\phi]$, can be written as $g[\phi]=\phi\circ S[\phi]^{(-1)}$, where $\phi\mapsto S[\phi]$ is analytic.

As we can see below, it is convenient to write the Cauchy-Riemann equations for a $\boldsymbol{R}^2$-valued function $f$ in terms of a linear operator acting on the $2\times2$ matrix of the partial derivatives of $f$. Thus we introduce the following Lemma, whose proof is straightforward.

3.1. LEMMA. *Let $L$ be the linear map of $M_2(\boldsymbol{R})$ into itself defined by*

$$L(A)=\begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}, \quad \forall A\equiv\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\in M_2(\boldsymbol{R}), \tag{3.2}$$

*and let $I$ be the identity map in $M_2(\boldsymbol{R})$. Then the following hold.*

(i) $L\circ L=I$.

(ii) $(I-L)(A)=0$ *if and only if* $a_{11}=a_{22}$, $a_{12}=-a_{21}$.

(iii) *If* $A$, $B\in M_2(\boldsymbol{R})$, *and* $(I-L)(A)=0$, *then* $(I-L)(AB)=A(I-L)(B)$.

3.3. REMARK. Let $\Omega$ be an open subset of $\boldsymbol{R}^2$. If $f\equiv(f_1, f_2)\in C^1(\Omega, \boldsymbol{R}^2)$, (which we identify with $f_1+if_2$), then both the first row and the first column of the $2\times 2$ real matrix

$$\frac{1}{2}(I-L)(Df)$$

equal $(\operatorname{Re}\bar{\partial}f, \operatorname{Im}\bar{\partial}f)$, where $\bar{\partial}f$ is defined by

$$\bar{\partial}f=\frac{1}{2}(\partial_{x_1}f+i\partial_{x_2}f)=\frac{1}{2}[(\partial_{x_1}f_1-\partial_{x_2}f_2)+i(\partial_{x_1}f_2+\partial_{x_2}f_1)]. \tag{3.4}$$

Furthermore, we have $(I-L)(Df)=0$ in $\Omega$, if and only if $f_1+if_2$ is holomorphic in $\Omega$.

We now introduce the operators $\phi\mapsto R[\phi]$, $\phi\mapsto S[\phi]$, which are basic both to decompose $\phi\mapsto g[\phi]$ and to prove the differentiability Theorems for $g[\cdot]$ of section 4.

3.5. THEOREM. *Let $m\in\boldsymbol{N}$, $\alpha\in]0, 1[$. Let $\phi\in C^{m+1,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)$, $l[\phi]>0$, then the following two statements hold.*

(i) *There exists a unique element $R\in C^{m+1,\alpha}(\mathrm{cl}\,\mathcal{D}, \mathrm{cl}\,\mathcal{D})$ such that*

$$l[R]>0, \tag{3.6a}$$

$$R_1^2+R_2^2=1 \quad \text{on } \partial\mathcal{D}, \tag{3.6b}$$

$$(I-L)(D(\phi\circ R))=0 \quad \text{in } \mathcal{D}, \tag{3.6c}$$

$$R(0)=0, \tag{3.6d}$$

$$(D(\phi\circ R)(0))_{12}=0, \tag{3.6e}$$

$$(D(\phi\circ R)(0))_{11}>0, \tag{3.6f}$$

*where $(D(\phi\circ R)(0))_{rs}$ denotes the $(r, s)$-th entry of the matrix $D(\phi\circ R)(0)$. Such $R$ is a homeomorphism of $\mathrm{cl}\,\mathcal{D}$ onto $\mathrm{cl}\,\mathcal{D}$, and satisfies $R(\mathcal{D})=\mathcal{D}$, $R(\partial\mathcal{D})=\partial\mathcal{D}$, $\phi\circ R=g[\phi]$. For each $\phi$, we denote such unique $R$ by $R[\phi]$.*

(ii) *There exists a unique element* $S \in C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$ *such that*

$$l[S] > 0, \tag{3.7a}$$

$$S_1^2 + S_2^2 = 1 \quad \text{on } \partial\mathscr{D}, \tag{3.7b}$$

$$(I-L)(D\phi\cdot(DS)^{-1}) = 0 \quad \text{in } \mathscr{D}, \tag{3.7c}$$

$$S(0) = 0, \tag{3.7d}$$

$$(DS(0)\cdot(D\phi(0))^{-1})_{12} = 0, \tag{3.7e}$$

$$(DS(0)\cdot(D\phi(0))^{-1})_{11} > 0. \tag{3.7f}$$

*Such unique* $S$ *is denoted by* $S[\phi]$. *Furthermore* $S[\phi] = (R[\phi])^{(-1)}$.

PROOF. By Lemma 2.2, $\phi$ is a homeomorphism of $\mathrm{cl}\,\mathscr{D}$ onto $\mathrm{cl}\,\phi(\mathscr{D})$. Then $\phi^{(-1)}$ exists and we can consider $R[\phi] \equiv \phi^{(-1)} \circ g[\phi]$. By the properties of the inverses of the elements of $C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2) \cap Y_0$, and of the composition of elements of class $C^{m+1,\alpha}$ (cf. Lanza (1991, Lemmas 3.1, 4.2)) and by Theorem 2.8, it easily follows that $R[\phi]$ is a homeomorphism of class $C^{m+1,\alpha}$ of $\mathrm{cl}\,\mathscr{D}$ onto $\mathrm{cl}\,\mathscr{D}$ and that conditions (3.6d, e, f) hold. Since $\det D\phi(x) \neq 0$, $\forall x \in \mathrm{cl}\,\mathscr{D}$, condition (2.10) implies that $\det D(R[\phi])(x) \neq 0$, $\forall x \in \mathrm{cl}\,\mathscr{D}$. Then by Lemma 2.1, we have $l[R[\phi]] > 0$. Since $R(\mathrm{cl}\,\mathscr{D}) = \mathrm{cl}\,\mathscr{D}$, Lemma 2.2 implies that $R[\phi](\partial\mathscr{D}) = \partial\mathscr{D}$. Since $g[\phi]$ is holomorphic in $\mathscr{D}$, Remark 3.3 implies that condition (3.6c) is satisfied. Conversely, let $R$ satisfy (3.6). By Lemma 2.2 and (3.6a), $R$ is a homeomorphism of $\mathrm{cl}\,\mathscr{D}$ onto $R(\mathrm{cl}\,\mathscr{D})$, and $R(\mathscr{D})$ is a nonempty bounded open subset of $\boldsymbol{R}^2$ such that $\partial R(\mathscr{D}) = R(\partial\mathscr{D})$. Then by (3.6b) we have $\partial R(\mathscr{D}) \subseteq \partial\mathscr{D}$, and thus the connectivity of $\mathscr{D}$ and (3.6d) imply that $R(\mathscr{D}) \cap \mathscr{D} = \mathscr{D}$. Since $R(\mathrm{cl}\,\mathscr{D}) \subseteq \mathrm{cl}\,\mathscr{D}$, the open set $R(\mathscr{D})$ coincides with $\mathscr{D}$ and $R(\mathrm{cl}\,\mathscr{D}) = R(\mathscr{D}) \cup R(\partial\mathscr{D}) = \mathscr{D} \cup \partial\mathscr{D}$. We also have $\phi\circ R(\mathscr{D}) = \phi(\mathscr{D})$, and by condition (3.6c) and Remark 3.3, $\phi\circ R$ is holomorphic and one to one from $\mathscr{D}$ to $\phi(\mathscr{D})$. By the argument principle, the complex derivative of $\phi\circ R$ cannot vanish in $\mathscr{D}$, and by the Inverse Function Theorem, the inverse of $\phi\circ R$ is also holomorphic. Then (3.6d, e, f) and the uniqueness of the Riemann map (cf. (2.3), (2.4)) imply that $R = R[\phi]$. Since $R[\phi]^{(-1)}$ is easily seen to satisfy the conditions for $S$ (we only note that (3.7e, f) follow from the combined use of (3.6c, e, f)), the existence of $S$ immediately follows from the first statement. To prove the uniqueness of $S$, let $S$ satisfy (3.7). By arguing as for $R$, with the only exception that the inclusion $S(\mathscr{D}) \subseteq \mathrm{cl}\,\mathscr{D}$ is now guaranteed by the boundedness of $S(\mathscr{D})$, by the connectivity of $\boldsymbol{R}^2 \backslash \mathrm{cl}\,\mathscr{D}$ and by condition $\partial S(\mathscr{D}) \cap (\boldsymbol{R}^2 \backslash \mathrm{cl}\,\mathscr{D}) = \emptyset$, we conclude that $S(\mathscr{D}) = \mathscr{D}$, that $S(\partial\mathscr{D}) = \partial\mathscr{D}$, and that $S$ is a homeomorphism of $\mathrm{cl}\,\mathscr{D}$ onto $\mathrm{cl}\,\mathscr{D}$. It is easily seen that $R \equiv S^{(-1)}$ satisfies (3.6) (we only note that (3.6e, f) follow from the combined use of (3.7c, e, f)). Then $S^{(-1)} = R[\phi]$, and the proof is complete. □

To study the regularity of $\phi \mapsto S[\phi]$, we need the following.

3.8. LEMMA. *Let* $\alpha\in]0, 1[$, $m\in\boldsymbol{N}$. *The set*

$$\mathcal{V}^{m,\alpha}\equiv\{v\in C^{m,\alpha}(\mathrm{cl}\,\mathcal{D}, M_2(\boldsymbol{R})):(I+L)(v)=0\},\tag{3.9}$$

*is a closed linear subspace of* $C^{m,\alpha}(\mathrm{cl}\,\mathcal{D}, M_2(\boldsymbol{R}))$. *For all elements* $v\in\mathcal{V}^{m,\alpha}$, *there exists* $\psi=(\psi_1, \psi_2)\in C^{m+1,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)$ *such that* $(I-L)(D\psi)=v$ *in* $\mathcal{D}$ *and* $\psi_1=0$ *on* $\partial\mathcal{D}$. *Such* $\psi$ *is uniquely determined up to an arbitrary purely imaginary constant.*

PROOF. It is trivial to verify that $\mathcal{V}^{m,\alpha}$ is a closed subspace of $C^{m,\alpha}(\mathrm{cl}\,\mathcal{D}, M_2(\boldsymbol{R}))$. Since $(I+L)(v)=0$, equation $(I-L)(D\psi)=v$ can be rewritten as

$$\begin{cases}\dfrac{\partial\psi_1}{\partial x_1}-\dfrac{\partial\psi_2}{\partial x_2}=v_{11}\\[2mm] \dfrac{\partial\psi_1}{\partial x_2}+\dfrac{\partial\psi_2}{\partial x_1}=v_{12}\end{cases}\quad\text{or as } 2\bar{\partial}(\psi_1+i\psi_2)=(v_{11}+iv_{12}).$$

Since $(v_{11}, v_{12})\in C^{m,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)$, it is well-known (cf. Vekua (1962, p. 56, Th. 1.32)) that there exists $h=h_1+ih_2$, $(h_1, h_2)\in C^{m+1,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)$ such that $2\bar{\partial}h=(v_{11}+iv_{12})$. Now by Privalov's Theorem (cf. *e.g.* Courant & Hilbert (1962, p. 401) together with Vekua (1962, p. 313)), there exists $(\gamma_1, \gamma_2)\in C^{m+1,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)$ such that $\bar{\partial}\gamma=0$ in $\mathcal{D}$ and $\gamma_1=-h_1$ on $\partial\mathcal{D}$. Then $\psi\equiv(\psi_1, \psi_2)\equiv(h_1+\gamma_1, h_2+\gamma_2)$ satisfies the requirements of the Lemma. The uniqueness statement is easily verified by standard properties of harmonic functions. □

3.10. THEOREM. *Let* $m\in\boldsymbol{N}$, $\alpha\in]0, 1[$. *The (nonlinear) map* $\phi\mapsto S[\phi]$ *is analytic from* $C^{m+1,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)\cap Y_0$ *into itself.*

PROOF. The idea is to apply the Implicit Function Theorem for analytic functions between real Banach spaces (cf. Prodi & Ambrosetti (1973, Th. 11.6)) to the equation

$$\Lambda[\phi, S]=0,$$

where $\Lambda$ is the map defined on the open subset

$$(\mathcal{F}_{m,\alpha}\times\mathcal{S}_{m,\alpha})^+\equiv\{(\phi, S)\in\mathcal{F}_{m,\alpha}\times\mathcal{S}_{m,\alpha}:(DS(0)\cdot(D\phi(0))^{-1})_{11}>0\}$$

of

$$\mathcal{F}_{m,\alpha}\times\mathcal{S}_{m,\alpha}\equiv(C^{m+1,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)\cap Y_0)\times(C^{m+1,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)\cap\{S\in Y_0: S(0)=0\})$$

to $\mathcal{V}^{m,\alpha}\times C^{m+1,\alpha}(\partial\mathcal{D})\times\boldsymbol{R}$, by the following equality

$$\Lambda[\phi, S]\equiv((I-L)(D\phi\cdot(DS)^{-1}),\ S_1^2+S_2^2-1,\ (DS(0)\cdot(D\phi(0))^{-1})_{12}).$$

Indeed, if $(\phi, S)$ belongs to the domain of $\Lambda$, then by Theorem 3.5, equality $S[\phi]=S$ holds if and only if $\Lambda[\phi, S]=0$. Furthermore, $(\phi, S[\phi])$ belongs to the domain of $\Lambda$ for all $\phi\in\mathcal{F}_{m,\alpha}$. We now check that the assumptions of the

Implicit Function Theorem are fulfilled. The membership of $(I-L)(D\phi\cdot(DS)^{-1})$ in $\mathcal{V}^{m,\alpha}$ when $(\phi, S)$ is in the domain of $\Lambda$ clearly follows from the identity $(I+L)\circ(I-L)=0$. The map $\Lambda$ is analytic because it is the composition of analytic maps (cf. Lemma 2.17). We still have to prove that for all $\phi_0\in\mathcal{F}_{m,\alpha}$, the Fréchet derivative $\Lambda_S[\phi_0, S[\phi_0]]$ of $S\mapsto\Lambda[\phi_0, S]$ at $S_0\equiv S[\phi_0]$ is a linear homeomorphism of $\mathcal{W}_{m,\alpha}\equiv C^{m+1,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)\cap\{S: S(0)=0\}$ onto $\mathcal{V}^{m,\alpha}\times C^{m+1,\alpha}(\partial\mathcal{D})\times\boldsymbol{R}$. By the open mapping Theorem, and by the continuity of $\Lambda_S[\phi_0, S[\phi_0]]$ it clearly suffices to show that $\Lambda_S[\phi_0, S[\phi_0]]$ is a bijection. By standard Calculus in Banach space (cf. Lemma 2.17), $\Lambda_S[\phi_0, S[\phi_0]]$ exists and is defined by

$$\text{(3.11)}\qquad \begin{aligned}\Lambda_S[\phi_0, S[\phi_0]](W) \equiv (&-(I-L)[D\phi_0\cdot(DS_0)^{-1}\cdot DW\cdot(DS_0)^{-1}],\\ &(2S_{01}W_1+2S_{02}W_2)_{|\partial\mathcal{D}}, (DW(0)\cdot(D\phi_0(0))^{-1})_{12}),\end{aligned}$$

where $S_0\equiv(S_{01}, S_{02})\equiv S[\phi_0]$, and $W\equiv(W_1, W_2)\in\mathcal{W}_{m,\alpha}$. We want to show that given $(V, h, c)\in\mathcal{V}^{m,\alpha}\times C^{m+1,\alpha}(\partial\mathcal{D})\times\boldsymbol{R}$, there exists a unique $W\in\mathcal{W}_{m,\alpha}$ such that

$$\Lambda_S[\phi_0, S[\phi_0]](W) = (V, h, c).$$

Now, equation

$$-(I-L)[D\phi_0\cdot(DS_0)^{-1}\cdot DW\cdot(DS_0)^{-1}] = V \quad \text{in } \mathrm{cl}\,\mathcal{D},$$

is equivalent to

$$\text{(3.12)}\qquad -(I-L)[D(\phi_0\circ S_0^{(-1)})\cdot D(W\circ S_0^{(-1)})] = V\circ S_0^{(-1)} \quad \text{in } \mathrm{cl}\,\mathcal{D}.$$

Since the function $\phi_0\circ S_0^{(-1)}(\cdot)$ is holomorphic in $\mathcal{D}$, and thus at least of class $C^2$ in $\mathcal{D}$, a direct computation shows that

$$\text{(3.13)}\qquad \begin{aligned}&(I-L)[D(\phi_0\circ S_0^{(-1)})\cdot D(W\circ S_0^{(-1)})]\\ &= (I-L)\{D[D(\phi_0\circ S_0^{(-1)})\cdot {}^t(W\circ S_0^{(-1)})]\} \quad \text{in } \mathcal{D}.\end{aligned}$$

Equation (3.12) holds in $\mathrm{cl}\,\mathcal{D}$ for some $W\in\mathcal{W}_{m,\alpha}$ if and only if it holds in $\mathcal{D}$, and thus it is equivalent to

$$\text{(3.14)}\qquad -(I-L)\{D[D(\phi_0\circ S_0^{(-1)})\cdot {}^t(W\circ S_0^{(-1)})]\} = V\circ S_0^{(-1)} \quad \text{in } \mathcal{D}.$$

Since $S_0(0)=0$ and $DS_0^{(-1)}(0)$ is invertible, we have

$$DW(0)\cdot(D\phi_0)^{-1}(0) = D(W\circ S_0^{(-1)})(0)\cdot(D(\phi_0\circ S_0^{(-1)})(0))^{-1}.$$

By definition of $S_0$, the map $\phi_0\circ S_0^{(-1)}$ is holomorphic in $\mathcal{D}$, and $(\phi_0\circ S_0^{(-1)})'(0)$ is a positive real number. Then

$$D(W\circ S_0^{(-1)})(0)\cdot(D(\phi_0\circ S_0^{(-1)})(0))^{-1} = \frac{1}{(\phi_0\circ S_0^{(-1)})'(0)}D(W\circ S_0^{(-1)})(0).$$

Now, we set

$$\varphi_0 = \varphi_1 + i\varphi_2, \quad (\varphi_1, \varphi_2) \equiv \phi_0 \circ S_0^{(-1)},$$

$$w = w_1 + iw_2, \quad (w_1, w_2) \equiv W \circ S_0^{(-1)},$$

$$v = v_1 + iv_2, \quad (v_1, v_2) \equiv \text{first row of the matrix } V \circ S_0^{(-1)},$$

$$h_0 \equiv h \circ S_0^{(-1)}.$$

The functions $\varphi_0$, $w$, $v$ can be regarded as complex-valued functions on the unit disk in $\boldsymbol{C}$. Then our linearized boundary-value problem takes the following complexified form

$$(3.15) \quad \begin{cases} 2\bar{\partial}\{\varphi_0'(z)w(z)\} = -v & \forall z \in \mathcal{D}, \\ 2\mathrm{Re}\dfrac{w(z)}{z} = h_0(z) & \forall z \in \partial\mathcal{D}, \\ w(0) = 0, & \\ (\partial_{x_2}\mathrm{Re}\, w)(0) = \varphi_0'(0)c. & \end{cases}$$

Since $(v, h_0, c) \in C^{m,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{C}) \times C^{m+1,\alpha}(\partial\mathcal{D}) \times \boldsymbol{R}$, $\varphi_0' \in C^{m,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{C})$, $\varphi_0' \neq 0$ in $\mathrm{cl}\,\mathcal{D}$, then by Lemma 3.8, there exists $\psi \in C^{m+1,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{C})$ such that

$$2\varphi_0'\bar{\partial}\psi = -v \quad \text{in } \mathcal{D}, \qquad \mathrm{Re}\,\psi = 0 \quad \text{on } \partial\mathcal{D}.$$

Now let $\tilde{\psi} \equiv \psi - \psi(0)$. Then we have

$$\begin{cases} 2\varphi_0'\bar{\partial}\tilde{\psi} = -v & \text{in } \mathcal{D}, \\ \mathrm{Re}\,\tilde{\psi} = -\mathrm{Re}\,\psi(0) & \text{on } \partial\mathcal{D}, \\ \tilde{\psi}(0) = 0, & \end{cases}$$

and the boundary-value problem for $w$ is clearly equivalent to the following boundary-value problem for $\tau(z) \equiv w(z) - \tilde{\psi}(z)$.

$$(3.16) \quad \begin{cases} \bar{\partial}\tau = 0 & \text{in } \mathcal{D}, \\ 2\mathrm{Re}\dfrac{\tau(z)}{z} = h_0(z) - 2\mathrm{Re}\dfrac{\tilde{\psi}(z)}{z} & \text{on } \partial\mathcal{D}, \\ \tau(0) = 0, & \\ (\partial_{x_2}\mathrm{Re}\,\tau)(0) = \varphi_0'(0)c - (\partial_{x_2}\mathrm{Re}\,\tilde{\psi})(0). & \end{cases}$$

Since $h_0(z) - 2\mathrm{Re}(\tilde{\psi}(z)/z) \in C^{m+1,\alpha}(\partial\mathcal{D})$, by Privalov's Theorem (cf. *e.g.* Courant & Hilbert (1962, p. 401) together with Vekua (1962, p. 313)) there exists a (unique) $\chi \in C^{m+1,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{C})$ such that

$$\begin{cases} \bar{\partial}\chi = 0 & \text{in } \mathcal{D}, \\ 2\mathrm{Re}\,\chi(z) = h_0(z) - 2\mathrm{Re}\dfrac{\tilde{\psi}(z)}{z} & \text{on } \partial\mathcal{D}, \\ -\mathrm{Im}\,\chi(0) = \varphi_0'(0)c - (\partial_{x_2}\mathrm{Re}\,\tilde{\psi})(0). & \end{cases}$$

By setting $\tau(z)=z\chi(z)$, it is easily seen that $\tau\in C^{m+1,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{C})$ solves (3.16) and consequently, that $w(z)\equiv z\chi(z)+\tilde{\phi}(z)$ solves (3.15). We now prove the uniqueness for (3.15). If $w^*, w^{**}\in C^{m+1,\alpha}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{C})$ are two solutions of (3.15), then the map $\sigma\equiv(w^*-w^{**})/z$ can be extended to a holomorphic function on $\mathcal{D}$, and $\mathrm{Re}\,\sigma=0$ on $\partial\mathcal{D}$. Then $\mathrm{Re}\,\sigma=0$ in $\mathrm{cl}\,\mathcal{D}$, and $\sigma=ik$ for some real constant $k$. Since $(\partial_{x_2}\mathrm{Re}(z\sigma))(0)=\partial_{x_2}(-kx_2)=-k$, we conclude that $\sigma=0$. □

## 4. Differentiability Theorems for the conformal representation.

We first introduce the following notation. Let $m\in\boldsymbol{N}$, $\alpha\in]0, 1[$. We denote $C^{m,\alpha,0}(\mathrm{cl}\,\mathcal{D})$ the (well-known) subspace of those $f\in C^{m,\alpha}(\mathrm{cl}\,\mathcal{D})$ such that

$$\lim_{\delta\to 0}\delta^{-\alpha}\sup\{|D^{\eta}f(x)-D^{\eta}f(y)| : |x-y|\leqq\delta,\ x, y\in\mathrm{cl}\,\mathcal{D}\}=0,$$

for all $\eta\in\boldsymbol{N}^n$ such that $|\eta|=m$. It can be proved (cf. Lanza (1994, (2.19c), Lemma 2.20 (iv))), that $C^{m,\alpha,0}(\mathrm{cl}\,\mathcal{D})$ coincides with the closure in $C^{m,\alpha}(\mathrm{cl}\,\mathcal{D})$ of the set $\mathcal{P}(\boldsymbol{R}^2)$ of the polynomials in two real variables. It is immediate to verify that if $0<\alpha<\beta\leqq 1$, then $C^{m,\beta}(\mathrm{cl}\,\mathcal{D})$ is contained in $C^{m,\alpha,0}(\mathrm{cl}\,\mathcal{D})$. Since $C^{m,\beta}(\mathrm{cl}\,\mathcal{D})$ is compactly imbedded in $C^{m,\alpha}(\mathrm{cl}\,\mathcal{D})$, the imbedding of $C^{m,\beta}(\mathrm{cl}\,\mathcal{D})$ into $C^{m,\alpha,0}(\mathrm{cl}\,\mathcal{D})$ is also compact. Since $g[\phi]=\phi\circ S[\phi]^{(-1)}$, and $S[\cdot]$ is analytic, it is clear that the regularity properties of $g[\cdot]$ are entirely determined by those of the operator which takes a function into its inverse and by that which takes a pair of functions into the composite of the two. The following two theorems state the differentiability properties for the composition and for the inversion operator that we need. We now introduce the differentiability Theorem for the composition operator. For a proof we refer to Lanza (1994, Lemma 2.20 (iv), (v), Theorems 3.3, 4.19).

4.1. THEOREM. *Let $\alpha, \beta\in]0, 1[$, $m\in\boldsymbol{N}$, $\gamma_m(\alpha, \beta)\equiv\min\{\alpha, \beta\}$ if $m>0$, $\gamma_0(\alpha, \beta)\equiv\alpha\beta$. Then the (composition) map $T$ defined by $T[f, h]=f\circ h$ satisfies the following.*

(i) *$T$ is continuous from $C^{m,\alpha,0}(\mathrm{cl}\,\mathcal{D})\times C^{m,\beta}(\mathrm{cl}\,\mathcal{D}, \mathrm{cl}\,\mathcal{D})$ to $C^{m,\gamma_m(\alpha,\beta)}(\mathrm{cl}\,\mathcal{D})$.*

(ii) *If $r\in\boldsymbol{N}\setminus\{0\}$, then $T$ as a map of $C^{m+r,\alpha,0}(\mathrm{cl}\,\mathcal{D})\times C^{m,\beta}(\mathrm{cl}\,\mathcal{D}, \mathrm{cl}\,\mathcal{D})$ to $C^{m,\gamma_m(\alpha,\beta)}(\mathrm{cl}\,\mathcal{D})$ is the restriction of an operator $\hat{T}$ of class $C^r$ defined in an open neighborhood of $C^{m+r,\alpha,0}(\mathrm{cl}\,\mathcal{D})\times C^{m,\beta}(\mathrm{cl}\,\mathcal{D}, \mathrm{cl}\,\mathcal{D})$ in the space $C^{m+r,\alpha,0}(\mathrm{cl}\,\mathcal{D})\times C^{m,\beta}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2)$. The differential of $\hat{T}$ at each $(f_0, h_0)\in C^{m+r,\alpha,0}(\mathrm{cl}\,\mathcal{D})\times C^{m,\beta}(\mathrm{cl}\,\mathcal{D}, \mathrm{cl}\,\mathcal{D})$ is delivered by the formula*

$$d\hat{T}[f_0, h_0](v, w)=v\circ h_0+\sum_{l=1}^{2}\frac{\partial f_0}{\partial y_l}(h_0)w_l, \tag{4.2}$$

$$\forall (v, w\equiv(w_1, w_2))\in C^{m+r,\alpha,0}(\mathrm{cl}\,\mathcal{D})\times C^{m,\beta}(\mathrm{cl}\,\mathcal{D}, \boldsymbol{R}^2).$$

We now have the following differentiability Theorem for the 'inversion' operator. For a proof, we refer the reader to Lanza (1994, Lemmas 2.20 (iv), (v), 5.7, Theorems 5.9, 5.13).

4.3. THEOREM. *Let $m\in\boldsymbol{N}$, $\alpha\in]0,1[$, $\delta>0$. Let $J$ be the (nonlinear) map of $C^{m,\alpha}(\mathrm{cl}\,\mathcal{D},\mathrm{cl}\,\mathcal{D})\cap\{f\in Y_0: f(\mathrm{cl}\,\mathcal{D})=\mathrm{cl}\,\mathcal{D}\}$ to $C^{m,\alpha}(\mathrm{cl}\,\mathcal{D},\mathrm{cl}\,\mathcal{D})$ defined by $J[f]\equiv f^{(-1)}$. Then the following two statements hold.*

(i) *If $m>0$, then $J$ maps bounded subsets of $C^{m,\alpha}(\mathrm{cl}\,\mathcal{D},\mathrm{cl}\,\mathcal{D})\cap\{f\in Y_\delta: f(\mathrm{cl}\,\mathcal{D})=\mathrm{cl}\,\mathcal{D}\}$ into bounded subsets of $C^{m,\alpha}(\mathrm{cl}\,\mathcal{D},\mathrm{cl}\,\mathcal{D})$, and is continuous from $C^{m,\alpha,0}(\mathrm{cl}\,\mathcal{D},\mathrm{cl}\,\mathcal{D})\cap\{f\in Y_0: f(\mathrm{cl}\,\mathcal{D})=\mathrm{cl}\,\mathcal{D}\}$ to $C^{m,\alpha,0}(\mathrm{cl}\,\mathcal{D},\mathrm{cl}\,\mathcal{D})$.*

(ii) *Let $r\in\boldsymbol{N}\setminus\{0\}$, $m\geqq 0$. Let $J$ be defined from $C^{m+r,\alpha,0}(\mathrm{cl}\,\mathcal{D},\mathrm{cl}\,\mathcal{D})\cap\{f\in Y_0: f(\mathrm{cl}\,\mathcal{D})=\mathrm{cl}\,\mathcal{D}\}$ to $C^{m,\alpha,0}(\mathrm{cl}\,\mathcal{D},\mathrm{cl}\,\mathcal{D})$ and let $f_0$ be in the domain of $J$. Then there exists an open neighborhood $\mathcal{W}_{f_0}$ of $f_0$ in the normed space $C^{m+r,\alpha,0}(\mathrm{cl}\,\mathcal{D},\boldsymbol{R}^2)$, and an operator $\hat{J}_{f_0}$ of class $C^r$ from $\mathcal{W}_{f_0}$ to $C^{m,\alpha,0}(\mathrm{cl}\,\mathcal{D},\boldsymbol{R}^2)$ such that*

$$\hat{J}_{f_0}[f]=J[f],\quad \forall f\in\mathcal{W}_{f_0}\cap\{f\in Y_0: f(\mathrm{cl}\,\mathcal{D})=\mathrm{cl}\,\mathcal{D}\}. \tag{4.4a}$$

*The differential of $\hat{J}$ at $f\in\mathcal{W}_{f_0}\cap\{f\in Y_0: f(\mathrm{cl}\,\mathcal{D})=\mathrm{cl}\,\mathcal{D}\}$ is delivered by the formula*

$$ {}^t[d\hat{J}_{f_0}[f](h)]=-[(Df)^{-1}\circ(f^{(-1)})]\cdot{}^t(h\circ f^{(-1)}),\quad \forall h\in C^{m+r,\alpha,0}(\mathrm{cl}\,\mathcal{D},\boldsymbol{R}^2). \tag{4.4b}$$

We now show that the following holds.

4.5. LEMMA. *Let $m\in\boldsymbol{N}\setminus\{0\}$, $\alpha\in]0,1[$. If $\phi\in C^{m,\alpha,0}(\mathrm{cl}\,\mathcal{D},\boldsymbol{R}^2)\cap Y_0$, then $S[\phi]\in C^{m,\alpha,0}(\mathrm{cl}\,\mathcal{D},\boldsymbol{R}^2)$.*

PROOF. If $\phi\in C^{m,\alpha,0}(\mathrm{cl}\,\mathcal{D},\boldsymbol{R}^2)\cap Y_0$, then as observed above, $\phi$ is a limit in the $\|\cdot\|_{m,\alpha}$-norm of a sequence $\{p_n\}$ of pairs of functions with polynomial components. By Theorem 3.5 and by the well-known inclusion of $C^\infty(\mathrm{cl}\,\mathcal{D})$ in $C^{k,\alpha}(\mathrm{cl}\,\mathcal{D})$, $\forall k\in\boldsymbol{N}$, we have $S[p_n]\in C^\infty(\mathrm{cl}\,\mathcal{D},\boldsymbol{R}^2)$. By Theorem 3.10, we have $S[\phi]=\lim_n S[p_n]$ in $C^{m,\alpha}(\mathrm{cl}\,\mathcal{D},\boldsymbol{R}^2)$. Since $\mathrm{cl}\,\mathcal{D}$ is a manifold with boundary of class $C^\infty$, all functions of class $C^{m+1}$ in $\mathrm{cl}\,\mathcal{D}$ are restriction of some element of $C^{m+1}(\boldsymbol{R}^2)$ (cf. *e.g.* Troianiello (1987, p. 13)). Then by Weierstrass Theorem (cf. *e.g.* Rohlin & Fuchs (1981, p. 185)), all elements of $C^\infty(\mathrm{cl}\,\mathcal{D})$ can be approximated in the $\|\cdot\|_{m+1}$-norm by polynomials. Since $C^{m+1}(\mathrm{cl}\,\mathcal{D})$ is compactly imbedded in $C^{m,\alpha}(\mathrm{cl}\,\mathcal{D})$, we conclude that $S[\phi]\in C^{m,\alpha,0}(\mathrm{cl}\,\mathcal{D},\boldsymbol{R}^2)$. □

By Theorem 3.5 we have

$$g[\phi]=T[\phi, J\circ S[\phi]],\quad \forall\phi\in C^{m,\alpha}(\mathrm{cl}\,\mathcal{D},\boldsymbol{R}^2)\cap Y_0, \tag{4.6}$$

when $m>0$. Furthermore, we note that by Theorem 4.1 (ii), and by the characterization of $C^{m,\alpha,0}(\mathrm{cl}\,\mathcal{D})$ as the closure of $\mathcal{P}(\boldsymbol{R}^2)$ in $C^{m,\alpha}(\mathrm{cl}\,\mathcal{D})$ mentioned above, we have $T[f, h] \in C^{m, r_m(\alpha, \beta), 0}(\mathrm{cl}\, \mathscr{D})$ if $(f, h) \in C^{m+r, \alpha, 0}(\mathrm{cl}\, \mathscr{D}) \times C^{m, \beta, 0}(\mathrm{cl}\, \mathscr{D}, \mathrm{cl}\, \mathscr{D})$. Then Theorems 3.10, 4.1, 4.3, and Lemma 4.5 imply the validity of the following.

4.7. THEOREM. *Let $r, m \in \boldsymbol{N}$, $\alpha \in ]0, 1[$. Then the following two statements hold.*

(i) *If $r > 0$, then $g[\cdot]$ is of class $C^r$ from $C^{r, \alpha, 0}(\mathrm{cl}\, \mathscr{D}, \boldsymbol{R}^2) \cap Y_0$ to $C^{0, \alpha^2, 0}(\mathrm{cl}\, \mathscr{D}, \boldsymbol{R}^2)$.*

(ii) *If $m > 0$, $r \geqq 0$, then $g[\cdot]$ is of class $C^r$ from $C^{m+r, \alpha, 0}(\mathrm{cl}\, \mathscr{D}, \boldsymbol{R}^2) \cap Y_0$ to $C^{m, \alpha, 0}(\mathrm{cl}\, \mathscr{D}, \boldsymbol{R}^2)$.*

By Theorem 4.7, we immediately deduce the validity of the following.

4.8. THEOREM. *Let $r, m \in \boldsymbol{N}$, $\alpha \in ]0, 1[$, and let $I$ be an open interval of $\boldsymbol{R}$. Let $\Phi(\cdot)$ be a map of class $C^r$ from $I$ to $C^{m+r, \alpha, 0}(\mathrm{cl}\, \mathscr{D}, \boldsymbol{R}^2) \cap Y_0$. Then the following hold.*

(i) *If $m = 0$ and $r > 0$, then $g[\Phi(\cdot)]$ is of class $C^r$ from $I$ to $C^{0, \alpha^2, 0}(\mathrm{cl}\, \mathscr{D}, \boldsymbol{R}^2)$.*

(ii) *If $m > 0$ and $r \geqq 0$, then $g[\Phi(\cdot)]$ is of class $C^r$ from $I$ to $C^{m, \alpha, 0}(\mathrm{cl}\, \mathscr{D}, \boldsymbol{R}^2)$.*

Then, by the inclusion of $C^\infty(\mathrm{cl}\, \mathscr{D})$ in $C^{m, \alpha, 0}(\mathrm{cl}\, \mathscr{D})$, we have following.

4.9. THEOREM. *Let $\alpha \in ]0, 1[$, $m \in \boldsymbol{N}$, and let $I$ be an open subset of $\boldsymbol{R}$. Let $\Phi$ be a map of $I$ to $C^\infty(\mathrm{cl}\, \mathscr{D}, \boldsymbol{R}^2) \cap Y_0$. Then the following hold.*

(i) *If $\Phi$ is of class $C^r$ from $I$ to $(C^\infty(\mathrm{cl}\, \mathscr{D}, \boldsymbol{R}^2) \cap Y_0, \|\cdot\|_{r, \alpha})$, $\forall r \in \boldsymbol{N} \setminus \{0\}$, then $g[\Phi(\cdot)]$ is of class $C^\infty$ from $I$ to $(C^\infty(\mathrm{cl}\, \mathscr{D}, \boldsymbol{R}^2) \cap Y_0, \|\cdot\|_{0, \alpha^2})$.*

(ii) *If $m > 0$, and if $\Phi$ is of class $C^r$ from $I$ to $(C^\infty(\mathrm{cl}\, \mathscr{D}, \boldsymbol{R}^2) \cap Y_0, \|\cdot\|_{m+r, \alpha})$, $\forall r \in \boldsymbol{N}$, then $g[\Phi(\cdot)]$ is of class $C^\infty$ from $I$ to $(C^\infty(\mathrm{cl}\, \mathscr{D}, \boldsymbol{R}^2) \cap Y_0, \|\cdot\|_{m, \alpha})$.*

4.10. REMARK. A simple example of $\Phi$ in Theorem 4.8 is given by $\Phi(\varepsilon) = \phi_0 + \varepsilon u_0$, where $\phi_0 \in C^{m+r, \alpha, 0}(\mathrm{cl}\, \mathscr{D}, \boldsymbol{R}^2) \cap Y_0$, $u_0 \in C^{m+r, \alpha, 0}(\mathrm{cl}\, \mathscr{D}, \boldsymbol{R}^2)$, and $I \equiv \{\varepsilon \in \boldsymbol{R} : \phi_0 + \varepsilon u_0 \in Y_0\}$. More general $\Phi$'s can be considered in the form $\Phi(\varepsilon) = m(\varepsilon, \cdot)$, where $m$ is a sufficiently regular function of $I \times \mathrm{cl}\, \mathscr{D}$ to $\boldsymbol{R}^2$. We do not pursue in this paper an investigation of the appropriate conditions on $m$ which ensure the fulfillment of the assumptions on $\Phi$ of Theorem 4.8. Similar considerations can be done for $\Phi$ in Theorem 4.9.

4.11. REMARK. We now briefly illustrate how the differentiability statements of this section for $\phi \mapsto g[\phi]$, can be used to study the dependence of the spectrum of the Laplacian upon perturbation of the domain. Let $\Omega$ be a Jordan domain bounded by a $C^{m, \alpha}$ curve with nonvanishing tangent vector. As shown in Lemma 2.13, we can represent $\Omega$ as $\phi(\mathscr{D})$, for some $\phi \in C^{m, \alpha}(\mathrm{cl}\, \mathscr{D}, \boldsymbol{R}^2)$. As well-known, $u$ is an eigenfunction with eigenvalue $\lambda$ for the Laplace operator in $\Omega$ with Dirichlet boundary conditions if $\Delta u - \lambda u = 0$ in $\Omega$, $u = 0$ on $\partial\Omega$. By setting $v=u\circ g[\phi]$, $v$ satisfies $\Delta v-\lambda|g'[\phi]|^2 v=0$ in $\mathscr{D}$, $v=0$ on $\partial\mathscr{D}$. Thus we have transformed a problem for $u$, $\lambda$ on the variable domain $\Omega$, into a problem for $v$, $\lambda$ on the fixed domain $\mathscr{D}$ and with coefficient $|g'[\phi]|^2$ determined by the conformal representation $g[\phi]$ of $\Omega$. It is then clear that the dependence of the eigenvalues of $\Delta$ in $\phi(\mathscr{D})$ on the function $\phi$ which represents it, is dictated by the dependence of $|g'[\phi]|^2$, and thus of $g[\phi]$ on $\phi$. We will pursue such investigation in a forthcoming paper.

## 5. A necessary condition for the conformal representation to be differentiable at a function.

According to Theorem 4.7, if $m>0$ the conformal representation $g[\cdot]$ is differentiable from $C^{m+1,\alpha,0}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)\cap Y_0$ to $C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$. Since by Theorem 2.8, $g[\cdot]$ maps elements of $C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)\cap Y_0$ to elements of $C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$, one may very well wonder whether the operator $g[\cdot]$ is differentiable from $C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)\cap Y_0$ to $C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$, and whether it is really necessary to restrict the domain of $g[\cdot]$ and to strengthen its topology in order to obtain a differentiability result as Theorem 4.7. The following "inverse result" somewhat clarifies the situation. We first introduce the following elementary Lemma, which is of immediate verification.

5.1. LEMMA. *Let $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ be normed spaces. Let $\mathscr{Y}$ be imbedded in $\mathscr{Z}$, and let $j$ be the identity map of $\mathscr{Y}$ into $\mathscr{Z}$. Let $\mathscr{O}$ be an open subset of $\mathscr{X}$, $p\in\mathscr{O}$, $f$ a map of $\mathscr{O}$ to $\mathscr{Y}$. If $j\circ f$ is differentiable from $\mathscr{O}$ to $\mathscr{Z}$ at $p$, with differential $L$, and if $f$ is differentiable at $p$, then $df(p)(u)=Lu$, $\forall u\in\mathscr{X}$.*

Then we have the following.

5.2. THEOREM. *Let $m\in\boldsymbol{N}\setminus\{0\}$, $\alpha\in]0,1[$. If $g[\cdot]$ is differentiable from $C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)\cap Y_0$ to $C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$ at $\phi_0$, then $\phi_0\circ R[\phi_0]\in C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$. (In other words, the differentiabillity of $g[\cdot]$ at $\phi_0$, implies that the conformal representation $g[\phi_0]$ of $\phi_0(\mathscr{D})$ is of class $C^{m+1,\alpha}$.)*

PROOF. We first show that the differential of $g[\cdot]$ at $\phi_0$ must be delivered by the formula.

$$(5.3)\qquad {}^t[dg[\phi_0](u)] = D\phi_0(R[\phi_0])\cdot {}^t[dR[\phi_0](u)]+{}^t u(R[\phi_0]),\quad \forall u\in C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2).$$

Let $0<\beta<\alpha$. If $m\geqq 2$, Theorem 4.7 (ii) implies that $g[\cdot]$ is differentiable from $C^{m,\beta,0}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)\cap Y_0$ to $C^{m-1,\beta}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$. Then (4.2), (4.4), (4.6) and the chain rule imply that (5.3) holds $\forall u\in C^{m,\beta,0}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$. Since $C^{m,\alpha}(\mathrm{cl}\,\mathscr{D})$ is imbedded in $C^{m,\beta,0}(\mathrm{cl}\,\mathscr{D})$, and $C^{m,\alpha}(\mathrm{cl}\,\mathscr{D})$ is imbedded in $C^{m-1,\beta}(\mathrm{cl}\,\mathscr{D})$, Lemma 5.1 implies that equality (5.3) holds when $g[\cdot]$ operates from $C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)\cap Y_0$ to $C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$. Similarly, if $m=1$, Theorem 4.7 (i) implies that $g[\cdot]$ is differentiable from $C^{1,\beta,0}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)\cap Y_0$ to $C^{0,\beta^2}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$, and Lemma 5.1 implies that (5.3) holds when $g[\cdot]$ operates from $C^{1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)\cap Y_0$ to $C^{1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$. Since the matrix $(DR[\phi_0])$ is nowhere singular and $u(R[\phi_0])\in C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$, (5.3) implies that for $m\geqq 1$

$$(5.4)\qquad D(\phi_0\circ R[\phi_0])\cdot\{(DR[\phi_0])^{-1}\cdot{}^t[dR[\phi_0](u)]\}\in C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2),\quad \forall u\in C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2).$$

Next, we show that

$$(5.5)\qquad (DR[\phi_0])^{-1}\cdot{}^t[dR[\phi_0](C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2))]\supseteqq\{{}^t\omega_\psi:\psi\in C^{m,\alpha}(\mathrm{cl}\,\mathscr{D})\},$$

$$\text{where}\quad \omega_\psi(x_1, x_2)\equiv(-x_2\psi(x_1, x_2),\ x_1\psi(x_1, x_2)).$$

We explain below how (5.5) implies that $\phi_0\circ R[\phi_0]$ belongs to $C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$. Since $R$ is the composite of $J$, $S$ the same argument above used to prove (5.3), based on Lemma 5.1, Theorems 3.10, 4.3, implies that

$$(5.6)\qquad {}^t\{dR[\phi_0](u)\}=-(DR[\phi_0])\cdot{}^t\{[dS[\phi_0](u)]\circ R[\phi_0]\},\quad \forall u\in C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2),$$

which in turn implies that (5.5) is equivalent to

$$(5.7)\qquad dS[\phi_0](C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2))\circ R[\phi_0]\supseteqq\{\omega_\psi:\psi\in C^{m,\alpha}(\mathrm{cl}\,\mathscr{D})\}.$$

As we have seen in the proof of Theorem 3.10, (5.7) is equivalent to the following. For all $\psi\in C^{m,\alpha}(\mathrm{cl}\,\mathscr{D})$ there exists $u\in C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$ such that

$$(5.8)\qquad \Lambda_S[\phi_0, S[\phi_0]](W)=-\Lambda_\phi[\phi_0, S[\phi_0]](u),$$

where $W\equiv\omega_\psi\circ R[\phi_0]^{(-1)}$. Clearly $W\in\mathscr{W}_{m-1,\alpha}$ and

$$S[\phi_0]\cdot W=[id_{\mathrm{cl}\,\mathscr{D}}\cdot\omega_\psi]\circ S[\phi_0]=0,$$

where $id_{\mathrm{cl}\,\mathscr{D}}$ denotes the identity map in $\mathrm{cl}\,\mathscr{D}$. Then, by (3.11) and by standard calculus, equation (5.8) is equivalent to

$$(5.9)\qquad (I-L)[D\phi_0\cdot(DS[\phi_0])^{-1}\cdot DW\cdot(DS[\phi_0])^{-1}]=(I-L)[Du\cdot(DS[\phi_0])^{-1}]\quad\text{in }\mathscr{D},$$

$$\{DW(0)\cdot(D\phi_0(0))^{-1}\}_{12}=\{DS[\phi_0](0)\cdot(D\phi_0(0))^{-1}\cdot(Du(0))\cdot(D\phi_0(0))^{-1}\}_{12}.$$

Since $D(\phi_0\circ S[\phi_0]^{(-1)})(0)$ is a diagonal invertible matrix, (5.9) is easily seen to be equivalent to

$$(5.10\text{a})\qquad (I-L)\{D(\phi_0\circ S[\phi_0]^{(-1)})\cdot D(W\circ S[\phi_0]^{(-1)})\}=(I-L)(D(u\circ S[\phi_0]^{(-1)}))\quad\text{in }\mathscr{D},$$

$$(5.10\text{b})\qquad \{D(W\circ S[\phi_0]^{(-1)})(0)-[D(\phi_0\circ S[\phi_0]^{(-1)})(0)]^{-1}\cdot D(u\circ S[\phi_0]^{(-1)})(0)\}_{12}=0.$$

As in the proof of Theorem 3.10, by setting

$$\varphi_0 \equiv \varphi_{01}+i\varphi_{02}, \quad (\varphi_{01}, \varphi_{02}) \equiv \phi_0 \circ S[\phi_0]^{(-1)},$$

$$w \equiv w_1+iw_2, \quad (w_1, w_2) \equiv W \circ S[\phi_0]^{(-1)},$$

$$\mu \equiv \mu_1+i\mu_2, \quad (\mu_1, \mu_2) \equiv u \circ S[\phi_0]^{(-1)},$$

and by exploiting assumption $\varphi_0'(0) \in ]0, \infty[$ it is easy to see that (5.10) can be rewritten as

$$\text{(5.11)} \qquad \begin{cases} \bar{\partial}(\varphi_0' w) = \bar{\partial}\mu & \text{in } \mathscr{D}, \\ \varphi_0'(0)\partial_{x_2}\operatorname{Re} w(0) = \partial_{x_2}\operatorname{Re}\mu(0). \end{cases}$$

Since $\bar{\partial}(\varphi_0' w)=\varphi_0'\bar{\partial}w \in C^{m-1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$, equation $\bar{\partial}f=\bar{\partial}(\varphi_0' w)$ has at least a solution $f \in C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$. In fact, we can choose such $f$ uniquely by imposing $\operatorname{Re} f=0$ on $\partial\mathscr{D}$, $\operatorname{Im} f(0)=0$ (cf. Lemma 3.8). We can easily obtain a solution of system (5.11), by setting $\mu(z)=icz+f(z)$ where $c$ is the unique real number determined by $\partial_{x_2}\operatorname{Re}[ic(x_1+ix_2)](0)=\varphi_0'(0)\partial_{x_2}\operatorname{Re} w(0)-\partial_{x_2}\operatorname{Re} f(0)$ and the proof of (5.7) is complete. We now show that (5.4) and (5.7) imply that $\phi_0 \circ R[\phi_0] \in C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$. A straightforward computation shows that the $2\times 1$ real matrix

$$\text{(5.12)} \qquad D(\phi_0 \circ R[\phi_0]) \cdot {}^t(-x_2\psi(x_1, x_2), x_1\psi(x_1, x_2))$$

equals the first column of the matrix

$$\text{(5.13)} \qquad \psi\left\{D(\phi_0 \circ R[\phi_0]) \cdot \begin{pmatrix} -x_2 & -x_1 \\ x_1 & -x_2 \end{pmatrix}\right\}.$$

Since $\phi_0 \circ R[\phi_0](\cdot)$ is holomorphic in $\mathscr{D}$, the entries of the second column of the matrix (5.13) are, up to a factor $(-1)$ equal to the entries of the first column of the same matrix. Then conditions (5.4) and (5.5) together with the arbitrariness of $\psi \in C^{m,\alpha}(\mathrm{cl}\,\mathscr{D})$ imply that

$$D(\phi_0 \circ R[\phi_0]) \cdot \begin{pmatrix} -x_2 & -x_1 \\ x_1 & -x_2 \end{pmatrix} \in C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, M_2(\boldsymbol{R})).$$

Since the determinant of the $C^\infty$ matrix $\begin{pmatrix} -x_2 & -x_1 \\ x_1 & -x_2 \end{pmatrix}$ vanishes only at $(x_1, x_2)=(0, 0)$, and since the jacobian matrix $D(\phi_0 \circ R[\phi_0])$ of the holomorphic function $\phi_0 \circ R[\phi_0]$ is of class $C^\infty$ in a neighborhood of $(0, 0)$, we conclude that $D(\phi_0 \circ R[\phi_0]) \in C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, M_2(\boldsymbol{R}))$, and that consequently $\phi_0 \circ R[\phi_0] \in C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$. □

Since we have seen that the differentiability of $g[\cdot]$ at $\phi_0$ implies that $\phi_0 \circ R[\phi_0]$ belongs to $C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$, we now confine our attention to a vector subspace of $C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$ of elements $\phi$ such that $\phi \circ R[\phi] \in C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$. Since by Theorem 2.8, $\phi \in C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$ and $l[\phi]>0$ imply that $g[\phi] \in$

$C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$, we now consider $C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$ and prove the following, which clarifies why the spaces $C^{m,\alpha,0}$ have been introduced.

5.14. PROPOSITION. *Let* $\alpha\in]0, 1[$, $m\in\boldsymbol{N}\setminus\{0\}$. *Let* $g[\cdot]$ *be differentiable at* $\phi_0\in C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)\cap Y_0$ *as a map of* $C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)\cap Y_0$ *to* $C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$. *Then* $\phi_0\circ R[\phi_0]\in C^{m+1,\alpha,0}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)\cap Y_0$.

PROOF. Let $u\in C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$ and $u\neq 0$ (*i.e.* $u$ not identically zero). Then the assumption of differentiability of $g[\cdot]$ at $\phi_0$ implies that

$$\lim_{\varepsilon\to 0}\left\|\frac{g[\phi_0+\varepsilon u]-g[\phi_0]}{\varepsilon\|u\|_{m+1,\alpha}}-dg[\phi_0]\Big(\frac{u}{\|u\|_{m+1,\alpha}}\Big)\right\|_{m,\alpha}=0.$$

By Theorem 2.8 and by assumption $\phi_0\in C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)\cap Y_0$ and $u\in C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$, we have

$$\frac{g[\phi_0+\varepsilon u]-g[\phi_0]}{\varepsilon\|u\|_{m+1,\alpha}}\in C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2),$$

when $\varepsilon$ is small and nonzero. Then $dg[\phi_0](u/\|u\|_{m+1,\alpha})$ can be approximated in the $\|\cdot\|_{m,\alpha}$-norm by elements of class $C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$, which can be approximated in the $\|\cdot\|_{m,\alpha}$-norm by pairs of polynomials by virtue of the imbedding of $C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$ in $C^{m,\alpha,0}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$. Then we have $dg[\phi_0](u)\in C^{m,\alpha,0}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$. By formula (5.3), we have

$$\begin{aligned}{}^t[dg[\phi_0](u)] &= D\phi_0(R[\phi_0])\cdot{}^t[dR[\phi_0](u)]+{}^tu(R[\phi_0]),\\ &\qquad\qquad\qquad\forall u\in C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2).\end{aligned}$$

As in the proof of Theorem 5.2, the membership of $D(\phi_0\circ R[\phi_0])\in C^{m,\alpha,0}(\mathrm{cl}\,\mathscr{D}, M_2(\boldsymbol{R}))$ can be deduced from the following

$$(5.15)\quad (DR[\phi_0])^{-1}\cdot{}^t\{dR[\phi_0](C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2))\}\supseteqq\{{}^t\omega_\psi:\psi\in C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)\}.$$

Indeed $\phi_0\circ R[\phi_0]$ is of class $C^\infty$ in a neighborhood of (0, 0) and $C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$ is imbedded in $C^{m,\alpha,0}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{R}^2)$. As in the proof of Theorem 5.2, the proof of (5.15) is easily reduced to the following. For all $\psi\in C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D})$, there exists $\mu\in C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{C})$ such that

$$\begin{cases}\bar\partial\mu=\bar\partial(\varphi_0'w) & \text{in } \mathscr{D},\\ \partial_{x_2}\mathrm{Re}\,\mu(0)=\varphi_0'(0)\partial_{x_2}\mathrm{Re}\,w(0), & \end{cases}$$

where

$$\varphi_0=\varphi_{01}+i\varphi_{02},\quad (\varphi_{01},\varphi_{02})\equiv\phi_0\circ S[\phi_0]^{(-1)}$$

$$w=w_1+iw_2,\qquad (w_1, w_2)\equiv(-x_2\psi(x_1, x_2), x_1\psi(x_1, x_2)).$$

The existence of such $\mu \in C^{m+1,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{C})$ follows as in the proof of Theorem 5.2 because $\bar\partial(\varphi_0' w) = \varphi_0' \bar\partial w \in C^{m,\alpha}(\mathrm{cl}\,\mathscr{D}, \boldsymbol{C})$. □

## References


Berger, M.S. (1977), Nonlinearity and Functional Analysis, Academic Press.

Coifman, R. and Meyer, Y. (1983), Lavrentiev's Curves and Conformal Mappings, Institut Mittag-Leffler, Report No. 5.

Courant R. and Hilbert, D. (1962), Methods of Mathematical Physics, II, Interscience.

Gaier, D. (1964), Konstruktive Methoden der konformen Abbildung, Springer-Verlag.

Glaeser, G. (1958), Étude de quelques algèbres Tayloriennes, J. Analyse Math., **6**, 1–124.

Hurewicz, W. and Wallman, H. (1948), Dimension theory, Princeton University Press.

Kantorovich, L.V. and Krylov, V.I. (1964), Approximate Methods of Higher Analysis, Interscience Publishers.

Kufner, A., John, O. and Fučik, S. (1977), Function Spaces, Noordhoff International Publishing.

Lanza de Cristoforis, M. (1991), Properties and Pathologies of the Composition and Inversion Operator in Schauder Spaces, Acc. Naz. delle Sci. detta dei XL, **15**, 93–109.

Lanza de Cristoforis, M. (1992), A Note on Conformal Representation in Sobolev Spaces, Complex Variables, **20**, 121–133.

Lanza de Cristoforis, M. (1993), The large deformation of nonlinearly elastic rings in a two-dimensional compressible flow, IMA Journal of Appl. Math., **50**, 253–283.

Lanza de Cristoforis, M. (1994), Higher order differentiability properties of the Composition and of the Inversion Operator, Indagationes Mathematicae, New Series, **5**, 457–482.

Lanza de Cristoforis, M. and Antman, S.S. (1991), The large deformation of nonlinearly elastic tubes in two-dimensional flows, SIAM J. Math. Anal., **22**, 1193–1221.

Pommerenke, C. (1975), Univalent Functions, Vandenhoeck and Ruprecht, Göttingen.

Prodi, G. and Ambrosetti, A. (1973), Analisi non lineare, Editrice tecnico scientifica, Pisa.

Radó, T. (1923), Sur la représentation conforme de domaines variables, Acta Univ. Szeged., **1**, 180–186.

Rohlin, V. and Fuchs, D. (1981), Premier cours de topologie—chapitres géométriques, Éditions Mir, Moscou.

Rosenblatt, A. (1936), Sur la représentation conforme de domaines plans, C.R. Acad. Sci. Paris, **202**, 1398–1400.

Rosenblatt, A. (1936), Sur la représentation conforme de domaines bornés limités par des courbes générales, C.R. Acad. Sci. Paris, **202**, 1832–1834.

Troianiello, G.M. (1987), Elliptic Differential Equations and Obstacle Problems, Plenum Publishing Co..

Vekua, I.N. (1962), Generalized analytic functions, Pergamon Press.

Warschawski, S.E. (1935), On the Higher Derivatives at the Boundary in Conformal Mapping, Trans. Amer. Math. Soc., **38**, 310–340.

Wu, S. (1993), Analytic Dependence of Riemann Mappings for Bounded Domains and Minimal Surfaces, Communications on Pure and Applied Mathematics, **XLVI**, 1303–1326.

Yoshikawa, H. (1960), On the conformal mapping of nearly circular domains, J. Math. Soc. Japan, **12**, 174–186.

Zeitlin, D. (1957), Behavior of conformal maps under analytic deformation of the domain, Thesis. University of Minnesota.

Massimo LANZA DE CRISTOFORIS
Dipartimento di Matematica
Pura ed Applicata
Università di Padova
Via Belzoni 7
35131 Padova
Italy