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# ON THE $L^2$ FORM SPECTRUM OF THE LAPLACIAN ON NONNEGATIVELY CURVED MANIFOLDS

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**Abstract.** Let $(M, g_o)$ be a complete, noncompact Riemannian manifold with a pole, and let $g = fg_o$ be a conformally related metric. We obtain conditions on the curvature of $g_o$ and on $f$ under which the Laplacian on $p$-forms on $(M, g)$ has no eigenvalues.

Let $(M, g_o)$ be a complete, noncompact Riemannian manifold with a pole $o$, and denote by $r(x)$ the corresponding distance function from $o$. We assume throughout that the radial sectional curvature satisfies the pinching condition

$$0 \le K_r \le \frac{B^2}{1 + r(x)^2} \tag{0.1}$$

for some constant $0 \le B \le 1/2$. In the sequel we will denote by $B'$ the constant related to $B$ by the formula

$$B' = \frac{1}{2}(1 + \sqrt{1 - 4B^2}) . \tag{0.2}$$

Let also $g = fg_o$ be a conformally related metric, where $f$ is a smooth positive function on $M$.

We denote by $\Lambda^p(M)$ the space of $p$-forms on $M$. Given $\omega$ and $\theta$ in $\Lambda^p(M)$ we define a pointwise inner product

$$g(\omega, \theta) = \frac{1}{p!} \sum_{i_1, \dots, i_p = 1}^{m} \omega(e_{i_1}, \dots, e_{i_p})\theta(e_{i_1}, \dots, e_{i_p}) ,$$

and denote by $|\cdot|$ the induced norm. The symbol $L^2\Lambda^p(M)$ denotes the space of square integrable $p$-forms, i.e., forms such that $|\omega|^2$ is integrable on $M$. If $X$ is a vector field on $M$, and $\omega$ is a $p$-form ($p \ge 1$) we define the interior product $X \lrcorner\, \omega \in \Lambda^{p-1}(M)$ by

$$X \lrcorner\, \omega(X_1, \dots, X_{p-1}) = \omega(X, X_1, \dots, X_{p-1}) .$$

For notational convenience, we extend the definition of inner product by setting $X \lrcorner\, \omega = 0$ if $\omega$ is a 0-form.

Finally, we denote by $\Delta^p = \Delta^p_g$ the Laplacian on $p$-forms of $(M, g)$, so that $\Delta^p = dd^* + d^*d$, where $d$ and $d^*$ are the exterior differential and codifferential. Note that when $p = 0$, $\Delta^0_g = -\operatorname{div}\operatorname{grad}$ is the positive definite Laplacian $\Delta$. It is well-known that the operator

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$\Delta^p$ is self-adjoint on $L^2\Lambda^p(M)$, indeed, essentially self-adjoint on the space $C_c^\infty \Lambda^p(M)$ of compactly supported smooth $p$-forms [St]. We denote the corresponding operator domain with the symbol $\mathcal{D}(\Delta^p)$.

The purpose of this note is to obtain conditions on the function $f$ and on the curvatures $K_r$ under which $\Delta_g^p$ has no point spectrum, i.e., there are no nonzero square integrable $p$-forms in $\mathcal{D}(\Delta^p)$ satisfying the eigenvalue equation $\Delta^p u = \lambda u$. Observe that, by elliptic regularity, solutions of the eigenvalue equation are necessarily smooth.

The negative curvature case was considered by Donnelly [Dn], and Donnelly and Xavier [DnX]. Our results improve and complement those obtained by Escobar and Freire in [EF1] and [EF2].

The case of harmonic $p$-forms ($\lambda = 0$) goes back to Dodziuk [D1], [D2], [D3], and Sealey [Se]. In this case, a direct application of [RS, Theorem 2.3] shows that if $1 \leq p < m/2$,

$$\frac{m-2p}{2} f^{-1}\frac{\partial f}{\partial r} + [(m-p)B' - p]r^{-1} \geq 0$$

and the left hand side is not identically zero, then there are no nonzero harmonic $p$-forms in $L^2(\Lambda^p M)$.

It is readily verified that if $f$ satisfies the above relation with $p = 0$, then the manifold $(M, g)$ has infinite volume, and therefore it does not carry any nonzero $L^2$ harmonic functions (see [Y]). Thus, the above statement holds for every $p < m/2$. Finally, the case where $p > m/2$ may be dealt with by Hodge duality.

It may be worth noting that, if $f \equiv 1$, so that there is no conformal deformation of the metric, the condition becomes $(m-p)B' - p > 0$, that is, $B' > p/(m-p)$, which improves somewhat the condition in [EF2] (we note that our $B'$ corresponds to their $c_n$).

Next, we consider the case of $p$-forms satisfying the eigenfunction equation with $\lambda > 0$. We consider separately the cases $p = 0$ or $m$, and $1 \leq p \leq m-1$. Our results are the following.

THEOREM A. *Assume that* (0.1) *holds with a constant $B$ such that $B' \geq (m-2)/m$ and that*

$$\left| f^{-1}\frac{\partial f}{\partial r}\right| \leq \frac{1}{m-1}[mB' - (m-2)]r^{-1}.$$

*If $u$ is in $\mathcal{D}(\Delta)$ and satisfies $\Delta u = \lambda u$ with $\lambda > 0$, then $u \equiv 0$.*

THEOREM B. *Assume that* (0.1) *holds with a constant $B$ such that $B' \geq (m-1)/(m+1)$ and that*

$$\left| f^{-1}\frac{\partial f}{\partial r}\right| \leq \frac{m+1}{m-2p+1}\left[B' - \frac{m-1}{m+1}\right]r^{-1} \quad \text{if } 2 \leq 2p < m,$$

$$f^{-1}\frac{\partial f}{\partial r} \geq -\frac{m+1}{2}\left[B' - \frac{m-1}{m+1}\right]r^{-1} \quad \text{if } 2p = m.$$

*If $u$ is in $\mathcal{D}(\Delta^p)$ $(1 \leq p \leq m/2)$ and satisfies $\Delta^p u = \lambda u$ with $\lambda > 0$, then $u \equiv 0$.*

As in the case of harmonic $p$-forms, the conclusion for $p > m/2$ follows from Theorems A and B by Hodge duality. We also remark that, when $f \equiv 1$, the conditions in Theorems A and B become $B' \geq (m-2)/m$ and $B' \geq (m-1)/(m+1)$, respectively. The former coincides with that obtained in [EF1], while the latter improves that in [EF2]. We stress, however, that the main new feature of our results is that in all cases we allow a controlled conformal deformation of the metric.

**1. Proof of the theorems.** We begin by noting that (0.1) and the Hessian comparison theorem imply that the estimate

$$\frac{\phi'}{\phi}(g_0 - dr \otimes dr) \leq \mathrm{Hess}_{g_0} r \leq \frac{1}{r}(g_0 - dr \otimes dr) \tag{1.1}$$

holds on $M$ in the sense of quadratic forms, where $\phi$ is the solution of the problem

$$\begin{cases} \phi'' + \dfrac{B^2}{1+t^2}\phi = 0 \quad \text{on } [0, +\infty)\,, \\ \phi(0) = 0 \;\; \phi'(0) = 1\,. \end{cases}$$

Standard comparison arguments show that

$$\frac{B'}{t} \leq \frac{\phi'}{\phi}(t) \leq \frac{1}{t} \quad \text{for any } t > 0\,,$$

where $B'$ is defined in (0.2).

The proofs of the theorems follow the lines of those in [EF1] and [EF2], and depend on appropriate integral formulae. These formulae may be obtained by applying the divergence theorem to suitable vector fields which are constructed in terms of the $p$-form $u$ and its exterior differential and codifferential.

Ultimately, the formulae we use coincide with those used by Escobar and Freire, but we find it convenient to express them in a form slightly different from theirs. For this reason, and for the convenience of the reader we outline below how they may be derived.

Given a $p$-form $\omega$ and a vector field $X$ on $M$, a generic vector field which is quadratic in the components of $\omega$ is a linear combination of the vector fields $T_i = T_i(\omega, X)$, $S_i = S_i(\omega)$ and $U_i = U_i(\omega)$ defined as follows:

$$\begin{aligned} &g(T_1, Y) = |\omega|^2 g(X, Y)\,, && g(T_2, Y) = g(X \lrcorner\, \omega, Y \lrcorner\, \omega)\,, \\ &g(S_1, Y) = g(Y \lrcorner\, d\omega, \omega)\,, && g(S_2, Y) = g(Y \lrcorner\, \omega, d^*\omega)\,, \\ &g(U_1, Y) = g(Y^\flat \wedge \omega, d\omega)\,, && g(U_2, Y) = g(Y^\flat \wedge d^*\omega, \omega)\,, \\ &g(U_3, Y) = g(X^\flat \wedge \omega, Y^\flat \wedge \omega)\,, \end{aligned}$$

and of those obtained replacing $\omega$ with $d\omega$ and $d^*\omega$. Here, $\flat : TM \to T^*M$ is the musical isomorphism. Simple computations show that in fact $U_1 = S_1$, $U_2 = S_2$, and that $U_3 = T_1 - T_2$, so that we only need to consider $T_i$ and $S_i$, $i = 1, 2$.

Let $\{e_i\}$ be a local orthonormal frame field which is normal at the point $q$, and denote by $L_X$ the Lie differentiation in the direction of $X$. Computing the divergence of the vector fields

$T_i$ and $S_i$, one finds, at the point $q$,

$$\begin{aligned}\operatorname{div}T_1 &= \frac{1}{2}|\omega|^2 \operatorname{tr}L_X g + 2g(\nabla_X\omega, \omega)\,,\\ \operatorname{div}T_2 &= g(\nabla_X\omega, \omega) - g(X \lrcorner\, \omega, d^*\omega) - g(X \lrcorner\, d\omega, \omega)\\ &\quad + \frac{1}{2}\sum_{s,t} g(e_s \lrcorner\, \omega, e_t \lrcorner\, \omega)L_X g(e_s, e_t)\,,\\ \operatorname{div}S_1 &= |d\omega|^2 - g(\omega, d^*d\omega)\,, \quad \operatorname{div}S_2 = -|d^*\omega|^2 + g(\omega, dd^*\omega)\,.\end{aligned}$$

If we further impose the requirement that the divergence of the vector fields depends only $\omega$, $d\omega$ and $d^*\omega$, then we see that the only possible combinations are $Z = T_1 - 2T_2$, $S_1$ and $S_2$. We explicitly note that

$$\begin{aligned}\operatorname{div}_g Z = \frac{1}{2}|\omega|^2 \operatorname{tr} L_X g - \sum_{s,t} g(e_s \lrcorner\, \omega, e_t \lrcorner\, \omega)L_X g(e_s, e_t)\\ +2g(\omega, X \lrcorner\, d\omega) + 2g(X \lrcorner\, \omega, d^*\omega)\,.\end{aligned}$$

The above considerations immediately yield the following Lemma.

LEMMA 1.1. *Let $u$ be a $p$-form ($p \geq 0$) satisfying $d^*u = 0$ and $\Delta^p u = \lambda u$ ($\lambda > 0$). Let also $X$ be a given vector field, and $k$ a constant in* $\mathbf{R}$. *For every compact domain $D$ with smooth boundary in $M$, we have*

$$\begin{aligned}&\int_D \Big\{ \frac{1}{2}|du|^2(\operatorname{tr} L_X g - k) - \sum_{s,t} g(e_s \lrcorner\, du, e_t \lrcorner\, du)L_X g(e_s, e_t)\\ &\qquad -\lambda\Big[\frac{1}{2}|u|^2(\operatorname{tr} L_X g - k) - \sum_{s,t} g(e_s \lrcorner\, u, e_t \lrcorner\, u)L_X g(e_s, e_t)\Big]\Big\}\\ &= \int_{\partial D}\Big\{(|du|^2 - \lambda|u|^2)g(X, \nu) - \frac{k}{2}\, g(\nu \lrcorner\, du, u)\\ &\qquad -2(g(X \lrcorner\, du, \nu \lrcorner\, du) - \lambda g(X \lrcorner\, u, \nu \lrcorner\, u))\Big\}\,,\end{aligned}$$

*where $\nu$ denotes the outward unit normal to $\partial D$.*

PROOF. Let $W$ be the vector field defined by

$$W = Z(du, X) - \lambda Z(u, X) - \frac{k}{2}S_1(u)\,.$$

Then

$$\begin{aligned}\operatorname{div}_g W &= \frac{1}{2}(|du|^2 - \lambda|u|^2)(\operatorname{tr} L_X g - k)\\ &\quad -\sum_{s,t}(g(e_s \lrcorner\, du, e_t \lrcorner\, du) - \lambda g(e_s \lrcorner\, u, e_t \lrcorner\, u))L_X g(e_s, e_t)\,.\end{aligned}$$

We integrate $\operatorname{div} W$, and apply the divergence theorem to obtain the required conclusion. □

We now specialize the discussion to the case where the metric $g$ is the conformal deformation of the background metric $g_o$, as specified in the Introduction, and obtain the following lemma.

LEMMA 1.2. *Assume that $u$ satisfies the hypotheses of Lemma* 1.1. *Suppose further that $g = fg_0$ and that $X = \gamma(r)\partial_r$, where $(r, \theta)$ are the geodesic polar coordinates centered at $o$ of the metric $g_o$, and $\gamma$ is a function satisfying $\gamma^{2k}(0) = 0$, $\gamma'(0) = 1$ and $\gamma(r) > 0$ for every $r > 0$. Finally, assume that there exists a constant $C > 0$ such that for every $\theta$ in $S^{m-1}$*

$$\text{(1.2)} \qquad \text{(i)}\ \liminf_{r\to+\infty} \gamma^2(r) f(r,\theta) \geq C > 0 \quad \textit{and} \quad \text{(ii)}\ \int_1^{+\infty} \frac{1}{\gamma(r)} dr = +\infty .$$

*Then, there exists a sequence $R_n \to +\infty$ such that, denoting by $B_R$ the $g_o$-geodesic ball of radius $R$ centered at $o$,*

$$\text{(1.3)} \qquad \begin{aligned} \lim_{n\to\infty} \int_{B_{R_n}} \Bigg\{ & \frac{1}{2}|du|^2(\operatorname{tr} L_X g - k) - \sum_{s,t} g(e_s \lrcorner\, du, e_t \lrcorner\, du) L_X g(e_s, e_t) \\ & -\lambda \Bigg[ \frac{1}{2}|u|^2 (\operatorname{tr} L_X g - k) - \sum_{s,t} g(e_s \lrcorner\, u, e_t \lrcorner\, u) L_X g(e_s, e_t) \Bigg] \Bigg\} = 0 . \end{aligned}$$

PROOF. We apply the identity of Lemma 1.1 taking as $D$ the $g_o$-geodesic ball $B_R$. Denoting by $S(R)$ the boundary term, the proof amounts to showing that there exists a sequence $R_n \to +\infty$ such that $S(R_n)$ tends to zero as $n \to +\infty$.

Since $u \in \mathcal{D}(\Delta^p)$, $|u|^2$ and $|du|^2$ are integrable on $M$. Also, the following identities are easily verified:

$$\nu = f^{-1/2}\partial_r \qquad |\nabla_g r|_g^2 = f^{-1}|\nabla_{g_o} r|_{g_o}^2 = f^{-1} ,$$

so that the co-area formula reads

$$\int_{B_R} \phi\, dV_g = \int_0^R dr \int_{\partial B_r} f^{1/2} \phi\, d\sigma_{g,r} \quad \text{for any } \phi \in C_c(M) ,$$

where $d\sigma_{g,r}$ denotes the surface measure induced by $dV_g$ on $\partial B_r$. Moreover, $g(\nu, X) = f^{1/2}\gamma$, $g(X \lrcorner\, du, \nu \lrcorner\, du) = f^{1/2}\gamma g(\nu \lrcorner\, du, \nu \lrcorner\, du)$ and $g(X \lrcorner\, u, \nu \lrcorner\, u) = f^{1/2}\gamma g(\nu \lrcorner\, u, \nu \lrcorner\, u)$. Using the Cauchy-Schwarz inequality and the assumption (1.2) (i), we estimate

$$\begin{aligned} |S_R| &\leq \gamma(R) \int_{\partial B_R} f^{1/2} \left\{ 3(|du|^2 + \lambda |u|^2) + \frac{k}{2\gamma(R) f^{1/2}} |u|\,|du| \right\} d\sigma_{g,R} \\ &\leq C\gamma(R) \int_{\partial B_R} f^{1/2}(|du|^2 + |u|^2) d\sigma_{g,R} . \end{aligned}$$

By the co-area formula,

$$\int_0^{+\infty} dR \int_{\partial B_R} f^{1/2}(|du|^2 + |u|^2) d\sigma_{g,R} = \int_M (|du|^2 + |u|^2) dV_g < +\infty ,$$

whence, using (1.2) (ii), we conclude that

$$\liminf_{R\to+\infty} S(R) = 0$$

as required. □

LEMMA 1.3. *Maintaining the notation of the previous lemma, assume that $\gamma(r) = r$. Then, for every p-form $\omega$ $(p \geq 1)$, and every $k \in \mathbf{R}$, we have*

$$r\left\{\frac{m-2p}{2}f^{-1}\frac{\partial f}{\partial r} - \left[\frac{k}{2} + p - (m-p)r\frac{\phi'}{\phi}\right]r^{-1}\right\}|\omega|^2$$
$$\leq \frac{1}{2}|\omega|^2(\operatorname{tr} L_X g - k) - \sum_{s,t} g(e_s \lrcorner\, \omega, e_t \lrcorner\, \omega) L_X g(e_s, e_t)$$
$$\leq r\left\{\frac{m-2p}{2}f^{-1}\frac{\partial f}{\partial r} - \left[\frac{k}{2} + pr\frac{\phi'}{\phi} - (m-p)\right]r^{-1}\right\}|\omega|^2 .$$

*If $\omega$ is a 0-form, then we also have*

$$\frac{1}{2}|\omega|^2(\operatorname{tr} L_X g - k) \geq r\left\{\frac{m}{2}f^{-1}\frac{\partial f}{\partial r} - \left[\frac{k}{2} - 1 - (m-1)r\frac{\phi'}{\phi}\right]r^{-1}\right\}|\omega|^2 .$$

PROOF. The proof is a modification of that of Lemma 2.2 in [RS] (see also [K]), and we outline it here for completeness. We consider the case $p \geq 1$. The statement relative to the case $p = 0$ can be proved in a similar way.

It is easy to show that in a neighbourhood of each point $q$ there is a local orthonormal frame $\{e_s\}$ which is normal at $q$, and diagonalizes $L_X g$. Further, if $Y$ is $g$-orthogonal to $\partial_r$, then $L_X g(Y, \partial_r) = 0$, and we may therefore arrange that one of the vectors, say $e_{s_r}$ be proportional to $\partial_r$. Let $\mu_s$ be the corresponding eigenvalues of $L_X g$, so that $L_X g(e_s, e_t) = \delta_{s,t}\mu_s$. We further assume that the indexing be chosen in such a way that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$. By definition of inner product in $\Lambda^p(M)$, we may write

$$\sum_s g(e_s \lrcorner\, \omega, e_s \lrcorner\, \omega)\mu_s = \frac{1}{(p-1)!}\sum_s \sum_{i_1,\dots,i_{p-1}} |\omega(e_s, e_{i_1}, \dots, e_{i_{p-1}})|^2 \mu_s$$
$$= \frac{1}{p!}\sum_{i_1,\dots,i_p} |\omega(e_{i_1}, \dots, e_{i_p})|^2 \sum_{j=1}^{p} \mu_{i_j} .$$

Since the eigenvalues are arranged in decreasing order,

$$\sum_{j=m-p+1}^{m} \mu_j \leq \sum_{j=1}^{p} \mu_{i_j} \leq \sum_{j=1}^{p} \mu_j ,$$

and we conclude that

$$|\omega|^2 \sum_{j=m-p+1}^{m} \mu_j \leq \sum_{s,t} g(e_s \lrcorner\, \omega, e_t \lrcorner\, \omega) L_X g(e_s, e_t) \leq |\omega|^2 \sum_{j=1}^{p} \mu_j . \tag{1.4}$$

Denoting by $Q$ the quantity to be estimated, and using the above inequalities we get

$$\begin{aligned}\frac{1}{2}|\omega|^2(\mu_{p+1} + \cdots + \mu_m - \mu_1 - \cdots - \mu_p - k) \leq Q \\ \leq \frac{1}{2}|\omega|^2(\mu_1 + \cdots + \mu_{m-p} - \mu_{m-p+1} - \cdots - \mu_m - k) .\end{aligned} \tag{1.5}$$

To estimate the eigenvalues $\mu_j$, we repeat the argument that led to [RS, formula (2.7)]. As mentioned earlier, the curvature assumption implies the bound (1.1) for the Hessian of $r(x)$ with respect to the background metric $g_o$.

Now, since $X$ is the gradient vector field $r\partial_r = 1/2\nabla_{g_o} r^2$, by definition of Lie differentiation we have $L_X g_o = \mathrm{Hess}_{g_o}(r^2) = 2r\,\mathrm{Hess}_{g_o} r + 2dr \otimes dr$, so that (1.1) is equivalent to

$$2r\left\{\frac{\phi'(r)}{\phi(r)} g_0 + \left[\frac{1}{r} - \frac{\phi'(r)}{\phi(r)}\right] dr \otimes dr\right\} \le L_X g_0 \le 2g_0 .$$

Since $L_X$ is a derivation, $L_X g = (Xf)g_0 + f L_X g_0$, and, in terms of the conformal metric $g = f g_0$, the last inequality implies

$$r\left\{\left(f^{-1}\frac{\partial f}{\partial r} + 2\frac{\phi'(r)}{\phi(r)}\right) g + 2\left(\frac{1}{r} - \frac{\phi'(r)}{\phi(r)}\right) f dr \otimes dr\right\} \le L_X g \le r\left(f^{-1}\frac{\partial f}{\partial r} + 2\frac{1}{r}\right) g .$$

Recalling that $e_{s_r} = f^{-1/2}\partial_r$, we therefore obtain

$$\mu_s = r\left[f^{-1}\frac{\partial f}{\partial r} + \frac{2}{r}\right] \quad \text{if } s = s_r ,$$

while

$$r\left[f^{-1}\frac{\partial f}{\partial r} + 2\frac{\phi'}{\phi}\right] \le \mu_s \le r\left[f^{-1}\frac{\partial f}{\partial r} + \frac{2}{r}\right] \quad \text{otherwise} .$$

The required conclusion now follows, substituting these estimates into (1.5). □

LEMMA 1.4. *Let $p$ be such that $0 \le 2p \le m$, and assume that the curvature bound (0.1) holds with a constant $B$ such that $B' \ge (m-2)/m$ if $p = 0$ and $B' \ge (m-1)/(m+1)$ if $p \ge 1$. Suppose also that*

$$\left|f^{-1}\frac{\partial f}{\partial r}\right| \le \frac{m}{m-1}\left[B' - \frac{m-2}{m}\right] r^{-1} \qquad \text{if } p = 0 ,$$

$$\left|f^{-1}\frac{\partial f}{\partial r}\right| \le \frac{m+1}{m-2p-1}\left[B' - \frac{m-1}{m+1}\right] r^{-1} \quad \text{if } 2 \le 2p < m-2 ,$$

$$f^{-1}\frac{\partial f}{\partial r} \ge -\frac{m+1}{2}\left[B' - \frac{m-1}{m+1}\right] r^{-1} \qquad \text{if } 2p = m-2 \text{ or } 2p = m ,$$

$$f^{-1}\frac{\partial f}{\partial r} \ge -(m+1)\left[B' - \frac{m-1}{m+1}\right] r^{-1} \qquad \text{if } 2p = m-1 .$$

*If $u \in \mathcal{D}(\Delta^p)$ is such that $d^*u = 0$ and $\Delta^p u = \lambda u$ $(\lambda > 0)$, then $u \equiv 0$.*

PROOF. Observe first of all that, since $B' \le 1$, our assumptions imply that $f^{-1}\partial f/\partial r \ge -2r^{-1}$, whence integrating this we deduce that there exists a constant $C > 0$ such that

$$f(r,\theta) \ge Cr^{-2} \quad (r \ge 1) .$$

We may therefore let $\gamma(r) = r$ in Lemma 1.2 and apply the integral identity with $X = r\partial_r$.

We consider the case $p \geq 1$. If $p = 0$, the argument is similar. Since $r\phi'/\phi \geq B'$, we deduce from Lemma 1.3 that

$$\begin{aligned} &\frac{1}{2}|du|^2(\operatorname{tr} L_X g - k) - \sum_{s,t} g(e_s \lrcorner\, du, e_t \lrcorner\, du) L_X g(e_s, e_t) \\ &\leq r\left\{\frac{m-2p-2}{2} f^{-1}\frac{\partial f}{\partial r} - \left[\frac{k}{2} + (p+1)B' - (m-p-1)\right] r^{-1}\right\} |du|^2, \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} &\frac{1}{2}|u|^2(\operatorname{tr} L_X g - k) - \sum_{s,t} g(e_s \lrcorner\, u, e_t \lrcorner\, u) L_X g(e_s, e_t) \\ &\geq r\left\{\frac{m-2p}{2} f^{-1}\frac{\partial f}{\partial r} - \left[\frac{k}{2} + p - (m-p) r\frac{\phi'}{\phi}\right] r^{-1}\right\} |u|^2 \\ &\geq r\left\{\frac{m-2p}{2} f^{-1}\frac{\partial f}{\partial r} - \left[\frac{k}{2} + p - (m-p)B'\right] r^{-1}\right\} |u|^2. \end{aligned} \tag{1.7}$$

Assume first that $2 \leq 2p < m-2$. We determine the constant $k$ in such a way that

$$\frac{2}{m-2p-2}\left[\frac{k}{2} + (p+1)B' - (m-p-1)\right] = -\frac{2}{m-2p}\left[\frac{k}{2} + p - (m-p)B'\right].$$

Then, a computation shows that the left hand side is equal to

$$\frac{1}{m-2p-1}[(m+1)B' - (m-1)],$$

which is nonnegative by our assumption on $B'$. Keeping into account the condition satisfied by $f$, we deduce that the right hand side of (1.6) is nonpositive, and that of (1.7) is nonnegative.

Arguing in a similar way, it is easily verified that the same conclusion holds if $2p$ is equal to $m-2$, $m-1$ or to $m$, provided we choose $k = 1 + B'$, $k = 0$, or $k = -1 - B'$, respectively.

In all cases, the integrand in the left hand side of (1.3) is of constant (nonpositive) sign, and the integrals over the balls $B_{R_n}$ tend to the integral over $M$ as $n$ tends to $\infty$. We conclude that the left hand side of (1.7) vanishes identically, and all inequalities are in fact equalities. In particular,

$$r\left\{\frac{m-2p}{2} f^{-1}\frac{\partial f}{\partial r} - \left[\frac{k}{2} + p - (m-p) r\frac{\phi'}{\phi}\right] r^{-1}\right\} |u|^2 \equiv 0 \quad \text{on } M.$$

Now, note that the quantity in braces on the left hand side is strictly positive in a neighbourhood of $o$. Indeed, we may rewrite it in the form

$$\left\{\frac{m-2p}{2} f^{-1}\frac{\partial f}{\partial r} - \left[\frac{k}{2} + p - (m-p)B'\right] r^{-1}\right\} + (m-p)\left(r\frac{\phi'}{\phi} - B'\right) r^{-1}.$$

If $B' < 1$, then the claim follows from the fact that $r\phi'/\phi \to 1$ as $r \to 0$. If $B' = 1$, then $B = 0$ and $\phi(r) = r$, so that the second term is identically zero. But then

$$-\left[\frac{k}{2} + p - (m-p)B'\right] = \frac{m-2p}{m-2p-1},$$

and since $f^{-1}\partial f/\partial r$ is bounded in a neighbourhood of $o$ ($f$ being smooth and positive on $M$), the first term is strictly positive near $o$.

It follows that $u$ must vanish in a neighbourhood of $o$. Since $u$ satisfies the equation $d^*du = \lambda u$, its components $u_I$ ($I = (i_1, \dots, i_p)$) with respect to a local orthonormal frame satisfy the linear system

$$\Delta u_I = \lambda u_I + L(u),$$

where $L$ is a linear differential operator of order $\leq 1$. By unique continuation (see [A, Remark 2], or [Kz, Theorem 1.8]) $u$ must vanish identically on $M$, as required to finish the proof. □

PROOF OF THE THEOREMS (see [EF1] and [EF2]). Theorem A follows immediately from the case $p = 0$ in Lemma 1.4. We prove Theorem B.

Thus, assume that $p \geq 1$, and let $u \in \mathcal{D}(\Delta^p)$ be such that $\Delta^p u = \lambda u$ with $\lambda > 0$. Then $v = d^*u$ belongs to $\mathcal{D}(\Delta^{p-1})$ and satisfies $\Delta^{p-1}v = \lambda v$, $d^*v = 0$. It is readily verified that $f$ satisfies the condition in Lemma 1.4 relative to $p - 1$, if $p \geq 2$, or that of Theorem A if $p = 1$, so that $v = d^*u = 0$. But $f$ also satisfies the condition of Lemma 1.4 relative to $p$, and therefore $u \equiv 0$, as required. □

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