Advanced Nonlinear Studies **9** (2009), 95–135

Connected Branches of Initial Points for Asymptotic BVPs, With Application to Heteroclinic and Homoclinic Solutions

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Received 5 April 2008
Communicated by Shair Ahmad

Abstract

We consider the second order nonlinear ODE $u'' - f(t,u) = 0$ and assume that $f(\cdot, v_0) \equiv 0$, for some $v_0 \in \mathbb{R}$. We prove the existence of closed connected sets $\Gamma \subseteq \mathbb{R}^2$ of initial points such that for each $(\alpha, \beta) \in \Gamma$ there exists a solution $u(\cdot)$ of the given differential equation, with $(u(t_0), u'(t_0)) = (\alpha, \beta)$ and $(u(t), u'(t)) \to (v_0, 0)$ as $t \to -\infty$ (or as $t \to +\infty$). These results are then applied to the search of heteroclinic and homoclinic solutions.

1991 Mathematics Subject Classification. 34B18, 34C30, 37C29.

Key words. Boundary value problems, connected branches of initial points, Leray–Schauder degree, heteroclinic solutions, homoclinic solutions.

*Supported by POCI/Mat/57258/2004 (POCI 2010 and FEDER), FEDER and FCT - Pluri-anual 2007.

†The author acknowledges the support of the PRIN–2005 project "Equazioni Differenziali Ordinarie e Applicazioni". The author also thanks the CMAF of Lisbon for the financial support and the pleasant hospitality.

1 Introduction

The search for heteroclinic or homoclinic solutions for second order non–autonomous differential equations plays a crucial role in many different areas of applied mathematics where these solutions represent relevant states for the systems under investigation. In the literature, different methods have been developed and applied for dealing with such problems (see, for instance [1, 4, 5, 10, 16, 21, 22, 32, 34] and the references therein).

Among these techniques, a possible approach already considered by some authors (cf. [8, 14, 17]) consists of gluing together the unstable and the stable manifolds of some critical points. More precisely, let us suppose that p and q are zeros of a given vector field and we want to find a solution $u(\cdot)$ of the corresponding second order ODE such that $(u(t), u'(t)) \to (p, 0)$ as $t \to -\infty$ and $(u(t), u'(t)) \to (q, 0)$ as $t \to +\infty$. In this situation, one could split the problem into the following three parts. As a first step, we fix $t_0 \in \mathbb{R}$ and look at the set $W_p^-(t_0)$ of the initial points $(u(t_0), u'(t_0))$ such that $(u(t), u'(t)) \to (p, 0)$ for $t \to -\infty$. Secondly, we fix $t_1 \in \mathbb{R}$, with $t_1 \geq t_0$ and look at the set $W_q^+(t_1)$ of the initial points $(u(t_1), u'(t_1))$ such that $(u(t), u'(t)) \to (q, 0)$ for $t \to +\infty$. Finally, if $t_0 = t_1$, we can try to check that $W_p^-(t_0) \cap W_q^+(t_0) \neq \emptyset$ (in some cases, like in the search of homoclinic solutions for $p = q$, one would like also to prove that the intersection is nontrivial). Otherwise, if $t_0 < t_1$, one is led to the study of a generalized Sturm - Liouville boundary value problem of the form

$$\begin{cases} \text{“ the given differential equation ”} \\ \\ (u(t_0), u'(t_0)) \in W_p^-(t_0), \quad (u(t_1), u'(t_1)) \in W_q^+(t_1) \end{cases}$$

with the aim of finding a solution connecting the unstable manifold of $(p, 0)$ to the stable manifold of $(q, 0)$. Results for generalized Sturm - Liouville type problems may be found in [33] and in [31], where the authors seek solutions joining two unbounded connected sets.

In some cases, the first two steps can be solved by using dynamical systems techniques. This occurs, for instance, if the differential system is asymptotically autonomous (like in [14, 17]). For the third step, one can study the displacement $\phi_{[t_0,t_1]}(W_p^-(t_0))$ of $W_p^-(t_0)$ under the action of the flow ϕ_t along the interval $[t_0, t_1]$ and check for the intersection of $\phi_{[t_0,t_1]}(W_p^-(t_0))$ with $W_q^+(t_1)$.

In other cases, the conditions on the non–autonomous vector field are so mild that the concept of unstable or stable manifolds must be considered in a topological sense, without reference to the usual smoothness assumptions associated to these sets. Hence, in general, proving the existence of (nontrivial) closed and connected subsets $\Gamma_p^-(t_0) \subseteq W_p^-(t_0)$ and $\Gamma_q^+(t_1) \subseteq W_q^+(t_1)$ is the best that one can hope to achieve in order to apply some topological properties and thus successfully deal with the third step.

In the present work we look for connected branches of initial points from which depart solutions having a prescribed asymptotic behavior for $t \to -\infty$ or for $t \to +\infty$. Even if the motivation of our study is that of applying these results to the search of heteroclinic and homoclinic solutions, the main part of the paper will be devoted just to the first two steps of the procedure described above. Indeed, we think that results in this direction may have some independent interest as they are linked to the problem of the detection of the stable and unstable manifolds in two-dimensional non-periodic time–dependent vector fields (see [20]).

In detail, we consider the second order ODE

$$u'' - f(t, u) = 0, \tag{1.1}$$

with $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying the Carathéodory assumptions and suppose that

$$f(t, v_0) = 0, \quad \text{for a.e. } t \in \mathbb{R},$$

for some $v_0 \in \mathbb{R}$. Our goal is to perform a careful analysis of the sets

$$\{(u(t_0), u'(t_0)) \in \mathbb{R}^2 \,:\, \exists\, u(\cdot)\,\text{solution of (1.1)} : u(-\infty) = v_0\,, u'(-\infty) = 0\}$$

$$\{(u(t_1), u'(t_1)) \in \mathbb{R}^2 \,:\, \exists\, u(\cdot)\,\text{solution of (1.1)} : u(+\infty) = v_0\,, u'(+\infty) = 0\},$$

or, more precisely, to prove that such sets contain closed connected subsets Γ^- and Γ^+ (respectively) satisfying suitable properties. In our analysis, we also discuss the behavior and the sign (with respect to v_0) of the solutions departing from Γ^- or from Γ^+.

The plan of this paper is the following. In Section 2 we present the main results about the existence of continua of initial points from which depart solutions of (1.1) possessing prescribed asymptotic properties. Our main results are Theorem 2.1 and its variants (theorems 2.2, 2.3 and 2.4). The argument in the proof makes use of an approach previously employed in [24] for a different asymptotic problem (namely, for the search of solutions presenting a blow up at the boundary of a certain interval). It seems to us that such a technique, which represents a rather classical tool in bifurcation theory [29, 30] and in the study of parameter dependent nonlinear operator equations [3, 23], has not been fully exploited yet in the search of branches of solutions possessing a prescribed asymptotic behavior. Theorem 2.5 deals with the motion of an unstable manifold along the flow in a compact time interval. Since we don't assume the uniqueness of the solutions for the associated Cauchy problems, we apply a "shooting without uniqueness" technique from [9] which is based on a refinement of an argument considered by Struwe in [33]. Section 3 contains an extension of our main results to the case of unbounded connected sets of initial points. In Section 4 we apply our theorems to the search of heteroclinic solutions. Among other results, we show also an extension of a theorem previously obtained by Conley [8] and motivated by a mathematical model in population genetics. The choice of such an example as a model where to test our technique is also motivated by the interest of putting in evidence the differences between our approach and those based on Ważewski type methods. Indeed, compared to [8], our approach is more functional–analytic and less geometric. With respect to [8], we don't need to find some Ważewski set in the extended phase–space $\{(t, x, x') \in \mathbb{R}^3\}$ where to check suitable egress point conditions on the boundary. In our setting, the existence of a continuum of initial points from which depart solutions with the desired asymptotic properties is a consequence of the Leray - Schauder continuation theorem for parameter depending operator equations, combined with some topological lemmas on metric continua. As a consequence, we can imitate some geometric features which are present in [8], but with less expense in terms of conditions on the vector field. See also [6] for yet another approach for the proof of heteroclinic solutions in the population genetics model investigated by Conley.

Section 5 deals with the case of homoclinic solutions. Besides an application to a second order non–autonomous ODE with a superlinear nonlinearity, we present some examples of piecewise autonomous systems where a detailed analysis of the time–map allows to provide precise existence and non–existence results. Finally, in the Appendix, we recall some technical lemmas which are used in the main proofs of Section 2.

We point out that our results can be applied to ODEs of the form

$$u'' + c(t)u' - g(t,u) = 0, \tag{1.2}$$

by reducing (1.2) to (1.1) through a suitable change of variable. Clearly, one has to adapt some integral conditions (like $(H1)$ and $(H2)$ of Theorem 2.1) to the new framework. For simplicity, however, we confine ourselves only to the analysis of equation (1.1) and don't give explicit applications to (1.2).

Throughout the article, the following notation is used: $\mathbb{R}^+$ (respectively, $\mathbb{R}^+_0$) denotes the subset of $\mathbb{R}$ consisting of the nonnegative (respectively, positive) real numbers. $\mathbb{N} = \{1, 2, \ldots\}$ is the set of positive integers. By $||\cdot||$ we mean any fixed norm in the plane $\mathbb{R}^2$ and denote by $B[R]$ the closed disc

$$B[R] := \{z \in \mathbb{R}^2 \,:\, ||z|| \le R\}.$$

For any given interval $[a,b] \subseteq \mathbb{R}$, we denote by $C^1([a,b])$ the Banach space of the continuously differentiable functions $u : [a,b] \to \mathbb{R}$ endowed with the norm

$$||u||_{1,\infty} := ||u||_\infty + ||u'||_\infty \,.$$

We say that a function $f = f(t,x) : J_1 \times J_2 \to \mathbb{R}$ satisfies the Carathéodory assumptions (where $J_1\,, J_2 \subseteq \mathbb{R}$ are arbitrary intervals) if $f(t,\cdot)$ is continuous for almost every $t \in J_1\,,\ f(\cdot,x)$ is measurable for every $x \in J_2$ and, for every pair of compact intervals $I_1 \subseteq J_1$ and $I_2 \subseteq J_2$ there exists a nonnegative measurable function $\rho = \rho_{_{I_1,I_2}} \in L^1(I_1, \mathbb{R}^+)$ such that

$$|f(t,x)| \le \rho(t), \qquad \text{for a.e. } t \in I_1\,,\ \forall\, x \in I_2\,.$$

In this case, solutions of $x'' = f(t,x)$ are considered in the generalized sense [12]. Of course, any continuous function satisfies the Carathéodory assumptions and, in such a situation, the solutions of the corresponding ODE are of class C^2.

By a continuum we mean any compact connected subset of a metric space.

2 Main results

We consider the asymptotic boundary value problem

$$(P_-) \qquad \begin{cases} u'' - f(t,u) = 0 \\ u(-\infty) = v_0 \\ u'(-\infty) = 0, \end{cases}$$

where $f : (-\infty, t_0] \times [v_0, v] \to \mathbb{R}$ satisfies the Carathéodory assumptions and, moreover,

$$f(\cdot, v_0) \equiv 0, \tag{2.1}$$

$$\forall\, s \in]v_0, v]\,, \quad f(t,s) \geq 0 \text{ for a.e. } t \in (-\infty, t_0]. \tag{2.2}$$

Theorem 2.1 *Assume* (2.1) *and* (2.2) *and suppose that the following conditions hold:*

> $(H1)$ *for each* $a \in]v_0, v[\,$, *there are* $\varepsilon > 0$, $t_\varepsilon \leq t_0$ *and a locally integrable function* $\gamma_\varepsilon = \gamma_{a,\varepsilon} : (-\infty, t_\varepsilon] \to \mathbb{R}^+$ *such that*
>
> $$\int_{-\infty}^{t_\varepsilon} \gamma_\varepsilon(\theta)\, d\theta = +\infty$$
>
> *and*
>
> $$f(t,s) \geq \gamma_\varepsilon(t), \quad \forall\, s \in [a, a+\varepsilon], \quad \text{for a.e. } t \leq t_\varepsilon\,,$$
>
> $(H2)$ *either* $f(t,v) > 0$ *in a subset of positive measure of* $(-\infty, t_0]$, *or* $f(t,v) = 0$ *for a.e.* $t \in (-\infty, t_0]$ *and there exist* $\delta > 0$ *and a locally integrable function* $\eta = \eta_\delta : (-\infty, t_0] \to \mathbb{R}^+$ *such that*
>
> $$\int_{-\infty}^{t_0} \eta(\theta)\, d\theta = +\infty$$
>
> *and*
>
> $$f(t,s) \geq \eta(t)(v - s), \quad \forall\, s \in [v-\delta, v], \quad \text{for a.e. } t \in (-\infty, t_0].$$

Then, there exists a continuum $\Gamma^- \subseteq [v_0, v] \times \mathbb{R}^+$ *satisfying the following properties:*

> (i_1) $\forall\, \alpha \in [v_0, v]\,, \exists\, \beta \geq 0$ *such that* $(\alpha, \beta) \in \Gamma^-$.
>
> (i_2) $\Gamma^- \cap \big([v_0, v] \times \{0\} \big) = \Gamma^- \cap \big(\{v_0\} \times \mathbb{R}^+ \big) = \{(v_0, 0)\}$.
>
> (i_3) *For each* $(\alpha, \beta) \in \Gamma^-$, *there exists a solution* $u(\cdot)$ *of* (P_-) *such that* $u(t_0) = \alpha$, $u'(t_0) = \beta$. *Moreover, if* $\alpha \in]v_0, v]$, *then there is a maximal interval* $]\tau_u, t_0]$ *such that* $u(t) \in]v_0, v]$ *and* $u'(t) > 0$, *for all* $t \in]\tau_u, t_0]$. *If* $\tau_u > -\infty$, *then* $u(t) = v_0$ *and* $u'(t) = 0$, $\forall\, t \leq \tau_u$.

Proof. The proof is split into several steps. First we will construct a sequence of continua Γ_n made of initial points of solutions of suitable auxiliary boundary value problems. Then we will show that these continua are equibounded and we will obtain the continuum Γ^- through a limit process on the continua Γ_n. Finally, we will show that Γ^- actually satisfies (i_1), (i_2) and (i_3).

We start by defining, for technical convenience, a suitable extension of f. Namely,

$$\tilde{f}(t,s) := f(t, P_{[v_0,v]}(s)), \tag{2.3}$$

where

$$P_{[v_0,v]}(s) := \max\{v_0, \min\{s,v\}\}$$

is the standard projection of $\mathbb{R}$ onto the interval $[v_0,v]$. Clearly, any solution of $u'' - \tilde{f}(t,u) = 0$, with $u(t) \in [v_0,v]$ for each $t \in \operatorname{dom} u(\cdot)$, is a solution of $u'' - f(t,u) = 0$ (and vice-versa).

Next, we consider the auxiliary two-points boundary value problems

$$(P_{n,\lambda}) \qquad \begin{cases} u'' - \tilde{f}(t,u) = 0 \\ u(t_0 - n) = v_0\,, \quad u(t_0) = \lambda, \end{cases}$$

where

$$v_0 \le \lambda \le v \quad \text{and} \quad n \in \mathbb{N}.$$

It is well known that, for each $n \in \mathbb{N}$ and $\lambda \in [v_0,v]$, the solutions of problem $(P_{n,\lambda})$ are fixed points in the Banach space $X_n := C^1[t_0-n,t_0]$, endowed with the standard C^1-norm, of the completely continuous operator $T_\lambda^{(n)} : X_n \to X_n$ defined by

$$T_\lambda^{(n)}(u)(t) := \psi_{\lambda,n}(t) + \int_{t_0-n}^{t_0} G_n(t,\theta)\tilde{f}(\theta,u(\theta))\,d\theta,$$

where G_n is the Green's function associated to the problem

$$\begin{cases} -u'' = w(t) \\ u(t_0-n) = 0\,, \quad u(t_0) = 0, \end{cases}$$

and

$$\psi_{\lambda,n}(t) := v_0 + \frac{\lambda - v_0}{n}(t - t_0 + n), \quad t \in [t_0-n,t_0].$$

We also observe that the image $T_\lambda^{(n)}(X_n)$ is bounded. In fact, by the Carathéodory assumption, there is an integrable function $\rho(\cdot)$ such that

$$|f(t,s)| \le \rho(t), \quad \text{for a.e. } t \in [t_0-n,t_0], \text{ and for every } s \in [v_0,v].$$

Hence, from $(P_{n,\lambda})$ and the definition of $T_\lambda^{(n)}$, we easily find that $||\frac{d}{dt}T_\lambda^{(n)}(u)||_\infty \le \frac{\lambda-v_0}{n} + ||\rho||_1 \le \frac{v-v_0}{n} + ||\rho||_1 := c_1^{(n)}$, as well as $||T_\lambda^{(n)}(u)||_\infty \le v_0 + nc_1^{(n)} = v + n\int_{t_0-n}^{t_0}\rho(\theta)\,d\theta := c_2^{(n)}$, so that we can conclude

$$||T_\lambda^{(n)}(u)||_{1,\infty} \le c_1^{(n)} + c_2^{(n)} := M_n\,.$$

As a consequence, for every $R > M_n$, there are no fixed points of the operator $T_\lambda^{(n)}$ on the boundary $\partial B(0,R)$ of the open ball, of center 0 and radius R, $B(0,R) \subseteq X_n$ and therefore, the Leray-Schauder degree

$$d := \deg(I - T_\lambda^{(n)}, B(0,R), 0)$$

is well defined. If we introduce now the auxiliary parameter, $\mu \in [0,1]$, by the above estimates, we know that

$$u \neq \mu T_\lambda^{(n)}(u), \quad \forall\, u \in \partial B(0,R),\ \mu \in [0,1]$$

and the invariance under homotopies of the Leray-Schauder degree guarantees that $d = \deg(I, B(0,R), 0) = 1$. We have thus proved that for every value of the parameter $\lambda \in [v_0, v]$,

$$\deg(I - T_\lambda^{(n)}, B(0,R), 0) = 1 = \text{constant w.r. to } \lambda$$

and the theory of topological degree (see Leray-Schauder [19, Théorème Fondamental] and also [23]) ensures the existence of a continuum

$$\mathcal{C}_n \subseteq [v_0, v] \times B(0,R) \subseteq \mathbb{R} \times X_n$$

such that

$$u = T_\lambda^{(n)}(u),$$

for every $(\lambda, u) \in \mathcal{C}_n$ and the projection of the set $\mathcal{C}_n$ on the first factor covers the interval $[v_0, v]$.

Now, for every fixed $n \in \mathbb{N}$, we can define the set

$$\Gamma_n := \{(u(t_0), u'(t_0)) :\ (u(t_0), u) \in \mathcal{C}_n\} = \{(\lambda, u'(t_0)) :\ (\lambda, u) \in \mathcal{C}_n\} \subseteq \mathbb{R} \times \mathbb{R}.$$

We note that, for each n, the set Γ_n is a continuum and we have:

- $\Gamma_n \subseteq [v_0, v] \times \mathbb{R}, \quad Pr_1(\Gamma_n) = [v_0, v]$

  (where we have denoted by $Pr_1 : (x,y) \mapsto x$ the projection of the plane onto its first coordinate);

- *for each $(\alpha, \beta) \in \Gamma_n$, there exists a solution $u = u_{n,\alpha,\beta}$ to the differential equation $u'' - \tilde{f}(t,u) = 0$ such that $u(t_0 - n) = v_0$ and $u(t_0) = \alpha$, $u'(t_0) = \beta$.*

Our next step is to prove the equiboundedness of the Γ_n. More precisely, we claim that there is $K > 0$ such that

$$\Gamma_n \subseteq [v_0, v] \times [0, K], \quad \forall\, n \in \mathbb{N}. \tag{2.4}$$

In order to check (2.4) we give some qualitative properties of the solutions of problem $(P_{n,\lambda})$. Let u be any solution of $(P_{n,\lambda})$. By (2.2) and the definition of $\tilde{f}$, we have that $u''(t) = \tilde{f}(t, u(t)) \geq 0$, for almost every $t \in [t_0 - n, t_0]$, and therefore u' is non-decreasing on the interval $[t_0 - n, t_0]$.

We claim that $u'(t_0 - n) \geq 0$. In fact, if, by contradiction, $u'(t_0 - n) < 0$, it follows that $u(t) < v_0$ for $t > t_0 - n$ in a neighborhood of $t_0 - n$. On the other hand, we know that $u(t_0) = \lambda \geq u(t_0 - n)$ and therefore, there exists an interval $[t_0 - n, t_1] \subseteq [t_0 - n, t_0]$ such that $u(t_1) = u(t_0 - n) = v_0$ and $u(t) < v_0$ for all $t \in]t_0 - n, t_1[$. By the definition of $\tilde{f}$ and (2.1) it follows that $u'' \equiv 0$ on $[t_0 - n, t_1]$ and hence, $u(t) = v_0$ for every $t \in [t_0 - n, t_1]$, a contradiction (remember that u' is absolutely continuous).

Now that we have proved that u' is non-decreasing with $u'(t_0 - n) \geq 0$, we can conclude that $u'(t) \geq 0, \forall\, t \in [t_0 - n, t_0]$ and therefore $v_0 \leq u(t) \leq v, \forall\, t \in [t_0 - n, t_0]$. Observe also that $u'(t_0) > 0$ when $u(t_0) = \lambda > v_0$.

It follows that if $(\alpha, \beta) \in \Gamma_n$, then $\alpha \in [v_0, v]$ and $\beta \geq 0$. Therefore, in order to complete the proof of (2.1) we must look for an upper bound for β.

Let u be a solution of $P_{n,\alpha}$ for some $n \in \mathbb{N}$ such that $u'(t_0) = \beta$. Since u is defined on $[t_0 - n, t_0]$, it is obviously defined also in $[t_0 - 1, t_0]$, thus we can consider a function $\rho \in L^1([t_0 - 1, t_0], \mathbb{R}^+)$ (which comes from the Carathéodory assumptions) such that $\tilde{f}(t, s) = |\tilde{f}(t, s)| = |f(t, s)| \leq \rho(t)$ for every $s \in [v_0, v]$ and almost every $t \in [t_0 - 1, t_0]$. From the equation in $(P_{n,\lambda})$, we then get

$$u'(t) \geq \beta - \int_{t_0-1}^{t_0} \tilde{f}(\theta, u(\theta))\, d\theta \geq \beta - \int_{t_0-1}^{t_0} \rho(\theta)\, d\theta := \beta - K_1\,,\ \forall\, t \in [t_0 - 1, t_0].$$

Integrating again on $[t_0 - 1, t_0]$, we have

$$v - v_0 \geq \alpha - u(t) \geq (\beta - K_1)(t_0 - t), \quad \forall\, t \in [t_0 - 1, t_0].$$

This gives immediately the estimate

$$\beta \leq v - v_0 + K_1 := K \tag{2.5}$$

with the constant K independent on $n \in \mathbb{N}$ and completes the proof of (2.4).

We are now in a position to apply a result about limits of continua (see, [2], [18, §47,II;p.171]) which ensures that

$$\Gamma^- := \limsup_{n \to \infty} \Gamma_n \subseteq [v_0, v] \times [0, K] \tag{2.6}$$

is a continuum. Note that

$$Pr_1(\Gamma^-) = [v_0, v],$$

and hence (i_1) holds.

Now we must prove (i_2) and (i_3). We start by analyzing the properties of some solutions of the equation

$$u'' - \tilde{f}(t, u) = 0 \tag{2.7}$$

with initial point in Γ^-. These properties will be used in the proof of both (i_2) and (i_3). More precisely, we will prove what follows:

(P_1) *for any* $(\alpha,\beta)\in\Gamma^-$ *there exists a solution of equation* (2.7) *with* $u(t_0)=\alpha$, $u'(t_0)=\beta$ *and such that* u *and* u' *are non-decreasing functions with*

$$v_0\le u(t)\le u(t_0)=\alpha,\quad 0\le u'(t)\le u'(t_0)=\beta,\quad \forall\, t\in(-\infty,t_0]. \tag{2.8}$$

Note that an immediate consequence of (P_1) is that

$$\lim_{t\to-\infty} u(t)=: u_{-\infty}\in[v_0,v] \quad\text{and}\quad \lim_{t\to-\infty} u'(t)=: u'_{-\infty}=0. \tag{2.9}$$

Now we start the proof of (P_1). Let $(\alpha,\beta)\in\Gamma^-$. By definition, there exists a sequence $(\alpha_n,\beta_n)\in\Gamma_n$ with $(\alpha_n,\beta_n)\to(\alpha,\beta)$. We also know that, for each $n\in\mathbb{N}$, there is at least one solution u_n to equation (2.7) with $(u_n(t_0),u_n'(t_0))=(\alpha_n,\beta_n)$ and $u_n(t_0-n)=v_0$. We have already proved that $v_0\le u_n(t)\le v$, as well as $0\le u_n'(t)\le\beta_n\le\beta+\varepsilon$ for every $t\in[t_0-n,t_0]$, where the last inequality holds, for any fixed ε and n sufficiently large (say $n\ge\bar n_\varepsilon$).

It is convenient now to extend u_n to $(-\infty,t_0]$ by setting

$$(-\infty,t_0]\ni t\mapsto\begin{cases} u_n(t_0-n), & \text{for } t\le t_0-n\\ u_n(t), & \text{for } t\ge t_0-n.\end{cases}$$

We continue to call u_n the new function as no confusion occurs. By Ascoli-Arzelà's theorem and a standard diagonal argument, we can conclude that there exists a C^1-function $u:(-\infty,t_0]\to\mathbb{R}$ which is the pointwise limit of a subsequence of the $(u_n)_n$ and the convergence is uniform on each compact subset of $(-\infty,t_0]$. Writing (2.7) in an integral form and passing to the limit as $n\to\infty$ on such a subsequence, we can also check that u is a solution of (2.7) on $(-\infty,t_0]$ and

$$u(t_0)=\alpha,\ \ u'(t_0)=\beta.$$

From the fact that u_n and u_n' are non-decreasing functions with $v_0\le u_n(t)\le v$ for all $t\in[t_0-n,t_0]$ and $0\le u_n'(t)\le\beta_n$ for all $t\in(-\infty,t_0]$, we conclude that u and u' are non-decreasing functions which satisfy property (2.8) (and, as a consequence, property (2.9)).

Having established these properties for some solutions of equation (2.7) originating from Γ^-, we are now ready to prove (i_2).

We note first that, if $\alpha=v_0$, then by (2.8), for the solutions constructed above we have necessarily $\beta=0$ and $u(t)=v_0$, for every $t\in(-\infty,t_0]$. This implies that $\Gamma^-\cap\big(\{v_0\}\times\mathbb{R}^+\big)=\{(v_0,0)\}$.

Suppose now that $(\alpha,0)\in\Gamma^-$ for some $\alpha\in\,]v_0,v[$ and let u be the corresponding solution of (2.7), with $(u(t_0),u'(t_0))=(\alpha,0)$, given by (P_1). Again by (2.8), we conclude that $u(\cdot)=\alpha$ on $(-\infty,t_0]$. Hence, $\tilde f(t,\alpha)=f(t,\alpha)=0$ for almost every $t\in(-\infty,t_0]$, and this contradicts $(H1)$. We have thus verified that $\Gamma^-\cap\big([v_0,v[\,\times\{0\}\big)=\{(v_0,0)\}$.

To end the proof of (i_2), it remains to check the case $(\alpha,\beta)=(v,\beta)\in\Gamma^-$ and prove that, in such a situation, $\beta>0$.

Assume, by contradiction, that $\beta = 0$. By the properties of the solutions given by (P_1), we obtain that $u(t) = v$ and $u'(t) = 0$, $\forall\, t \leq t_0$. Hence,

$$f(t, v) = 0, \quad \text{for a.e. } t \in (-\infty, t_0].$$

We claim that there are $\bar\alpha$ and $\bar\beta$ with $v_0 < \bar\alpha < v$ and $\bar\beta > 0$, such that every solution u of equation (2.7) with $u(t_0) = \alpha \in]\bar\alpha, v[$, $u'(t_0) \in]0, \bar\beta[$ and such that $u(t) \in [v_0, v]$ $\forall\, t \leq t_0$, satisfies $u'(t^*) < 0$ for a suitable $t^* < t_0$. Since, for each point $(\alpha, \beta) \in \Gamma^-$, with $\alpha \in]v_0, v[$ and $\beta > 0$, we have found solutions of (2.7) originating from that point at time t_0 and we know that those solutions remain in the interval $[v_0, v]$ with nonnegative derivative, we'll then infer that $\Gamma^- \cap \big( ]\bar\alpha, v[\times]0, \bar\beta[\big) = \emptyset$. On the other hand, Γ^- is a continuum which projects onto $[v_0, v]$ and thus we'll conclude that $(v, 0) \notin \Gamma^-$.

To prove our claim, we assume by contradiction that there is a sequence $x_n(\cdot)$ of solutions to equation (2.7) defined on $(-\infty, t_0]$ satisfying the following properties:

$$v_0 \leq x_n(t) \leq \alpha_n := x_n(t_0) \nearrow v, \quad 0 \leq x_n'(t) \leq x_n'(t_0) := \beta_n \searrow 0^+ .$$

Arguing as in [8], we use polar coordinates around $(v, 0)$ and evaluate the angular displacement of the solutions in the phase–plane. We take the point $(v, 0)$ as origin and set

$$x(t) := v + \rho(t)\cos\varphi(t), \quad x'(t) = y(t) := \rho(t)\sin\varphi(t),$$

in order to obtain

$$-\varphi'(t) = \sin^2(\varphi(t)) + \frac{\tilde f(t, x(t))}{v - x(t)}\cos^2(\varphi(t)). \tag{2.10}$$

By taking n sufficiently large, we can suppose that $\alpha_n > v - \frac{\delta}{2}$ (for $\delta > 0$ as in $(H2)$) and therefore we can define $t_n^* < t_0$ such that $]t_n^*, t_0]$ is the maximal interval such that $x_n(t) > v - \delta$ for all $t \in]t_n^*, t_0]$. One can see that $t_n^* \to -\infty$. In fact, either $t_n^* = -\infty$, or $\frac{\delta}{2} \leq x_n(t_0) - x_n(t_n^*) \leq (t_0 - t_n^*)\beta_n$, so that $t_n^* \leq t_0 - \frac{\delta}{2\beta_n}$. From $(H2)$ and (2.10) we can now write

$$-\varphi'(t) \geq \sin^2(\varphi(t)) + \eta(t)\cos^2(\varphi(t)), \quad \text{for a.e. } t \in]t_n^*, t_0] \tag{2.11}$$

and thus we see that φ is strictly decreasing as long as $x' > 0$. The length of the time-interval on which $\varphi(t) \in [\frac{\pi}{2}, \frac{3\pi}{4}]$ is at most $\frac{\pi}{2}$. Therefore, we can take

$$\tau_n^* \in [t_0 - \frac{\pi}{2}, t_0]$$

such that

$$\varphi(t) \in [\frac{3\pi}{4}, \pi], \forall\, t \in [t_n^*, \tau_n^*].$$

Now, from (2.11), we have

$$-\varphi'(t) \geq \frac{1}{2}\,\eta(t), \quad \text{for a.e. } t \in]t_n^*, \tau_n^*].$$

An integration of this inequality on $[t_n^*, \tau_n^*]$ yields

$$\int_{t_n^*}^{\tau_n^*} \eta(t)\,dt \le \pi.$$

As a consequence, we find

$$\int_{t_n^*}^{t_0} \eta(t)\,dt \le \pi + M, \quad \text{with } M := \int_{t_0-\frac{\pi}{2}}^{t_0} \eta(t)\,dt.$$

Letting $n \to \infty$ and recalling that $t_n^* \to -\infty$, we get a contradiction with respect to $(H2)$. This ends the proof of (i_2).

Now we complete the proof of (i_3). We check first that $u_{-\infty} = v_0$ in the case when $u(t_0) = \alpha \in]v_0, v]$ and (by (i_2)) $u'(t_0) = \beta > 0$. To this aim, we argue by contradiction as follows. Assume $u_{-\infty} > v_0$. If $\alpha = v$, as $\beta > 0$ we have $u_{-\infty} < v$. Hence, we have $v_0 < u_{-\infty} < v$. With respect to $a = u_{-\infty}$, there are $\varepsilon, t_\varepsilon$ and γ_ε as in assumption $(H1)$. We also take $\varepsilon < v - a$ and denote by τ_ε the largest $t \le t_\varepsilon$ such that $u(\theta) \in [a, a+\varepsilon]$ for every $\theta \le t$. An integration of (2.7) gives the following estimate:

$$\beta \ge u'(\tau_\varepsilon) = u'(t) + \int_t^{\tau_\varepsilon} \tilde{f}(\theta, u(\theta))\,d\theta \ge \int_t^{\tau_\varepsilon} \gamma_\varepsilon(\theta)\,d\theta := \mathcal{G}_\varepsilon(t)$$

and the first condition in $(H1)$ leads to a contradiction. In fact, $\mathcal{G}_\varepsilon(-\infty) = +\infty$.

At last, for each $(\alpha, \beta) \in \Gamma^-$ with $\alpha \in]v_0, v]$ and $\beta > 0$, let u be any solution to equation (2.7) and let us denote by $]\tau_u^1, t_0]$ the maximal interval where $u(t) > v_0$ and by $]\tau_u^2, t_0]$ the maximal interval where $u'(t) > 0$, respectively. We have already proved that $\beta > 0$ so that both the intervals are well defined. By definition, we have that, if $\tau_u^2 > -\infty$, then $u'(t) = 0$ and also $u(t) = u(\tau_u^2)$ for every $t \le \tau_u^2$. Hence, $u(\tau_u^2) = v_0$ and therefore, $\tau_u^2 \le \tau_u^1$. Conversely, if $\tau_u^1 > -\infty$, since we know that $u(t) \ge v_0$, then $u(t) = v_0$ for every $t \le \tau_u^1$. Hence we conclude that $u'(t) = 0$ for every $t \le \tau_u^1$ which proves that $\tau_u^1 \le \tau_u^2$. This suffices to guarantee that $\tau_u^1 = \tau_u^2 := \tau_u$ and $u(t) > v_0$ as well as $u'(t) > 0$ for all $t \in]\tau_u, t_0]$ with $]\tau_u, t_0]$ the maximal interval with such property. Moreover, if $\tau_u > -\infty$, then $u(t) = v_0$ and $u'(t) = 0$, $\forall\, t \le \tau_u$.

We have thus proved that all the properties (i_1)-(i_2)-(i_3) hold with respect to the solutions of the truncated equation (2.7) emanating from Γ^- at $t = t_0$. Since we have found solutions with range in the interval $[v_0, v]$ we can conclude that the same is true also with respect to the solutions of $u'' - f(t, u) = 0$. □

Remark 2.1 In property (i_3) of the theorem above we claim that for every $(\alpha, \beta) \in \Gamma^-$ there is *at least* one solution with the desired asymptotic properties. Since we don't assume the uniqueness of the solutions to the Cauchy problems, we cannot guarantee that the same is true for every solution of $u'' - f(t, u) = 0$ with initial values in Γ^-. In fact, one could imagine the situation in which there is a solution starting at $(\alpha, \beta) \in \Gamma^-$ for $t = t_0$, reaching the level v_0 in finite time (say $t = \tau_u < t_0$) and then being extended for $t < \tau_u$ in any (nonconstant) manner compatible to the fact of being a solution of the differential equation.

Remark 2.2 Condition $(H2)$ in Theorem 2.1 is adapted (in a more general setting) from a similar assumption considered by Conley in [8] for a particular model. Since $(H2)$ looks rather technical, a natural question is if such a condition may be avoided. Indeed, from a first look, the convexity and monotonicity of the positive solutions, with respect to v_0, of problem (P_-) would lead to the conjecture that the continuum Γ^- we find is the graph of a monotonically non-decreasing function with respect to $u(t_0)$ and therefore it cannot end in $(v, 0)$ even in the case when $f(\cdot, v) \equiv 0$. This conjecture is indeed true in the autonomous case and it may be verified in various interesting situations; however, it is not true in general. The next example shows that if $f(\cdot, v) \equiv 0$, then some extra condition like $(H2)$ must be imposed in order to prevent the possibility that there exists an unstable manifold of $(v_0, 0)$ which ends in $(v, 0)$. The same example also shows that the monotonicity of Γ^- as a graph of the $u(t_0)$–variable is not guaranteed.

Example 2.1 Let $\nu : \mathbb{R} \to \mathbb{R}$ be a C^1–function with

$$\nu(0) = 0, \;\; \nu(1) = 1, \;\; \nu'(s) > 0, \; \forall\, s \in [0, 1].$$

Observe that $\nu([0, 1]) = [0, 1]$ and define the continuously differentiable map

$$\Psi = (\Psi_1, \Psi_2) : \mathbb{R}^2 \to \mathbb{R}^2, \quad \Psi : (t, s) \mapsto (t, x), \text{ with } x = \nu(s) \exp\bigl((1 - s)t\bigr).$$

It is easy to check that Ψ is a bijection of $(-\infty, 0] \times [0, 1]$ onto itself and also a C^1–diffeomorphism on a neighborhood of $(-\infty, 0] \times [0, 1]$. Let $\Phi = (\Phi_1, \Phi_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1–map such that

$$\Phi(t, x) = \Psi^{-1}(t, x), \quad \forall\, (t, x) \in \, (-\infty, 0] \times [0, 1].$$

Now we define

$$f(t, x) := x(1 - \Phi_2(t, x))^2.$$

By construction,

$$f(t, 0) = f(t, 1) = 0, \quad \forall\, t \in (-\infty, 0]$$

and the continuum

$$\mathcal{C}^- := \bigl\{ \bigl(\nu(s), \nu(s)(1 - s)\bigr) : \; s \in [0, 1] \bigr\} \subseteq \mathbb{R}^2 \tag{2.12}$$

is such that, for each $(\alpha, \beta) \in \mathcal{C}^- \setminus \{(1, 0)\}$, there exists a solution $u(\cdot)$ of (P_-), for $v_0 = 0$, satisfying $u(0) = \alpha$, $u'(0) = \beta$. Indeed, for $(\alpha, \beta) = (\nu(s), \nu(s)(1 - s))$, the solution is given by

$$u(t) = u_s(t) = \nu(s) \exp\bigl((1 - s)t\bigr). \tag{2.13}$$

Of course, for $(\alpha, \beta) = (1, 0) \in \mathcal{C}^-$, the unique solution departing at time $t = t_0 := 0$ from that point is the constant one $u \equiv 1$.

Finally, we point out that, as a consequence of results on exponential dichotomy and hyperbolic critical sets for non-autonomous systems (see [36, Th. 3.6.3, p.56]), it is possible to prove that the manifold of initial points in $[0, 1] \times \mathbb{R}^+$ which are asymptotic at $-\infty$ to $(0, 0)$ is unique and thus it coincides with the set $\mathcal{C}^-$ defined in

(2.12). Therefore, for this particular example, there is no hope to find a continuum of initial points of solutions of (P_-), for $v_0 = 0$, which ends at $v = 1$ with $u'(0) > 0$. □

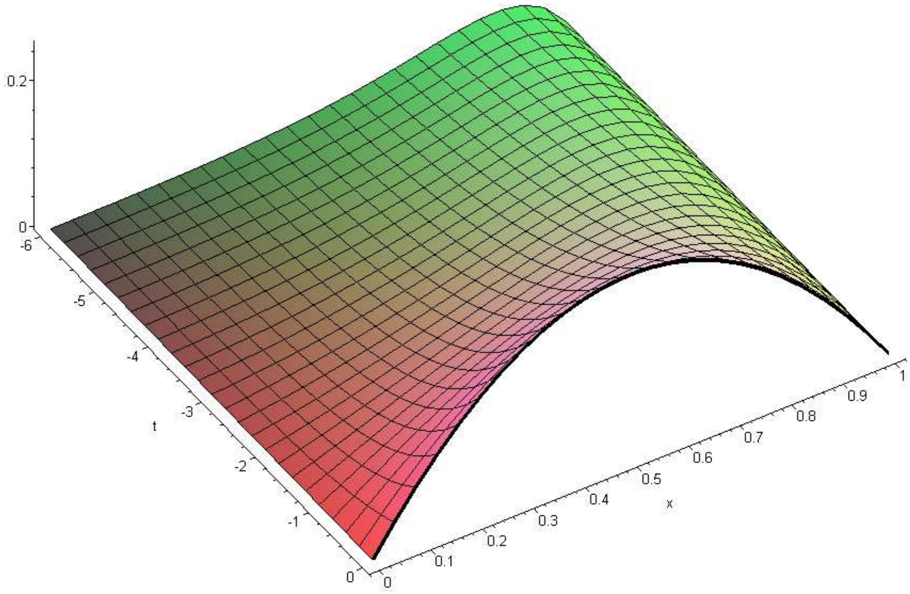

Figure 1. Graph of the parametric surface $(t, s) \mapsto u_s(t)$ with $u_s(t)$ defined as in (2.13), for $\nu(s) := s$. The set $\mathcal{C}^-$ in (2.12) corresponds to the thicker line at the level $t = 0$.

We can also give examples in which Γ^- is not necessarily the graph of a continuous function in the $u(t_0)$–variable. With this respect, we refer to Example 3.1 of the next section where the case of unbounded sets of initial points is also considered.

Theorem 2.1 guarantees the existence of a continuum Γ^- made by initial points of solutions of (P_-) with $u(t) \geq v_0$ and $u'(t) \geq 0$. In a completely similar manner, we can produce a continuum Υ^- made by initial points of solutions of (P_-) with $u(t) \leq v_0$ and $u'(t) \leq 0$. For this variant of Theorem 2.1 we assume $v < v_0$ as well as that $f : (-\infty, t_0] \times [v, v_0] \to \mathbb{R}$ satisfies the Carathéodory assumptions and, moreover, (2.1) with

$$\forall\, s \in [v, v_0[, \quad f(t, s) \leq 0 \text{ for a.e. } t \in (-\infty, t_0]. \tag{2.14}$$

Now we are ready to present our result. The precise statement reads as follows:

Theorem 2.2 *Assume* (2.1) *and* (2.14) *and suppose that the following conditions hold:*

$(H1^*)$ *for each* $a \in]v, v_0[$, *there are* $\varepsilon > 0$, $t_\varepsilon \le t_0$ *and a locally integrable function* $\gamma_\varepsilon = \gamma_{a,\varepsilon} : (-\infty, t_\varepsilon] \to \mathbb{R}^-$ *such that*

$$\int_{-\infty}^{t_\varepsilon} \gamma_\varepsilon(\theta)\, d\theta = -\infty$$

and

$$f(t,s) \le \gamma_\varepsilon(t), \quad \forall\, s \in [a - \varepsilon, a], \quad \text{for a.e. } t \le t_\varepsilon\,,$$

$(H2^*)$ *either* $f(t,v) < 0$ *in a subset of positive measure of* $(-\infty, t_0]$, *or* $f(t,v) = 0$ *for a.e.* $t \in (-\infty, t_0]$ *and there exist* $\delta > 0$ *and a locally integrable function* $\eta = \eta_\delta : (-\infty, t_0] \to \mathbb{R}^-$ *such that*

$$\int_{-\infty}^{t_0} \eta(\theta)\, d\theta = -\infty$$

and

$$f(t,s) \le \eta(t)(s - v), \quad \forall\, s \in [v, v+\delta], \ \text{for a.e. } t \in (-\infty, t_0].$$

Then, there exists a continuum $\Upsilon^- \subseteq [v, v_0] \times \mathbb{R}^-$ *satisfying the following properties:*

(i_1^*) $\forall\, \alpha \in [v, v_0]\,, \exists\, \beta \le 0$ *such that* $(\alpha, \beta) \in \Upsilon^-$.

(i_2^*) $\Upsilon^- \cap \big([v, v_0] \times \{0\}\big) = \Upsilon^- \cap \big(\{v_0\} \times \mathbb{R}^-\big) = \{(v_0, 0)\}$.

(i_3^*) *For each* $(\alpha, \beta) \in \Upsilon^-$, *there exists a solution* $u(\cdot)$ *of* (P_-) *such that* $u(t_0) = \alpha$, $u'(t_0) = \beta$. *Moreover, if* $\alpha \in [v, v_0[$, *then there is a maximal interval* $]\tau_u, t_0]$ *such that* $u(t) \in [v, v_0[$ *and* $u'(t) < 0$, *for all* $t \in]\tau_u, t_0]$. *If* $\tau_u > -\infty$, *then* $u(t) = v_0$ *and* $u'(t) = 0$, $\forall\, t \le \tau_u$.

The proof of Theorem 2.2 follows, line by line, as that of Theorem 2.1 (with obvious changes in some inequalities) and therefore it is omitted.

As a next step, we now study the problem of the search of solutions tending to an equilibrium point in the forward time. Therefore, we consider the following symmetric version of problem (P_-) :

$$(P_+) \qquad \begin{cases} u'' - f(t,u) = 0 \\ u(+\infty) = v_1 \\ u'(+\infty) = 0, \end{cases}$$

where $f : [t_1, +\infty) \times [v, v_1] \to \mathbb{R}$ satisfies the Carathéodory assumptions and, moreover,

$$f(\cdot, v_1) \equiv 0, \tag{2.15}$$

$$\forall\, s \in [v, v_1[\,, \quad f(t,s) \le 0 \ \text{ for a.e. } \ t \in [t_1, +\infty). \tag{2.16}$$

Consequently, we have also a symmetric version of Theorem 2.1, namely, the next result.

Theorem 2.3 *Assume* (2.15) *and* (2.16) *and suppose that the following conditions hold:*

$(K1)$ *for each* $a \in]v, v_1[$, *there are* $\varepsilon > 0$, $t_\varepsilon \geq t_1$ *and a locally integrable function* $\gamma_\varepsilon = \gamma_{a,\varepsilon} : [t_\varepsilon, +\infty) \to \mathbb{R}^-$ *such that*

$$\int_{t_\varepsilon}^{+\infty} \gamma_\varepsilon(\theta)\, d\theta = -\infty$$

and

$$f(t,s) \leq \gamma_\varepsilon(t), \quad \forall\, s \in [a-\varepsilon, a], \quad \textit{for a.e. } t \geq t_\varepsilon\,,$$

$(K2)$ *either* $f(t,v) < 0$ *in a subset of positive measure of* $[t_1, +\infty)$, *or* $f(t,v) = 0$ *for a.e.* $t \in [t_1, +\infty)$ *and there exist* $\delta > 0$ *and a locally integrable function* $\eta = \eta_\delta : [t_1, +\infty) \to \mathbb{R}^-$ *such that*

$$\int_{t_1}^{+\infty} \eta(\theta)\, d\theta = -\infty$$

and

$$f(t,s) \leq \eta(t)(s-v), \quad \forall\, s \in [v, v+\delta], \;\; \textit{for a.e. } t \in [t_1, +\infty).$$

Then, there exists a continuum $\Gamma^+ \subseteq [v, v_1] \times \mathbb{R}^+$ *satisfying the following properties:*

(j_1) $\forall\, \alpha \in [v, v_1]\,, \exists\, \beta \geq 0$ *such that* $(\alpha, \beta) \in \Gamma^+$.

(j_2) $\Gamma^+ \cap \big([v, v_1] \times \{0\} \big) = \Gamma^+ \cap \big(\{v_1\} \times \mathbb{R}^+ \big) = \{(v_1, 0)\}$.

(j_3) *For each* $(\alpha, \beta) \in \Gamma^+$, *there exists a solution* $u(\cdot)$ *of* (P_+) *such that* $u(t_1) = \alpha$, $u'(t_1) = \beta$. *Moreover, if* $\alpha \in [v, v_1[$, *then there is a maximal interval* $[t_1, \tau_u[$ *such that* $u(t) \in [v, v_1[$ *and* $u'(t) > 0$, *for all* $t \in [t_1, \tau_u[$. *If* $\tau_u < +\infty$, *then* $u(t) = v_1$ *and* $u'(t) = 0$, $\forall\, t \geq \tau_u$.

This result guarantees the existence of a continuum Γ^+ made by initial points of solutions of (P_+) with $u(t) < v_1$. Similarly, we can also produce a continuum Υ^+ to the right of $(v_1, 0)$. For this variant of Theorem 2.3 we assume $v > v_1$ as well as that $f : [t_1, +\infty) \times [v_1, v] \to \mathbb{R}$ satisfies the Carathéodory assumptions and, moreover, (2.15) with

$$\forall\, s \in]v_1, v]\,, \quad f(t,s) \geq 0 \;\text{ for a.e. }\; t \in [t_1, +\infty). \tag{2.17}$$

Now we can present our result which is a symmetric version of Theorem 2.2.

Theorem 2.4 *Assume* (2.15) *and* (2.17) *and suppose that the following conditions hold:*

($K1^*$) for each $a \in]v_1, v[$, *there are* $\varepsilon > 0$, $t_\varepsilon \geq t_1$ *and a locally integrable function* $\gamma_\varepsilon = \gamma_{a,\varepsilon} : [t_\varepsilon, +\infty) \to \mathbb{R}^+$ *such that*

$$\int_{t_\varepsilon}^{+\infty} \gamma_\varepsilon(\theta)\, d\theta = +\infty$$

and

$$f(t,s) \geq \gamma_\varepsilon(t), \quad \forall\, s \in [a, a+\varepsilon], \quad \text{for a.e. } t \geq t_\varepsilon\,,$$

($K2^*$) either $f(t,v) > 0$ *in a subset of positive measure of* $[t_1, +\infty)$, *or* $f(t,v) = 0$ *for a.e.* $t \in [t_1, +\infty)$ *and there exist* $\delta > 0$ *and a locally integrable function* $\eta = \eta_\delta : [t_1, +\infty) \to \mathbb{R}^+$ *such that*

$$\int_{t_1}^{+\infty} \eta(\theta)\, d\theta = +\infty$$

and

$$f(t,s) \geq \eta(t)(v - s), \quad \forall\, s \in [v - \delta, v], \quad \text{for a.e. } t \in [t_1, +\infty).$$

Then, there exists a continuum $\Upsilon^+ \subseteq [v_1, v] \times \mathbb{R}^-$ *satisfying the following properties:*

(j_1^*) $\forall\, \alpha \in [v_1, v]\,, \exists\, \beta \leq 0$ *such that* $(\alpha, \beta) \in \Upsilon^+$.

(j_2^*) $\Upsilon^+ \cap \big([v_1, v] \times \{0\} \big) = \Upsilon^+ \cap \big(\{v_1\} \times \mathbb{R}^- \big) = \{(v_1, 0)\}$.

(j_3^*) *For each* $(\alpha, \beta) \in \Upsilon^+$, *there exists a solution* $u(\cdot)$ *of* (P_+) *such that* $u(t_1) = \alpha$, $u'(t_1) = \beta$. *Moreover, if* $\alpha \in]v_1, v]$, *then there is a maximal interval* $[t_1, \tau_u[$ *such that* $u(t) \in]v_1, v]$ *and* $u'(t) < 0$, *for all* $t \in [t_1, \tau_u[$. *If* $\tau_u < +\infty$, *then* $u(t) = v_1$ *and* $u'(t) = 0$, $\forall\, t \geq \tau_u$.

Note that Theorem 2.1 and Theorem 2.4 (as well as Theorem 2.2 and Theorem 2.3, respectively) correspond to each other through the change of variable $t \mapsto -t$.

The next result shows how the continuum Γ^- moves along the flow of the differential equation from $t = t_0$ to $t = t_1$. A similar theorem holds for Γ^+ in backward time. We give our result in Theorem 2.5 below for a continuum Γ which is not necessarily the Γ^- of Theorem 2.1.

Theorem 2.5 *Let* $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfy the Carathéodory assumptions. Suppose* $v_0 < v_1$ *are such that, for* $i = 0, 1$, *the following conditions hold:*

(E) $f(\cdot, v_i) \equiv 0$;

(L) *there are* $\delta > 0$ *and* $\eta \in L^1([t_0, t_1], \mathbb{R}^+)$ *such that,* $|f(t,s)| \leq \eta(t)|s - v_i|$, $\forall\, s : |s - v_i| < \delta$ *with* $s \in [v_0, v_1]$ *and for a.e.* $t \in [t_0, t_1]$.

Assume that $\Gamma \subseteq [v_0, v_1] \times \mathbb{R}$ *is a continuum such that*

$$\Gamma \cap \big(\{v_0\} \times \mathbb{R} \big) = \{(v_0, 0)\}$$

and

$$\Gamma \cap \big(\{v_1\} \times \mathbb{R}\big) = \Gamma \cap \big(\{v_1\} \times \mathbb{R}_0^+\big) \neq \emptyset.$$

Then, there is a continuum $\hat{\Gamma} \subseteq [v_0, v_1] \times \mathbb{R}$ *with*

$$\hat{\Gamma} \cap \big(\{v_0\} \times \mathbb{R}\big) = \hat{\Gamma} \cap \big(\{v_0\} \times \mathbb{R}^-\big) \neq \emptyset,$$

$$\hat{\Gamma} \cap \big(\{v_1\} \times \mathbb{R}\big) = \hat{\Gamma} \cap \big(\{v_1\} \times \mathbb{R}_0^+\big) \neq \emptyset$$

and, moreover, for every $\hat{\xi} \in \hat{\Gamma}$ *there is a solution* $u(\cdot)$ *of*

$$u'' - f(t,u) = 0$$

defined on $[t_0, t_1]$ *such that*

$$(u(t_0), u'(t_0)) \in \Gamma, \quad (u(t_1), u'(t_1)) = \hat{\xi} \qquad \textit{and } u(t) \in [v_0, v_1], \ \forall\, t \in [t_0, t_1].$$

Proof. First of all, we consider the truncation $\tilde{f}$ of the field f, following the same procedure like in (2.3) and take the differential equation (2.7) with initial point in Γ, that is, we have

$$\begin{cases} u'' - \tilde{f}(t,u) = 0 \\ (u(t_0), u'(t_0)) = \xi \in \Gamma. \end{cases} \tag{2.18}$$

Notice that for the truncated equation we have the global existence (backward and forward) of the solutions. Also, uniqueness in forward time (respectively in backward time) for solutions with initial condition $u(t_0) > v_1$ and $u'(t_0) > 0$ ($u(t_0) < v_0$ and $u'(t_0) > 0$) holds.

By Lemma 6.3 in the Appendix there is a continuum $S \subseteq \Gamma$, with $S \cap \big(\{v_0\} \times \mathbb{R}\big) = \{(v_0, 0)\}$ and $S \cap \big(\{v_1\} \times \mathbb{R}\big) \neq \emptyset$ and there exists a continuum $\mathcal{C} \subseteq S \times C^1([t_0, t_1])$ such that $Pr_1(\mathcal{C}) = S$ and, for every $(\xi, u) \in \mathcal{C}$, $u(\cdot)$ is a solution of $u'' - \tilde{f}(t,u) = 0$, defined on $[t_0, t_1]$ with $(u(t_0), u'(t_0)) = \xi \in S$. We define now the continuous map

$$\Pi : \mathcal{C} \to \mathbb{R}^2, \quad \Pi : (\xi, u(\cdot)) \mapsto (u(t_1), u'(t_1)).$$

Observe that $\Pi(\mathcal{C})$ is a continuum of the plane.

We claim that

$$(v_0, 0) \in \Pi(\mathcal{C})$$

and, moreover, there exists $(v, w) \in \Pi(\mathcal{C})$ with $v > v_1$ and $w > 0$. To prove the first part of the claim, it is sufficient to note that if $u(\cdot)$ is a solution to (2.18) with $\xi = (v_0, 0)$, then, necessarily $u(t) = v_0$ and $u'(t) = 0$, for all $t \in [t_0, t_1]$. This, in turn, follows from (E) and (L) with $i = 0$. In fact, assume, by contradiction, that there exists a solution $z(\cdot) = (x(\cdot), y(\cdot)) = (u(\cdot), u'(\cdot))$ to system

$$\begin{cases} x' = y \\ y' = \tilde{f}(t,x) \end{cases} \tag{2.19}$$

with $z(t_0) = (v_0, 0)$ and such that $z(t) \neq (v_0, 0)$ for some $t \in]t_0, t_1]$. We denote by $\tau \in [t_0, t_1[$ the maximal time such that $z(t) = (v_0, 0)$ for all $t \in [t_0, \tau]$. Let $\varepsilon \in]0, t_1 - \tau]$ be such that $|x(t) - v_0| < \delta$ for every $t \in [\tau, \tau + \varepsilon]$. Integrating (2.19) on $[\tau, \tau + \varepsilon]$, and using (L), we obtain

$$||z(t) - z(\tau)|| = ||z(t) - (v_0, 0)|| \leq \int_\tau^t (1 + \eta(\theta))||z(\theta) - (v_0, 0)||\, d\theta.$$

The Gronwall inequality [12] implies that

$$||z(t) - (v_0, 0)|| \leq 0 \times \exp\Big(\int_\tau^t (1 + \eta(\theta))\, d\theta \Big) = 0, \quad \forall\, t \in [\tau, \tau + \varepsilon],$$

which contradicts the maximality of τ.

To prove the second part of the claim, let us consider any solution $z(\cdot) = (x(\cdot), y(\cdot)) = (u(\cdot), u'(\cdot))$ to system (2.19) with $z(t_0) = (v_1, y_1) \in S \cap \big(\{v_1\} \times \mathbb{R}\big)$ (we recall that the uniqueness of the solutions is not assumed). From $x'(t_0) = y(t_0) = y_1 > 0$ and $x(t_0) = v_1$ we know that there is $\varepsilon \in]0, t_1 - t_0]$ such that $x(t) > v_1$ and $y(t) > 0$, for every $t \in]t_0, t_0 + \varepsilon]$. Hence, by the definition of the truncated function, we have that $y'(t) = \tilde{f}(t, x(t)) = \tilde{f}(t, v_1) = f(t, v_1) = 0$, for every $t \in [t_0, t_0 + \varepsilon]$, which implies that $(x(t), y(t)) = (v_1 + (t - t_0)y_1, y_1)$ for each $t \in [t_0, t_0 + \varepsilon]$. By the uniqueness in forward time mentioned above we obtain $(x(t_1), y(t_1)) = (v_1 + (t_1 - t_0)y_1, y_1)$ and the second part of the claim is proved.

Now, as a consequence of the Whyburn lemma (see Lemma 6.1 in the Appendix) we can get a sub-continuum $\hat{\Gamma}$ of $\Pi(\mathcal{C})$ such that

$$\hat{\Gamma} \cap \big(\{v_0\} \times \mathbb{R} \big) \neq \emptyset, \quad \hat{\Gamma} \cap \big(\{v_1\} \times \mathbb{R} \big) \neq \emptyset$$

and

$$\hat{x} \in [v_0, v_1], \;\; \forall\, (\hat{x}, \hat{y}) = \hat{\xi} \in \hat{\Gamma}.$$

By definition, for each $\hat{\xi} \in \hat{\Gamma}$, there is a $\xi \in \Gamma$ and a solution $u(\cdot)$ to (2.18) such that $(u(t_1), u'(t_1)) = \hat{\xi}$. Clearly, for such a solution $u(\cdot)$, we have $v_0 \leq u(t_0) \leq v_1$.

Suppose $(v_0, \hat{y}) = \hat{\xi} \in \hat{\Gamma} \cap \big(\{v_0\} \times \mathbb{R}\big)$. If we assume that $\hat{y} > 0$, arguing as above but now using the uniqueness in backward time for solutions with initial condition $u(t_1 - \varepsilon) < v_0$ and $u'(t_1 - \varepsilon) > 0$, we arrive at a contradiction with the fact that $u(t_0) \geq v_0$. In this way we conclude that $\hat{y} \leq 0$.

Suppose that $(v_1, \hat{y}) = \hat{\xi} \in \hat{\Gamma} \cap \big(\{v_1\} \times \mathbb{R}\big)$. We claim that $\hat{y} > 0$. The case $\hat{y} < 0$ is excluded by arguing as above. The case $\hat{y} = 0$ is excluded by assumptions (E) and (L) for $i = 1$, using the Gronwall inequality.

To conclude our proof, we need to show that if $u(\cdot)$ is a solution to (2.18) such that $(u(t_0), u'(t_0)) = \xi \in \Gamma$ and $(u(t_1), u'(t_1)) = \hat{\xi} \in \hat{\Gamma}$, then

$$v_0 \leq u(t) \leq v_1\,, \quad \forall\, t \in [t_0, t_1],$$

so that $u(\cdot)$ is actually a solution of $u'' - f(t, u) = 0$. The proof of this last claim follows, mutatis mutandis, the same arguments given above and it is omitted. □

Clearly, one can obtain an analogous result by moving the continuum Υ^- in forward time, or the continuum Υ^+ in backward time.

3 Unbounded connected branches of initial points

In this section we just outline a further variant of Theorem 2.1 in which we find a closed connected and unbounded set of initial points for problem

$$(P_-) \qquad \begin{cases} u'' - f(t,u) = 0 \\ u(-\infty) = v_0 \\ u'(-\infty) = 0. \end{cases}$$

We suppose that $f : (-\infty, t_0] \times [v_0, +\infty) \to \mathbb{R}$ satisfies the Carathéodory assumptions and, moreover,

$$f(\cdot, v_0) \equiv 0, \tag{3.1}$$

$$\forall\, s \in]v_0, +\infty)\,, \quad f(t,s) \geq 0 \ \text{ for a.e. } \ t \in (-\infty, t_0]. \tag{3.2}$$

Then we have:

Theorem 3.1 *Assume* (3.1) *and* (3.2) *and suppose that the following condition holds:*

> $(J1)$ *for each* $a > v_0$ *, there are* $\varepsilon > 0$, $t_\varepsilon \leq t_0$ *and a locally integrable function* $\gamma_\varepsilon = \gamma_{a,\varepsilon} : (-\infty, t_\varepsilon] \to \mathbb{R}^+$ *such that*
>
> $$\int_{-\infty}^{t_\varepsilon} \gamma_\varepsilon(\theta)\, d\theta = +\infty$$
>
> *and*
>
> $$f(t,s) \geq \gamma_\varepsilon(t), \quad \forall\, s \in [a, a+\varepsilon], \quad \textit{for a.e. } \ t \leq t_\varepsilon\,.$$

Then, there exists a closed connected set $\Gamma^- \subseteq [v_0, +\infty) \times \mathbb{R}^+$ *satisfying the following properties:*

> (l_1) $\forall\, \alpha \geq v_0\,, \exists\, \beta \geq 0$ *such that* $(\alpha, \beta) \in \Gamma^-$.
>
> (l_2) $\Gamma^- \cap \big([v_0, +\infty) \times \{0\}\big) = \Gamma^- \cap \big(\{v_0\} \times \mathbb{R}^+\big) = \{(v_0, 0)\}$.
>
> (l_3) *For each* $(\alpha, \beta) \in \Gamma^-$, *there exists a solution* $u(\cdot)$ *of* (P_-) *such that* $u(t_0) = \alpha$, $u'(t_0) = \beta$. *Moreover, if* $\alpha > v_0$, *then there is a maximal interval* $]\tau_u, t_0]$ *such that* $u(t) > v_0$ *and* $u'(t) > 0$, *for all* $t \in]\tau_u, t_0]$. *If* $\tau_u > -\infty$, *then* $u(t) = v_0$ *and* $u'(t) = 0$, $\forall\, t \leq \tau_u$.

Proof. For each positive integer m, we apply Theorem 2.1 choosing

$$v = v_m := v_0 + m,$$

and find a continuum $\Gamma^-(m)$ of initial points satisfying $(i_1), (i_2), (i_3)$. From the proof of Theorem 2.1 (see (2.6)) we also know that there exists a constant $K = K(m)$ such that

$$\Gamma^-(m) \subseteq [v_0, v_m] \times [0, K(m)]$$

and, furthermore, from the definition of K in (2.5), it follows that

$$\Gamma^-(m+1) \cap [v_0, v_m] \times \mathbb{R}^+ \subseteq [v_0, v_m] \times [0, K(m)], \quad \forall m.$$

Hence we can consider a continuous function $\mu : [v_0, +\infty) \to \mathbb{R}_0^+$ with $\mu(v) > \mu(v_0) = 0$, $\forall\, v > v_0$, such that

$$\Gamma^-(m) \subseteq [v_0, v_m] \times [0, \mu(v_m)], \quad \forall\, m.$$

Let $\mathcal{A}_\infty := \mathcal{A} \cup \{p_\infty\}$ be the one-point compactification of the locally compact set

$$\mathcal{A} := \{(x, y) \in [v_0, +\infty) \times \mathbb{R}^+ \ : y \leq \mu(x)\}.$$

The result in [18, §47,II;p.171] about limits of continua guarantees the existence of a set

$$\Gamma^-_\infty := \limsup_{m\to\infty} \Gamma(m) \subseteq \mathcal{A}_\infty$$

which is compact and connected with respect to the topology of $\mathcal{A}_\infty$. By a result on irreducible continua (see [2, 18]) it follows that $\Gamma^-_\infty \setminus \{p_\infty\}$ contains a connected set Γ^- with $(v_0, 0) \in \Gamma^-$ and such that Γ^- is closed relatively to $\mathcal{A}$ and the closure of Γ^- in $\mathcal{A}_\infty$ is given by $\Gamma^- \cup \{p_\infty\}$. Thus

$$\Gamma^- \subseteq [v_0, +\infty) \times \mathbb{R}^+$$

is a closed connected set satisfying (l_1). Since every $(\alpha, \beta) \in \Gamma^-$ is a limit of a sequence of points from which depart solutions satisfying (2.8), it is immediate to verify that the property (P_1) obtained in the proof of Theorem 2.1 also holds for the unbounded branch Γ^- that we have obtained above. From condition (P_1) the properties (l_2) and (l_3) easily follow (just repeating an analogous argument as that in the proof of Theorem 2.1). □

In the same manner, one could obtain analogous versions of theorems 2.2, 2.3, or 2.4, getting unbounded branches of initial points Υ^-, Γ^+, or Υ^+, respectively. For the sake of conciseness, we leave to the interested reader the task of formulating the precise statements of the corresponding theorems, having Theorem 3.1, as a model.

A possible question related to the above results is whether the connected sets of initial points that we have found have some special structure. In particular, one could be interested to know whether such sets are the graphs of a continuous function in the $u(t_0) = \alpha$–variable. For this reason, we present now an example (see Example 3.1 below) which shows that the answer is negative, in general. To this aim, we give first two propositions which may have some independent interest as they show how to construct examples of ODEs possessing connected branches of initial points satisfying some desired properties.

Proposition 3.1 *Let $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ be a C^1–function with $\nu'(s) > 0$ for every $s \geq 0$ and such that*

$$\nu(0) = 0, \quad \nu(+\infty) = +\infty.$$

Let also $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing C^1–function satisfying

$$\chi(s) > 0, \quad \forall\, s > 0.$$

Then, for every $\tau \in \mathbb{R}$, there exists a locally Lipschitz continuous function $f : (-\infty, \tau] \times \mathbb{R}^+ \to \mathbb{R}^+$, with

$$f(t, 0) = 0, \quad \forall\, t \leq \tau,$$

such that, for every $s \in \mathbb{R}^+$, there exists a (unique) solution $u_s(\cdot)$ to

$$\begin{cases} u'' - f(t, u) = 0 \\ u(\tau) = \nu(s), \quad u'(\tau) = \nu(s)\chi(s) \end{cases}$$

and $(u_s(t), u_s'(t)) \to (0, 0)$ for $t \to -\infty$.

Proof. Following a modification of the argument employed in the construction of Example 2.1, we define the continuously differentiable map

$$\Psi = (\Psi_1, \Psi_2) : \mathbb{R}^2 \to \mathbb{R}^2, \quad \Psi : (t, s) \mapsto (t, x), \text{ with } x = \nu(s) \exp\big(\chi(s)(t - \tau)\big),$$

where ν and χ have been extended to the whole real line as C^1–functions. It is easy to check that Ψ is a bijection of $(-\infty, \tau] \times \mathbb{R}^+$ onto itself and also a C^1–diffeomorphism on a neighborhood of $(-\infty, \tau] \times \mathbb{R}^+$. Let $\Phi = (\Phi_1, \Phi_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1–map such that

$$\Phi(t, x) = \Psi^{-1}(t, x), \quad \forall\, (t, x) \in (-\infty, \tau] \times \mathbb{R}^+.$$

Now we define

$$f(t, x) := x \left(\chi\big(\Phi_2(t, x)\big) \right)^2.$$

By construction,

$$f(t, 0) = 0, \quad \forall\, t \in (-\infty, \tau]$$

and the unbounded closed connected set

$$\Gamma_\tau^- := \{ \big(\nu(s), \nu(s)\chi(s) \big) : s \in \mathbb{R}^+ \}$$

is such that for each $(\alpha, \beta) \in \Gamma_\tau^-$, there exists a solution $u(\cdot)$ of (P_-), for $v_0 = 0$, satisfying $u(\tau) = \alpha$, $u'(\tau) = \beta$. By direct investigation, one easily sees that the solution $u_s(\cdot)$ corresponding to the initial point $(\nu(s), \nu(s)\chi(s)) \in \Gamma_\tau^-$ is

$$u_s(t) = \nu(s) \exp\big(\chi(s)(t - \tau)\big).$$

This ends the proof of our result. □

Proposition 3.2 *Let $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ be a C^1–function which is strictly increasing and satisfies*

$$\nu(0) = 0, \quad \nu(+\infty) = +\infty.$$

Let also $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ be a C^1–function such that

$$\chi(s) > 0, \quad \forall\, s > 0.$$

Suppose that $f : [\tau_1, \tau_2] \times \mathbb{R}^+ \to \mathbb{R}^+$ is a locally Lipschitz continuous function with

$$f(t, 0) = 0, \qquad \forall\, t \in [\tau_1, \tau_2]$$

and there exists a constant $M > 0$ such that

$$f(t, x) \leq M, \qquad \forall\, (t, x) \in [\tau_1, \tau_2] \times \mathbb{R}^+\,.$$

Let $u_s(\cdot)$ be the unique solution of

$$\begin{cases} u'' - f(t, u) = 0 \\ u(\tau_1) = \nu(s), \quad u'(\tau_1) = \nu(s)\chi(s)\,. \end{cases}$$

Then the map $\mathbb{R}^+ \ni s \mapsto \gamma(s) := (u_s(\tau_2), u_s'(\tau_2)) \in \mathbb{R}^+ \times \mathbb{R}^+$ is continuous, with $\gamma(0) = (0, 0)$, $u_s(\tau_2) \geq \nu(s) > 0$ and $u_s'(\tau_2) \geq \nu(s)\chi(s) > 0$, for every $s > 0$.

Proof. The locally Lipschitz condition and the boundedness of f, together with the fact that $f \geq 0$, imply that, for every $(\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+$, there exists a unique solution $u(\cdot) = u_{(\alpha,\beta)}$ of $x'' - f(t, x) = 0$ satisfying $(u(\tau_1), u'(\tau_1)) = (\alpha, \beta)$ and, moreover, such a solution is defined on $[\tau_1, \tau_2]$ and such that

$$u'(t) \geq u'(\tau_1), \quad u(t) \geq u(\tau_1) + u'(\tau_1)(t - \tau_1), \quad \forall\, t \in [\tau_1, \tau_2].$$

Suppose now that $\alpha = \nu(s)$, $\beta = \nu(s)\chi(s)$ and set

$$u_s(\cdot) := u_{(\nu(s), \nu(s)\chi(s))}(\cdot).$$

From the above inequalities, evaluated at $t = \tau_2$, we get that $u_s'(\tau_2) \geq \nu(s)\chi(s)$ and $u_s(\tau_2) \geq \nu(s)$. The theorem of continuous dependence of the solutions from initial data also imply that the map $s \mapsto \gamma(s)$ is continuous. Finally, as a consequence of $f(t, 0) = 0$ for every $t \in [\tau_1, \tau_2]$, we get $\gamma(0) = (0, 0)$. This completes the proof. □

Example 3.1 Let $\nu, \chi : \mathbb{R}^+ \to \mathbb{R}^+$ be two continuously differentiable maps satisfying the following conditions:

$$\nu(0) = 0, \;\; \nu(+\infty) = +\infty, \quad \nu'(s) > 0, \;\; \chi(s) > 0, \;\; \chi'(s) < 0, \quad \forall\, s \geq 0. \tag{3.3}$$

Let $\varepsilon \in]0, 1[$ and define, for $0 < \delta \leq 1$, the function

$$g(t, x) := \begin{cases} \dfrac{x}{\delta} + \dfrac{(x/\delta) - x\,\chi(\nu^{-1}(x))^2}{\varepsilon}\,(t - \varepsilon), & \text{for} \quad 0 \leq x \leq \delta, \quad t \in [0, \varepsilon] \\[2ex] 1 + \dfrac{1 - x\,\chi(\nu^{-1}(x))^2}{\varepsilon}\,(t - \varepsilon), & \text{for} \quad x \geq \delta, \quad t \in [0, \varepsilon] \end{cases}$$

$$g(t,x) := \begin{cases} x/\delta, & \text{for} \quad 0 \le x \le \delta, \qquad t \in [\varepsilon,1] \\ 1, & \text{for} \qquad x \ge \delta, \qquad t \in [\varepsilon,1] \end{cases}$$

where we assume also that

$$\sup_{s \ge 0} \nu(s)\,\chi(s)^2 < \infty. \tag{3.4}$$

By Proposition 3.1 (applied with $\tau = 0$) and Proposition 3.2 (applied with $\tau_1 = 0$ and $\tau_2 = 1$), we can define a locally Lipschitz continuous function $f : (-\infty, 1] \times \mathbb{R}^+ \to \mathbb{R}^+$ with $f(t,x) = g(t,x)$ for $(t,x) \in [0,1] \times \mathbb{R}^+$ such that, for every $s \in \mathbb{R}^+$ there exists a (unique) solution $u_s(\cdot)$ for problem

$$\begin{cases} u'' - f(t,u) = 0 \\ u(0) = \nu(s), \quad u'(0) = \nu(s)\chi(s) \end{cases}$$

with $u_s(\cdot)$ defined on $(-\infty, 1]$ and such that $(u_s(t), u_s'(t)) \to (0,0)$ as $t \to -\infty$. Moreover, the map

$$\gamma = (\gamma_1, \gamma_2) : \mathbb{R}^+ \ni s \mapsto (u_s(1), u_s'(1))$$

is continuous and satisfies the properties described at the end of Proposition 3.2. In particular, note also that $\gamma_i(s) > 0$ for $s > 0$ and $i = 1, 2$.

Let us consider now the set

$$\Gamma_1^- := \{\gamma(s) : s \in \mathbb{R}^+\} \subseteq \mathbb{R}^+ \times \mathbb{R}^+$$

which is closed, connected unbounded (at least in its first component) and is also a set of initial points for problem (P_-) for $t_0 = 1$ and $v_0 = 0$.

The definition of $f = g$ for $t \in [0,1]$ allows us to estimate $u_s(1) = \gamma_1(s)$ for $s \ge 1$. Indeed, we have:

$$\gamma_1(s) = \nu(s) + \nu(s)\chi(s) + \frac{1}{2} + O(\varepsilon).$$

Now, if there exist s_1, s_2, with

$$\nu(s_1) + \nu(s_1)\chi(s_1) > \nu(s_2) + \nu(s_2)\chi(s_2), \qquad \text{for } 1 \le s_1 < s_2, \tag{3.5}$$

and we have that (for $\varepsilon > 0$ small enough) the map γ_1 takes all the values from 0 to $\gamma_1(s_1)$ for $s \in [0, s_1]$, then it takes all the values from $\gamma_1(s_2)$ to $\gamma_1(s_1)$ (with $0 < \gamma_1(s_2) < \gamma_1(s_1)$) and, finally, it takes again all the values in $[\gamma_1(s_2), +\infty)$ (including again the value $\gamma_1(s_1)$) for $s \in [s_2, +\infty)$. This shows that Γ_1^- is a S–shaped curve and it is not the graph of a continuous function in the $u(1)$–variable.

To conclude the example, we have to show that it is possible to find functions ν and χ satisfying assumptions (3.3), (3.4) and (3.5). A possible example is given by the choice:

$$\nu(s) := s, \quad \chi(s) := \frac{2ck}{1 + k^2 s^2}, \qquad (c, k > 0),$$

which satisfies (3.3) and (3.4) and is such that $\nu(s)\chi(s) = \dfrac{2cks}{1 + k^2 s^2}$ has c as maximum value for $s = \frac{1}{k}$. Choosing c and k suitably, we can find intervals included in $[1, +\infty)$ where the function $\nu(s)(1 + \chi(s))$ is strictly decreasing and thus have (3.5) satisfied for an appropriate choice of $s_1 < s_2$. □

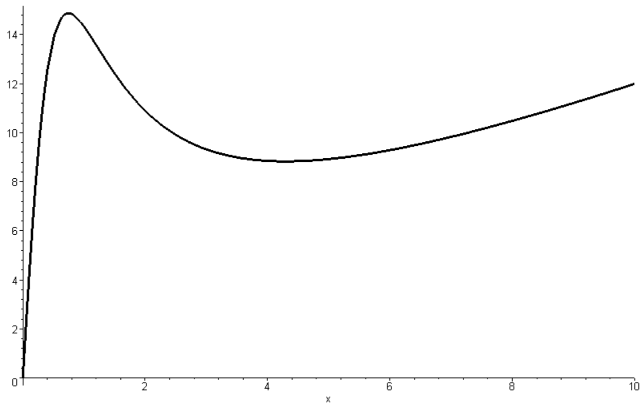

Figure 2. Graph of the function $\nu(s)(1+\chi(s)) = s + \dfrac{2cks}{1+k^2s^2}$ with $k=2$ and $c=10$.

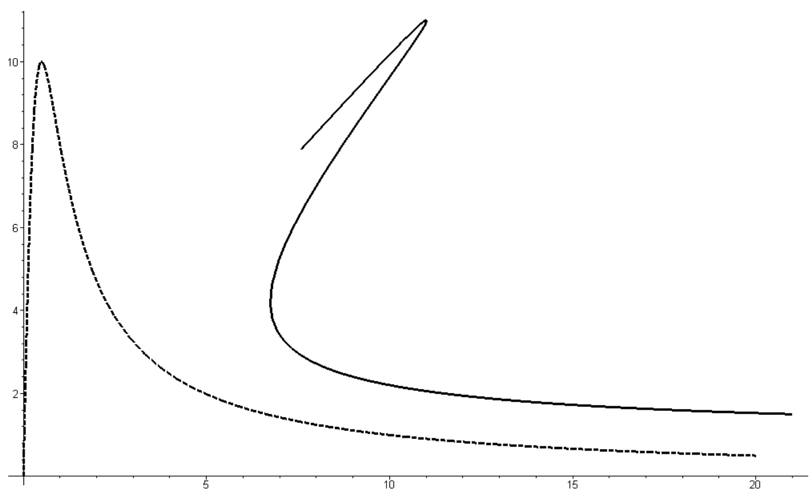

Figure 3. Image of the path $\gamma(s) = (u_s(1), u_s'(1))$ with $s \in [\delta, 20]$ for $\varepsilon > 0$ sufficiently small and $\delta = 0.2$ (thick line), compared to the path $s \mapsto (\nu(s), \nu(s)\chi(s))$ with $s \in [0, 20]$ (dashed line). The functions $\nu(s)$ and $\chi(s)$ are those chosen for Figure 2.

We end this section with a result (Lemma 3.1, below) which provides a useful tool in order to join two unbounded connected sets in the phase–plane by means of large and oscillating solutions of a first order differential system. Lemma 3.1 is taken from [27] and [28] and is adapted from previous works by Butler [7], Hartman [13] and Struwe [33]. It has been applied in a different context in [24, Theorem 2], looking for solutions of a generalized Sturm - Liouville problem joining two sets of initial points from which depart solutions which blow up at the boundary of a certain interval. For the sake of simplicity, we confine the presentation of our result to the case

$$f(t, x) = -w(t)g(x).$$

We define the half–planes

$$H^+ := \{(x, y) \in \mathbb{R}^2 : x > 0\} \cup \{(0, y) \in \mathbb{R}^2 : y > 0\} \quad \text{and} \quad H^- := -H^+ .$$

Lemma 3.1 *Let $w : [t_0, t_1] \to \mathbb{R}$ be a continuous and piecewise monotone function such that*

$$w \geq 0 \ \text{ and } \ w \not\equiv 0 \ \text{ on } \ [t_0, t_1].$$

Assume that $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$\lim_{s \to \pm\infty} \frac{g(s)}{s} = +\infty.$$

Suppose also that, for some $v_0 \in \mathbb{R}$ such that $g(v_0) = 0$, it holds that

$$\left| \frac{g(s)}{s - v_0} \right| \quad \text{is bounded in a neighborhood of } \ v_0 .$$

Let $\Gamma_0 \subseteq \bar{A}$ and $\Gamma_1 \subseteq \bar{B}$ be two unbounded closed and connected sets, where $\bar{A}$ and $\bar{B}$ are two (not necessarily distinct) closed quadrants of the plane with origin in $P_0 := (v_0, 0)$; let also $R > 0$ be such that

$$\Gamma_i \cap B[R] \neq \emptyset, \qquad \text{for } \ i = 0, 1.$$

Then there is $n^* = n^*_R$ such that the following hold:

- *if Γ_0 and Γ_1 are both in $P_0 + H^+$ or both in $P_0 + H^-$ then, for every $n \geq n^*$ and n even, there is at least one solution $z(\cdot) = (x(\cdot), y(\cdot))$ of*

$$x' = y, \quad y' = -w(t)g(x) \tag{3.6}$$

  such that

$$z(t_0) \in \Gamma_0, \quad z(t_1) \in \Gamma_1, \quad |z(t)| \geq R, \quad \forall\, t \in [t_0, t_1]$$

  and $x(\cdot) - v_0$ has exactly n zeros on $]t_0, t_1]$;

- *if $\Gamma_0 \subseteq P_0 + H^+$ and $\Gamma_1 \subseteq P_0 + H^-$ (or vice versa) then, for every $n \geq n^*$ and n odd, there is at least one solution $z(\cdot)$ of (3.6) such that*

$$z(t_0) \in \Gamma_0, \quad z(t_1) \in \Gamma_1, \quad |z(t)| \geq R, \quad \forall\, t \in [t_0, t_1]$$

  and $x(\cdot) - v_0$ has exactly n zeros on $]t_0, t_1]$.

We refer to [28] for the case in which the weight function $w(\cdot)$ may change sign on $[t_0, t_1]$.

4 Heteroclinic orbits

In this section we present a few results which show how Theorem 2.1 (as well as its variants) and Theorem 2.5 permit us to obtain the existence of heteroclinic solutions by intersecting the continua of initial points. The applications we consider are inspired by the work of Conley in [8]. The technique that we have developed in the previous sections allows us to deal with more general examples which could be the topic of a future research. We refer to [1, 11, 32, 34] for other interesting results in this area.

Let $v_0 < v_1$ and suppose that $f : \mathbb{R} \times [v_0, v_1] \to \mathbb{R}$ satisfies the Carathéodory conditions and, moreover,

$$f(t, v_0) \equiv 0, \quad f(t, v_1) \equiv 0$$

and for every $s \in]v_0, v_1[$,

$$f(t, s) \geq 0 \text{ for a.e. } t \leq t_0\,, \quad f(t, s) \leq 0 \text{ for a.e. } t \geq t_1\,.$$

Then, the following theorem holds:

Theorem 4.1 *Assume* $(H1)$–$(H2)$ *with* $v = v_1$ *and* $(K1)$–$(K2)$ *with* $v = v_0$. *Suppose also that* (L) *is satisfied. Then, there exists a solution* $\tilde{u}$ *to equation*

$$u'' - f(t, u) = 0 \tag{4.1}$$

such that

$$\tilde{u}(-\infty) = v_0\,, \quad \tilde{u}(+\infty) = v_1\,, \quad v_0 < \tilde{u}(t) < v_1\,, \;\; \forall\, t \in \mathbb{R}$$

and

$$\tilde{u}'(t) > 0, \quad \forall\, t \in (-\infty, t_0] \cup [t_1, +\infty).$$

Proof. We apply Theorem 2.1 and Theorem 2.3 to get two compact connected sets Γ^- and Γ^+ satisfying (i_1), (i_2), (i_3) and (j_1), (j_2), (j_3), respectively. We observe that in (i_3) we always have $\tau_u = -\infty$ (respectively $\tau_u = +\infty$ in (j_3)), due to the local Lipschitz type condition (L). Next we apply Theorem 2.5 and obtain a continuum $\hat{\Gamma}$ from Γ^-.

Both the continua Γ^+ and $\hat{\Gamma}$ are contained in a rectangle $[v_0, v_1] \times [-M, M]$, where $M > 0$ is a sufficiently large constant. We set

$$A_0 := \hat{\Gamma} \cap \big(\{v_0\} \times \mathbb{R}\big) \neq \emptyset, \quad B_0 := \hat{\Gamma} \cap \big(\{v_1\} \times \mathbb{R}\big) \neq \emptyset,$$

$$A_1 := \Gamma^+ \cap \big(\{v_0\} \times \mathbb{R}\big) \neq \emptyset, \quad B_1 := \Gamma^+ \cap \big(\{v_1\} \times \mathbb{R}\big) \neq \emptyset$$

and, observe that, as a consequence of Theorem 2.5 and Theorem 2.3, we have

$$y \leq 0, \; \forall\, (x, y) \in A_0 \quad \text{and} \quad y > 0, \; \forall\, (x, y) \in B_0\,,$$

$$y > 0, \; \forall\, (x, y) \in A_1 \quad \text{and} \quad y \leq 0, \; \forall\, (x, y) \in B_1\,.$$

By Lemma 6.2 in the Appendix it follows that there exists $(\alpha, \beta) \in \hat{\Gamma} \cap \Gamma^+$. By definition of $\hat{\Gamma}$ there exists a point $(\alpha^-, \beta^-) \in \Gamma^-$ and a solution $\tilde{u}(\cdot)$ to (4.1) defined on $[t_0, t_1]$, such that

$$\tilde{u}(t_0) = \alpha^-, \;\; \tilde{u}'(t_0) = \beta^-, \;\; \tilde{u}(t_1) = \alpha, \;\; \tilde{u}'(t_1) = \beta$$

and $v_0 < \tilde{u}(t) < v_1$ for every $t \in [t_0, t_1]$. Moreover, from $(\alpha^-, \beta^-) \in \Gamma^-$ departs a solution (that we still denote by $\tilde{u}$) defined on $(-\infty, t_0]$ for problem (P_-). Similarly, from $(\alpha, \beta) \in \Gamma^+$ departs a solution (that we still denote by $\tilde{u}$) defined on $[t_1, +\infty)$ for problem (P_+). Patching together these three solutions to a solution $\tilde{u}$ of (4.1) defined on the whole $\mathbb{R}$ and recalling the qualitative properties of such a solution coming from (i_3) and (j_3) we achieve the thesis. □

As we have just seen, in order to prove Theorem 4.1 we have performed a link between Γ^- and Γ^+ through the intermediate continuum $\hat{\Gamma}$. We can achieve the same conclusion in a more direct way when some additional symmetry conditions are imposed on f. A result in this direction is the following:

Let $v_0 < v_1$ and set

$$p := \frac{v_0 + v_1}{2}\,.$$

Suppose that $f : \mathbb{R} \times [v_0, v_1] \to \mathbb{R}$ satisfies the Carathéodory conditions and, moreover,

$$f(t, v_0) \equiv 0, \quad f(t, v_1) \equiv 0,$$

as well as there exists $t_0 \in \mathbb{R}$ such that, for every $s \in \,]v_0, p[\,$,

$$f(t, s) \geq 0 \;\text{ for a.e. }\; t \leq t_0\,.$$

We also define

$$\hat{f}(t, s) := f(t + t_0, p + s).$$

The following result holds:

Theorem 4.2 *Assume* $(H1)$–$(H2)$ *with* $v = p$. *Suppose also that* (L) *is satisfied and that* $\hat{f}(t, s)$ *is even with respect to* t *and odd with respect to* s. *Then, there exists a solution* $\tilde{u}$ *to equation* (4.1) *such that*

$$\tilde{u}(-\infty) = v_0\,, \quad \tilde{u}(+\infty) = v_1\,, \quad v_0 < \tilde{u}(t) < v_1\,, \;\; \forall\, t \in \mathbb{R}$$

and

$$\tilde{u}'(t) > 0, \quad \forall\, t \in \mathbb{R}.$$

Proof. We apply Theorem 2.1 to equation (4.1) in order to have a continuum

$$\Gamma^- \subseteq [v_0, p] \times \mathbb{R}^+.$$

If we take a point $(p, \hat{y}) \in \Gamma^-_* \cap \left(\{p\} \times \mathbb{R}^+_0\right)$, we have that there exists a solution $\hat{u}(\cdot)$ to equation (4.1) such that

$$\hat{u}(-\infty) = v_0\,, \quad \hat{u}(t_0) = p, \quad \hat{u}'(t_0) = \hat{y}\,, \quad v_0 < \hat{u}(t) < p\,, \;\; \forall\, t < t_0$$

and

$$\hat{u}'(t) > 0, \quad \forall\, t \in (-\infty, t_0].$$

By the symmetry of $\hat{f}$ we can easily see that the function

$$\tilde{u}(t) := \begin{cases} \hat{u}(t), & t \le t_0, \\ v_1 + v_0 - \hat{u}(2t_0 - t), & t \ge t_0 \end{cases}$$

is a solution of (4.1) with the desired properties. □

We now show some applications of the above results to the simplified equation

$$u'' + w(t)g(u) = 0,$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function and $w : \mathbb{R} \to \mathbb{R}$ is a measurable function with $w \in L^1_{\rm loc}(\mathbb{R})$. We start by assuming the existence of at least a zero for the function g, that is

$$g(v_0) = 0.$$

A consequence of Theorem 2.1 for

$$f(t,u) := -w(t)g(u) \tag{4.2}$$

is the following:

Corollary 4.1 *Let $v > v_0$ and suppose that*

$$g(s) > 0, \quad \forall\, s \in]v_0, v[$$

and, either $g(v) > 0$, or $g(v) = 0$ and $g(s)$ is differentiable at $s = v$ with $g'(v) < 0$. Assume also that there exists $t_0 \in \mathbb{R}$ such that

$$w(t) \le 0, \text{ for a.e. } t \le t_0 \text{ and } \int_{-\infty}^{t_0} w(t)\, dt = -\infty.$$

Then, there exists a continuum $\Gamma^- \subseteq [v_0, v] \times \mathbb{R}^+$ satisfying $(i_1), (i_2), (i_3)$ of Theorem 2.1.

Proof. The proof follows straightforwardly from Theorem 2.1 by (4.2). We just observe that $(H2)$ is trivially satisfied when $g(v) > 0$, while, if $g(v) = 0$, it holds by taking $\eta(t) = -w(t)g'(v)/2$. On the other hand, if $a \in]v_0, v[$ we can take any $\varepsilon > 0$ with $a + \varepsilon < v$ and $(H1)$ holds with $\gamma_\varepsilon(t) := -w(t) \min_{[a,a+\varepsilon]} g$. □

Similar corollaries may be easily obtained from Theorem 2.2, Theorem 2.3 and Theorem 2.4.

Now we are in position to obtain a result on the existence of an heteroclinic connection for equation (4.2) which follows immediately from Theorem 4.1.

Corollary 4.2 *Let $v_0 < v_1$ be such that*

$$g(v_0) = g(v_1) = 0 \quad \text{and} \;\; g(s) > 0, \;\; \forall\, s \in]v_0, v_1[\,.$$

Suppose also that $g(s)$ is differentiable at $s = v_0$ and $s = v_1$ with

$$g'(v_0) > 0 > g'(v_1).$$

Assume there are $t_0\,, t_1$ with $t_0 \le t_1$ such that

$$w(t) \le 0, \text{ for a.e. } t \le t_0\,, \quad w(t) \ge 0, \text{ for a.e. } t \ge t_1$$

and

$$\int_{-\infty}^{t_0} w(t)\,dt = -\infty, \qquad \int_{t_1}^{+\infty} w(t)\,dt = +\infty.$$

Then, there exists a solution $\tilde{u}$ to equation (4.2) such that

$$\tilde{u}(-\infty) = v_0\,, \quad \tilde{u}(+\infty) = v_1\,, \quad v_0 < \tilde{u}(t) < v_1\,, \;\; \forall\, t \in \mathbb{R}$$

and

$$\tilde{u}'(t) > 0, \quad \forall\, t \in (-\infty, t_0] \cup [t_1, +\infty).$$

5 Homoclinic orbits

In this section we obtain homoclinic orbits for equation (4.1) by suitably connecting the unbounded branches of initial points Γ^- and Υ^+ produced in Section 3. We also present various examples of piecewise autonomous equations where an accurate computation of the time map allows a thorough discussion of the existence of a connection between these continua leading to a positive homoclinic orbit.

5.1 Multibump solutions

We give an application of the results obtained in the previous section, by proving the existence of homoclinic solutions which possess a large number of zeros. For simplicity, we restrict ourselves to the case of equation

$$u'' + w(t)g(u) = 0, \tag{5.1}$$

where, as in Section 4, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function and $w : \mathbb{R} \to \mathbb{R}$ is a measurable function with $w \in L^1_{\text{loc}}(\mathbb{R})$. We also assume that there exists $v_0 \in \mathbb{R}$ such that

$$g(v_0) = 0.$$

Our first result extends Corollary 4.1 by producing unbounded connected branches in the phase–plane emanating from $(v_0, 0)$.

Corollary 5.1 *Let*

$$g(s) > 0, \quad \forall s > v_0\,.$$

Suppose there exists $t_0 \in \mathbb{R}$ such that

$$w(t) \leq 0, \text{ for a.e. } t \leq t_0 \text{ and } \int_{-\infty}^{t_0} w(t)\,dt = -\infty. \tag{5.2}$$

Then, there exists an unbounded closed connected set $\Gamma^- \subseteq [v_0, +\infty) \times \mathbb{R}^+$, with

$$\Gamma^- \cap \big([v_0, +\infty) \times \{0\}\big) = \Gamma^- \cap \big(\{v_0\} \times \mathbb{R}^+\big) = \{(v_0, 0)\},$$

such that for each $(\alpha, \beta) \in \Gamma^-$, there exists a solution $u(\cdot)$ of (5.1) *such that $u(t_0) = \alpha$, $u'(t_0) = \beta$ and $(u(t), u'(t)) \to (v_0, 0)$ for $t \to -\infty$, with $u(\cdot) \geq v_0$ convex and non-decreasing on $(-\infty, t_0]$.*

Suppose there exists $t_1 \in \mathbb{R}$ such that

$$w(t) \leq 0, \text{ for a.e. } t \geq t_1 \text{ and } \int_{t_1}^{+\infty} w(t)\,dt = -\infty. \tag{5.3}$$

Then, there exists an unbounded closed connected set $\Upsilon^+ \subseteq [v_0, +\infty) \times \mathbb{R}^-$, with

$$\Upsilon^+ \cap \big([v_0, +\infty) \times \{0\}\big) = \Upsilon^+ \cap \big(\{v_0\} \times \mathbb{R}^-\big) = \{(v_0, 0)\},$$

such that for each $(\alpha, \beta) \in \Upsilon^+$, there exists a solution $u(\cdot)$ of (5.1) *such that $u(t_0) = \alpha$, $u'(t_0) = \beta$ and $(u(t), u'(t)) \to (v_0, 0)$ for $t \to +\infty$, with $u(\cdot) \geq v_0$ convex and non-increasing on $[t_1, +\infty)$.*

The proof is a straightforward application of Theorem 3.1 and its variants indicated in Section 3.

Putting together Corollary 5.1 with Lemma 3.1, we obtain:

Theorem 5.1 *Let g be differentiable in v_0 with $g(v_0) = 0$. Suppose also that $g(s) > 0$, $\forall s > v_0$ and*

$$\lim_{s \to \pm\infty} \frac{g(s)}{s} = +\infty.$$

Assume there are $t_0 < t_1$ such that $w(\cdot)$ is continuous and piecewise monotone on $[t_0, t_1]$,

$$w(\cdot) \geq 0, \quad \text{and } w \not\equiv 0 \text{ on } [t_0, t_1]$$

and with $w(\cdot)$ satisfying also (5.2) *and* (5.3). *Then, there is n^* such that, for each even integer $n \geq n^*$, there exists at least one solution $u(\cdot)$ of* (5.1), *with $u(\cdot) - v_0$ having exactly n zeros on $]t_0, t_1]$ and such that $(u(t), u'(t)) \to (v_0, 0)$ for $t \to \pm\infty$, with $u(\cdot) \geq v_0$ convex and non-decreasing on $(-\infty, t_0]$ and convex non-increasing on $[t_1, +\infty)$.*

For some special nonlinearities and weights it is possible to provide some estimates of n^*. In particular, if $n^* = 0$, Theorem 5.1 would allow to prove the existence of solutions with $u(\cdot) - v_0$ positive, as well as with $u(\cdot) - v_0$ having any even number of zeros. Some specific examples for positive homoclinics are given in the next subsection.

5.2 Examples

We consider the equation

$$u'' + h(t,u) = 0 \tag{5.4}$$

with

$$h(t,u) = a_+(t)h(u) - a_-(t)g(u)$$

and

$$a_+(t) = a_+\,\phi_{[t_0,t_1]}(t), \quad a_-(t) = a_-\,\phi_{]-\infty,t_0]\cup[t_1,+\infty[}(t),$$

where $a_+, a_- > 0$ and ϕ_I is the characteristic function of the interval I. With this choice, equation (5.4) splits into two autonomous equations:

$$u'' + a_+h(u) = 0, \quad \text{on } [t_0,t_1] \tag{5.5}$$

$$u'' - a_-g(u) = 0, \quad \text{on }]-\infty,t_0] \cup [t_1,+\infty). \tag{5.6}$$

In what follows, we let for simplicity $a_+ = a_- = 1$ and assume that g and h are continuous functions on $\mathbb{R}$ satisfying

$$sg(s) > 0 \text{ and } sh(s) > 0, \quad s \neq 0.$$

In this case $(0,0)$ is the unique equilibrium point in the phase–plane $(x,y) = (u,u')$ of the first order system associated to (5.4). We look for positive homoclinic orbits in $(0,0)$. For this aim, we will make use of phase–plane analysis to investigate when it is possible to glue the trajectories corresponding to the autonomous equations (5.5) and (5.6) in order to get these homoclinic orbits.

We start by defining

$$G(s) := \int_0^s g(\xi)\,d\xi.$$

Setting $(v_0,0) = (v_1,0) = (0,0)$, by the energy conservation law applied to equation (5.6) we have that

$$\Gamma^- = \{(x,y) : y = \sqrt{2G(x)},\ x \geq 0\} \quad \text{and} \quad \Upsilon^+ = \{(x,y) : y = -\sqrt{2G(x)},\ x \geq 0\}.$$

That is, Γ^- and Υ^+ coincide, respectively, with the branches of the unstable and of the stable manifolds of $(0,0)$ which lie in the first and in the fourth quadrants.

As a consequence, we note that positive homoclinic orbits in $(0,0)$ correspond to positive solutions $u(t)$ of equation (5.5) such that $(u(t_0),u'(t_0)) \in \Gamma^-$ and $(u(t_1),u'(t_1)) \in \Upsilon^+$. Equivalently, positive homoclinics correspond to trajectories of (5.5) which start at time t_0 from $(u_0,v_0) \in \Gamma^-$ and that meet the x-axis after a travel of duration $\tau(u_0,v_0) = \frac{t_1-t_0}{2}$. Therefore, under our assumptions, a positive homoclinic exists for equation (5.4) if and only if $\frac{t_1-t_0}{2}$ belongs to the image of the time map τ restricted to Γ^-. Our aim in what follows is to describe such an image when the functions $g(u)$ and $h(u)$ are powers of u.

Let

$$H(s) := \int_0^s h(\xi)\, d\xi$$

and consider a point of the form $(c,0)$, $c>0$. The trajectory associated to $u''+h(u)=0$ which passes through $(c,0)$ is defined by $\frac{y^2}{2}+H(x)=H(c)$. Considering the initial point $(c,0)$, this trajectory will intersect Γ^- in backward time at a point that we denote by $(s(c),\sqrt{2G(s(c))})$, after a travel of duration $\tau(c)$. Then, $s=s(c)$ satisfies

$$K(s) := G(s)+H(s)=H(c), \tag{5.7}$$

where K is a continuous increasing function. It follows that

$$0 < s = s(c) := K^{-1}(H(c)) < c$$

is a continuous increasing function, too. Moreover, by the energy conservation law for equation (5.5), we get

$$\tau(c) = \int_{s(c)}^{c} \frac{du}{\sqrt{2(H(c)-H(u))}} \tag{5.8}$$

and $\tau :]0,+\infty[\to]0,+\infty[$, $c \to \tau(c)$, is a continuous function of c.

In what follows, in order to study the behavior of the function τ when $c\to 0^+$ and $c\to+\infty$ we specialize the functions g and h. More precisely, we start with a first class of examples where we set for $u\geq 0$

$$g(u)=au^{\alpha}, \quad h(u)=bu^{\beta}, \quad \text{with } \ a,b,\alpha,\beta>0, \quad \alpha\neq\beta.$$

Then, of course, $G(u)=\dfrac{au^{\alpha+1}}{\alpha+1}$, $H(u)=\dfrac{bu^{\beta+1}}{\beta+1}$ and equations (5.8) and (5.7) take the form

$$\tau(c) = \int_{s(c)}^{c} \frac{du}{\sqrt{\frac{2b}{\beta+1}(c^{\beta+1}-u^{\beta+1})}},$$

and

$$\frac{as^{\alpha+1}}{\alpha+1}+\frac{bs^{\beta+1}}{\beta+1}=\frac{bc^{\beta+1}}{\beta+1}. \tag{5.9}$$

In order to draw our conclusions about the behavior of the function τ, we assume that $\alpha>\beta>1$. If we consider the change of variables $u=tc$ and $s=\theta c$, then the equation (5.8) will be transformed in

$$\tau(c)=\sqrt{\frac{\beta+1}{2b}}\,c^{\frac{1-\beta}{2}}\int_{\theta}^{1}\frac{dt}{\sqrt{1-t^{\beta+1}}}, \tag{5.10}$$

and from equation (5.9) we can explicit c as a function of θ obtaining:

$$c=c(\theta)=\left[\frac{b(\alpha+1)}{a(\beta+1)}\,\frac{1-\theta^{\beta+1}}{\theta^{\alpha+1}}\right]^{\frac{1}{\alpha-\beta}}. \tag{5.11}$$

From this last equation we see that $0 < \theta < 1$ and that $c \to 0^+ \Leftrightarrow \theta \to 1^-$ and $c \to +\infty \Leftrightarrow \theta \to 0^+$. Finally, substituting (5.11) into (5.10), we obtain

$$\tau(\theta) = M \left[\frac{\theta^{\alpha+1}}{1-\theta^{\beta+1}}\right]^{\frac{\beta-1}{2(\alpha-\beta)}} \int_\theta^1 \frac{dt}{\sqrt{1-t^{\beta+1}}}, \tag{5.12}$$

where the constant M is given by

$$M = \sqrt{\frac{\beta+1}{2b}} \left[\frac{b(\alpha+1)}{a(\beta+1)}\right]^{\frac{1-\beta}{2(\alpha-\beta)}}.$$

Then, from equation (5.12) it follows immediately that $\lim_{\theta\to 0^+} \tau(\theta) = 0$. Moreover, it is not hard to show that $\lim_{\theta\to 1^-} \tau(\theta) \in \mathbb{R} \iff \alpha \geq 2\beta - 1$. Taking into account the continuity of the function τ, we can conclude that if $\beta > 1$ and $\alpha \geq 2\beta - 1$, then equation (5.4) does not admit any positive homoclinic orbit whenever $t_1 - t_0 > \sup_{\mathbb{R}_0^+} \tau(c)$, whereas it admits positive homoclinic solutions for any (t_0, t_1) if $\alpha > \beta > 1$ and $\alpha < 2\beta - 1$.

A second class of examples may be obtained if we consider

$$h(u) = \sigma g(u), \ \sigma > 0.$$

In this case equation (5.4) (in which we still set $a_+ = a_- = 1$) takes the form

$$u'' + w(t)g(u) = 0, \tag{5.13}$$

with

$$w(t) := \begin{cases} \sigma, & t \in [t_0, t_1], \\ -1, & t \in (-\infty, t_0[\cup]t_1, +\infty). \end{cases} \tag{5.14}$$

Here we also have $H(u) = \sigma G(u)$, and, from equations (5.7) and (5.8), we get, respectively,

$$G(s) = \frac{\sigma}{1+\sigma} G(c) \tag{5.15}$$

and

$$\tau(c) = \frac{1}{\sqrt{\sigma}} \int_s^c \frac{du}{\sqrt{2(G(c) - G(u))}}. \tag{5.16}$$

The change of variable $z = \sqrt{G(u)/G(c)}$ transforms (5.16) into

$$\tau(c) = \sqrt{\frac{2}{\sigma}} \int_{\sqrt{\frac{\sigma}{1+\sigma}}}^1 \frac{\sqrt{G(u)}}{g(u)} \frac{dz}{\sqrt{1-z^2}}, \tag{5.17}$$

where $u = u(c, z) = G^{-1}(z^2 G(c))$. Then, from (5.17) we obtain the following estimate for the map τ:

$$k_\sigma \min_{s \leq u \leq c} \frac{\sqrt{G(u)}}{g(u)} \leq \tau(c) \leq k_\sigma \max_{s \leq u \leq c} \frac{\sqrt{G(u)}}{g(u)}, \tag{5.18}$$

where

$$k_\sigma = \sqrt{\frac{2}{\sigma}}\left(\frac{\pi}{2} - \arcsin\sqrt{\frac{\sigma}{1+\sigma}}\right).$$

From (5.18) we see that the behavior of the function τ depends on the behavior of the function $\psi(u) := \sqrt{G(u)}/g(u)$. As a consequence, if $\lim_{u\to 0^+}\psi(u) = +\infty$ and $\lim_{u\to+\infty}\psi(u) = 0$ with ψ decreasing, then for any $\sigma > 0$ we get the existence of a positive homoclinic orbit for our equation. This is the case, for example, when $g(u) = u^\alpha$, with $\alpha > 1$. In fact, with such a choice we obtain $\psi(u) = u^{\frac{1-\alpha}{2}}/\sqrt{\alpha+1}$.

On the other hand if we assume that $g(u)$ behaves like $u^\alpha,\ \ \alpha > 1$, at $u = 0$ and like $u^\beta,\ \ 0 < \beta < 1$, for $u \to +\infty$ then

$$\min_{s=s(c)\le u\le c} \frac{\sqrt{G(u)}}{g(u)} \to +\infty$$

both for $c \to 0^+$ and $c \to +\infty$. This implies that $\tau(c)$ has a positive minimum in $\mathbb{R}_0^+$ and therefore if $\frac{t_1-t_0}{2}$ is less than this minimum there are no positive homoclinic orbits for our equation.

With reference to this last remark, we conclude the paper by presenting in detail the analysis of equation (5.13) with

$$g(x) := \min(x^\alpha, x^\beta) = \begin{cases} x^\alpha, & \text{for } 0 \le x < 1 \\ x^\beta, & \text{for } x \ge 1, \end{cases} \tag{5.19}$$

for $0 < \beta < 1 < \alpha$. The weight function $w(t)$ is defined as in (5.14). In this situation, we have

$$G(x) := \int_0^x g(s)\,ds = \begin{cases} \dfrac{x^{\alpha+1}}{\alpha+1}, & \text{for } 0 \le x < 1 \\ \dfrac{x^{\beta+1}}{\beta+1} + \dfrac{\beta-\alpha}{(\alpha+1)(\beta+1)}, & \text{for } x \ge 1. \end{cases}$$

For any $c > 0$, we can now consider the value $s(c) \in]0, c[$ corresponding to the abscissa of the points at which the trajectory through $(c, 0)$ of the system

$$x' = y, \quad y' = -\sigma g(x),$$

intersects the unstable and the stable manifolds of the origin for system

$$x' = y, \quad y' = g(x).$$

As previously observed, to find $s(c)$ we have to solve the equation (5.15) for $s > 0$.

This yields

$$s(c) = \begin{cases} \left(\dfrac{\sigma}{\sigma+1}\right)^{\frac{1}{\alpha+1}} c, & \text{for } \ 0 < c \le 1 \\[2ex] \left(\dfrac{\sigma}{\sigma+1}\right)^{\frac{1}{\alpha+1}} \left(\dfrac{\alpha+1}{\beta+1}\, c^{\beta+1} - \dfrac{\alpha-\beta}{\beta+1}\right)^{\frac{1}{\alpha+1}}, & \text{for } \ 1 < c \le c^* \\[2ex] \left(\dfrac{\sigma}{\sigma+1}\right)^{\frac{1}{\beta+1}} \left(c^{\beta+1} + \dfrac{\alpha-\beta}{\sigma(\alpha+1)}\right)^{\frac{1}{\beta+1}}, & \text{for } \ c > c^*, \end{cases}$$

where

$$c^* := \left(1 + \frac{\beta+1}{\sigma(\alpha+1)}\right)^{\frac{1}{\beta+1}}$$

is the special value of $c > 0$ such that $s(c) = 1$.

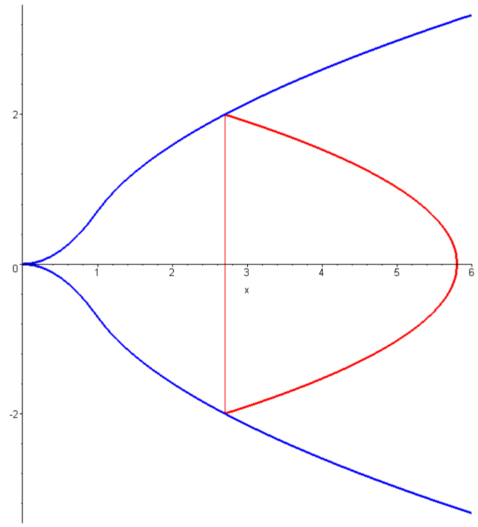

Figure 4. Example of unstable and stable manifolds in $[0, +\infty) \times \mathbb{R}$ for $u'' - g(u) = 0$, joined by an arc of trajectory of $u'' + \sigma g(u) = 0$ (with $\sigma = 0.6$). This example corresponds to the case of equation (5.13) with $w(t)$ defined as in (5.14). The unstable manifold Γ^- is the line of equation $y = \sqrt{2G(x)}$ contained in the first quadrant, while the stable manifold Υ^+ is the line of equation $y = -\sqrt{2G(x)}$ contained in the fourth quadrant. For our example, we have taken $g(x) = \min(x^\alpha, x^\beta)$ for $x \ge 0$, with $\alpha = 3$ and $\beta = 1/20$, in order to consider the case in which $g(u) \sim u^\alpha$ near zero and $g(u) \sim u^\beta$ near infinity, with $\alpha > 1 > \beta > 0$. The orbit path of $u'' + \sigma g(u) = 0$ goes from $P := (s(c), \sqrt{2G(s(c))}) \in \Gamma^-$ to $Q := (s(c), -\sqrt{2G(s(c))}) \in \Upsilon^+$, moving in the clockwise sense and passing through $(c, 0)$. In our picture, we have chosen $c := 5.8$. The thin vertical line represents the segment joining Q with P.

As a last task, we can compute $\tau(c)$ according to (5.16) and by taking into account the choice of g in (5.19), as well as the corresponding form of G. In this manner, we find a critical constant

$$\tau_* := \min\{\tau(c) : c > 0\},$$

such that there exists a homoclinic solution $u(\cdot)$ of (5.13) with $u(t) > 0$ for every $t \in \mathbb{R}$, if and only if $t_1 - t_0 \geq 2\tau_*$.

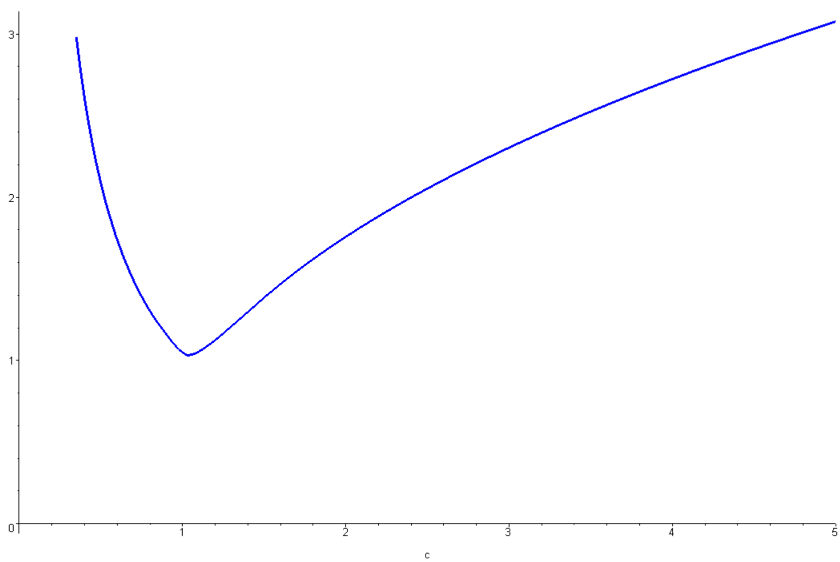

Figure 5. Graph of the time–mapping function $\tau(c)$ considered in (5.16). The function $g(x) = \min(x^\alpha, x^\beta)$ for $x \geq 0$, and the coefficients α, β and σ are those taken in Figure 4.

6 Appendix

We collect in this Section some technical results that we employ in the proof of some theorems of this paper. Our first result is a consequence of the Kuratowsky - Whyburn lemma [18, 35]. It allows us to cut any continuum to a sub-continuum which lies between two level sets of a given continuous map (see also [25]).

Lemma 6.1 *Let $\mathcal{S}$ be a continuum in a metric space X. Let $\jmath : X \to \mathbb{R}$ be a continuous function and let $\alpha, \beta \in \mathbb{R}$, with $\alpha < \beta$, be such that*

$$\mathcal{S} \cap \jmath^{-1}\big((-\infty, \alpha]\big) \neq \emptyset, \quad \mathcal{S} \cap \jmath^{-1}\big([\beta, +\infty)\big) \neq \emptyset.$$

Then, there is a continuum $\mathcal{S}^* \subseteq \mathcal{S}$ *such that*

$$\jmath(\mathcal{S}^*) = [\alpha, \beta].$$

Proof. The connectedness of $\mathcal{S}$ implies that $\jmath(\mathcal{S}) \supseteq [\alpha, \beta]$. Then we define the compact set

$$Z := \{x \in \mathcal{S} : \jmath(x) \in [\alpha, \beta]\}$$

and the nonempty disjoint compact sets

$$A := \mathcal{S} \cap \jmath^{-1}\big(\{\alpha\}\big) \subseteq Z, \quad B := \mathcal{S} \cap \jmath^{-1}\big(\{\beta\}\big) \subseteq Z.$$

Whyburn lemma ensures that, either there exists a continuum $\mathcal{S}^* \subseteq Z$ with $\mathcal{S}^* \cap A \neq \emptyset$ and $\mathcal{S}^* \cap B \neq \emptyset$, or A and B are separated in Z, that is, there exist compact sets F_1 and F_2 with $F_1 \supseteq A$, $F_2 \supseteq B$, with $F_1 \cap F_2 = \emptyset$ and $F_1 \cup F_2 = Z$. If the first of the two alternatives occurs, we have $\jmath(\mathcal{S}^*) \subseteq [\alpha, \beta]$ (as $\mathcal{S}^* \subseteq Z$) and $\alpha, \beta \in \jmath(\mathcal{S}^*)$. Hence, $\jmath(\mathcal{S}^*) = [\alpha, \beta]$ and the thesis follows. Thus we have only to show that the second alternative does not occur. But, this is obvious, because if F_1 and F_2 as above do exist, then the sets $E_1 := F_1 \cup \big(\mathcal{S} \cap \jmath^{-1}\big((-\infty, \alpha]\big)\big)$ and $E_2 := F_2 \cup \big(\mathcal{S} \cap \jmath^{-1}\big([\beta, +\infty)\big)\big)$ disconnect $\mathcal{S}$. □

Next, we recall a useful result of plane topology (see, for instance [26, Lemma 3, p.702]).

Lemma 6.2 *Let* $\mathcal{W}_0$ *and* $\mathcal{W}_1$ *be two nonempty continua of a compact metric space* Z *which is homeomorphic to a rectangle and let* $h : Z \to \mathcal{R}$ *be a homeomorphism of* Z *onto the rectangle* $\mathcal{R} := [a, b] \times [-M, M] \subseteq \mathbb{R}^2$. *Assume that*

$$A_0 := h(\mathcal{W}_0) \cap \big(\{a\} \times \mathbb{R}\big) \neq \emptyset, \quad B_0 := h(\mathcal{W}_0) \cap \big(\{b\} \times \mathbb{R}\big) \neq \emptyset,$$

as well as

$$A_1 := h(\mathcal{W}_1) \cap \big(\{a\} \times \mathbb{R}\big) \neq \emptyset, \quad B_1 := h(\mathcal{W}_1) \cap \big(\{b\} \times \mathbb{R}\big) \neq \emptyset$$

and, moreover,

$$y \leq 0, \ \forall\, (x, y) \in A_0 \quad \text{and} \quad y > 0, \ \forall\, (x, y) \in B_0\,,$$

$$y > 0, \ \forall\, (x, y) \in A_1 \quad \text{and} \quad y \leq 0, \ \forall\, (x, y) \in B_1\,.$$

Then

$$\mathcal{W}_0 \cap \mathcal{W}_1 \neq \emptyset.$$

Proof. Let $\varepsilon > 0$ be such that $y \geq 2\varepsilon$ for every $(x, y) \in A_1 \cup B_0$. Let also U^0_ε and U^1_ε be, respectively, an ε-neighborhood of $h(\mathcal{W}_0)$ and an ε-neighborhood of $h(\mathcal{W}_1)$ in $\mathcal{R}$. Since $\mathcal{R}$ is an arcwise connected and locally arcwise connected space, we can find two continuous paths $\gamma^0_\varepsilon, \gamma^1_\varepsilon : [0, 1] \to \mathcal{R}$ with $\gamma^i_\varepsilon(t) = (x^i_\varepsilon(t), y^i_\varepsilon(t)) \in U^i_\varepsilon$, for every $t \in [0, 1]$, for $i = 0, 1$ and such that $\gamma^i_\varepsilon(0) \in \{a\} \times \mathbb{R}$ and $\gamma^i_\varepsilon(1) \in \{b\} \times \mathbb{R}$, for $i = 0, 1$. By the assumptions, we also have that $y^0_\varepsilon(0) < y^1_\varepsilon(0)$ and $y^0_\varepsilon(1) > y^1_\varepsilon(1)$. By the properties of Peano spaces [15] we know that the image sets $\gamma^i_\varepsilon([0, 1])$ contain arcs (i.e., homeomorphic images of a compact interval) with the same extreme points, respectively (see, for instance, [15, pp.115–131] or [35]). Hence, we can suppose that there are arcs $\omega_\varepsilon^0 \subseteq U_\varepsilon^0 \cap \mathcal{R}$ and $\omega_\varepsilon^1 \subseteq U_\varepsilon^1 \cap \mathcal{R}$ such that

$$\omega_\varepsilon^0 \cap \big(\{a\} \times \mathbb{R}\big) =: \{(a, a_\varepsilon^0)\}, \quad \omega_\varepsilon^0 \cap \big(\{b\} \times \mathbb{R}\big) =: \{(b, b_\varepsilon^0)\},$$

$$\omega_\varepsilon^1 \cap \big(\{a\} \times \mathbb{R}\big) =: \{(a, a_\varepsilon^1)\}, \quad \omega_\varepsilon^1 \cap \big(\{b\} \times \mathbb{R}\big) =: \{(b, b_\varepsilon^1)\},$$

with

$$a_\varepsilon^0 < a_\varepsilon^1, \qquad \text{and} \quad b_\varepsilon^0 > b_\varepsilon^1.$$

Without loss of generality (taking M larger if necessary), we can also suppose that

$$\omega_\varepsilon^{i,\#} := \omega_\varepsilon^i \setminus \{(a, a_\varepsilon^i), (b, b_\varepsilon^i)\} \subseteq \mathrm{int}\mathcal{R}, \tag{6.1}$$

for $i = 0, 1$.

We now consider the simple closed curve (Jordan curve) J obtained by gluing together the following arcs: ω_ε^1 , the vertical segment joining (b, b_ε^1) to $(b, -M)$, the horizontal segment joining $(b, -M)$ to $(a, -M)$ and the vertical segment joining $(a, -M)$ to (a, a_ε^1). Since the points of ω_ε^0 belonging to a sufficiently small neighborhood of (a, a_ε^0) are in the internal side of the curve J and the point (b, b_ε^0) lies in the external side of J, as a consequence of the Jordan curve theorem we see that $\omega_\varepsilon^{0,\#}$ must intersect J and, actually, by (6.1) and the definition of J we have that

$$\omega_\varepsilon^{0,\#} \cap \omega_\varepsilon^{1,\#} = \omega_\varepsilon^0 \cap \omega_\varepsilon^1 \neq \emptyset.$$

Letting $\varepsilon \to 0^+$ and using a standard compactness argument, we find that $h(\mathcal{W}_0) \cap h(\mathcal{W}_1) \neq \emptyset$ and hence we can conclude that $\mathcal{W}_0 \cap \mathcal{W}_1 \neq \emptyset$. □

Our last result is used in the proof of Theorem 2.5.

Lemma 6.3 *(cf. [9, Corollary 2.4]). Let $g : [\tau_0, \tau_1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfy the Carathéodory assumptions and suppose that $K \subseteq \mathbb{R}^2$ is a nonempty closed and connected set such that, for each $(x_0, y_0) \in K$, all the solutions to the Cauchy problem*

$$\begin{cases} u'' + g(t, u, u') = 0 \\ u(\tau_0) = x_0\,, \quad u'(\tau_0) = y_0\,, \end{cases}$$

are defined on $[\tau_0, \tau_1]$. Then, for each pair of points $P, Q \in K$, with $P \neq Q$, there is a continuum $S \subseteq K$, with $P, Q \in S$ and a continuum $\mathcal{C} \subseteq S \times C^1([\tau_0, \tau_1])$ such that $Pr_1(\mathcal{C}) = S$ and, for every $(\xi, u) \in \mathcal{C}$, $u(\cdot)$ is a solution of $u'' + g(t, u, u') = 0$, defined on $[\tau_0, \tau_1]$ with $(u(\tau_0), u'(\tau_0)) = \xi \in S$.

Proof. The proof is a simple translation of [9, Corollary 2.4] where the corresponding result is stated for a first order system in $\mathbb{R}^N$. In the space $C^1[\tau_0, \tau_1]$ we consider the standard C^1-norm $||u||_{1,\infty} := ||u||_\infty + ||u'||_\infty$ (like in the proof of Theorem 2.1). By Pr_1 we mean the projection of the product space $S \times C^1([\tau_0, \tau_1])$ onto its first factor S. □

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