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Some results on stability and continuous dependence in Green-Naghdi thermoelasticity of Cosserat bodies

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Abstract

In our paper, we consider the mixed problem in the context of the Green-Naghdi theory of thermoelastic Cosserat media. Using very accessible mathematical calculations, we prove two qualitative results on the solutions of the formulated mixed problem. Thus, in the first of these approaches, we obtain a result of stability, of the Hölder type, with regards to the loads. In the second main result, we prove a continuous dependence result regarding the initial values from the mixed problem. It should be noted that we obtain these results without imposing very restrictive conditions on the thermoelastic tensors in the constitutive equations. In fact, imposed restrictions are commonly used in Mechanics of Continuous Media.

Keywords: Green-Naghdi theory; Thermoelasticity; Cosserat bodies; Stability of Hölder type; Continuous dependence

1 Introduction

Generalized theories of continuous media aim to eliminate a shortcoming of classical elasticity: the waves of heat propagate with infinite speed, the fact which is in contradiction with real experiments. This paradox has two explanations: first, the heat conduction energy is of parabolic type and, second, the energy equation does not contain any elastic term.

One of the best-known theories to eliminate these inaccuracies is the Green and Naghdi theory, which approached in [1–3] the so-called type I, type II, and type III theory. In our study, we consider the type III theory of Green-Naghdi for the following two reasons: firstly, in this context, the dissipation of energy takes place and, secondly, the heat flux is a combination of the flux of heat from the theory of type I and of the theory type II. Of course, in the Green-Naghdi theory of type III, heat waves propagate with a finite speed.

Other studies, such as [4], replace the Fourier equation with a generalized form of this law in which the thermal conductivity tensor, the conductivity rate tensor, and the function of thermal displacement are taken into account. It should be noted that the three types of theories of Green and Naghdi can be included in this new theory highlighted in [4]. To support this statement, we can use, for example, the Taylor approximations.

The theory of media having microstructure also falls within those theories dedicated to eliminating the two discrepancies. This theory, first considered by Eringen (see for instance [5, 6]), highlights the impossibility of the classical theory of elasticity to consider the effect of microstructure in the case of deformation of modern materials, such as graphite, ceramics, polymers, and even in the case of human bones. This explains the large number of studies dedicated to this theory, of which we list [7–9]. We can find some results in which the problem of waves propagation is approached, for instance, in [10–12]. Other results regarding the microstructure can be found in [13–17]. The theory of Cosserat thermoelastic bodies occupies an important place among the theories dedicated to microstructure. Many studies dedicated to Cosserat bodies are published, of which we mention the following: [18–21].

To describe the evolution of a Cosserat thermoelastic body in the context of the Green-Naghdi theory of type III, we will use a vector of the displacement vector of components v_m, a vector of the couple displacement of components ϕ_m, the variation of the temperature θ and the thermal displacement, denoted by τ and defined by:

$$\tau(t,x) = \int_{t_0}^{t} \theta(x,s)\,ds + \tau_0(x).$$

Also, in the formulation of the mixed initial-boundary value problem in the above context, we will use the density of mass ϱ, a thermal capacity c, a tensor of the conductivity of heat, and some tensors that characterize the thermoelastic properties of the media. We will assume that the above-mentioned quantities are enough regular functions, regarding the variable of position, i.e., of the form $f = f(x)$. During the description, we will impose some restrictions, which are, in fact, common, on the mass density, the heat capacity, the thermoelastic coefficients, and the tensor of the thermal conductivity, restrictions that will ensure the well-posedness of the mixed problem in the context of the Green-Naghdi thermoelasticity of type III. We must emphasize that these restrictions will guarantee stability in the Lyapunov sense of the solutions to the mixed problem. We can anticipate that if we do not impose restrictions that will ensure a positive definition of the tensors of thermoelastic coefficients, then the mixed problem, considering the usual initial values and boundary relations, becomes ill-posed.

First results regarding the continuous dependence of solutions, in the sense of Hölder, were published by John, see for instance, [22]. Here it is clear that the solutions depend continuously only if there are functions that belong to a class of constraints conveniently chosen. The concept of continuous dependence, in the classical formulation, is stronger than the continuous dependence concept that is proposed by Hölder. Based on the positivity of the tensors of thermoelastic coefficients, Quintanilla proved a stability result of Hölder type in [23]. For this, he invoked an argument of logarithmic convexity. Results can be obtained regarding the stability of the solutions of Hölder type, even if the tensors of the thermoelastic coefficients are not positively definite. To obtain some results regarding the Hölder stability of the solutions, it is necessary to use the method of the Lagrange identity, which we will address in our later considerations. Many studies have been published that address the Hölder stability, of which we mention Ames and Straughan [24], Ames and Payne [25], and Wilkes [26].

Our work is organized as follows: To have a good formulation of the mixed problem for the type III thermoelasticity of Cosserat bodies, we will introduce the main notations, the basic equations, the initial values, and the boundary relations in Sect. 2. Our basic results are considered in Sect. 3. So, in our first two theorems, we obtain two estimations, on the basis of which we deduce the two important results. In Theorem 2, we prove the first result regarding the stability of the Hölder type. From the proof of this theorem, it is clear that, in fact, this result ensures that the solution depends continuously on the loads. An extension of the analysis of the stability of the Hölder type is included in the last theorem. It can be easily deduced that it covers the continuous dependence of solutions regarding the initial values.

2 Basic notations, equations and conditions

In the following, we will consider a three-dimensional domain Ω, as a part of the Euclidean space R^3 and assume that it is occupied, at the initial time t_0, by a Cosserat thermoelastic media. The boundary of Ω is denoted by $\partial\Omega$, and it is a piecewise regular surface. All points of the domain Ω are identifiable by three coordinates in form $x = (x_1, x_2, x_3)$, and the functions we will use are dependent on the time variable t and position x, in the form $f = f(t,x)$. The domain of definition for a function of the form $f = f(t,x)$ will be the cylinder $(0,\infty) \times \bar{\Omega}$, in which $\bar{\Omega}$ is $\bar{\Omega} = \Omega \cup \partial\Omega$. We will use the summation rule over repeated subscripts. In order to designate the differentiation with respect to the time variable, t, we will use a superposed dot, that is, $\dot{f} = \partial f/\partial t$. The partial differentiation regarding a spatial variable used a comma followed by a subscript, i.e., $f_{,j} = \partial f/\partial x_j$.

We want to introduce a mathematical model consisting of a system of partial differential equations in the thermoelasticity of Cosserat media in the context of a linear theory. Addressing the technique proposed by Green and Rivlin, we can consider a new deformation of the media, different from the given deformation by a rotation having a uniform angular velocity that is superposed over the initial motion. Based on procedure Green and Rivlin, we must assure that the other characteristics of the media remain unchanged as a result of this overlap. Due to this technique, we deduce the geometric equations by which we can express the strain tensors e_{mn} and ε_{mn} with the help of the motion variables (see Eringen [5]):

$$e_{mn} = u_{n,m} + \epsilon_{mnk}\phi_k, \quad \varepsilon_{mn} = \phi_{n,m}. \tag{1}$$

In the following, we will rely on the hypothesis that the vector of the deformation, the vector of the couple deformation, the temperature variation, and the derivatives of all these functions are small.

Let us consider our Cosserat body has a center of symmetry. Also, we suppose that in the initial state of the media. there is no stress, and the values of the intrinsic body forces and body couples are zero. In the context of a linear theory, it is normal to assume that the internal energy has a quadratic form regarding all constitutive intern variables and denote it by W. By expanding this energy as a series regarding the given undeformed state and considering the principle of energy conservation, we can write the density of energy in the form that follows:

$$\begin{aligned} W &= \frac{1}{2}A_{mnkl}e_{mn}e_{kl} + B_{mnkl}e_{mn}\varepsilon_{kl} \\ &\quad + \frac{1}{2}C_{mnkl}\varepsilon_{mn}\varepsilon_{kl} - a_{mn}e_{mn}\theta - b_{mn}\varepsilon_{mn}\theta - \frac{1}{2}c\theta^2. \end{aligned} \tag{2}$$

If we take into account the above form of the internal energy and use the Clausius-Duhem inequality, that is, the inequality of the entropy production, we obtain the constitutive relations, which give the expression of the stress tensors and the entropy with the help of the strain tensors:

$$
\begin{aligned}
t_{mn} &= \frac{\partial W}{\partial e_{mn}} = A_{mnkl}e_{kl} + B_{mnkl}\varepsilon_{kl} - a_{mn}\theta, \\
\tau_{mn} &= \frac{\partial W}{\partial \varepsilon_{mn}} = B_{mnkl}e_{mn} + C_{mnkl}\varepsilon_{kl} - b_{mn}\theta, \\
\eta &= -\frac{\partial W}{\partial \theta} = a_{mn}e_{mn} + b_{mn}\varepsilon_{mn} + c\theta.
\end{aligned}
\tag{3}
$$

It is clear that the above tensors of thermoelastic coefficients satisfy, in the domain Ω, the relations of symmetry of the following form:

$$
A_{mnkl} = A_{klmn}, \qquad B_{mnkl} = B_{klmn}, \qquad C_{klmn} = C_{mnkl}. \tag{4}
$$

In Eq. (4), we used the notations t_{mn} and τ_{mn} for the tensors of stress. We also denoted the entropy by η.

According to [3], the vector of heat flux has the elements q_m, which satisfy the following equation:

$$
q_m = \kappa_{mn}\theta_{,n} + c_{mn}\beta_{,n}, \tag{5}
$$

where κ_{mn} are the components of the thermal tensor of the conductivity of heat, and c_{mn} are the components of the tensor, which connect the thermal deformation with the flux of heat. It is important to note that the tensor c_{mn} was first included in the theories of Green-Naghdi of type II and type III.

By using the first law of thermodynamics, we deduce the motion equations in the following form:

$$
\begin{aligned}
& t_{mn,n} + \varrho F_m = \varrho \ddot{v}_m, \\
& \tau_{mn,n} + \epsilon_{mnk}t_{nk} + \varrho G_m = I_{mn}\ddot{\phi}_n.
\end{aligned}
\tag{6}
$$

The equation of energy also receives the next form:

$$
\dot{\eta} = q_{m,m} + \varrho r. \tag{7}
$$

With the help of the geometric Eqs. (1) and the constitutive relations (3) and (5), and the kinetic relations (1), we obtain another form of the system of Eqs. (6) and (7), namely:

$$
\begin{aligned}
\ddot{v}_m &= \frac{1}{\varrho}\big[(A_{mnkl}e_{kl})_{,n} + (B_{mnkl}\varepsilon_{kl})_{,n} - (a_{mn}\theta)_{,n}\big] + F_m, \\
\ddot{\phi}_m &= \frac{1}{I_{mn}}\big[(B_{mnkl}e_{kl})_{,m} + (C_{mnkl}\varepsilon_{kl})_{,m} - (b_{mn}\theta)_{,m} \\
&\quad + \varepsilon_{mnj}(A_{mjkl}e_{kl} + B_{mjkl}\varepsilon_{kl} - a_{mj}\theta) + G_n\big], \\
a_{mn}\dot{e}_{mn} &+ b_{mn}\dot{\varepsilon}_{mn} + c\dot{\theta} = \frac{\varrho}{T_0}(\kappa_{mn}\theta_{,n})_{,m} + \frac{1}{T_0}r.
\end{aligned}
\tag{8}
$$

3 Main results

Let us specify that the main hypotheses necessary to obtain the proposed results are the following:

i) c, the function of the capacity of heat, and ϱ, the density of mass, are positive with regards to the variable of position, i.e.,

$$c(x) \geq c_0 > 0, \qquad \varrho(x) \geq \varrho_0 > 0, \quad x \in \Omega;$$

ii) all tensors of thermoelastic coefficients are bounded.

iii) the conductivity of heat tensor, k_{ij}, is positively definite, i.e., $\exists k_0 > 0$ so that

$$\kappa_{mn}\xi_m\xi_n \geq k_0\xi_m\xi_m, \quad \forall \xi_m. \tag{9}$$

Let us designate by c^* the biggest value of the proper values for the matrix of components c_{mn} and, also, by k^* the smallest of the proper values of the matrix of components κ_{mn}. Then, for any vector ξ_m, from Eq. (10), we obtain the following estimate:

$$\kappa_{mn}\xi_m\xi_n \geq k_1|c_{mn}\xi_m\xi_n|, \quad k_1 = \frac{k^*}{c^*}. \tag{10}$$

Now, we consider that k_m is the minimum value of the eigenvalues for the matrix of components κ_{mn}, and k^M is the maximum value for the same matrix.

If we use the notation $k_2 = k^M/k_m$ and consider any function τ, which satisfies the relation $\tau(0) = 0$, we can deduce the next Poincaré type estimate:

$$k_2 t^2 \int_0^t \kappa_{mn}\dot{\tau}_{,m}\dot{\tau}_{,n}\,ds \geq \frac{\pi^2}{4}\int_0^t \kappa_{mn}\tau_{,m}\tau_{,n}\,ds. \tag{11}$$

This estimate is useful in obtaining our main results.

To complete the mixed initial-boundary value problem in the above context, we adjoin to the system of Eqs. (8) the following initial values:

$$\begin{aligned} &v_m(0,x) = v_m^0(x), \qquad \dot{v}_m(0,x) = v_m^1(x), \qquad \phi_m(0,x) = \phi_m^0(x), \\ &\dot{\phi}_m(0,x) = \phi_m^1(x), \qquad \theta(0,x) = \theta^0(x), \qquad \dot{\theta}(0,x) = \theta^1(x), \quad x \in \Omega, \end{aligned} \tag{12}$$

and the boundary conditions:

$$\begin{aligned} &v_m(t,x) = \tilde{v}_m(t,x), \qquad \phi_m(t,x) = \tilde{\phi}_m(t,x), \\ &\theta(t,x) = \tilde{\theta}(t,x), \quad (t,x) \in [0,t_0) \times \partial\Omega. \end{aligned} \tag{13}$$

We will denote by $\mathcal{P}$ the mixed problem consisting of differential Eqs. (8), the initial values (12), and the boundary relations (13).

Let us consider two systems of loads,

$$\left(F_m^{(\nu)}, G_m^{(\nu)}, r^{(\nu)}\right), \quad \nu = 1, 2,$$

and denote by

$$\left(v_m^{(\nu)}, \phi_m^{(\nu)}, \tau^{(\nu)}\right), \quad \nu = 1, 2$$

the solutions that correspond to each previous loads.

We will use the notations:

$$\begin{aligned} &v_m = v_m^{(2)} - v_m^{(1)}, \qquad \phi_m = \phi_m^{(2)} - \phi_m^{(1)}, \qquad \tau = \tau^{(2)} - \tau^{(1)}, \\ &\mathcal{F}_m = F_m^{(2)} - F_m^{(1)}, \qquad \mathcal{G}_m = G_m^{(2)} - G_m^{(1)}, \qquad \mathcal{R} = r^{(2)} - r^{(1)} \end{aligned} \tag{14}$$

relative to the difference of the two solutions and of the two loads, respectively.

As such, because of the linearity, the ordered array (v_m, ϕ_m, τ) is also a solution to the mixed problem $\mathcal{P}$, which now corresponds to:

– the partial differential equations:

$$\begin{aligned} &\varrho \ddot{v}_m = (A_{mnkl}e_{kl} + B_{mnkl}\varepsilon_{kl})_{,n} - (a_{mn}\theta)_{,n} + \varrho \mathcal{F}_m, \\ &I_{mn}\ddot{\phi}_m = (B_{mnkl}e_{kl} + C_{mnkl}\varepsilon_{kl})_{,m} - (b_{mn}\theta)_{,m} \\ &\qquad + \varepsilon_{mnj}(A_{mjkl}e_{kl} + B_{mjkl}\varepsilon_{kl} - a_{mj}\theta) + \varrho \mathcal{G}_n, \\ &c\dot{\theta} = -a_{mn}\dot{e}_{mn} - b_{mn}\dot{\varepsilon}_{mn} + (\kappa_{mn}\theta_{,n})_{,m} + \varrho \mathcal{R}. \end{aligned} \tag{15}$$

– the null initial values:

$$v_m(0,x) = \dot{v}_m(0,x) = \phi_m(0,x) = \dot{\phi}_m(0,x) = \theta(0,x) = \dot{\theta}(0,x) = 0, \quad \forall x \in \Omega, \tag{16}$$

– the null relations to the limit:

$$v_m(t,x) = \phi_m(t,x) = \theta(t,x) = 0, \quad \forall (t,x) \in [0,t_0) \times \partial\Omega. \tag{17}$$

The following two theorems are devoted to some estimations that will help us prove the main results.

Theorem 1 *If* (v_m, ϕ_m, τ) *is a solution to the mixed problem consisting of Eqs.* (15) *and conditions* (16), (17), *then we have the following identity*:

$$\begin{aligned} &\int_\Omega \bigl(\varrho \dot{v}_m \dot{v}_m + I_{mn}\dot{\phi}_m\dot{\phi}_n + c\theta^2 + A_{mnkl}e_{mn}e_{kl} + 2B_{mnkl}e_{mn}\varepsilon_{kl} + C_{mnkl}\varepsilon_{mn}\varepsilon_{kl}\bigr)\, dV \\ &\quad - \int_0^t \int_\Omega (\varrho \mathcal{F}_m \dot{v}_m + \varrho \mathcal{G}_m \dot{\phi}_m + \varrho \mathcal{R}\theta)\, dV\, ds + \int_0^t \int_\Omega \kappa_{mn}\theta_{,m}\theta_{,n}\, dV\, ds = 0. \end{aligned} \tag{18}$$

Proof It is easy to prove this estimation if we take into account the equation of energy and consider the homogeneous Dirichlet boundary relations and zero initial values. □

The Lagrange identity, which we prove in the following theorem, is a useful tool in obtaining the Hölder type stability.

Theorem 2 *For any solution* (v_m, ϕ_m, τ) *of the mixed problem consisting of Eqs.* (15) *and conditions* (16), (17), *then we have the following identity*:

$$\begin{aligned} &\int_\Omega (\varrho \dot{v}_m \dot{v}_m + I_{mn}\dot{\phi}_m\dot{\phi}_n + c_{mn}\tau_{,m}\tau_{,n})\, dV + \int_0^t \int_\Omega \kappa_{mn}\theta_{,m}\theta_{,n}\, dV\, ds \\ &\quad = \int_0^t \int_\Omega \varrho(\mathcal{F}_m \dot{v}_m + \mathcal{G}_m \dot{\phi}_m + \mathcal{R}\theta)\, dV\, ds \end{aligned}$$

$$+\frac{1}{2}\int_0^t\int_\Omega \varrho\big[\mathcal{F}_m(s)\dot{v}_m(2t-s)-\mathcal{F}_m(2t-s)\dot{v}_m(s)\big]\,dV\,ds$$
$$+\frac{1}{2}\int_0^t\int_\Omega \varrho\big[\mathcal{G}_m(s)\dot{\phi}_m(2t-s)-\mathcal{G}_m(2t-s)\dot{\phi}_m(s)\big]\,dV\,ds$$
$$+\frac{1}{2}\int_0^t\int_\Omega \varrho\big[\mathcal{R}(2t-s)\theta(s)-\mathcal{R}_{ij}(s)\theta(2t-s)\big]\,dV\,ds. \tag{19}$$

Proof Based on the basic rule of deriving a product of functions, it is easy to show that the following relations take place:

$$\begin{aligned}
&\frac{d}{ds}\big[\varrho\dot{v}_m(s)\dot{v}_m(2t-s)\big]=\varrho\ddot{v}_m(s)\dot{v}_m(2t-s)-\varrho\dot{v}_m(s)\ddot{v}_m(2t-s),\\
&\frac{d}{ds}\big[I_{mn}\dot{\phi}_m(s)\dot{\phi}_n(2t-s)\big]=I_{mn}\ddot{\phi}_m(s)\dot{\phi}_n(2t-s)-I_{mn}\dot{\phi}_m(s)\ddot{\phi}_n(2t-s),\\
&\frac{d}{ds}\big[\theta(s)\theta(2t-s)\big]=c\dot{\theta}(s)\theta(2t-s)-c\theta(s)\dot{\theta}(2t-s).
\end{aligned} \tag{20}$$

By summing up these three equalities, term by term we obtain an equality in which we consider the differential Eqs. (15). Then, we introduce the constitutive Eqs. (3), and the obtained relation is integrated into cylinder $[0,t]\times\Omega$. If we take into account that we have zero initial values and null boundary relations, we get the following identity:

$$\begin{aligned}
&\int_\Omega\big(\varrho\dot{v}_m\dot{v}_m+I_{mn}\dot{\phi}_m\dot{\phi}_n+c_{mn}\tau_{,m}\tau_{,n}-c\theta^2\\
&\qquad -A_{mnkl}e_{mn}e_{kl}-2B_{mnkl}e_{mn}\varepsilon_{kl}-C_{mnkl}\varepsilon_{mn}\varepsilon_{kl}\big)\,dV\\
&=\int_0^t\int_\Omega \varrho\big[\mathcal{F}_m(s)\dot{v}_m(2t-s)-\mathcal{F}_m(2t-s)\dot{v}_m(s)\big]\,dV\,ds\\
&\qquad +\int_0^t\int_\Omega \varrho\big(\mathcal{G}_m(s)\dot{\phi}_m(2t-s)-\mathcal{G}_m(2t-s)\dot{\phi}_m(s)\big)\,dV\,ds\\
&\qquad -\frac{1}{2}\int_0^t\int_\Omega \varrho\big[\mathcal{R}(2t-s)\theta(s)-\mathcal{R}(s)\theta(2t-s)\big]\,dV\,ds.
\end{aligned} \tag{21}$$

Finally, using the identities (18) and (21), we ge the desired estimate (19). □

Our first main result will be proven in the next theorem. The inequality that will be obtained ensures the stability for the solutions to the problem $\mathcal{P}$ of Hölder type, in relation to the external loads.

Theorem 3 *We suppose the the assumptions i)-iii) are satisfied and consider a solution* (v_m,ϕ_m,τ) *to the mixed problem* $\mathcal{P}$, *but consisting of Eqs.* (15), *conditions* (16), *and* (17). *Then, we have the next inequality*:

$$\begin{aligned}
&\int_\Omega(\varrho\dot{v}_m\dot{v}_m+I_{mn}\dot{\phi}_m\dot{\phi}_n+c_{mn}\tau_{,m}\tau_{,n})\,dV+\int_0^t\int_\Omega\kappa_{mn}\theta_{,m}\theta_{,n}\,dV\,ds\\
&\quad\le\frac{3}{2}\sqrt{t_0}M\left[\int_0^{t_0}\int_\Omega\varrho\big(\mathcal{F}_m\mathcal{F}_m+\mathcal{G}_m\mathcal{G}_m+\mathcal{R}^2\big)\,dV\,ds\right]^{1/2},
\end{aligned} \tag{22}$$

which takes place for any $t\in[0,t_0/2]$.

The positive constant M is chosen so that

$$\sup_{t\in[0,t_0]}\int_\Omega\left(\varrho\dot{v}_m\dot{v}_m+I_{mn}\dot{\phi}_m\dot{\phi}_n+c\theta^2\right)dV\leq M^2. \tag{23}$$

Proof In order to estimate the integrals from the identity (19), we will use a simple inequality of the form:

$$\begin{aligned}&\sqrt{a}\sqrt{p}+\sqrt{b}\sqrt{q}+\sqrt{c}\sqrt{r}\\&\quad\leq\sqrt{a+b}\sqrt{p+q}+\sqrt{a+c}\sqrt{p+r}+\sqrt{b+c}\sqrt{q+r},\end{aligned}$$

which is true for the real numbers $a,b,c,p,q,r\geq 0$.

Based on this elementary inequality and considering the first integral of (19), we immediately obtain the estimation:

$$\begin{aligned}&\int_0^t\int_\Omega\varrho[\mathcal{F}_m\dot{v}_m+\mathcal{G}_m\dot{\phi}_m+\mathcal{R}\theta]\,dV\,d\tau\\&\quad\leq\left[\int_0^t\int_\Omega\varrho\mathcal{F}_m\mathcal{F}_m\,dV\,ds\right]^{\frac{1}{2}}\left[\int_0^t\int_\Omega\varrho\dot{v}_m\dot{v}_m\,dV\,ds\right]^{\frac{1}{2}}\\&\qquad+\left[\int_0^t\int_\Omega\varrho\mathcal{G}_m\mathcal{G}_m\,dV\,ds\right]^{\frac{1}{2}}\left[\int_0^t\int_\Omega\varrho\dot{\phi}_m\dot{\phi}_m\,dV\,ds\right]^{\frac{1}{2}}\\&\qquad+\left[\int_0^t\int_\Omega\varrho\mathcal{R}^2\,dV\,ds\right]^{\frac{1}{2}}\left[\int_0^t\int_\Omega\varrho\theta^2\,dV\,ds\right]^{\frac{1}{2}}\\&\quad\leq\left[\int_0^t\int_\Omega\varrho\left(\mathcal{F}_m\mathcal{F}_m+\mathcal{G}_m\mathcal{G}_m+\mathcal{R}^2\right)dV\,ds\right]^{\frac{1}{2}}\\&\qquad\times\left[\int_0^t\int_\Omega\varrho\left(\dot{v}_m\dot{v}_m+\dot{\phi}_m\dot{\phi}_m+\theta^2\right)dV\,ds\right]^{\frac{1}{2}}.\end{aligned} \tag{24}$$

Similarly, the second integral in (19) leads to the following estimate:

$$\begin{aligned}&\int_0^t\int_\Omega\varrho\left[\mathcal{F}_m(s)\dot{v}_m(2t-s)-\mathcal{F}_m(2t-s)\dot{v}_m(s)\right]dV\,ds\\&\quad\leq\left[\int_0^t\int_\Omega\varrho\mathcal{F}_m\mathcal{F}_m\,dV\,ds\right]^{\frac{1}{2}}\left[\int_t^{2t}\int_\Omega\varrho\dot{v}_m\dot{v}_m\,dV\,ds\right]^{\frac{1}{2}}\\&\qquad+\left[\int_t^{2t}\int_\Omega\varrho\mathcal{F}_m\mathcal{F}_m\,dV\,ds\right]^{\frac{1}{2}}\left[\int_0^t\int_\Omega\varrho\dot{v}_m\dot{v}_m\,dV\,ds\right]^{\frac{1}{2}}\\&\quad\leq\left[\int_0^{2t}\int_\Omega\varrho\mathcal{F}_m\mathcal{F}_m\,dV\,ds\right]^{\frac{1}{2}}\left[\int_0^{2t}\int_\Omega\varrho\dot{v}_m\dot{v}_m\,dV\,ds\right]^{\frac{1}{2}}.\end{aligned} \tag{25}$$

With an analogous procedure, using the third integral from (19), we deduce:

$$\begin{aligned}&\int_0^t\int_\Omega\varrho\left[\mathcal{G}_m(s)\dot{\phi}_m(2t-s)-\mathcal{G}_m(2t-s)\dot{\phi}_m(s)\right]dV\,ds\\&\quad\leq\left[\int_0^{2t}\int_\Omega\varrho\mathcal{G}_m\mathcal{G}_m\,dV\,ds\right]^{\frac{1}{2}}\left[\int_0^{2t}\int_\Omega\varrho\dot{\phi}\dot{\phi}_m\,dV\,ds\right]^{\frac{1}{2}}.\end{aligned} \tag{26}$$

Ultimately, the fourth integral in (19) leads to the following estimate:

$$\int_0^t \int_\Omega \varrho\big[\mathcal{R}(2t-s)\theta(s) - \mathcal{R}(s)\theta(2t-s)\big]\,dV\,ds \\ \leq \left[\int_0^{2t}\int_\Omega \varrho\mathcal{R}^2\,dV\,ds\right]^{\frac{1}{2}}\left[\int_0^{2t}\int_\Omega \varrho\theta^2\,dV\,ds\right]^{\frac{1}{2}}. \tag{27}$$

Now, we take into account all estimations from (24), (25), (26), and (27), so that considering the identity (19), we obtain the inequality:

$$\int_\Omega (\varrho\dot{v}_m\dot{v}_m + I_{mn}\dot{\phi}_m\dot{\phi}_n + c_{mn}\tau_{,m}\tau_{,n})\,dV + \int_0^t\int_\Omega \kappa_{mn}\theta_{,m}\theta_{,n}\,dV\,d\tau \\ \leq \frac{3}{2}\left[\int_0^{2t}\int_\Omega \varrho\big(\mathcal{F}_m\mathcal{F}_m + \mathcal{G}_m\mathcal{G}_m + \mathcal{R}^2\big)\,dV\,ds\right]^{\frac{1}{2}} \\ \times\left[\int_0^{2t}\int_\Omega \big(\varrho\dot{v}_m\dot{v}_m + I_{mn}\dot{\phi}_m\dot{\phi}_n + \varrho\theta^2\big)\,dV\,ds\right]^{\frac{1}{2}}. \tag{28}$$

Finally, we consider the estimate (23) so that from (28), we observe that the estimate (22) takes place for any $t \in [0, t_0]$, which concludes the demonstration of this theorem. □

Remark It is easy to see that the estimation (22) with M from (23) assures the stability of the solution in the sense of the Hölder regarding the supply terms.

Finally, we want to study the stability of solutions, in the sense of the Hölder regarding the initial values. In this regard, we will take the partial differential Eqs. (8) without the charges, that is, $F_m = 0$, $G_m = 0$, $r = 0$, i.e., in its homogeneous form. Let us consider two solutions $(v_m^{(1)}, \phi_m^{(1)}, \tau^{(1)})$, $(v_m^{(2)}, \phi_m^{(2)}, \tau^{(2)})$ of (8), corresponding to equal boundary relations but to different initial values. Then, the difference of these two solutions (v_m, ϕ_m, τ) satisfies the system (16) with null loads, that is, $\mathcal{F}_m = 0$, $\mathcal{G}_m = 0$, $\mathcal{R} = 0$, also, with null to the limit values (17), but the initial values of the following form:

$$\begin{aligned} v_m(x,0) &= v_m^{(2)}(x,0) - v_m^{(1)}(x,0) = v_m^0(x), \\ \dot{v}_m(x,0) &= \dot{v}_m^{(2)}(x,0) - \dot{v}_m^{(1)}(x,0) = w_m^0(x), \\ \phi_m(x,0) &= \phi_m^{(2)}(x,0) - \phi_m^{(1)}(x,0) = \phi_m^0(x), \\ \dot{\phi}_m(x,0) &= \dot{\phi}_m^{(2)}(x,0) - \dot{\phi}_m^{(1)}(x,0) = \psi_m^0(x), \\ \tau(x,0) &= \tau^{(2)}(x,0) - \tau^{(1)}(x,0) = \tau^0(x), \\ \dot{\tau}(x,0) &= \dot{\tau}^{(2)}(x,0) - \dot{\tau}^{(1)}(x,0) = \theta^0(x). \end{aligned} \tag{29}$$

Theorem 4 *We suppose that the assumptions i)-iii) are satisfied and consider the difference of the two solutions* (v_m, ϕ_m, τ) *to the mixed problem* $\mathcal{P}$*, but consisting of the homogeneous Eqs.* (15)*, null boundary conditions, and the initial conditions in the form* (30)*. Then the stability of the solution is ensured, in the sense of the Hölder regarding the initial values.*

Proof Recalling the form of the internal energy:

$$\begin{aligned} W(t) &= \int_\Omega \big(\varrho \dot{v}_m \dot{v}_m + I_{mn} \dot{\phi}_m \dot{\phi}_n + c\theta^2 + c_{mn}\tau_{,m}\tau_{,n} \\ &\quad + A_{mnkl} e_{mn} e_{kl} + 2B_{mnkl} e_{mn}\varepsilon_{kl} + C_{mnkl}\varepsilon_{mn}\varepsilon_{kl}\big)\, dV \\ &\quad + 2\int_0^t \int_\Omega \kappa_{mn}\theta_{,m}\theta_{,n}\, dV\, ds, \end{aligned} \tag{30}$$

we can deduce the following law of conservation:

$$W(t) = W(0), \tag{31}$$

where

$$\begin{aligned} W(0) &= \int_\Omega \big(\varrho w^0_m w^0_m + I_{mn}\psi^0_m \psi^0_n + c\big(\theta^0\big)^2 + c_{mn}\tau^0_{,m}\tau^0_{,n} \\ &\quad + A_{mnkl} e^0_{mn} e^0_{kl} + 2B_{mnkl} e^0_{mn}\varepsilon^0_{kl} + C_{mnkl}\varepsilon^0_{mn}\varepsilon^0_{kl}\big)\, dV. \end{aligned} \tag{32}$$

However, we consider only the solutions to the homogeneous system of partial differential Eqs. (15), so that, with help of the Lagrange identity, we obtain the following identity:

$$\begin{aligned} &\int_\Omega \big(\varrho \dot{v}_m \dot{v}_m + I_{mn}\dot{\phi}_m\dot{\phi}_n + c_{mn}\tau_{,m}\tau_{,n} - c\theta^2 \\ &\qquad - A_{mnkl} e_{mn} e_{kl} - 2B_{mnkl} e_{mn}\varepsilon_{kl} - C_{mnkl}\varepsilon_{mn}\varepsilon_{kl}\big)\, dV \\ &\quad = \int_\Omega \big[\varrho w^0_m \dot{v}_m(2t) + I_{mn}\psi^0_m \dot{\phi}_n(2t) + c_{mn}\tau^0_{,m}\tau_{,n}(2t) - c\theta^0\theta(2t) \\ &\qquad - A_{mnkl} e^0_{mn} e_{kl}(2t) - 2B_{mnkl} e^0_{mn}\varepsilon_{kl}(2t) - C_{mnkl}\varepsilon^0_{mn}\varepsilon_{kl}(2t)\big]\, dV. \end{aligned} \tag{33}$$

Now, we take into account the expression (30) for the energy $W(t)$ and (32) for $W(0)$. As such, taking into account (31) and (33), we obtain the identity:

$$\begin{aligned} &\int_\Omega (\varrho \dot{v}_m \dot{v}_m + I_{mn}\dot{\phi}_m\dot{\phi}_n + c_{mn}\tau_{,m}\tau_{,n})\, dV + \int_0^t \int_\Omega \kappa_{mn}\theta_{,m}\theta_{,n}\, dV\, ds \\ &\quad = \frac{E(0)}{2} + \frac{1}{2}\int_\Omega \big[\varrho w^0_m \dot{v}_m(2t) + I_{mn}\psi^0_m\dot{\phi}_n(2t) + c_{mn}\tau^0_{,m}\tau_{,n}(2t) \\ &\qquad - A_{mnkl} e^0_{mn} e_{kl}(2t) - 2B_{mnkl} e^0_{mn}\varepsilon_{kl}(2t) - C_{mnkl}\varepsilon^0_{mn}\varepsilon_{kl}(2t) - c\theta^0\theta(2t)\big]\, dV. \end{aligned} \tag{34}$$

It is not difficult to obtain the next inequalities:

$$\begin{aligned} &\left|\int_\Omega \varrho w^0_m \dot{v}_m(2t)\, dV\right| \le \left(\int_\Omega \varrho w^0_m w^0_m\, dV\right)^{\frac{1}{2}} \left(\int_\Omega \varrho \dot{v}_m(2t)\dot{v}_m(2t)\, dV\right)^{\frac{1}{2}}, \\ &\left|\int_\Omega I_{mn}\psi^0_m \dot{\phi}_n(2t)\, dV\right| \le \left(\int_\Omega I_{mn}\psi^0_m\psi^0_n\, dV\right)^{\frac{1}{2}} \left(\int_\Omega I_{mn}\dot{\phi}_m(2t)\dot{\phi}_n(2t)\, dV\right)^{\frac{1}{2}}, \\ &\left|\int_\Omega c\theta^0\theta(2t)\, dV\right| \le \left(\int_\Omega c\big(\theta^0\big)^2\, dV\right)^{\frac{1}{2}} \left(\int_\Omega c\theta^2(2t)\, dV\right)^{\frac{1}{2}}, \end{aligned}$$

$$\left|\int_\Omega c_{mn}\tau^0_{,m}\tau_{,n}(2t)\,dV\right| \le k_3\left(\int_\Omega \tau^0_{,m}\tau^0_{,m}\,dV\right)^{\frac{1}{2}}\left(\int_\Omega \tau_{,n}(2t)\tau_{,n}(2t)\,dV\right)^{\frac{1}{2}}, \tag{35}$$

$$\left|\int_\Omega A_{mnkl}e^0_{mn}e_{kl}(2t)\,dV\right| \le k_4\left(\int_\Omega e^0_{mn}e^0_{mn}\,dV\right)^{\frac{1}{2}}\left(\int_\Omega e_{mn}(2t)e_{mn}(2t)\,dV\right)^{\frac{1}{2}},$$

$$\left|\int_\Omega B_{mnkl}e^0_{mn}\varepsilon_{kl}(2t)\,dV\right| \le k_5\left(\int_\Omega e^0_{mn}e^0_{mn}\,dV\right)^{\frac{1}{2}}\left(\int_\Omega \varepsilon_{kl}(2t)\varepsilon_{kl}(2t)\,dV\right)^{\frac{1}{2}},$$

$$\left|\int_\Omega C_{mnkl}\varepsilon^0_{mn}\varepsilon_{kl}(2t)\,dV\right| \le k_6\left(\int_\Omega \varepsilon^0_{mn}\varepsilon^0_{mn}\,dV\right)^{\frac{1}{2}}\left(\int_\Omega \varepsilon_{kl}(2t)\varepsilon_{kl}(2t)\,dV\right)^{\frac{1}{2}},$$

in which k_m takes the values 3, 4, 5, 6.

These constants are expressed only with the help of the thermoelastic tensor situated in the same line. As an example, the constant k_5 depends only on the themoelastic tensor B_{mnkl}.

Now, we take an arbitrary $t \in [0, t_0/2]$ and suppose that

$$\sup_{t\in[0,t_0]}\int_\Omega\big(\varrho\dot{v}_m\dot{v}_m + I_{mn}\dot{\phi}_m\dot{\phi}_n + k_3\tau_{,m}\tau_{,m} + N_1e_{mn}e_{mn} + N_2\varepsilon_{mn}\varepsilon_{mn} + c\theta^2\big)\,dV \le M_1^2, \tag{36}$$

in which the constants $N_1 > 0$, $N_2 > 0$ and $N_3 > 0$ can be expressed as functions of $k_3, k_4, \dots, k_6$.

Then, with the help of (34), we deduce the estimate:

$$\int_\Omega(\varrho\dot{v}_m\dot{v}_m + I_{mn}\dot{\phi}_m\dot{\phi}_n + c_{mn}\tau_{,m}\tau_{,n})\,dV + \int_0^t\int_\Omega \kappa_{mn}\theta_{,m}\theta_{,n}\,dV\,ds \le \frac{1}{2}E(0) + \frac{t_0}{2}M_1\sqrt{M_2}, \tag{37}$$

in which the constant M_2 has the expression:

$$M_2 = \int_\Omega\big(\varrho w^0_m w^0_m + I_{mn}\psi^0_m\psi^0_n + k_3\tau^0_{,m}\tau^0_{,m} + N_1e^0_{mn}e^0_{mn} + N_2\varepsilon^0_{mn}\varepsilon^0_{mn} + c\big(\theta^0\big)^2\big)\,dV. \tag{38}$$

With this, we end the proof of Theorem. □

Remark It is not difficult to notice that the estimation (37) with M_2 from (38) guarantees the stability of the solution, in the sense of the Hölder with regard to the initial values.

4 Conclusions

Using the same technique, which was used in the case of simple thermoelastic media, we obtain the stability, in the sense of the Hölder, for the solutions to the mixed problem $\mathcal{P}$ defined in the context of thermoelasticity of type III for the Cosserat bodies. As we have already shown, this is actually the case of the solutions that are continuously depending on the external loads. Taking into account that the known concept of continuous dependence is louder than the stability in the sense of the Hölder, we also obtain this type of stability with respect to the initial values.

We need to outline that the imposed conditions are not very restrictive. Thus, we imposed the positivity of the thermal capacity and density, which are commonly used in Continuum Mechanics. It also is usual to impose a positive definition of thermoelastic tensors. Clearly, in theory of thermoelasticity of type III for Cosserat bodies, we have a larger number of equations, and these are more complicated. The same goes for the initial values and boundary relations. Nevertheless, these complications do not disturb the stability, in the sense of the Hölder, for the solutions to the mixed problem constructed in our context.

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MM, EC, SV wrote the main manuscript text and MMB supervised all calculations. All authors reviewed the manuscript. All authors read and approved the final manuscript.

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