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A study of sharp coefficient bounds for a new subfamily of starlike functions

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Abstract

In this article, by employing the hyperbolic tangent function $\tanh z$, a subfamily $\mathcal{S}^*_{\tanh}$ of starlike functions in the open unit disk $\mathbb{D} \subset \mathbb{C}$:

$$\mathbb{D} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

is introduced and investigated. The main contribution of this article includes derivations of sharp inequalities involving the Taylor–Maclaurin coefficients for functions belonging to the class $\mathcal{S}^*_{\tanh}$ of starlike functions in $\mathbb{D}$. In particular, the bounds of the first three Taylor–Maclaurin coefficients, the estimates of the Fekete–Szegö type functionals, and the estimates of the second- and third-order Hankel determinants are the main problems that are proposed to be studied here.

Keywords: Analytic (or regular or holomorphic) functions; Univalent functions; Starlike functions; Principle of subordination; Schwarz function; Hyperbolic and trigonometric functions; Coefficient bounds; Fekete–Szegö functional; The quantum or basic (or $\mathfrak{q}$-) calculus and its trivial $(\mathfrak{p}, \mathfrak{q})$-variation

1 Introduction, definitions, and preliminaries

Let us represent the family of analytic (or regular or holomorphic) functions in $\mathbb{D}$ by the notation $\mathcal{H}(\mathbb{D})$ and suppose that $\mathcal{A}$ is the subclass of $\mathcal{H}(\mathbb{D})$ defined as follows:

$$\mathcal{A} := \left\{ f : f \in \mathcal{H}(\mathbb{D}) \text{ and } f(z) = \sum_{k=1}^{\infty} a_k z^k \, (a_1 = 1) \right\}. \tag{1}$$

Further, all normalized univalent functions in $\mathbb{D}$ are contained in the set $\mathcal{S} \subset \mathcal{A}$. For two given functions $g_1, g_2 \in \mathcal{H}(\mathbb{D})$, we say that g_1 is subordinate to g_2, written symbolically as $g_1 \prec g_2$, if there exists a Schwarz function w, which is analytic in $\mathbb{D}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1,$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{D}).$$

Moreover, if the function g_2 is univalent in $\mathbb{D}$, then the following equivalence holds true:

$$g_1(z) \prec g_2(z), \qquad (z \in \mathbb{D}) \quad \Longleftrightarrow \quad g_1(0) = g_2(0) \quad \text{and} \quad g_1(\mathbb{D}) \subset g_2(\mathbb{D}).$$

Though the subject of function theory was founded in 1851, the coefficient conjecture presented by Bieberbach [13] in 1916 led to the field's emergence as a promising area of new research. This conjecture was proved by de Branges [18] in 1985. Between 1916 and 1985, many of the finest scholars of the day sought to prove or disprove this Bieberbach conjecture. As a consequence, they discovered numerous sub-families of the class $\mathcal{S}$ of normalized univalent functions connected to distinct image domains. The families of starlike and convex functions, respectively, denoted by $\mathcal{S}^*$ and $\mathcal{K}$, are the most fundamental and significant subclasses of the set $\mathcal{S}$. In 1992, Ma and Minda [36] considered the general form of the family as follows:

$$\mathcal{S}^*(\phi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\},$$

where ϕ is a holomorphic function with $\phi'(0) > 0$ and has a positive real part in $\mathbb{D}$. Also, the function ϕ maps $\mathbb{D}$ onto a star-shaped region with respect to $\phi(0) = 1$ and is symmetric about the real axis. They addressed some specific results such as distortion, growth, and covering theorems. In recent years, several sub-families of the normalized analytic function class $\mathcal{A}$ were studied as a special case of the class $\mathcal{S}^*(\phi)$. For example, we have:

(i) If we choose

$$\phi(z) = \frac{1 + Lz}{1 + Mz} \quad (-1 \leqq M < L \leqq 1),$$

then we achieve the class given by

$$\mathcal{S}^*[L, M] \equiv \mathcal{S}^*\left(\frac{1 + Lz}{1 + Mz}\right),$$

which is described as the functions of the Janowski starlike class investigated in [22]. Furthermore, the class $\mathcal{S}^*(\xi)$ given by

$$\mathcal{S}^*(\xi) := \mathcal{S}^*[1 - 2\xi, -1]$$

is the familiar starlike function family of order ξ with $0 \leqq \xi < 1$.

(ii) The following family:

$$\mathcal{S}^*_{\mathcal{L}} := \mathcal{S}^*\big(\phi(z)\big) \quad \big(\phi(z) = \sqrt{1 + z}\big)$$

was studied in [49] by Sokól and Stankiewicz. The function $\phi(z) = \sqrt{1 + z}$ maps the region $\mathbb{D}$ onto the image domain which is bounded by $|w^2 - 1| < 1$.

(iii) The class given by

$$\mathcal{S}^*_{car} := \mathcal{S}^*\big(\phi(z)\big) \quad \left(\phi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2\right)$$

was examined by Sharma *et al.* [46]. It consists of functions $f \in \mathcal{A}$ in such a manner that

$$\frac{zf'(z)}{f(z)}$$

is located in the region bounded by the cardioid given by

$$\left(9x^2 + 9y^2 - 18x + 5\right)^2 - 16\left(9x^2 + 9y^2 - 6x + 1\right) = 0.$$

(iv) By selecting $\phi(z) = 1 + \sin z$, the class $\mathcal{S}^*(\phi(z))$ leads to the family $\mathcal{S}^*_{\sin}$, which was investigated by Cho *et al.* [17]. On the other hand, the function class given by

$$\mathcal{S}^*_e \equiv \mathcal{S}^*\left(e^z\right)$$

was studied in [38] and, subsequently, in [48]. This function class was recently generalized by Srivastava *et al.* [56] in which the authors determined an upper bound of the Hankel determinant of the third order.

(v) The following families:

$$\mathcal{S}^*_{\cos} := \mathcal{S}^*(\cos z)$$

and

$$\mathcal{S}^*_{\cosh} := \mathcal{S}^*(\cosh z)$$

were considered, respectively, by Raza and Bano [9] and Alotaibi *et al.* [2]. In both of these papers, the authors studied some interesting properties of the families which they studied.

(vi) By choosing $\phi(z) = 1 + \sin z$, we obtain the following class:

$$\mathcal{S}^*_{\sin} := \mathcal{S}^*\left(\phi(z)\right),$$

which was investigated in [17]. The authors in [17] addressed the radii problems for the defined class $\mathcal{S}^*_{\sin}$.

(vii) By considering the function $\phi(z) = 1 + \sinh^{-1} z$, we get the recently-examined family given by

$$\mathcal{S}^*_\rho := \mathcal{S}^*\left(1 + \sinh^{-1} z\right),$$

which was introduced by Kumar and Arora [29]. They discussed relationships of this class with the already known classes. In 2021, Barukab *et al.* [12] derived sharp bounds for the Hankel determinant of the third order for the following function class:

$$\mathcal{R}_s := \left\{f : f \in \mathcal{A} \text{ and } f'(z) \prec 1 + \sinh^{-1} z \ (z \in \mathbb{D})\right\}.$$

In the present paper, we consider the following hyperbolic function:

$$\varphi_1(z) := 1 + \tanh z \quad \bigl(\varphi_1(0) = 1\bigr).$$

Also, one can easily find that $\Re(\varphi_1(z)) > 0$.

Definition 1 ([59]) By using the above-defined hyperbolic function $\varphi_1(z)$, we define the following family of functions:

$$\mathcal{S}^*_{\tanh} := \left\{ f : f \in \mathcal{S} \text{ and } \frac{zf'(z)}{f(z)} \prec 1 + \tanh z \ (z \in \mathbb{D}) \right\}. \tag{2}$$

In other words, a function f is in the class $\mathcal{S}^*_{\tanh}$ if and only if there exists a holomorphic function q, fulfilling $q(z) \prec q_0(z) := 1 + \tanh z$, such that

$$f(z) = z \exp\left(\int_0^z \frac{q(t) - 1}{t} \, \mathrm{d}t \right). \tag{3}$$

By taking

$$q(z) = q_0(z) = 1 + \tanh z$$

in (3), we get the function that plays the role of the extremal function in many problems of the class $\mathcal{S}^*_{\tanh}$, given by

$$f_0(z) = z \exp\left(\int_0^z \frac{\tanh t}{t} \, \mathrm{d}t \right) = z + z^2 + \frac{1}{2} z^3 + \frac{1}{18} z^4 + \cdots. \tag{4}$$

Definition 2 The Hankel determinant

$$\mathcal{HD}_{q,n}(f) \quad \bigl(q, n \in \mathbb{N} := \{1, 2, 3, \ldots\}; a_1 = 1\bigr)$$

for a function $f \in \mathcal{S}$ of the series form (1) was given by Pommerenke [40, 41] as follows:

$$\mathcal{HD}_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

In particular, the following determinants are known as the first-, the second-, and the third-order Hankel determinants, respectively:

$$\mathcal{HD}_{2,1}(f) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2, \tag{5}$$

$$\mathcal{HD}_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2, \tag{6}$$

and

$$\mathcal{HD}_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3\left(a_2a_4 - a_3^2\right) - a_4(a_4 - a_2a_3) + a_5\left(a_3 - a_2^2\right). \tag{7}$$

In the literature, there are just a few references to the Hankel determinant for functions belonging to the general family $\mathcal{S}$. For the function $f \in \mathcal{S}$, the best established sharp inequality is given by

$$\left|\mathcal{HD}_{2,n}(f)\right| \leqq \lambda\sqrt{n},$$

where λ is an absolute constant. This result is due to Hayman [21]. Further, for the same class $\mathcal{S}$, it was derived in [39] as follows:

$$\left|\mathcal{HD}_{2,2}(f)\right| \leqq \lambda \quad \left(1 \leqq \lambda \leqq \frac{11}{3}\right)$$

and

$$\left|\mathcal{HD}_{3,1}(f)\right| \leqq \mu \quad \left(\frac{4}{9} \leqq \mu \leqq \frac{32 + \sqrt{285}}{15}\right).$$

The challenge of finding the sharp bounds of Hankel determinants for a particular family of functions drew the attention of numerous researchers. For example, the sharp bounds of $|\mathcal{HD}_{2,2}(f)|$ for the sub-families $\mathcal{K}$, $\mathcal{S}^*$, and $\mathcal{R}$ (the family of bounded turning functions) of the class $\mathcal{S}$ were calculated by Janteng *et al.* [23, 24]. These estimates are given by

$$\left|\mathcal{HD}_{2,2}(f)\right| \leqq \begin{cases} \frac{1}{8} & (f \in \mathcal{K}), \\ 1 & (f \in \mathcal{S}^*), \\ \frac{4}{9} & (f \in \mathcal{R}). \end{cases}$$

For the families

$$\mathcal{S}^*(\beta) \quad (0 \leqq \beta < 1)$$

of starlike functions of order β and

$$\mathcal{SS}^*(\beta) \quad (0 < \beta \leqq 1)$$

of strongly starlike functions of order β, the authors in [15, 16] showed that $|\mathcal{HD}_{2,2}(f)|$ is bounded by $(1-\beta)^2$ and β^2, respectively. The exact bound for the family $\mathcal{S}^*(\phi)$ of the Ma–Minda type starlike functions was derived in [33] (see also [19]). For other works involving $|\mathcal{HD}_{2,2}(f)|$, see (for example) [4, 10, 14, 25, 35].

It is quite clear from the formulas given in (5), (6), and (7) that the calculation of the bound for $|\mathcal{HD}_{3,1}(f)|$ is far more challenging in comparison with the finding of the bound for $|\mathcal{HD}_{2,2}(f)|$. In the year 2010, Babalola [8] investigated the bounds for the third-order

Hankel determinant for the families of $\mathcal{K}$, $\mathcal{S}^*$, and $\mathcal{R}$. Subsequently, by using the same or analogous approach, several authors in [3, 11, 28, 43, 45] derived bounds for the third-order Hankel determinant $|\mathcal{HD}_{3,1}(f)|$ for various sub-families of analytic and univalent functions. On the other hand, in the year 2017, Zaprawa [61] improved the findings of Babalola [8] by applying a new methodology to show that

$$\left|\mathcal{HD}_{3,1}(f)\right| \leqq \begin{cases} \frac{49}{540} & (f \in \mathcal{K}), \\ 1 & (f \in \mathcal{S}^*), \\ \frac{41}{60} & (f \in \mathcal{R}). \end{cases}$$

Zaprawa [61] remarked that such limits were indeed not the best ones. Later in the year 2018, Kwon *et al.* [31] strengthened Zaprawa's result for $f \in \mathcal{S}^*$ and showed that $|\mathcal{HD}_{3,1}(f)| \leqq \frac{8}{9}$, and this bound was further improved by Zaprawa *et al.* [62] by showing in 2021 that

$$\left|\mathcal{HD}_{3,1}(f)\right| \leqq \frac{5}{9} \quad \left(f \in \mathcal{S}^*\right).$$

In recent years, the following sharp bounds for the third-order Hankel determinant $|\mathcal{HD}_{3,1}(f)|$ were given by Kowalczyk *et al.* [27] and Lecko *et al.* [32]:

$$\left|\mathcal{HD}_{3,1}(f)\right| \leqq \begin{cases} \frac{4}{135} & (f \in \mathcal{K}), \\ \frac{1}{9} & (f \in \mathcal{S}^*(\frac{1}{2})), \end{cases}$$

where $\mathcal{S}^*(\frac{1}{2})$ represents the family of starlike functions of order $\frac{1}{2}$ in $\mathbb{D}$. The interested readers may also refer to the research provided by Mahmood *et al.* [37] in which they calculated bounds for the third-order Hankel determinant for the basic (or $\mathfrak{q}$-) starlike functions in $\mathbb{D}$.

For more contributions in this direction, the interested reader should see, for example, [20, 44, 47, 52–55]. In particular, Arif *et al.* [6], Srivastava *et al.* [55], Arif *et al.* [5], and Wang *et al.* [60] successfully investigated bounds for the fourth-order Hankel determinant for different subclasses of analytic functions.

In the present article, our aim is to calculate the sharp bounds of the coefficient inequalities, Fekete–Szegö type functional, and the Hankel determinants of order two and order three for the subclass $\mathcal{S}^*_{\tanh}$ of starlike functions.

2 A set of lemmas

Definition 3 A function p is said to be in the class $\mathcal{P}$ if and only if it has the following series expansion:

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{D}) \tag{8}$$

and satisfies the inequality given by

$$\Re\bigl(p(z)\bigr) \geqq 0 \quad (z \in \mathbb{D}).$$

Lemma 1 Let the function $p \in \mathcal{P}$ have the series form (8). Then, for $x, \delta, \rho \in \overline{\mathbb{D}} = \mathbb{D} \cup \{1\}$,

$$2c_2 = c_1^2 + (4 - c_1^2)x, \tag{9}$$

$$4c_3 = c_1^3 + 2c_1x(4 - c_1^2) - x^2c_1(4 - c_1^2) + 2(1 - |x|^2)(4 - c_1^2)\delta \tag{10}$$

and

$$\begin{aligned} 8c_4 &= c_1^4 + x\big[c_1^2(x^2 - 3x + 3) + 4x\big](4 - c_1^2) - 4(4 - c_1^2)(1 - |x|^2) \\ &\quad \times \big[c(x - 1)\delta + \overline{x}\delta^2 - (1 - |\delta|^2)\rho\big](4 - c_1^2). \end{aligned} \tag{11}$$

Remark 2 *In Lemma 1 and elsewhere in this paper, for the formula for c_2, see [42]. The formula for c_3 is due to Libera and Złotkiewicz [34]. The formula for c_4 was proved in [30].*

Lemma 3 If the function $p \in \mathcal{P}$ has the series form (8), then

$$|c_{n+k} - \mu c_n c_k| \leqq 2\max(1, |2\mu - 1|) \tag{12}$$

and

$$|c_n| \leqq 2 \quad (n \geqq 1). \tag{13}$$

If $B \in [0,1]$ with $B(2B - 1) \leqq D \leqq B$, then

$$\left|c_3 - 2Bc_1c_2 + Dc_1^3\right| \leqq 2. \tag{14}$$

Remark 4 *Inequalities (12), (13), and (14) in Lemma 3 are taken from [26, 42] and [6, 7, 47], respectively.*

3 Coefficient inequalities for the function class $\mathcal{S}^*_{\tanh}$

The first two findings, Theorem 5 and Theorem 6, are special cases of the results established in the paper [1], and that is why we omitted both the proofs.

Theorem 5 *Let the function f of the form (1) be in the class $\mathcal{S}^*_{\tanh}$. Then*

$$|a_2| \leqq 1,$$

$$|a_3| \leqq \frac{1}{2},$$

$$|a_4| \leqq \frac{1}{3}.$$

Each of these bounds is sharp.

Theorem 6 *Let the function f of the form (1) be in the class $\mathcal{S}^*_{\tanh}$. Then*

$$\left|a_3 - \lambda a_2^2\right| \leqq \max\left\{\frac{1}{2}, \frac{1}{2}|2\lambda - 1|\right\}.$$

This inequality is sharp.

Theorem 7 *Let the function f of the form* (1) *be in the class $\mathcal{S}^*_{\tanh}$. Then*

$$|a_2a_3 - a_4| \leqq \frac{1}{3}.$$

This result is sharp.

Proof Let $f \in \mathcal{S}^*_{\tanh}$. Then equation (2) can be written in the form of a hyperbolic function w as follows:

$$\frac{zf'(z)}{f(z)} = 1 + \tanh w(z).$$

Let $p \in \mathcal{P}$. Then, in terms of the Schwarz function w, we have

$$p(z) = \frac{1 + (w(z))}{1 - (w(z))} := 1 + c_1z + c_2z^2 + c_3z^3 + c_4z^4 + \cdots \tag{15}$$

or, equivalently,

$$w(z) := \frac{p(z) - 1}{p(z) + 1} = \frac{c_1z + c_2z^2 + c_3z^3 + c_4z^4 + \cdots}{2 + c_1z + c_2z^2 + c_3z^3 + c_4z^4 + \cdots},$$

where

$$\begin{aligned} w(z) &= \frac{1}{2}c_1z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)z^2 + \left(\frac{1}{8}c_1^3 - \frac{1}{2}c_1c_2 + \frac{1}{2}c_3\right)z^3 \\ &\quad + \left(\frac{1}{2}c_4 - \frac{1}{2}c_1c_3 - \frac{1}{4}c_2^2 - \frac{1}{16}c_1^4 + \frac{3}{8}c_1^2c_2\right)z^4 + \cdots. \end{aligned} \tag{16}$$

By using (1), we obtain

$$\begin{aligned} \frac{zf'(z)}{f(z)} &:= 1 + a_2z + \left(2a_3 - a_2^2\right)z^2 + \left(a_2^3 - 3a_2a_3 + 3a_4\right)z^3 \\ &\quad + \left(4a_5 - a_2^4 + 4a_2^2a_3 - 4a_2a_4 - 2a_3^2\right)z^4 + \cdots. \end{aligned} \tag{17}$$

After some calculation and by using the series expansion given by (16), we get

$$\begin{aligned} 1 + \tanh\bigl(w(z)\bigr) &= 1 + \frac{1}{2}c_1z + \left(-\frac{1}{4}c_1^2 + \frac{1}{2}c_2\right)z^2 + \left(\frac{1}{12}c_1^3 - \frac{1}{2}c_1c_2 + \frac{1}{2}c_3\right)z^3 \\ &\quad + \left(\frac{1}{2}c_4 + \frac{1}{4}c_1^2c_2 - \frac{1}{2}c_1c_3 - \frac{1}{4}c_2^2\right)z^4 + \cdots. \end{aligned} \tag{18}$$

Now, if we compare (17) and (18), we get

$$a_2 = \frac{1}{2}c_1, \tag{19}$$

$$a_3 = \frac{1}{4}c_2, \tag{20}$$

$$a_4 = \frac{1}{6}c_3 - \frac{1}{72}c_1^3 - \frac{1}{24}c_1c_2, \tag{21}$$

and

$$a_5 = \frac{1}{8}c_4 + \frac{5}{576}c_1^4 - \frac{1}{32}c_2^2 - \frac{1}{24}c_1c_3 - \frac{1}{48}c_1^2c_2. \tag{22}$$

By using (19), (20), and (21), we obtain

$$|a_2a_3 - a_4| = \frac{1}{72}\left|c_1^3 + 12c_1c_2 - 12c_3\right|,$$

which, in view of (9) and (10), together with $c_1 = c \in [0,1]$, yields

$$|a_2a_3 - a_4| = \frac{1}{72}\left|4c^3 + 3c\left(4-c^2\right)x^2 - 6\left(4-c^2\right)\left(1-|x|^2\right)\delta\right|.$$

Now, upon applying $|\delta| \leqq 1$ and $|x| = b \leqq 1$, and using the triangle inequality, we get

$$|a_2a_3 - a_4| \leqq \frac{1}{72}\left[4c^3 + 3\left(4-c^2\right)(c-2)b^2 + 6\left(4-c^2\right)\right] = F(c,b).$$

It is a simple exercise to differentiate $F(c,b)$ with respect to b and show that $F'(c,b) \leqq 0$ on the rectangle $[0,2] \times [0,1]$. So, by putting $b = 0$, we obtain

$$\max\left\{F(c,b)\right\} = F(c,0).$$

We thus find that

$$|a_2a_3 - a_4| \leqq \frac{1}{72}\left[4c^3 + 6\left(4-c^2\right)\right] = G(c).$$

Finally, upon taking $G'(c) = 0$, we obtain $c = 0, 1$. Thus, clearly, $G(c)$ has its maximum value at $c = 0$, so that

$$|a_2a_3 - a_4| \leqq \frac{1}{72}(24) = \frac{1}{3},$$

in which the equality holds true for the extremal function given by

$$\begin{aligned} f_3(z) &= z\exp\left(\int_0^z \frac{(1+\tanh t^3)-1}{t}\,\mathrm{d}t\right) \\ &= z + \frac{1}{3}z^4 + \frac{1}{18}z^7 - \frac{5}{162}z^{10} + \cdots. \end{aligned} \tag{23}$$

This evidently completes our demonstration of Theorem 7. □

Theorem 8 *Let the function f of the form (1) be in the class $\mathcal{S}^*_{\tanh}$. Then*

$$\left|\mathcal{HD}_{2,2}(f)\right| \leqq \frac{1}{4}.$$

This inequality is sharp.

Proof We can write $\mathcal{HD}_{2,2}(f)$ as follows:

$$\mathcal{HD}_{2,2}(f) = \left|a_2a_4 - a_3^2\right|.$$

From (19), (20), and (21), we have

$$\left|a_2a_4 - a_3^2\right| = \frac{1}{144}\left|-c_1^4 - 3c_1^2c_2 + 12c_1c_3 - 9c_2^2\right|.$$

Now, by using (9) and (10) in order to express c_2 and c_3 in terms of c_1 and also $c_1 = c$ $(0 \leqq c \leqq 2)$, we obtain

$$\left|a_2a_4 - a_3^2\right| = \frac{1}{144}\left|-\frac{7}{4}c^4 - 3\left(4-c^2\right)c^2x^2 + 6c\left(4-c^2\right)\left(1-|x|^2\right)\delta - \frac{9}{4}\left(4-c^2\right)^2x^2\right|.$$

By using $|\delta| \leqq 1$ and $|x| = b \leqq 1$ and applying the triangle inequality, if we take $c \in [0,2]$, we obtain

$$\begin{aligned}\left|a_2a_4 - a_3^2\right| &\leqq \frac{1}{144}\left[\frac{7}{4}c^4 + 3\left(4-c^2\right)c^2b^2 + 6c\left(4-c^2\right)\left(1-b^2\right) + \frac{9}{4}\left(4-c^2\right)^2b^2\right] \\ &=: \Xi(c,b).\end{aligned}$$

Upon differentiating with respect to b, we have

$$\frac{\partial\,\Xi(c,b)}{\partial b} = \frac{1}{144}\left(\frac{3}{2}\left(4-c^2\right)\left(c^2 - 8c + 12\right)b\right).$$

It is a simple exercise to show that $\Xi'(c,b) \geqq 0$ on $[0,1]$, so that

$$\Xi(c,b) \leqq \Xi(c,1).$$

Putting $b = 1$, we get

$$\left|a_2a_4 - a_3^2\right| \leqq \frac{1}{144}\left(\frac{7}{4}c^4 + 3\left(4-c^2\right)c^2 + \frac{9}{4}\left(4-c^2\right)^2\right) := G(c).$$

As $G'(c) \leqq 0$, so $G(c)$ is a decreasing function of c, so that it gives the maximum value at $c = 0$:

$$\left|\mathcal{HD}_{2,2}(f)\right| \leqq \frac{1}{144}(36) = \frac{1}{4}.$$

Finally, the above bound for $\mathcal{HD}_{2,2}(f)$ is sharp and is achieved by the following extremal function:

$$\begin{aligned}f_2(z) &= z\exp\left(\int_0^z \frac{(1+\tanh t^2) - 1}{t}\,\mathrm{d}t\right) \\ &= z + \frac{1}{2}z^3 + \frac{1}{8}z^5 - \frac{5}{144}z^7 + \cdots.\end{aligned} \tag{24}$$

We have thus completed the proof of Theorem 8. □

4 The third Hankel determinant

In this section, we determine the bounds of $|\mathcal{HD}_{3,1}(f)|$ for the function $f \in \mathcal{S}^*_{\tanh}$.

Theorem 9 *Let the function f of the form (1) be in the class $\mathcal{S}^*_{\tanh}$. Then*

$$\left|\mathcal{HD}_{3,1}(f)\right| \leqq \frac{1}{9}.$$

This result is sharp.

Proof The third-order Hankel determinant can be written as follows:

$$\mathcal{HD}_{3,1}(f) = 2a_2a_3a_4 - a_2^2a_5 - a_3^3 + a_3a_5 - a_4^2.$$

By using (19), (20), (21), and (22), together with $c_1 = c \in [0,2]$, we have

$$\begin{aligned}\mathcal{HD}_{3,1}(f) = \frac{1}{20{,}736}\big(&-49c^6 + 57c^4c_2 + 312c^3c_3 - 198c^2c_2^2 - 648c^2c_4 + 936cc_2c_3 \\ &- 486c_2^3 + 648c_2c_4 - 576c_3^2\big).\end{aligned} \tag{25}$$

For simplifying the computation, we let $t = 4 - c^2$ in (9), (10), and (11). Then, by using the simplified form of these formulas, we have

$$\begin{aligned}
57c^4c_2 &= \frac{57}{2}\big(c^6 + c^4tx\big),\\
312c^3c_3 &= 78c^6 + 156c^4tx - 78c^4tx^2 + 156c^3t\big(1-|x|^2\big)\delta,\\
198c^2c_2^2 &= \frac{99}{2}\big(c^6 + 2c^4tx + c^2t^2x^2\big),\\
648c^2c_4 &= 81c^4tx^3 - 324c^2t\bar{x}\big(1-|x|^2\big)\delta^2 - 324c^3tx\big(1-|x|^2\big)\delta - 243c^4tx^2\\
&\quad + 324c^2t\big(1-|x|^2\big)\big(1-|\delta|^2\big)\rho + 324c^3t\big(1-|x|^2\big)\delta + 243c^4tx\\
&\quad + 81c^6 + 324c^2ax^2,\\
936cc_2c_3 &= -117c^2t^2x^3 - 117c^4tx^2 + 234cxt^2\big(1-|x|^2\big)\delta + 243c^2t^2x^2 + 243c^3t\big(1-|x|^2\big)\delta\\
&\quad + 117c^6 + 351c^4tx,\\
486c_2^3 &= \frac{243}{4}\big(t^3x^3 + 3c^2t^2x^2 + 3c^4tx + c^6\big),\\
648c_2c_4 &= \frac{81}{2}\big(4c^2tx^2 + 4t^2x^3 + c^6 + 4c^4tx + 4c^3t\big(1-|x|^2\big)\delta + 4c^2t\big(1-|x|^2\big)\big(1-|\delta|^2\big)\rho\\
&\quad + 3c^2t^2x^2 + 4ct^2x\big(1-|x|^2\big)\delta + 4t^2x\big(1-|x|^2\big)\big(1-|\delta|^2\big)\rho - 3c^4tx^2\\
&\quad - 4c^3tx\big(1-|x|^2\big)\delta - 4c^2t\bar{x}\big(1-|x|^2\big)\delta^2 - 3c^2t^2x^3 - 4ct^2x^2\big(1-|x|^2\big)\delta\\
&\quad - 4t^2x\bar{x}\big(1-|x|^2\big)\delta^2 + c^4tx^3 + c^2t^2x^4\big)
\end{aligned}$$

and

$$576c_3^2 = 36c^2t^2x^4 - 144ct^2x^2\big(1-|x|^2\big)\delta - 144c^2t^2x^3 - 72c^4tx^2 + 144t^2\big(1-|x|^2\big)^2\delta^2$$

$$+ 288ct^2x\left(1-|x|^2\right)\delta + 144c^3t\left(1-|x|^2\right)\delta + 144c^2t^2x^2 + 144c^4tx + 36c^6.$$

Upon substituting these expressions into (25) and simplifying, we get

$$\begin{aligned}\mathcal{HD}_{3,1}(f) = \frac{1}{20{,}736}\Bigg[&-\frac{49}{4}c^6 + 84c^3t\left(1-|x|^2\right)\delta + 162t^2x^3 - 162c^2t\left(1-|x|^2\right)\left(1-|\delta|^2\right)\rho \\ &+ 108ct^2x\left(1-|x|^2\right)\delta - \frac{243}{4}t^3x^3 + 162t^2x\left(1-|x|^2\right)\left(1-|\delta|^2\right)\rho \\ &+ 162c^3tx\left(1-|x|^2\right)\delta + 162c^2t\bar{x}\left(1-|x|^2\right)\delta^2 - 18ct^2x^2\left(1-|x|^2\right)\delta \\ &- 162t^2x\bar{x}\left(1-|x|^2\right)\delta^2 - 144t^2\left(1-|x|^2\right)^2\delta^2 - \frac{3}{2}c^4x^2t - \frac{81}{4}c^2t^2x^2 \\ &- \frac{189}{2}c^2t^2x^3 - 162c^2tx^2 + \frac{9}{2}c^2t^2x^4 - \frac{81}{2}c^4tx^3 + \frac{117}{4}c^4tx\Bigg].\end{aligned}$$

Now, since $t = (4-c^2)$, we have

$$\mathcal{HD}_{3,1}(f) = \frac{1}{20{,}736}\left[v_1(c,x) + v_2(c,x)\delta + v_3(c,x)\delta^2 + \Phi(c,x,\delta)\rho\right],$$

where

$$\begin{aligned}v_1(c,x) &= -\frac{3}{4}\left(4-c^2\right)x\Big[3\left(4-c^2\right)x\left(-2x^2c^2 + 15xc^2 + 9c^2 + 36x\right) \\ &\quad + 54c^4x^2 + 2c^4x - 39c^4 + 216xc^2\Big] - \frac{49}{4}c^6, \\ v_2(c,x) &= -6\left(4-c^2\right)\left(1-|x|^2\right)c\left[\left(3x^2-18x\right)\left(4-c^2\right) - 27xc^2 - 14c^2\right], \\ v_3(c,x) &= -18\left(4-c^2\right)\left(1-|x|^2\right)\left[\left(x^2+8\right)\left(4-c^2\right) - 9\bar{x}c^2\right],\end{aligned}$$

and

$$\Phi_4(c,x,\delta) = 162\left(4-c^2\right)\left(1-|x|^2\right)\left(1-|\delta|^2\right)\left[\left(4-c^2\right)x - c^2\right].$$

Thus, upon setting $|\delta| = y$ and $|x| = x$, and by taking $|\rho| \leqq 1$, we obtain

$$\begin{aligned}\left|\mathcal{HD}_{3,1}(f)\right| &\leqq \frac{1}{20{,}736}\left(\left|v_1(c,x)\right| + \left|v_2(c,x)\right|y + \left|v_3(c,x)\right|y^2 + \left|\Phi(c,x,\delta)\right|\right), \\ &\leqq \frac{1}{20{,}736}\left[H(c,x,y)\right],\end{aligned} \tag{26}$$

where

$$H(c,x,y) = \left(h_1(c,x) + h_2(c,x)y + h_3(c,x)y^2 + h_4(c,x)\left(1-y^2\right)\right), \tag{27}$$

with

$$\begin{aligned}h_1(c,x) &= \frac{3}{4}\left(4-c^2\right)x\Big[3\left(4-c^2\right)x\left(-2x^2c^2 + 15xc^2 + 9c^2 + 36x\right) \\ &\quad + 54c^4x^2 + 2c^4x - 39c^4 + 216xc^2\Big] + \frac{49}{4}c^6,\end{aligned}$$

$$h_2(c,x) = 6\left(4-c^2\right)\left(1-x^2\right)c\left[\left(-3x^2+18x\right)\left(4-c^2\right)+27xc^2+14c^2\right],$$
$$h_3(c,x) = 18\left(4-c^2\right)\left(1-x^2\right)\left[\left(x^2+8\right)\left(4-c^2\right)+9xc^2\right],$$

and

$$h_4(c,x) = 162\left(4-c^2\right)\left(1-x^2\right)\left[\left(4-c^2\right)x+c^2\right].$$

Let the closed cuboid be of the following form:

$$\Delta : [0,2] \times [0,1] \times [0,1].$$

We need to find the points of maxima inside this closed cuboid Δ, inside the six faces, and on the twelve edges in order to maximize the function $H(c,x,y)$ given by (27). For this objective in view, we consider the following three cases.

I. Let $c,x,y \in (0,2) \times (0,1) \times (0,1)$. In order to find the points of maxima inside Δ, we take partial derivative of (27) with respect to y, so that we achieve

$$\begin{aligned}\frac{\partial H}{\partial y} &= 6\left(4-c^2\right)\left(1-x^2\right)\left[6y(x-1)\left[(x-8)\left(4-c^2\right)+9c^2\right]\right.\\&\quad\left.+c\left(3x\left(4-c^2\right)(6-x)+c^2(27x+14)\right)\right],\end{aligned} \tag{28}$$

which can be seen to vanish when

$$y = \frac{c[3x(4-c^2)(x-6)-c^2(27x+14)]}{6(x-1)[(4-c^2)(x-8)+9c^2]}.$$

If y_0 is a critical point inside Δ, then $y_0 \in (0,1)$, which is possible only if

$$\begin{aligned}&c\left(3x\left(4-c^2\right)(6-x)+c^2(27x+14)\right)-6(1-x)\left(4-c^2\right)(8-x)\\&\quad< -54(1-x)c^2\end{aligned} \tag{29}$$

and

$$c^2 > \frac{4(8-x)}{17-x}. \tag{30}$$

We now have to get the solutions which satisfy both of inequalities (29) and (30) for the existence of the critical points. Let us set

$$h(x) = \frac{4(8-x)}{17-x}.$$

Since $h'(x) < 0$ for $(0,1)$, the function $h(x)$ is decreasing in $(0,1)$. Hence $c^2 > \frac{7}{4}$, and a simple exercise shows that (29) does not hold true in this case for all values of $x \in (0,1)$ and there is no critical point of $H(c,x,y)$ in $(0,2) \times (0,1) \times (0,1)$.

II. In order to find the points of maxima inside the six faces of the cuboid Δ, we deal with each face individually. On $c = 0$, $H(c,x,y)$ reduces to

$$q_1(x,y) = H(0,x,y) = 1296x^3 + 72\left(1-x^2\right)\left(4x^2+32\right)y^2$$

$$+ 2592x\big(1-x^2\big)\big(1-y^2\big) \quad \big(x,y \in (0,1)\big). \tag{31}$$

Clearly, q_1 has no optimal points in $(0,1) \times (0,1)$ since

$$\frac{\partial q_1}{\partial y} = 144\big(1-x^2\big)\big(4x^2+32\big)y - 5184\big(1-x^2\big)xy \neq 0 \quad \big(x,y \in (0,1)\big). \tag{32}$$

On $c = 2$, $H(c,x,y)$ reduces to

$$H(2,x,y) = 784 \quad \big(x,y \in (0,1)\big). \tag{33}$$

On $x = 0$, $H(c,x,y)$ reduces to

$$\begin{aligned} q_2(c,y) = H(c,0,y) &= \frac{49}{4}c^6 + 14\big(24-6c^2\big)c^3y + \big(72-18c^2\big)\big(32-8c^2\big)y^2 \\ &\quad + c^2\big(648-162c^2\big)\big(1-y^2\big), \end{aligned} \tag{34}$$

where $y \in (0,1)$ and $c \in (0,2)$. We now solve

$$\frac{\partial q_2}{\partial y} = 0 \quad \text{and} \quad \frac{\partial q_2}{\partial c} = 0$$

in order to find the points of maxima. On solving

$$\frac{\partial q_2}{\partial y} = 0,$$

we obtain

$$y = \frac{7c^3}{3(17c^2-32)} =: y_1 \tag{35}$$

for the given range of y, y_1 that should belong to $(0,1)$. This is possible only if

$$c > c_0 \quad (c_0 \approx 1.54572016538129).$$

A calculation shows that

$$\frac{\partial q_2}{\partial c} = 0$$

implies that

$$\frac{147}{2}c^5 - 420c^4y + 1008c^2y + 1224c^3y^2 - 3600cy^2 - 648c^3 + 1296c = 0. \tag{36}$$

By substituting from equation (35) into equation (36) and simplifying, we have

$$9c\big(2499c^8 - 47{,}888c^6 + 239{,}904c^4 - 460{,}800c^2 + 294{,}912\big) = 0. \tag{37}$$

A further calculation gives the solution of (37) in $(0,2)$, that is, $c \approx 1.16653673056906$. Thus q_2 has no optimal point in $(0,2) \times (0,1)$.

On $x = 1$, $H(c,x,y)$ reduces to

$$q_3(c,y) = H(c,1,y) = 49c^6 - 426c^4 + 792c^2 + 1296 \quad \bigl(c \in (0,2)\bigr). \tag{38}$$

Solving

$$\frac{\partial q_3}{\partial c} = 0,$$

we obtain the critical points given by

$$c = c_0 = 0 \quad \text{and} \quad c = c_1 \approx 1.07838082301303.$$

Since c_0 is the minimum point of q_3, q_3 attains its maximum value at c_1, that is, at $c = 1717.98045$.

On $y = 0$, $H(c,x,y)$ reduces to

$$\begin{aligned} q_4(c,x) &= H(c,x,0) \\ &= \frac{49}{4}c^6 + \left(3 - \frac{3}{4}c^2\right)x\bigl[\bigl(12 - 3c^2\bigr)x\bigl(15c^2x - 2x^2c^2 + 9c^2 + 36x\bigr) \\ &\quad + 54c^4x^2 + 2c^4x - 39c^4 + 216c^2x\bigr] \\ &\quad + \bigl(648 - 162c^2\bigr)\bigl(1 - x^2\bigr)\bigl[\bigl(4 - c^2\bigr)x + c^2\bigr]. \end{aligned}$$

A computation reveals that the following system of equations has no solution:

$$\frac{\partial q_4}{\partial x} = 0 \quad \text{and} \quad \frac{\partial q_4}{\partial c} = 0$$

in $(0,2) \times (0,1)$.

On $y = 1$, $H(c,x,y)$ reduces to

$$\begin{aligned} q_5(c,x) &= H(c,x,1) \\ &= \frac{49}{4}c^6 + \left(3 - \frac{3}{4}c^2\right)x\bigl[\bigl(12 - 3c^2\bigr)x\bigl(15c^2x - 2x^2c^2 + 9c^2 + 36x\bigr) \\ &\quad + 54c^4x^2 + 2c^4x - 39c^4 + 216c^2x\bigr] \\ &\quad + \bigl(24 - 6x^2\bigr)(1 - x)c\bigl[\bigl(18x - 3x^2\bigr)\bigl(4 - c^2\bigr) + 27xc^2 + 14c^2\bigr] \\ &\quad + \bigl(72 - 18c^2\bigr)\bigl(1 - x^2\bigr)\bigl[\bigl(x^2 + 8\bigr)\bigl(4 - c^2\bigr) + 9c^2x\bigr]. \end{aligned}$$

A computation reveals that the following system of equations has no solution:

$$\frac{\partial q_5}{\partial x} = 0 \quad \text{and} \quad \frac{\partial q_5}{\partial c} = 0$$

in $(0,2) \times (0,1)$.

III. In this case, we find the maxima of $H(c,x,y)$ on the edges of Δ. By putting $y=0$ in (34), we have

$$H(c,0,0)=m_1(c)=\frac{49}{4}c^6-162c^4+648c^2.$$

Clearly, $m_1'(c)=0$ for $c=\eta_0=0$ and $c=\eta_1=1.75122868295016$ in $[0,2]$, where η_0 is the minimum point and the maximum point of $m_1(c)$ is attained at η_1. This implies that

$$H(c,0,0)\leqq 816.973630 \quad \bigl(c\in[0,2]\bigr).$$

Solving equation (34) at $y=1$, we get

$$H(c,0,1)=m_2(c)=\frac{49}{4}c^6-84c^5+144c^4+336c^3-1152c^2+2304.$$

Since $m_2'(c)<0$ for $c\in[0,2]$, $m_2(c)$ is decreasing in $[0,2]$ and hence the maximum is obtained at $c=0$. Thus

$$H(c,0,1)\leqq 2304 \quad \bigl(c\in[0,2]\bigr).$$

By putting $c=0$ in (34), we get

$$H(0,0,y)=2304y^2.$$

A simple calculation gives

$$H(0,0,y)=2304 \quad \bigl(y\in[0,1]\bigr).$$

Equation (38) is independent of y, so we have

$$H(c,1,1)=H(c,1,0)=m_3(c)=49c^6-426c^4+792c^2+1296.$$

Now $m_3'(c)=0$ for $c=\eta_0=0$ and $c=\eta_1=1.07838082301303$ in $[0,2]$, where η_0 is the minimum point and the maximum point of $m_3(c)$ is attained at η_1. We conclude that

$$H(c,1,1)=H(c,1,0)\leqq 1717.98045 \quad \bigl(c\in[0,2]\bigr).$$

By putting $c=0$ in (38), we obtain

$$H(0,1,y)=1296.$$

As (33) is independent of c, x, and y, we find that

$$H(2,1,y)=H(2,0,y)=H(2,x,0)=H(2,x,1)=784 \quad \bigl(x,y\in[0,1]\bigr).$$

By putting $y=0$ in (31), we have

$$H(0,x,0)=m_4(x)=-1296x^3+2592x.$$

Now $m_4'(x) = 0$ for $x = x_0 = 0.8164965809$ in $[0,1]$. Therefore, the function $m_4(x)$ is increasing for $x \leqq x_0$ and decreasing for $x_0 \leqq x$. Hence $m_4(x)$ has its maximum at $x = x_0$. We conclude that

$$H(0,x,0) \leqq 1410.906092 \quad \bigl(x \in [0,1]\bigr).$$

By putting $y = 1$ in (31), we get

$$H(0,x,1) = m_5(x) = -288x^4 + 1296x^3 - 2016x^2 + 2304.$$

Since $m_5'(x) < 0$ for $[0,1]$, therefore the function $m_4(x)$ is decreasing in $[0,1]$ and hence attains its maximum value at $x = 0$, so that

$$H(0,x,1) \leqq 2304 \quad \bigl(x \in [0,1]\bigr).$$

Thus, from the above cases, we conclude that

$$H(c,x,y) \leqq 2304 \quad \text{on } [0,2] \times [0,1] \times [0,1].$$

From equation (26), we can write

$$\bigl|\mathcal{HD}_{3,1}(f)\bigr| \leqq \frac{1}{20{,}736}\bigl(H(c,x,y)\bigr) \leqq \frac{1}{9}.$$

If $f \in \mathcal{S}^*_{\tanh}$, then the equality is achieved by the function given by

$$\begin{aligned} f_3(z) &= z\exp\left(\int_0^z \frac{(1+\tanh t^3)-1}{t}\,\mathrm{d}t\right) \\ &= z + \frac{1}{3}z^4 + \frac{1}{18}z^7 - \frac{5}{162}z^{10} + \cdots. \end{aligned} \tag{39}$$

Theorem 9 has thus been proved as asserted. □

5 Concluding remarks and observations

In the present article, we have introduced and studied a new subfamily of starlike functions in the open unit disk $\mathbb{D}$, which involves the hyperbolic function $\tanh z$. For functions belonging to such a class of starlike functions, we have considered some interesting problems such as the bounds of the first three Taylor–Maclaurin coefficients, the estimates of the Fekete–Szegö type functional, and the estimates of the second- and third-order Hankel determinants. All of the bounds which we have investigated in this article have been shown to be sharp.

A potential direction for further research based upon our present investigation would involve the use of the familiar quantum or basic (or $\mathfrak{q}$-) calculus as (for example) in the related recent works [37, 44, 50, 53, 54, 56], [57], and [58]. However, as clearly pointed out in the survey-cum-expository review articles by Srivastava (see, for details, [50, p. 340]; see also [51, pp. 1511–1512]), any attempt to translate these suggested $\mathfrak{q}$-results in terms of the so-called trivial and inconsequential $(\mathfrak{p},\mathfrak{q})$-calculus would obviously lead to a shallow research, because the additional forced-in parameter $\mathfrak{p}$ is obviously redundant or superfluous.

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