Journal of Nonlinear Mathematical Physics

AN INVERTIBLE TRANSFORMATION AND SOME OF ITS APPLICATIONS

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Published online: 04 Mar 2013.

PLEASE SCROLL DOWN FOR ARTICLE

AN INVERTIBLE TRANSFORMATION AND SOME OF ITS APPLICATIONS

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Several applications of an *explicitly invertible* transformation are reported. This transformation is elementary and therefore all the results obtained via it might be considered trivial; yet the findings highlighted in this paper are generally far from appearing trivial until the way they are obtained is revealed. Various contexts are considered: algebraic and Diophantine equations, nonlinear Sturm–Liouville problems, dynamical systems (with continuous and with discrete time), nonlinear partial differential equations, analytical geometry, functional equations. While this transformation, in one or another context, is certainly known to many, it does not seem to be as universally known as it deserves to be, for instance it is not routinely taught in basic University courses (to the best of our knowledge). The main purpose of this paper is to bring about a change in this respect; but we also hope that some of the findings reported herein — and the multitude of analogous findings easily obtainable via this technique — will be considered remarkable by the relevant experts, in spite of their elementary origin.

Keywords: Invertible transformations; isochronous systems; solvable algebraic and Diophantine equations; solvable nonlinear Sturm–Liouville problems; solvable dynamical systems; solvable Hamiltonian systems; solvable discrete-time dynamical systems; solvable functional equations.

Mathematics Subject Classification: 26B10, 11Dxx, 34C20, 34K17, 51Mxx, 70H15

1. Introduction

The main purpose of this paper is to advertise an *explicitly invertible* change of variables, the potentialities of which seem to us remarkable. This transformation is certainly known, in specific contexts, to many, but it is too elementary to make a search for its *first* appearance(s) a feasible enterprise for us: identifying the inventor(s) of this wonderful trick is a task for the mathematical historian rather than the mathematical physicist. Yet it seems to us that the fantastic potential of this *explicitly invertible* transformation, in a multitude of quite different mathematical and applicative contexts, has not been adequately recognized

so far; as indicated by the fact that it is not — to the best of our knowledge — routinely taught in basic university course (except perhaps in specific contexts, such as, for instance, Hamiltonian dynamics). We hope that this paper will produce a change in this respect.

In this paper we outline a number of possible applications of this transformation, in a variety of different contexts: algebraic and Diophantine equations, nonlinear Sturm–Liouville problems, dynamical systems (with continuous and with discrete time), nonlinear Partial Differential Equations (PDEs), analytical geometry, functional equations. In each of these cases we exhibit examples the general character of which is to show how, via this transformation, a *trivial* result (typically, an obviously solvable problem) gets transformed into a finding (typically another, just as solvable, problem) that many an expert in the specific field under consideration might well classify, on the face of it, as interesting and *nontrivial*; only to change opinion and reclassify it as *trivial* after discovering how that finding has been obtained from a *trivial* result via an *elementary* transformation. In each of these cases it will be clear to the reader that the examples we display provide merely a glimpse of the much wider applicability of the approach we describe. And undoubtedly there also is a vast range of possible applications in other fields of pure and applied mathematics.

The general philosophy subtending this paper is that progress in science is achieved not only by identifying a specific (*interesting*!) problem and then finding a technique to solve it, but also by focusing instead on some specific mathematical technique and then exploring the multitude of problems that can be mastered by it. A much quoted formulation of this research strategy — in the context of trying to solve linear PDEs by separation of variables — goes back to Carl Jacobi: "The main difficulty to integrate these differential equations is to find the appropriate change of variables. There is no rule to discover it. Hence we need to follow the inverse path, namely to introduce some convenient change of variables and investigate to which problems it can be successfully applied." [1] (The reader puzzled by Jacobi's drastic statement that "There is no rule to discover it" should remember that Jacobi died in 1851, when Sophus Lie was 9 years old ...)

The transformation treated herein is described, in its simplest formulation, in the following Sec. 2. Generalizations are outlined in Sec. 3. Various applications are treated in Sec. 4: Algebraic and Diophantine equations in Subsec. 4.1, nonlinear Sturm–Liouville problems in Subsec. 4.2, standard (continuous-time) dynamical systems (including Hamiltonian systems) in Subsec. 4.3, *discrete-time* dynamical systems (or, equivalently, maps) in Subsec. 4.4, nonlinear PDEs in Subsec. 4.5, analytical geometry in Subsec. 4.6, functional equations in Subsec. 4.7. Their presentation — while avoiding unnecessary repetitions — is tailored to allow the reader wishing to do so, to jump directly to the relevant subsection of Sec. 4, after having digested the material of Sec. 2 and perhaps relevant aspects of Sec. 3; the level of detail of the treatment is however quite different in different subsections. Experts in fields different from those mentioned herein are of course welcome to try and apply the trick in other contexts.

2. The Explicitly Invertible Transformation

In this section we introduce first of all the simplest version of the *explicitly invertible* transformation: it consists of a change of variables, involving 2 *arbitrary* functions $F_1(w), F_2(w)$,

from 2 quantities u_1, u_2 to 2 quantities x_1, x_2 and vice versa. It reads as follows:

$$x_1 = u_1 + F_1(u_2), \quad x_2 = u_2 + F_2(x_1) = u_2 + F_2(u_1 + F_1(u_2)); \tag{2.1a}$$

$$u_2 = x_2 - F_2(x_1), \quad u_1 = x_1 - F_1(u_2) = x_1 - F_1(x_2 - F_2(x_1)). \tag{2.1b}$$

The most remarkable aspect of this transformation is its *explicitly invertible* character: note that both the *direct* respectively the *inverse* changes of variables, (2.1a) respectively (2.1b), involve *only* (albeit also in a nested manner) the 2 *arbitrary* functions $F_1(w), F_2(w)$, and *not* their inverses. This in particular entails that, if the 2 functions $F_1(w), F_2(w)$ are *one-valued* (as we hereafter assume), both the direct and inverse changes of variables are *one-valued*; if the 2 functions $F_1(w), F_2(w)$ are *entire*, this property is inherited by both the direct and inverse changes of variables; if the 2 functions $F_1(w), F_2(w)$ are *polynomials* (of arbitrary degree), both the expressions of x_1 and x_2 in terms of u_1 and u_2, and the expressions of u_1 and u_2 in terms of x_1 and x_2, are as well *polynomial*.

For instance for

$$F_1(w) = c_1 w^2, \quad F_2(w) = c_2 w^2, \tag{2.2a}$$

the direct and inverse transformations (2.1) read as follows:

$$
\begin{aligned}
x_1 &= u_1 + c_1 u_2^2, \\
x_2 &= u_2 + c_2 x_1^2 = u_2 + c_2(u_1 + c_1 u_2^2)^2 \\
&= u_2 + c_2(u_1^2 + 2c_1 u_1 u_2^2 + c_1^2 u_2^4);
\end{aligned}
\tag{2.2b}
$$

$$
\begin{aligned}
u_1 &= x_1 - c_1 u_2^2 = x_1 - c_1(x_2 - c_2 x_1^2)^2 \\
&= x_1 - c_1(x_2^2 - 2c_2 x_2 x_1^2 + c_2 x_1^4), \\
u_2 &= x_2 - c_2 x_1^2.
\end{aligned}
\tag{2.2c}
$$

In the following section we outline various generalizations of the *explicitly invertible* transformation (2.1).

3. Generalizations

In this Sec. 3, we provide various generalizations of the transformation (2.1). Firstly we provide, in Subsec. 3.1 — always in the context of transformations from 2 variables to 2 variables — the "multinested" generalization of (2.1) involving more than 2 arbitrary functions. Next we consider, in Subsec. 3.2, transformations involving more than 2 variables (and more than 2 arbitrary functions). Combinations of these two generalizations are clearly possible, but we leave their, rather obvious, development to the initiative of the interested reader.

3.1. *A multinested approach: 2 variables, more than 2 arbitrary functions*

In this Subsec. 3.1 we firstly introduce, without any comment, generalizations of the *invertible transformation* (2.1) involving 3 respectively 4 *arbitrary* functions, $F_n(w)$ with $n = 1, 2, 3$ respectively $n = 1, 2, 3, 4$; next we indicate how these generalizations can be extended to involve N *arbitrary* functions (with $N \geq 2$ an *arbitrary*, of course finite, *positive* integer). Note that in this Subsec. 3.1 attention is always restricted to transformations from 2 variables to 2 variables.

3.1.1. 3 *arbitrary functions*

$$x_1 = u_1 + F_1(u_2) + F_3(u_2 + F_2(u_1 + F_1(u_2))),$$
$$x_2 = u_2 + F_2(u_1 + F_1(u_2)); \tag{3.1a}$$

$$u_1 = x_1 - F_3(x_2) - F_1(x_2 - F_2(x_1 - F_3(x_2))),$$
$$u_2 = x_2 - F_2(x_1 - F_3(x_2)). \tag{3.1b}$$

3.1.2. 4 *arbitrary functions*

$$x_1 = u_1 + F_1(u_2) + F_3(u_2 + F_2(u_1 + F_1(u_2))),$$
$$x_2 = u_2 + F_2(u_1 + F_1(u_2)) + F_4(u_1 + F_1(u_2) + F_3(u_2 + F_2(u_1 + F_1(u_2)))); \tag{3.2a}$$

$$u_1 = x_1 - F_3(x_2 - F_4(x_1)) - F_1(x_2 - F_4(x_1) - F_2(x_1 - F_3(x_2 - F_4(x_1)))),$$
$$u_2 = x_2 - F_4(x_1) - F_2(x_1 - F_3(x_2 - F_4(x_1))). \tag{3.2b}$$

3.1.3. N *arbitrary functions*

In this case, with *arbitrary* (positive integer) $N \geq 2$, the relevant formulas must be written recursively. The direct transformation (from u_1, u_2 to x_1, x_2) reads as follows:

$$x_1^{(0)} = u_1, \quad x_2^{(0)} = u_2, \tag{3.3a}$$

$$x_1^{(j)} = x_1^{(j-1)} + F_j(x_2^{(j-1)}), \quad x_2^{(j)} = x_2^{(j-1)}, \qquad j = 1, 3, 5, \ldots, N-1 \text{ or } N, \tag{3.3b}$$

$$x_1^{(j)} = x_1^{(j-1)}, \quad x_2^{(j)} = x_2^{(j-1)} + F_j(x_1^{(j-1)}), \qquad j = 2, 4, 6, \ldots, N-1 \text{ or } N, \tag{3.3c}$$

$$x_1 = x_1^{(N)}, \quad x_2 = x_2^{(N)}, \qquad N = 2, 3, \ldots, \tag{3.3d}$$

where the recursion must be applied with $j = 1, 2, \ldots, N$, using alternatively (3.3b) or (3.3c) as appropriate. Likewise, the inverse transformation (from x_1, x_2 to u_1, u_2) reads as follows:

$$u_1^{(0)} = x_1, \quad u_2^{(0)} = x_2, \tag{3.4a}$$

then, if N is *even*,

$$u_1^{(j)} = u_1^{(j-1)}, \quad u_2^{(j)} = u_2^{(j-1)} - F_{N+1-j}(u_1^{(j-1)}), \qquad j = 1, 3, 5, \ldots, N-1, \tag{3.4b}$$

$$u_1^{(j)} = u_1^{(j-1)} - F_{N+1-j}(u_2^{(j-1)}), \quad u_2^{(N-j)} = u_2^{(j-1)}, \qquad j = 2, 4, 6, \ldots, N, \tag{3.4c}$$

while if N is *odd*,

$$u_1^{(j)} = u_1^{(j-1)} - F_{N+1-j}(u_2^{(j-1)}), \quad u_2^{(j)} = u_2^{(j-1)}, \qquad j = 1, 3, 5, \ldots, N, \tag{3.4d}$$

$$u_1^{(j)} = u_1^{(j-1)}, \quad u_2^{(N-j)} = u_2^{(j-1)} - F_{N+1-j}(u_1^{(j-1)}), \qquad j = 2, 4, 6, \ldots, N-1, \tag{3.4e}$$

and finally

$$u_1 = u_1^{(N)}, \quad x_2 = x_2^{(N)}, \qquad N = 2, 3, \ldots. \tag{3.4f}$$

The verification that these formulas are consistent — in particular, that the transformation (3.4) is the inverse of the transformation (3.3) — can be left to the diligent reader.

3.2. *More variables*

In this Subsec. 3.2 we outline two approaches generalizing the *invertible transformation* (2.1) so that it involves more than 2 variables.

The first approach is to consider the (direct and inverse) transformation of an *arbitrary* number N of scalar variables into as many scalar variables. This transformation involves N *arbitrary* functions, each of them depending on $N-1$ variables.

The second approach is to consider the original change of variables (2.1), but for the (direct and inverse) transformation of 2 *matrices* into 2 *matrices*, rather than 2 *scalars* into 2 *scalars*. Restricting attention for simplicity only to square matrices, say $N \times N$ with N an *arbitrary* integer, this transformation transforms $2N^2$ scalar variables into $2N^2$ scalar variables, and involves $2N^2$ *arbitrary* functions, each of these functions depending on an $N \times N$ matrix, hence on N^2 scalar variables.

These two approaches are outlined in the next two subsections. Their combination, and also the combination of these approaches with the multinested approach of Subsec. 3.1, are rather obvious; this development is again left to the initiative of the interested reader.

3.2.1. *A direct approach*

This approach is introduced by providing, without any comment, the formulas for the $N = 3$ and $N = 4$ cases. The extension to arbitrary N is then sufficiently evident not to require explicit display.

$N = 3$:

$$x_1 = u_1 + F_1(u_2, u_3), \quad x_2 = u_2 + F_2(x_1, u_3), \quad x_3 = u_3 + F_3(x_1, x_2); \tag{3.5a}$$

$$u_3 = x_3 - F_3(x_1, x_2), \quad u_2 = x_2 - F_2(x_1, u_3), \quad u_1 = x_1 - F_1(u_2, u_3); \tag{3.5b}$$

or, more explicitly

$$x_1 = u_1 + F_1(u_2, u_3), \quad x_2 = u_2 + F_2(u_1 + F_1(u_2, u_3), u_3),$$
$$x_3 = u_3 + F_3(u_1 + F_1(u_2, u_3), \quad u_2 + F_2(u_1 + F_1(u_2, u_3), u_3)); \tag{3.6a}$$

$$u_1 = x_1 - F_1(x_2 - F_2(x_1, x_3 - F_3(x_1, x_2)), \quad x_3 - F_3(x_1, x_2)),$$
$$u_2 = x_2 - F_2(x_1, x_3 - F_3(x_1, x_2)), \quad u_3 = x_3 - F_3(x_1, x_2). \tag{3.6b}$$

$N = 4$:

$$x_1 = u_1 + F_1(u_2, u_3, u_4), \quad x_2 = u_2 + F_2(x_1, u_3, u_4),$$
$$x_3 = u_3 + F_3(x_1, x_2, u_4), \quad x_4 = u_4 + F_4(x_1, x_2, x_3); \tag{3.7a}$$

$$u_4 = x_4 - F_4(x_1, x_2, x_3), \quad u_3 = x_3 - F_3(x_1, x_2, u_4),$$
$$u_2 = x_2 - F_2(x_1, u_3, u_4), \quad u_1 = x_1 - F_1(u_2, u_3, u_4). \tag{3.7b}$$

We refrain from reformulating the last relations, (3.7), in more explicit form, since it is quite obvious how to do so (if in doubt, look at (3.6)).

3.2.2. *Matrices*

This generalization of (2.1) is quite straightforward, amounting to a systematic replacement of scalars with $N \times N$ matrices. Of course while doing so appropriate account must be taken of the noncommutativity of matrices.

So the matrix analog of (2.1) reads as follows:

$$\underline{x}_1 = \underline{u}_1 + \underline{F}_1(\underline{u}_2), \quad \underline{x}_2 = \underline{u}_2 + \underline{F}_2(\underline{x}_1) = \underline{u}_2 + \underline{F}_2(\underline{u}_1 + \underline{F}_1(\underline{u}_2)); \tag{3.8a}$$

$$\underline{u}_1 = \underline{x}_1 - \underline{F}_1(\underline{x}_2 - \underline{F}_2(\underline{x}_1)), \quad \underline{u}_2 = \underline{x}_2 - \underline{F}_2(\underline{x}_1). \tag{3.8b}$$

Here the underlined symbols denote $N \times N$ matrices; in particular the two symbols $\underline{F}_1(\underline{w}), \underline{F}_2(\underline{w})$ denote 2 arbitrary $N \times N$ matrices featuring N^2 matrix elements each of which is an *arbitrary* function of the N^2 matrix elements of the $N \times N$ matrix $\underline{w}$. And of course the nesting of functions must now take into account the noncommutativity of matrices.

For instance for the simple assignment

$$\underline{F}_1(\underline{w}) = \underline{w}\,\underline{C}_1\,\underline{w}, \quad \underline{F}_2(\underline{w}) = \underline{w}\,\underline{C}_2\,\underline{w}, \tag{3.9}$$

where $\underline{C}_1$ and $\underline{C}_2$ are 2 arbitrary $N \times N$ matrices (independent of $\underline{w}$), the direct transformation reads

$$\underline{x}_1 = \underline{u}_1 + \underline{u}_2\underline{C}_1\underline{u}_2, \tag{3.10a}$$

$$\begin{aligned}
\underline{x}_2 &= \underline{u}_2 + \underline{x}_1\underline{C}_2\underline{x}_1 = \underline{u}_2 + (\underline{u}_1 + \underline{u}_2\underline{C}_1\underline{u}_2)\underline{C}_2(\underline{u}_1 + \underline{u}_2\underline{C}_1\underline{u}_2) \\
&= \underline{u}_2 + \underline{u}_1\underline{C}_2\underline{u}_1 + \underline{u}_2\underline{C}_1\underline{u}_2\underline{C}_2\underline{u}_1 + \underline{u}_1\underline{C}_2\underline{u}_2\underline{C}_1\underline{u}_2 + \underline{u}_2\underline{C}_1\underline{u}_2\underline{C}_2\underline{u}_2\underline{C}_1\underline{u}_2;
\end{aligned} \tag{3.10b}$$

and the inverse transformation reads

$$\begin{aligned}
\underline{u}_1 &= \underline{x}_1 - \underline{u}_2\underline{C}_1\underline{u}_2 = \underline{x}_1 - (\underline{x}_2 - \underline{x}_1\underline{C}_2\underline{x}_1)\underline{C}_1(\underline{x}_2 - \underline{x}_1\underline{C}_2\underline{x}_1) \\
&= \underline{x}_1 - \underline{x}_2\underline{C}_1\underline{x}_2 + \underline{x}_1\underline{C}_2\underline{x}_1\underline{C}_1\underline{x}_2 + \underline{x}_2\underline{C}_1\underline{x}_1\underline{C}_2\underline{x}_1 - \underline{x}_1\underline{C}_2\underline{x}_1\underline{C}_1\underline{x}_1\underline{C}_2\underline{x}_1,
\end{aligned} \tag{3.11a}$$

$$\underline{u}_2 = \underline{x}_2 - \underline{x}_1\underline{C}_2\underline{x}_1. \tag{3.11b}$$

4. Applications

In this section we outline various applications of the *invertible transformation* described in the previous two sections. Generally we illustrate typical possibilities via quite elementary examples, mainly based on simple instances of the basic transformation (2.1). The reader will have no difficulty in imaging and exploring a multitude of additional examples, possibly also using the generalized transformations introduced in Sec. 3. Let us reemphasize that the level of detail of the treatment is different in the following subsections, whose contents are indicated by their titles.

4.1. *Algebraic and Diophantine equations*

In this section we indicate — via a simple example — how highly nonlinear yet explicitly solvable algebraic equations can be manufactured using the *invertible transformation* (2.1).

And in a subsection we show how in an analogous manner nontrivial *Diophantine* equations can be identified, namely nonlinear algebraic equations featuring *integer* solutions.

Assume that the 2 quantities u_1, u_2 are characterized as the solutions of the following 2 quite trivial algebraic equations:

$$u_1 = u_2, \quad u_1 u_2 + \alpha u_1 + \beta u_2 + \gamma = 0, \tag{4.1a}$$

obviously implying

$$u_1 = u_2 = u_\pm, \tag{4.1b}$$

with

$$u_\pm = \frac{1}{2}(-\alpha - \beta \pm \sqrt{(\alpha + \beta)^2 - 4\gamma}). \tag{4.1c}$$

Here and below α, β, γ are 3 arbitrary numbers.

Then we can conclude that the following system of 2 equations in the 2 unknowns x_1, x_2, see (2.1b),

$$x_1 - F_1(x_2 - F_2(x_1)) = x_2 - F_2(x_1), \tag{4.2a}$$

$$[x_1 - F_1(x_2 - F_2(x_1))][x_2 - F_2(x_1)] + \alpha[x_1 - F_1(x_2 - F_2(x_1))] + \beta[x_2 - F_2(x_1)] + \gamma = 0, \tag{4.2b}$$

where $F_1(w), F_2(w)$ are 2 *arbitrary* functions, has the two solutions (see (2.1a))

$$x_1 = u_\pm + F_1(u_\pm), \quad x_2 = u_\pm + F_2(u_\pm + F_1(u_\pm)) \tag{4.3}$$

with $u_\pm$ given by (4.1c). Even the fact that the, generally highly nonlinear, system (4.2) admits only these two solutions (corresponding to the two determinations of $u_\pm$) is far from obvious. Indeed some readers might amuse themselves by implementing the equations (4.2) for specific assignments of the 2 functions $F_1(w), F_2(w)$, and by then — after rewriting the resulting nonlinear algebraic equations in some standardized manner that does not display their origin — challenging others (be they humans or computers) to find the solutions — which are of course known to them, see (4.3) — of these algebraic equations; and the same could be done in didactic contexts, or to check the efficiency of computer packages.

4.1.1. *Diophantine equations*

Again, we limit our presentation merely to exhibiting 3 (or rather 4, see below) representative examples.

The first case we consider is the single *linear* equation in *two* variables reading

$$a_1 u_1 + a_2 u_2 = a_1 v_1 + a_2 v_2, \tag{4.4a}$$

where the 4 numbers a_1, a_2, v_1, v_2 are 4 *arbitrary* (coprime) *integers*. The general *Diophantine* solution of this equation reads (see, for instance, [2])

$$u_1 = v_1 + a_2 n, \quad u_2 = v_2 - a_1 n, \tag{4.4b}$$

where n is an *arbitrary integer*.

Via the transformation (2.1) the relation (4.4a) becomes

$$a_1[x_1 - F_1(x_2 - F_2(x_1))] + a_2[x_2 - F_2(x_1)] = a_1 v_1 + a_2 v_2, \tag{4.5}$$

and its *Diophantine* character is maintained provided the 2 functions $F_1(w)$, $F_2(w)$ are 2 *arbitrarily assigned polynomials* with *integer* coefficients. Indeed the solution of this *Diophantine* equation reads (see (2.1a) and (4.4b))

$$x_1 = v_1 + a_2 n + F_1(v_2 - a_1 n), \tag{4.6a}$$

$$x_2 = v_2 - a_1 n + F_2(v_1 + a_2 n + F_1(v_2 - a_1 n)). \tag{4.6b}$$

For instance via the assignment

$$F_1(w) = c_{10} + c_{11}w + c_{12}w^2, \quad F_2(w) = c_{20} + c_{21}w + c_{22}w^2, \tag{4.7}$$

where the 6 parameters $c_{jk}, j = 1, 2, k = 1, 2, 3$ are *arbitrarily assigned integers*, the *Diophantine* equation (4.5) becomes

$$p_1 + p_2 x_1 + p_3 x_2 + p_4 x_1^2 + p_5 x_1 x_2 + p_6 x_2^2 + p_7 x_1^3 + p_8 x_1^2 x_2 + p_9 x_1^4 = 0, \tag{4.8a}$$

where the 9 coefficients p_ℓ are expressed in terms of the 10 arbitrary *integers* $a_j, v_j, c_{jk}, j = 1, 2, k = 1, 2, 3$, as follows:

$$p_1 = -a_1(c_{10} - c_{11}c_{20} + c_{12}c_{20}^2 + v_1) - a_2(c_{20} + v_2), \tag{4.8b}$$

$$p_2 = a_1(1 + c_{11}c_{21} - 2c_{12}c_{20}c_{21}) - a_2 c_{21}, \tag{4.8c}$$

$$p_3 = -a_1(c_{11} - 2c_{12}c_{20}) + a_2, \tag{4.8d}$$

$$p_4 = a_1(c_{11}c_{22} - c_{12}c_{21}^2 - 2c_{12}c_{20}c_{22}) - a_2 c_{22}, \tag{4.8e}$$

$$p_5 = 2a_1 c_{12}c_{21}, \tag{4.8f}$$

$$p_6 = -a_1 c_{12}, \tag{4.8g}$$

$$p_7 = -2a_1 c_{12}c_{21}c_{22}, \tag{4.8h}$$

$$p_8 = 2a_1 c_{12}c_{22}, \tag{4.8i}$$

$$p_9 = -a_1 c_{12}c_{22}^2. \tag{4.8j}$$

It is remarkable that the *general* solution of this fairly *general* algebraic equation can be expressed in explicit form (see (4.6) and (4.7)):

$$x_1 = v_1 + a_2 n + c_{10} + c_{11}(v_2 - a_1 n) + c_{12}(v_2 - a_1 n)^2, \tag{4.9a}$$

$$x_2 = v_2 - a_1 n + c_{20} + c_{21}[v_1 + a_2 n + c_{10} + c_{11}(v_2 - a_1 n) + c_{12}(v_2 - a_1 n)^2]$$
$$+ c_{22}[v_1 + a_2 n + c_{10} + c_{11}(v_2 - a_1 n) + c_{12}(v_2 - a_1 n)^2]^2; \tag{4.9b}$$

and the *Diophantine* character of this solution — when all the coefficients are *integer* numbers, and the *arbitrary* parameter n is as well an *integer* number — is plain.

The second example of *Diophantine* equation we consider starts again from a quite trivial system, constituted now by the 2 equations

$$a_1 u_1 + a_2 u_2 = 2a_1 a_1, \quad u_1 u_2 = a_1 a_2, \tag{4.10a}$$

which is easily seen to admit the *unique* solution

$$u_1 = a_2, \quad u_2 = a_1. \tag{4.10b}$$

Here we assume a_1, a_2 to be 2 *arbitrary integers*, hence the system (4.10a) can be considered a *Diophantine* system (of course, quite a trivial one).

Via the transformation (2.1), the system (4.10a) gets reformulated as the following system for the 2 unknowns x_1, x_2:

$$a_1[x_1 - F_1(x_2 - F_2(x_1))] + a_2[x_2 - F_2(x_1)] - 2a_1 a_2 = 0, \tag{4.11a}$$

$$[x_1 - F_1(x_2 - F_2(x_1))][x_2 - F_2(x_1)] - a_1 a_2 = 0, \tag{4.11b}$$

and its explicit solution reads

$$x_1 = a_2 + F_1(a_1), \quad x_2 = a_1 + F_2(a_2 + F_1(a_1)). \tag{4.12}$$

Hence if, for instance,

$$F_1(w) = k_1 w^2, \quad F_2(w) = k_2 w^2, \tag{4.13}$$

with k_1 and k_2 two arbitrary *integers*, the system (4.11) becomes the following highly nonlinear system,

$$k_1 k_2^3 x_1^6 - 3k_1 k_2^2 x_1^4 x_2 + 3k_1 k_2 x_1^2 x_2^2 - k_2 x_1^3 - k_1 x_2^3 + x_1 x_2 - a_1 a_2 = 0, \quad \tag{4.14a}$$

$$a_1 k_1 k_2^2 x_1^4 - 2a_1 k_1 k_2 x_1^2 x_2 + a_2 k_2 x_1^2 + a_1 k_1 x_2^2 - a_1 x_1 - a_2 x_2 + 2a_1 a_2 = 0, \quad \tag{4.14b}$$

which is *Diophantine* whenever the 4 numbers a_1, a_2, k_1, k_2 are (*arbitrary*!) *integers*. Indeed clearly its *unique* solution reads as follows:

$$x_1 = a_2 + k_1 a_1^2, \quad x_2 = a_1 + k_2(a_2 + k_1 a_1^2)^2. \tag{4.15}$$

The fact that this system, (4.14), is *Diophantine* (and *explicitly solvable*, see (4.15)!), is remarkable — as long as one does not know how it has been manufactured. And note that even the two simpler cases with either k_1 or k_2 vanishing seem far from trivial.

The third example we report is perhaps the most unlikely one to be considered trivial. We formulate it as the following

Proposition 4.1.1.1. *Consider the cubic equation*

$$k_1(x_1 + k_2 x_2)^3 + k_3(x_1 + k_2 x_2)^2 + k_4 x_1 + k_5 x_2 + k_6 = 0, \tag{4.16}$$

where x_1, x_2 are the 2 "unknown variables" and, here and below, all the coefficients k's (and $\tilde{k}$'s, see below) are integer *numbers*, arbitrary *except for the few conditions detailed below.*

Let k_0 be the greatest common divisor of k_1 and k_3 (including possibly $k_0 = 1$), namely

$$k_0 = \gcd(k_1, k_3), \quad k_1 = k_0 \tilde{k}_1, \quad k_3 = k_0 \tilde{k}_3, \tag{4.17a}$$

and let there holds the relations

$$\gcd(k_0, k_4) = 1, \tag{4.17b}$$

i.e. k_0 *and* k_4 *are* coprime, *and*

$$k_5 = k_2 k_4 + k_0. \tag{4.17c}$$

Then the cubic equation (4.16) *has the following* integer *solutions featuring the* arbitrary integer n:

$$x_1 = u_1 - k_2(u_2 - \tilde{k}_3 u_1^2 - \tilde{k}_1 u_1^3), \tag{4.18a}$$

$$x_2 = u_2 - \tilde{k}_3 u_1^2 - \tilde{k}_1 u_1^3. \tag{4.18b}$$

Here u_1, u_2 *are the following* integers,

$$u_1 = -k_6 \tilde{u}_1 + n k_0, \quad u_2 = -k_6 \tilde{u}_2 - n k_4, \tag{4.19a}$$

where $\tilde{u}_1, \tilde{u}_2$ *are the* integer *solutions of the following* Diophantine *equation*:

$$k_0 \tilde{u}_2 + k_4 \tilde{u}_1 = 1, \tag{4.19b}$$

and are therefore computable via the Euclid algorithm (*see, for instance,* [3]).

Corollary 4.1.1.1. *If* k_1 *and* k_3 *are* coprime, *namely, if* $k_0 = 1$, *then* (4.17b) *is identically satisfied hence the* only *constraint on the coefficients* k's *featured by the Diophantine equation* (4.16) *is* (*see* (4.17c))

$$k_5 - k_2 k_4 = 1. \tag{4.20}$$

The proof of Proposition 4.1.1.1 is via the relations (2.1) with

$$F_1(w) = k_2 w, \quad F_2(w) = (\tilde{k}_3 + \tilde{k}_1 w) w^2. \tag{4.21}$$

Indeed, via (2.1a) with (4.21), (4.16) becomes

$$k_0 u_2 + k_4 u_1 + k_6 = 0, \tag{4.22}$$

which is well-known (see [3]) to feature the *integer* solution (4.19). And, to complete the proof, one notes that (4.18) is yielded by (2.1a) with (4.21).

Let us end this subsection by pointing out that the algebraic system (4.2) discussed above provides itself a fourth *Diophantine* example, if $\alpha + \beta \equiv K_1$ is an arbitrary *integer*, $(\alpha + \beta)^2 - 4\gamma \equiv K_2^2$ is an arbitrary *squared integer*, and the 2 functions $F_1(w), F_2(w)$ are again assigned as arbitrary *polynomials* with *integer* coefficients; as clearly demonstrated by its explicit solution (4.3).

4.2. *Nonlinear Sturm–Liouville problems*

In this Subsec. 4.2 we limit our presentation to exhibiting a single example of highly nonlinear Sturm–Liouville problem. It reads as follows:

$$\begin{aligned}
x_1' = {}& x_2 + x_1[(4\alpha\lambda - \beta)x_1 + 4\alpha z x_2] - 4\alpha(\beta z x_1^3 + 2\alpha\lambda x_1 x_2^2 + \alpha z\, x_2^3) \\
&+ 4\alpha^2 x_2(4\beta\lambda x_1^3 + 3\beta z x_1^2 x_2 + \alpha\lambda x_2^3) - 4\alpha^2\beta x_1^2(2\beta\lambda x_1^3 + 3\beta z x_1^2 x_2 + 4\alpha\lambda x_2^3) \\
&+ 4\alpha^2\beta^2 x_1^4(\beta z x_1^2 + 6\alpha\lambda x_2^2) - 16\alpha^3\beta^3\lambda x_1^6 x_2 + 4\alpha^3\beta^4\lambda x_1^8, \tag{4.23a}
\end{aligned}$$

$$x_2' = 2(\lambda x_1 + z x_2 - \beta z x_1^2 - \alpha\lambda x_2^2 + 2\alpha\beta\lambda x_1^2 x_2 - \alpha\beta^2\lambda x_1^4) + 2\beta x_1 x_1'. \tag{4.23b}$$

Here and below z is the (*real*) variable, $x_1 \equiv x_1(z), x_2 \equiv x_2(z)$ are two functions of this variable, appended primes indicate differentiation with respect to this variable, α, β are two arbitrary constants and λ is the *eigenvalue* of the nonlinear Sturm–Liouville problem characterized by these two first-order ODEs and the requirement that the *eigenfunctions* $x_1(z), x_2(z)$ be *polynomials* in the variable z. The solution of this problem is that the eigenvalues λ are the *nonnegative integers*,

$$\lambda = \ell, \quad \ell = 0, 1, 2, \ldots, \tag{4.24a}$$

and the corresponding eigenfunctions are

$$x_1(z) = H_\ell(z) + 4\alpha[\ell H_{\ell-1}(z)]^2, \tag{4.24b}$$

$$x_2(z) = 2\ell H_{\ell-1}(z) + \beta\{H_\ell(z) + 4\alpha[\ell H_{\ell-1}(z)]^2\}^2. \tag{4.24c}$$

Here and below $H_\ell(z)$ is the Hermite polynomial of degree ℓ in the variable z (see for instance [4]).

These findings are obtained from the standard linear Sturm–Liouville problem characterizing Hermite polynomials, reading

$$u'' - 2zu' - 2\lambda u = 0, \quad u \equiv u(z), \tag{4.25a}$$

and (via the requirement that $u(z)$ be a polynomial in z) entailing $\lambda = \ell$ and $u(z) = H_\ell(z)$. Indeed by setting

$$u_1(z) = u(z) = H_\ell(z), \quad u_2(z) = u'(z) = H'_\ell(z) = 2\ell H_{\ell-1}(z), \tag{4.26a}$$

(4.25a) becomes

$$u'_1(z) = u_2(z), \quad u'_2(z) = 2zu'(z) + 2\lambda u(z); \tag{4.26b}$$

and by then using the transformation (2.2) to replace $u_1(z), u_2(z)$ with $x_1(z), x_2(z)$ one obtains (4.23), while this transformation (2.2) yields (4.24) from (4.26a).

4.3. *Dynamical systems*

Let us recall that a dynamical system is characterized by a finite but otherwise *a priori arbitrary* number N of (first-order) Ordinary Differential Equations (ODEs), say

$$\dot{x}_n = f_n(\underline{x}), \quad n = 1, \ldots, N, \tag{4.27}$$

where the N (real) "dependent variables" $x_n \equiv x_n(t)$ are functions of the (real) "independent variable" t ("time"), superimposed dots indicate time-differentiations, $\underline{x}$ is the N-vector of components x_n, $\underline{x} \equiv (x_1, \ldots, x_N)$, and the N functions $f_n(\underline{x})$ are (*a priori* arbitrarily) assigned.

Here for simplicity we restrict attention to systems with $N = 2$ and $N = 3$.

Consider the following elementary dynamical system characterizing the evolution of the 2 variables $u_1(t), u_2(t)$ depending on the independent variable t ("time"):

$$\dot{u}_1 = u_2, \quad \dot{u}_2 = -u_1. \tag{4.28a}$$

The *general* solution of this trivial system of 2 ODEs is

$$u_1(t) = A\sin(t+\theta), \quad u_2(t) = A\cos(t+\theta), \tag{4.28b}$$

where A and θ are 2 arbitrary constants — clearly expressible as follows via the *initial* data $u_1(0), u_2(0)$,

$$A = \sqrt{u_1^2(0) + u_2^2(0)}, \quad \theta = \arctan\left[\frac{u_1(0)}{u_2(0)}\right], \tag{4.28c}$$

to obtain the solution of the *initial-value* problem.

Let us now apply the *invertible transformation* (2.1). Clearly (2.1a) yields

$$\dot{x}_1 = \dot{u}_1 + F_1'(u_2)\dot{u}_2, \quad \dot{x}_2 = \dot{u}_2 + F_2'(x_1)\dot{x}_1. \tag{4.29}$$

Here and throughout appended primes denote differentiations with respect to the argument of the function they are appended to.

Via (4.28a) this becomes

$$\dot{x}_1 = u_2 - F_1'(u_2)u_1, \quad \dot{x}_2 = -u_1 + F_2'(x_1)\dot{x}_1, \tag{4.30}$$

and via (2.1b) this becomes the following, new dynamical system for the two dependent variables $x_1(t), x_2(t)$:

$$\dot{x}_1 = x_2 - F_2(x_1) - F_1'(x_2 - F_2(x_1))[x_1 - F_1(x_2 - F_2(x_1))], \tag{4.31}$$

$$\dot{x}_2 = -[x_1 - F_1(x_2 - F_2(x_1))] + F_2'(x_1)\dot{x}_1, \tag{4.32a}$$

namely

$$\dot{x}_2 = -x_1 + F_1(x_2 - F_2(x_1))$$
$$+ F_2'(x_1)\{x_2 - F_2(x_1) - F_1'(x_2 - F_2(x_1))[x_1 - F_1(x_2 - F_2(x_1))]\}. \tag{4.32b}$$

Clearly via (2.1a) one gets from (4.28b) the *general* solution of this dynamical system, (4.31) and (4.32):

$$x_1(t) = A\sin(t+\theta) + F_1(A\cos(t+\theta)), \tag{4.33a}$$

$$x_2(t) = A\cos(t+\theta) + F_2(A\sin(t+\theta) + F_1(A\cos(t+\theta))). \tag{4.33b}$$

And moreover, it is clear how to obtain, via (4.28c) and (2.1b), the *explicit* solution of the *initial-value* problem for this nonlinear dynamical system (4.31), (4.32).

The readers might amuse themselves by writing explicit versions of the dynamical system (4.31), (4.32) corresponding to different assignments of the 2 functions $F_1(w), F_2(w)$, and will thereby notice that even quite simple choices of these 2 functions (such as (2.2a)) yield apparently quite nontrivial dynamical systems, whose property to be *isochronous* with period 2π (namely, to *only* feature solutions which are *periodic* with the fixed period 2π) is *a priori* far than evident. Yet this property is clearly implied by the solution (4.33).

Another simple and instructive case is that obtained, via the *invertible transformation* (3.5), from the elementary dynamical system

$$\dot{u}_1 = u_2, \quad \dot{u}_2 = u_3, \quad \dot{u}_3 = -(\rho\omega^2 u_1 + \omega^2 u_2 + \rho u_3), \tag{4.34}$$

the *general* solution of which reads as follows:

$$u_1(t) = A\sin(\omega t + \theta) + B\exp(-\rho t), \tag{4.35a}$$

$$u_2(t) = A\omega\cos(\omega t + \theta) - B\rho\exp(-\rho t), \tag{4.35b}$$

$$u_3(t) = -A\omega^2\sin(\omega t + \theta) + B\rho^2\exp(-\rho t). \tag{4.35c}$$

Here ω and ρ are two arbitrary constants, and if they are both *real* and ρ is *positive*, $\rho > 0$, then the *general* solution (4.35) — where A, B, θ are 3 *arbitrary* constants — is *asymptotically isochronous*, namely it becomes periodic in the remote future with the fixed period $T = 2\pi/\omega$, up to corrections vanishing exponentially, of order $\exp(-\rho t)$.

Clearly the same property of *asymptotic isochrony* is inherited by the *general* solution of the dynamical system that obtains from (4.34) via (3.5), which reads as follows,

$$\dot{x}_1 = u_2 + F_{1,1}(u_2, u_3)u_3 - F_{1,2}(u_2, u_3)(\rho\omega^2 u_1 + \omega^2 u_2 + \rho u_3), \tag{4.36a}$$

$$\dot{x}_2 = u_3 + F_{2,1}(x_1, u_3)\dot{x}_1 - F_{2,1}(x_1, u_3)(\rho\omega^2 u_1 + \omega^2 u_2 + \rho u_3), \tag{4.36b}$$

$$\dot{x}_3 = -(\rho\omega^2 u_1 + \omega^2 u_2 + \rho u_3) + F_{3,1}(x_1, x_2)\dot{x}_1 + F_{3,2}(x_1, x_2)\dot{x}_2, \tag{4.36c}$$

with

$$F_{j,k}(w_1, w_2) \equiv \frac{\partial F_j(w_1, w_2)}{\partial w_k}, \quad j = 1, 2, 3, \;\; k = 1, 2, \tag{4.36d}$$

and u_1, u_2, u_3 given in terms of x_1, x_2, x_3 by (3.6b). Indeed clearly the general solution of this system is yielded by the formulas (3.6a) with $u_1 \equiv u_1(t), u_2 \equiv u_2(t), u_3 \equiv u_3(t)$ given by (4.35).

4.3.1. *Hamiltonian systems*

The class of Hamiltonian dynamical systems is characterized by the system of 2 ODEs

$$\dot{q} = \frac{\partial h(p, q)}{\partial p}, \quad \dot{p} = -\frac{\partial h(p, q)}{\partial q}, \tag{4.37}$$

where the Hamiltonian function $h(p, q)$ is *a priori* arbitrary. Note that we are restricting here, for simplicity, attention to the case of a single canonical coordinate q and correspondingly a single canonical momentum p; the extension of the following treatment to several canonical variables is rather obvious.

The fact to be highlighted is that, in this context, the transformation (2.1a) — say, from $q = u_1$ and $p = u_2$ to new canonical variables $Q = x_1$ and $P = x_2$ (or, for that matter, $Q = x_2$ and $P = x_1$) — is *canonical*, namely it leads to new equations of motion which retain the Hamiltonian form (4.37),

$$\dot{Q} = \frac{\partial H(P, Q)}{\partial P}, \quad \dot{P} = -\frac{\partial H(P, Q)}{\partial Q} \tag{4.38}$$

with

$$H(P, Q) = h(p(P, Q), q(P, Q)), \tag{4.39}$$

where the functions $p(P, Q), q(P, Q)$ are of course provided by the formulas (2.1b) with the identification indicated above, i.e., $q = u_1, p = u_2$ and $Q = x_1, P = x_2$ (or, as the case may

be, $Q = x_2, P = x_1$). The verification of this fact is left as an easy exercise for the diligent reader.

Let us emphasize that this opens the option to employ in the Hamiltonian context the *invertible transformation* discussed in this paper. For instance it allows to manufacture a multitude of *isochronous* Hamiltonian systems, such as

$$H(P, Q) = \frac{1}{2}\{(P - \beta_0 - \beta_1 Q - \beta_2 Q^2)^2 + [Q - \alpha_0 - \alpha_1(P - \beta_0 - \beta_1 Q - \beta_2 Q^2)$$
$$- \alpha_2(P - \beta_0 - \beta_1 Q - \beta_2 Q^2)^2]^2\}, \tag{4.40}$$

which clearly obtains from the harmonic oscillator Hamiltonian $h(p, q) = (p^2 + q^2)/2$ via the canonical transformation (2.1) with $u_1 = q$, $u_2 = p$, $x_1 = Q$, $x_2 = P$ and

$$F_1(w) = \alpha_0 + \alpha_1 w + \alpha_2 w^2, \quad F_1(w) = \beta_0 + \beta_1 w + \beta_2 w^2, \tag{4.41}$$

hence it features solutions which are *all* periodic with period 2π (their explicit display can be left to the interested reader).

Remark 4.3.1.1. An interesting related issue is whether these Hamiltonians, after quantization, feature an *equispaced* spectrum. This may well depend on the specific prescription employed to make the transition from the *classical* to the *quantal* Hamiltonian.

For various examples where this very question is explored see [5–7], and the references quoted in these papers.

Remark 4.3.1.2. The property of the transformation (2.1a) to be canonical — as discussed above — does not necessarily extend to the more general transformations discussed in Sec. 3.

4.4. *Discrete-time dynamical systems* (*in particular, isochronous and asymptotically isochronous examples*)

The expediency of the *invertible transformation* (2.1) came to the attention of two of us (FC and FL) in the context of a search of techniques suitable to identify *discrete-time* dynamical systems displaying the *isochronous* — as well as the *asymptotically isochronous* — phenomenology. This Subsec. 4.4 reflects those findings, and their novelty justifies the more extended treatment (in comparison with that of other sections).

Let us recall that a *discrete-time* dynamical system — or, equivalently, a multidimensional map — is characterized by a set of N "dependent variables" x_n, which are functions of a "*discrete-time*" independent variable ℓ taking (here and hereafter) *nonnegative integer* values, $x_n \equiv x_n(\ell)$, $\ell = 0, 1, 2, \ldots$, and which evolve according to the discrete analog of the dynamical system (4.27):

$$\tilde{x}_n = f_n(\underline{x}), \quad n = 1, \ldots, N. \tag{4.42a}$$

The notation here is analogous to that used in Subsec. 4.3, except that now (and hereafter) superimposed tildes indicate that the independent variable has been advanced by one unit,

$$\tilde{x}_n \equiv \tilde{x}_n(\ell) \equiv x_n(\ell + 1), \quad n = 1, \ldots, N. \tag{4.42b}$$

A *discrete-time dynamical system* is called *isochronous* if there exists an open, fully-dimensional set of initial data $\underline{x}(0)$ yielding solutions which are *completely periodic* with a *fixed* period L,

$$x_n(\ell + L) = x_n(\ell), \quad n = 1, \ldots, N, \tag{4.43}$$

where L is of course now a *fixed positive integer* (independent of the initial data). The *isochronous discrete-time dynamical systems* that we consider below are such that this relation, (4.43), holds for *arbitrary* initial data.

Several techniques to manufacture *isochronous continuous-time* dynamical systems are available, in addition to those based on our *invertible transformation* (2.1), see for instance (4.31), (4.32). Hence a plethora of such *isochronous* models is known: see for instance [8–20] and the literature quoted there. But no analogous techniques are — to the best of our knowledge — available to manufacture *discrete-time isochronous* dynamical systems (see, however, the very recent paper [21]). The invertible transformations of Secs. 2 and 3 allow us to manufacture such systems by taking as starting point a trivial *discrete-time isochronous* system, in analogy to the procedure applied in the preceding section in the context of *continuous-time* dynamical systems. Again, in order to illustrate this approach in the simplest context, below we mainly restrict consideration to two-dimensional systems, which moreover allow neat graphical representations of their evolution, taking place in a plane. But we also discuss below — in analogy to what we did in the preceding section — a tridimensional example, displaying the possibility to manufacture solvable *discrete-time* systems that are *asymptotically isochronous*.

So, to begin with let us start from the following quite simple *discrete-time* 2-dimensional dynamical system:

$$\tilde{u}_1 = cu_1 - su_2, \quad \tilde{u}_2 = su_1 + cu_2. \tag{4.44}$$

Here and throughout this Subsec. 4.4

$$c = \cos\left(\frac{2\pi}{\lambda}\right), \quad s = \sin\left(\frac{2\pi}{\lambda}\right), \tag{4.45}$$

so that, at every time step, the two-vector $\vec{u}(\ell)$ of which the 2 dependent variables $u_1(\ell)$ and $u_2(\ell)$ are the 2 components makes a counterclockwise rotation, by the angle $2\pi/\lambda$, in the u_1u_2 Cartesian plane. This evolution is obviously *solvable*, since clearly the two-vector $\vec{u}(\ell)$ obtains from the two-vector $\vec{u}(0)$ by performing a counterclockwise rotation, in the u_1u_2 Cartesian plane, by the angle $2\pi\ell/\lambda$: hence the corresponding formulas expressing $u_1(\ell)$ and $u_2(\ell)$ in terms of the initial data $u_1(0)$ and $y_2(0)$ are simply

$$u_1(\ell) = \cos\left(\frac{2\pi\ell}{\lambda}\right) u_1(0) - \sin\left(\frac{2\pi\ell}{\lambda}\right) u_2(0), \tag{4.46a}$$

$$u_2(\ell) = \sin\left(\frac{2\pi\ell}{\lambda}\right) u_1(0) + \cos\left(\frac{2\pi\ell}{\lambda}\right) u_2(0). \tag{4.46b}$$

Clearly this evolution is *isochronous* with period L if $\lambda = L$ with L a *positive integer*. More generally, this evolution is as well *isochronous* with period L if λ is *rational*, $\lambda = L/M$ with L and M *coprime integers* (and, say, L *positive*). Indeed it is easily seen that

the evolution (4.44) implies that each of the 2 dependent variables $u_1(\ell)$, $u_2(\ell)$ satisfies separately the same, *linear, second-order discrete-time* evolution equation:

$$\widetilde{\widetilde{u}}_n - 2c\tilde{u}_n + u_n = 0, \quad n = 1, 2, \tag{4.47a}$$

the general solution of which is

$$u_n(\ell) = u_n(0)\cos\left(\frac{2\pi\ell}{\lambda}\right) + \frac{u_n(1) - u_n(0)\cos(2\pi/\lambda)}{\sin(2\pi/\lambda)}\sin\left(\frac{2\pi\ell}{\lambda}\right), \quad n = 1, 2. \tag{4.47b}$$

This confirms the *isochronous* character of $u_1(\ell)$ and $u_2(\ell)$ whenever $\lambda = L/M$ with L and M *coprime integers*, and it is of course easily seen to coincide with (4.46).

The treatment reported herein could be easily extended by replacing the original evolution (4.44) with the more general *discrete-time* equation of motion

$$\widetilde{\vec{u}} = \underline{U}\vec{u}, \tag{4.48}$$

where $\underline{U}$ is a constant (i.e., ℓ-independent) 2×2 matrix acting on the vector $\vec{u} \equiv \vec{u}(\ell)$. Note in particular that also this evolution is, rather trivially, *solvable*. We take as starting point of our treatment the simpler evolution (4.44) in the interest of simplicity, as well as to focus on more interesting evolutions, in particular the *isochronous* ones.

Let us now perform the change of dependent variables (2.1), from the 2 variables $u_1(\ell)$, $u_2(\ell)$ to the 2 variables $x_1(\ell)$ and $x_2(\ell)$. Then the *discrete-time* dynamical system characterizing the evolution of the 2 new dependent variables $x_1(\ell)$ and $x_2(\ell)$ clearly reads as follows:

$$\tilde{x}_1 = cx_1 - sx_2 + sF_2(x_1) - cF_1(x_2 - F_2(x_1))$$
$$+ F_1(sx_1 + cx_2 - cF_2(x_1) - sF_1(x_2 - F_2(x_1))), \tag{4.49}$$

$$\tilde{x}_2 = sx_1 + cx_2 - cF_2(x_1) - sF_1(x_2 - F_2(x_1)) + F_2(\tilde{x}_1), \tag{4.50a}$$

or, more explicitly,

$$\tilde{x}_2 = sx_1 + cx_2 - cF_2(x_1) - sF_1(x_2 - F_2(x_1)) + F_2(cx_1 - sx_2 + sF_2(x_1)$$
$$- cF_1(x_2 - F_2(x_1)) + F_1(sx_1 + cx_2 - cF_2(x_1) - sF_1(x_2 - F_2(x_1)))). \tag{4.50b}$$

These 2 "equations of motion", (4.49) and (4.50), constitute the new *discrete-time* dynamical system. This more complicated system is, via (2.1), just as solvable as the original, trivial system (4.44), and obviously as well *isochronous* with period L,

$$\vec{x}(\ell \pm L) = \vec{x}(\ell), \tag{4.51}$$

if $\lambda = L/M$ with L a *positive integer* and M a *coprime integer*. Clearly the freedom to choose arbitrarily the 2 functions $F_1(w)$ and $F_2(w)$ entails that this class of discrete dynamical systems is quite vast. But the multiple convolution of these functions that occurs in these equations of motion (note, in particular, that (4.50b) features a 4-fold nesting, see the last term in its right-hand side) entails that these equations are generally not very simple (but see below).

For $\lambda = L$ with L a *very small integer* the *isochronous* character of the *discrete-time* dynamical system (4.49) and (4.50) is evident. Indeed clearly for $L = 1$ entailing $c = 1$,

$s = 0$ (see (4.45)) this dynamical system amounts merely to the identity, yielding $\tilde{\vec{x}} = \vec{x}$ entailing $\vec{x}(\ell) = \vec{x}(0)$; for $L = 2$ entailing $c = -1$, $s = 0$ (see (4.45)) it amounts to the evolution

$$\tilde{x}_1 = -x_1 + F_1(x_2 - F_2(x_1)) + F_1(-x_2 + F_2(x_1)), \tag{4.52a}$$

$$\tilde{x}_2 = -x_2 + F_2(x_1) + F_2(\tilde{x}_1), \tag{4.52b}$$

entailing (as can be fairly easily verified) $\tilde{\tilde{\vec{x}}} = \vec{x}$, i.e. $\vec{x}(\ell + 2) = \vec{x}(\ell)$; likewise with $L = 3$ one arrives eventually at $\tilde{\tilde{\tilde{\vec{x}}}} = \vec{x}$, i.e. $\vec{x}(\ell + 3) = \vec{x}(\ell)$; and so on. But such a *direct verification* of the property of *isochrony* (4.51) becomes more and more cumbersome for $L = 4, 5, \ldots$ (as the diligent reader will verify).

Clearly the quantity

$$K = u_1^2 + u_2^2 \tag{4.53a}$$

is a "constant of motion" for the evolution of the original system (4.44). Likewise the image of this constant under the transformation (2.1) is a constant of motion for the new evolution (4.49) and (4.50), reading

$$K = [x_1 - F_1(x_2 - F_2(x_1))]^2 + [x_2 - F_2(x_1)]^2. \tag{4.53b}$$

Note that it depends on the assignment of the 2 functions $F_1(w)$ and $F_2(w)$, but *not* on the number λ which also characterizes, via (4.45), the *discrete-time* dynamical system (4.49) and (4.50). The specific value of this constant K depends of course on the initial data $x_1(0)$ and $x_2(0)$. Plotting in the $x_1 x_2$ Cartesian plane for various values of K the curves defined by (4.53b) — which might be quite complicated, but cannot feature any crossing — yields a qualitative assessment of the behavior of the dynamical system under consideration, each (discrete) trajectory of which is of course confined to lie on the curve characterized by the value of K determined by the initial data. Indeed, if λ is a large number, the fact that the discrete trajectory shall lie on the curve with the relevant value of K — as determined by the initial data — entails that the discrete evolution shall closely mimic a continuous evolution along that curve: of course the evolution of the *discrete-time* dynamical system (4.49) and (4.50) shall be *periodic* if λ is *rational* and *nonperiodic* if λ is *irrational*, and in the latter case it will eventually seem to completely cover the relevant constant-K curve, although of course it shall, after a finite time ℓ, only cover ℓ different points on that curve. For small values of λ the discrete character of the evolution (4.49) and (4.50) may instead have a more dramatic connotation, with the moving point characterized by the Cartesian coordinates $x_1(\ell), x_2(\ell)$ jumping sequentially from one point of the constant-K curve to another point on that curve possibly quite far from the previous one. Of course this phenomenology shall be particularly striking in the case of a *rational* value of $\lambda = L/M$ with L a small (positive) integer (and M a coprime integer), entailing periodicity of the motion with a short period L, in which case the entire motion will involve only L points on the relevant constant-K curve; and the behavior shall be, at least initially, somewhat similar — albeit of course *not* periodic — if λ is an *irrational* number quite close to a *rational* number L/M (with L and M again *coprime integers* and L *small*).

These phenomenologies are illustrated by the examples presented below. But before doing so let us mention a few special cases that obtain for quite simple assignments of the 2 *a priori* arbitrary functions $F_1(w)$ and $F_2(w)$.

If one of these 2 functions vanishes the model becomes rather trivial. Indeed if, say, $F_2(w) = 0$, then (4.49) and (4.50) become

$$\tilde{x}_1 = c[x_1 - F_1(x_2)] - sx_2 + F_1(s[x_1 - F_1(x_2)] + cx_2), \tag{4.54a}$$

$$\tilde{x}_2 = s[x_1 - F_1(x_2)] + cx_2, \tag{4.54b}$$

or, equivalently,

$$\tilde{x}_2 = s[x_1 - F_1(x_2)] + cx_2, \tag{4.55a}$$

$$\tilde{x}_1 = \frac{c\tilde{x}_2 - x_2}{s} + F_1(\tilde{x}_2). \tag{4.55b}$$

But it is easily seen that this system implies for x_2 the following *linear* second-order discrete evolution equation:

$$\tilde{\tilde{x}}_2 - 2c\tilde{x}_2 + x_2 = 0. \tag{4.56}$$

Note the disappearance of the function $F_1(w)$. This conclusion is of course consistent, see (4.47a), with the fact that when $F_2(w)$ vanishes x_2 coincides with u_2, see (2.1).

If the function $F_1(w)$ is linear, $F_1(w) = aw$ (with a an arbitrary constant), then (4.49) and (4.50) become

$$\tilde{x}_1 = (c + as)x_1 - (1 + a^2)s[x_2 - F_2(x_1)], \tag{4.57}$$

$$\tilde{x}_2 = sx_1 + (c - as)[x_2 - F_2(x_1)] + F_2(\tilde{x}_1), \tag{4.58a}$$

or, more explicitly,

$$\tilde{x}_2 = sx_1 + (c - as)[x_2 - F_2(x_1)] + F_2([c + as]x_1 - (1 + a^2)s[x_2 - F_2(x_1)]). \tag{4.58b}$$

If the function $F_2(w)$ is linear, $F_2(w) = bw$ (with b an arbitrary constant), then (4.49) and (4.50) become

$$\tilde{x}_1 = (c + bs)x_1 - sx_2 - cF_1(x_2 - bx_1) + F_1([s - bc]x_1 + cx_2 - sF_1(x_2 - bx_1)), \tag{4.59}$$

$$\tilde{x}_2 = (s - bc)x_1 + cx_2 - sF_1(x_2 - bx_1) + b\tilde{x}_1, \tag{4.60a}$$

or, more explicitly,

$$\tilde{x}_2 = (1 + b^2)sx_1 + (c - bs)x_2 - (bc + s)F_1(x_2 - bx_1)$$
$$+ bF_1([s - bc]x_1 + cx_2 - sF_1(x_2 - bx_1)). \tag{4.60b}$$

Clearly the case when $F_2(w)$ vanishes, see (4.54), is too simple to be of interest: the evolution of x_2 is independent of $F_1(w)$ and is described by the second-order constant-coefficient *linear* equation (4.56), and the evolution of x_1, while depending of $F_1(w)$, is slave to the evolution of x_2 and quite simple, see (4.55b). The other two special cases considered above are instead somewhat less trivial, see (some of) the examples described below.

First example. Let us consider the model characterized by the equations of motion (4.57) and (4.58) with $F_2(w) = \beta w^2$ where β is an *a priori* arbitrary function. Then the equations of motion read

$$\tilde{x}_1 = (c + as)x_1 - (1 + a^2)s(x_2 - \beta x_1^2), \tag{4.61}$$

$$\tilde{x}_2 = sx_1 + (c - as)(x_2 - \beta x_1^2) + \beta \tilde{x}_1, \tag{4.62a}$$

or, more explicitly,

$$\tilde{x}_2 = sx_1 + (c - as)(x_2 - \beta x_1^2) + \beta[(c + as)x_1 - (1 + a^2)s(x_2 - \beta x_1^2)]^2. \tag{4.62b}$$

Note that the evolution equation for the first dependent variable, x_1, features in its right-hand side a *second-order* polynomial of the dependent variables, and the evolution equation for the second dependent variable, x_2, features in its right-hand side a *fourth-order* polynomial of the dependent variables. In this case the constant of motion K (see (4.53b)) reads

$$K = [x_1 - a(x_2 - \beta x_1^2)]^2 + (x_2 - \beta x_1^2)^2. \tag{4.63}$$

Second example. Let us consider the model characterized by the equations of motion (4.59) and (4.60) with $F_1(w) = \alpha w^2$ where α is an *a priori* arbitrary number. Then the equations of motion read

$$\tilde{x}_1 = (c + bs)x_1 - sx_2 - \alpha c(x_2 - bx_1)^2 + \alpha[(s - bc)x_1 + cx_2 - \alpha s(x_2 - bx_1)^2]^2, \tag{4.64}$$

$$\tilde{x}_2 = (s - bc)x_1 + cx_2 - \alpha s(x_2 - bx_1)^2 + b\tilde{x}_1, \tag{4.65a}$$

or, more explicitly,

$$\tilde{x}_2 = (s - bc)x_1 + cx_2 - \alpha s(x_2 - bx_1)^2 + b\{(c + bs)x_1 - sx_2 - \alpha c(x_2 - bx_1)^2$$
$$+ \alpha[(s - bc)x_1 + cx_2 - \alpha s(x_2 - bx_1)^2]^2\}. \tag{4.65b}$$

Note that the evolution equations for both dependent variables, x_1 and x_2, feature now in their right-hand sides a *fourth-order* polynomial of the dependent variables. In this case the constant of motion K (see (4.53b)) reads

$$K = [x_1 - \alpha(x_2 - bx_1)^2]^2 + (x_2 - bx_1)^2. \tag{4.66}$$

Third example. Let us make the simple assignments (see (2.2a))

$$F_1(w) = \alpha w^2, \quad F_2(w) = \beta w^2, \tag{4.67}$$

where α and β are 2 *a priori* arbitrary numbers. Then the new *discrete-time* dynamical system (4.49) and (4.50) reads as follows:

$$\tilde{x}_1 = c[x_1 - \alpha(x_2 - \beta x_1^2)^2] - s[x_2 - \beta x_1^2] + \alpha\{s[x_1 - \alpha(x_2 - \beta x_1^2)^2] + c(x_2 - \beta x_1^2)\}^2,$$
$$\tag{4.68}$$

and

$$\tilde{x}_2 = s[x_1 - \alpha(x_2 - \beta x_1^2)^2] + c(x_2 - \beta x_1^2) + \beta(\tilde{x}_1)^2, \tag{4.69a}$$

or, equivalently,

$$\tilde{x}_2 = s[x_1 - \alpha(x_2 - \beta x_1^2)^2] + c(x_2 - \beta x_1^2) + \beta\{c[x_1 - \alpha(x_2 - \beta x_1^2)^2] - s[x_2 - \beta x_1^2]$$
$$+ \alpha\{s[x_1 - \alpha(x_2 - \beta x_1^2)^2] + c(x_2 - \beta x_1^2)\}^2\}^2. \tag{4.69b}$$

This *discrete-time* dynamical system is fairly complicated: indeed the right-hand side of the last equation features a term with the dependent variable x_1 raised to the 16th power, and the right-hand side of (4.68) features a term with the dependent variable x_1 raised to the 8th power. In this case the constant of motion K (see (4.53b)) reads

$$K = [x_1 - \alpha(x_2 - \beta x_1^2)^2]^2 + (x_2 - \beta x_1^2)^2. \tag{4.70}$$

The dynamical evolution entailed by this model is fairly rich, while being *solvable*, and of course *isochronous* whenever λ is an *integer* (or a *rational* number). See Fig. 1 for a display of the trajectory in the $x_1 x_2$ Cartesian plane of this system (with $\alpha = \beta = 1, \lambda = 6$ and initial data $x_1(0) = 0.8$, $x_2(0) = -0.8$ implying $K = 3.69566$, see (4.70)).

Let us also mention that, if the initial values assigned are somewhat larger than those of the trajectory reported in Fig. 1, the corresponding trajectory can reach quite large values before returning to the initial values (for instance the initial values $x_1(0) = 3$, $x_2(0) = 3$ yield $x_1(3) = 69$, $x_2(3) = 4767$, with $K = 1125$).
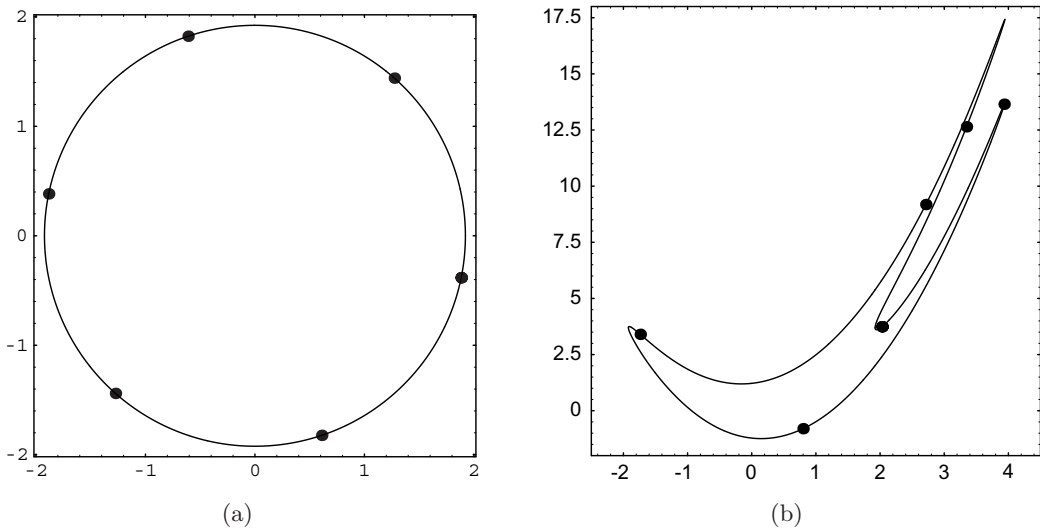

Fig. 1. Curve K and orbit in the $u_1 u_2$-plane for system (4.46) and the corresponding curve and orbit in the $x_1 x_2$-plane via transformation (2.1) with (4.67). (a) The curve (4.53a) (solid line) and the orbit generated by (4.46) (dots), with $u_1(0) = -1.2736$, $u_2(0) = -1.44$ and $\lambda = 6$, in the $u_1 u_2$-plane. (b) The curve (4.70) (solid line) and the orbit generated by (4.68, 4.69b) (dots), with $x_1(0) = 0.8$, $x_2(0) = -0.8$ and $\alpha = \beta = 1$, $\lambda = 6$, in the $x_1 x_2$-plane.

Fourth example. Let us make the simple assignments

$$F_1(w) = \alpha \sin(aw), \quad F_2(w) = \beta \sin(bw). \tag{4.71}$$

Then the new *discrete-time* dynamical system (4.49) and (4.50) reads as follows:

$$\tilde{x}_1 = c[x_1 - \alpha \sin(a[x_2 - \beta \sin(\delta x_1)])] - s[x_2 - \beta \sin(\delta x_1)]$$
$$+ \alpha \sin(a\{s[x_1 - \alpha \sin(a\{x_2 - \beta \sin(\delta x_1)\})] + c[x_2 - \beta \sin(\delta x_1)]\}), \tag{4.72}$$

$$\tilde{x}_2 = s[x_1 - \alpha \sin(a[x_2 - \beta \sin(\delta x_1)])] + c[x_2 - \beta \sin(\delta x_1)] + \beta \sin(\delta \tilde{x}_1), \tag{4.73a}$$

or, more explicitly,

$$\tilde{x}_2 = s[x_1 - \alpha \sin(a[x_2 - \beta \sin(\delta x_1)])] + c[x_2 - \beta \sin(\delta x_1)]$$
$$+ \beta \sin(\delta\{c[x_1 - \alpha \sin(a[x_2 - \beta \sin(\delta x_1)])] - s[x_2 - \beta \sin(\delta x_1)]$$
$$+ \alpha \sin(a\{s[x_1 - \alpha \sin(a\{x_2 - \beta \sin(\delta x_1)\})] + c[x_2 - \beta \sin(\delta x_1)]\})\}) \tag{4.73b}$$

In this case the constant of motion K (see (4.53b)) reads

$$K = [x_1 - \alpha \sin\{a[x_2 - \beta \sin(bx_1)]\}]^2 + [x_2 - \beta \sin(bx_1)]^2. \tag{4.74}$$

Fifth example. Let us make the simple assignments

$$F_1(w) = \alpha\theta(a - w), \quad F_2(w) = \beta\theta(b - w). \tag{4.75a}$$

Here and throughout $\theta(w)$ is the standard step function,

$$\theta(w) = 0 \quad \text{for } w < 0, \qquad \theta(w) = 1 \quad \text{for } w \geq 0. \tag{4.75b}$$

Then the new *discrete-time* dynamical system (4.49) and (4.50) reads as follows:

$$\tilde{x}_1 = cx_1 - sx_2 + s\beta\theta(b - x_1) - c\alpha\theta(a - x_2 + \beta\theta(b - x_1))$$
$$+ \alpha\theta(sx_1 + cx_2 - c\beta\theta(b - x_1) - s\alpha\theta(a - x_2 + \beta\theta(b - x_1))), \tag{4.76}$$

$$\tilde{x}_2 = sx_1 + cx_2 - c\beta\theta(b - x_1) - sF(x_2 - \alpha\theta(a - x_2 + \beta\theta(b - x_1))) + \beta\theta(b - \tilde{x}_1), \tag{4.77a}$$

or, more explicitly,

$$\tilde{x}_2 = sx_1 + cx_2 - c\beta\theta(b - x_1) - sF(x_2 - \alpha\theta(a - x_2 + \beta\theta(b - x_1)))$$
$$+ \beta\theta(b - [cx_1 - sx_2 + s\beta\theta(b - x_1) - c\alpha\theta(a - x_2 + \beta\theta(b - x_1))$$
$$+ \alpha\theta(sx_1 + cx_2 - c\beta\theta(b - x_1) - s\alpha\theta(a - x_2 + \beta\theta(b - x_1)))]). \tag{4.77b}$$

In this case the constant of motion K (see (4.53b)) reads

$$K = [x_1 - \alpha\theta(a - x_2 + \beta\theta(b - x_1))]^2 + [x_2 - \beta\theta(b - x_1)]^2, \tag{4.78}$$

or, equivalently,

$$K = x_1^2 + x_2^2 \quad \text{for } x_1 > b, \ x_2 > a, \tag{4.79a}$$

$$K = (x_1 - \alpha)^2 + x_2^2 \quad \text{for } x_1 > b, \ x_2 \leq a, \tag{4.79b}$$

$$K = x_1^2 + (x_2 - \beta)^2, \quad \text{for } x_1 \le b, \ x_2 > a + \beta, \tag{4.79c}$$

$$K = (x_1 - \alpha)^2 + (x_2 - \beta)^2, \quad \text{for } x_1 \le b, \ x_2 \le a + \beta. \tag{4.79d}$$

Note however that this function $K \equiv K(x_1, x_2)$ is *not* a continuous function of its 2 arguments x_1, x_2.

Sixth example. To remedy the "defect" we just mentioned, let us modify the previous assignment so that it now reads

$$F_1(w) = \alpha(a - w)\theta(a - w), \quad F_2(w) = \beta(b - w)\theta(b - w). \tag{4.80}$$

In this case the constant of motion K (see (4.53b)) reads

$$K = \{x_1 - \alpha[a - x_2 + \beta(b - x_1)\theta(b - x_1)]\theta(a - x_2 + \beta(b - x_1)\theta(b - x_1))\}^2$$
$$+ [x_2 - \beta(b - x_1)\theta(b - x_1)]^2, \tag{4.81}$$

or, equivalently,

$$K = x_1^2 + x_2^2, \quad x_1 > b, \quad x_2 > a, \tag{4.82a}$$

$$K = z_1^2 + x_2^2, \quad x_1 > b, \quad x_2 \le a, \tag{4.82b}$$

$$K = z_1^2 + z_2^2, \quad x_1 \le b, \quad z_2 \le a, \tag{4.82c}$$

$$K = x_1^2 + z_2^2, \quad x_1 \le b, \quad z_2 > a, \tag{4.82d}$$

where

$$z_1 \equiv x_1 + \alpha(x_2 - a), \quad z_2 \equiv x_2 + \beta(x_1 - b). \tag{4.82e}$$

Note that our treatment implies that, for arbitrary values of the 4 numbers α, β, a, b, the constant-K curve in the Cartesian $x_1 x_2$ plane corresponding to these formulas is *closed*, for any positive value of K (see Fig. 2).

As for the *discrete-time* equations of motion in this case, we feel their explicit display can also be left as an easy exercise for the diligent reader.

Finally we report a *discrete-time* dynamical system involving 3 dependent variables and displaying the *asymptotically isochronous* phenomenology (see the somewhat analogous *continuous-time* dynamical system (4.36) discussed at the end of Subsec. 4.3).

An asymptotically isochronous discrete-time system involving 3 *dependent variables.* Let

$$\tilde{u}_1 = u_2, \quad \tilde{u}_2 = u_3, \quad \tilde{u}_3 = \alpha u_1 + \beta u_2 + \gamma u_3 \tag{4.83a}$$

with

$$\beta = -1 - 2\alpha \cos\left(\frac{2\pi}{\lambda}\right), \tag{4.83b}$$

$$\gamma = \alpha + 2\cos\left(\frac{2\pi}{\lambda}\right), \tag{4.83c}$$

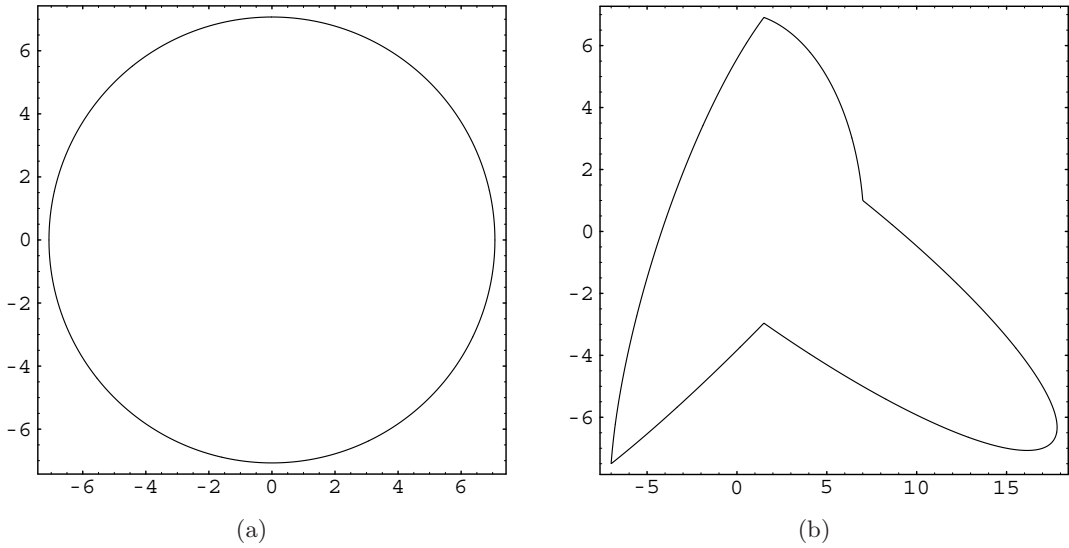where α and λ are two (*a priori* arbitrary) *real* numbers.

Fig. 2. The curve K for system (4.46) and the corresponding curve in the $x_1 x_2$-plane via transformation (2.1) with (4.80). (a) The curve (4.53a) in the $u_1 u_2$-plane for $K = 50$. (b) The curve (4.81) in the $x_1 x_2$-plane for $K = 50$ and $a = 1$, $b = \frac{3}{2}$, $\alpha = 2$ and $\beta = -1$.

It is easily seen that the solution of the initial-value problem for this *discrete-time* model reads as follows:

$$u_1(\ell) = A\alpha^\ell + C\cos\left(\frac{2\pi\ell}{\lambda}\right) + S\sin\left(\frac{2\pi\ell}{\lambda}\right),$$

$$u_2(\ell) = \tilde{u}_1(\ell) = u_1(\ell+1), \quad u_3(\ell) = \tilde{u}_2(\ell) = u_1(\ell+2), \tag{4.84a}$$

with

$$A = \left[u_1(0) - 2u_2(0)\cos\left(\frac{2\pi}{\lambda}\right) + u_3(0)\right] \Big/ D, \tag{4.84b}$$

$$C = \frac{1}{D}\left\{2u_2(0)\cos\left(\frac{2\pi}{\lambda}\right) - u_3(0) - 2\alpha u_1(0)\cos\left(\frac{2\pi}{\lambda}\right) + \alpha^2 u_1(0)\right\}, \tag{4.84c}$$

$$S = \frac{1}{D\sin\left(\frac{2\pi}{\lambda}\right)}\left\{u_3(0)\cos\left(\frac{2\pi}{\lambda}\right) - u_2(0)\left(\frac{4\pi}{\lambda}\right)\right.$$

$$\left. + \alpha\left[u_1(0)\cos\left(\frac{4\pi}{\lambda}\right) - u_3(0)\right] - \alpha^2\left[u_1(0)\cos\left(\frac{2\pi}{\lambda}\right) - u_2(0)\right]\right\}, \tag{4.84d}$$

$$D = 1 - 2\alpha\cos\left(\frac{2\pi}{\lambda}\right) + \alpha^2. \tag{4.84e}$$

Hence the solution is, for arbitrary initial data, *isochronous* with period L (i.e., $u(\ell + L) = u(\ell)$) if λ is *rational*, $\lambda = L/M$ with L and M *coprime integers* (and, say, L *positive*) and $\alpha = 1$ (if $\alpha = -1$, it is as well *isochronous*, with period L if L is *even* and $2L$ if L is *odd*), and it is *asymptotically isochronous* (i.e., $u(\ell + L) - u(\ell) = O(|\alpha|^\ell)$ as $\ell \to \infty$) if $\lambda = L/M$ is *rational* and $|\alpha| < 1$.

A class of apparently quite less trivial *discrete-time* dynamical systems is then obtained via the invertible change of variables (3.5). It involves the 3 *arbitrary* functions $F_n(w_1, w_2), n = 1, 2, 3$, and it reads:

$$\tilde{x}_1 = x_2 - G_2(x_1, x_2, x_3) + G_4(x_1, x_2, x_3), \tag{4.85a}$$

$$\tilde{x}_2 = x_3 - G_1(x_1, x_2, x_3) + G_5(x_1, x_2, x_3), \tag{4.85b}$$

$$\tilde{x}_3 = J(x_1, x_2, x_3) + F_3(\tilde{x}_1, \tilde{x}_2), \tag{4.85c}$$

where the 5 functions G_1, G_2, G_3, G_4, G_5 are recursively given, one in terms of the other, as follows:

$$G_1(x_1, x_2, x_3) = F_3(x_1, x_2), \tag{4.86a}$$

$$G_2(x_1, x_2, x_3) = F_2(x_1, x_3 - G_1(x_1, x_2, x_3)), \tag{4.86b}$$

$$G_3(x_1, x_2, x_3) = F_1(x_2 - G_2(x_1, x_2, x_3), x_3 - G_1(x_1, x_2, x_3)), \tag{4.86c}$$

$$G_4(x_1, x_2, x_3) = F_1(x_3 - G_1(x_1, x_2, x_3), J(x_1, x_2, x_3)), \tag{4.86d}$$

$$G_5(x_1, x_2, x_3) = F_2(x_2 - G_2(x_1, x_2, x_3) + G_4(x_1, x_2, x_3), J(x_1, x_2, x_3)), \tag{4.86e}$$

and the function J depends on G_1, G_2 and G_3,

$$J(x_1, x_2, x_3) = \alpha[x_1 - G_3(x_1, x_2, x_3)] + \beta[x_2 - G_2(x_1, x_2, x_3)] + \gamma[x_3 - G_1(x_1, x_2, x_3)]. \tag{4.86f}$$

The *general* solution of (4.85) reads as follows:

$$x_1(\ell) = u_1(\ell) + F_1(u_1(\ell + 1), u_1(\ell + 2)), \tag{4.87a}$$

$$x_2(\ell) = u_1(\ell + 1) + F_2(x_1(\ell), u_1(\ell + 2)), \tag{4.87b}$$

$$x_3(\ell) = u_1(\ell + 2) + F_3(x_1(\ell), x_2(\ell)), \tag{4.87c}$$

where $u_1(\ell)$ is given in (4.84). And clearly this system inherits the same phenomenology (see Fig. 3) — concerning *isochrony* and *asymptotic isochrony* — of the simple linear model (4.83), as described above (after (4.84e)).

4.5.   *Solvable and integrable nonlinear PDEs*

In this section, via the presentation of a simple example, we show how the *invertible transformation* (2.1) allows to manufacture highly nonlinear, yet explicitly solvable, PDEs.

Let us take as starting point the elementary pair of PDEs

$$\varphi_{1,t} = \varphi_{2,x}, \quad \varphi_{2,t} = \varphi_{1,x}, \tag{4.88a}$$

where the 2 functions $\varphi_n \equiv \varphi_n(x, t)$ depend on the 2 variables x and t, and subscripted variables indicate partial differentiation with respect to them, $\varphi_{n,t}(x, t) \equiv \partial \varphi_n(x, t)/\partial t, \varphi_{n,x}(x, t) \equiv \partial \varphi_n(x, t)/\partial x$. Clearly this system of 2 linear PDEs has the following *general* solution:

$$\varphi_1(x, t) = f_1(x + t) + f_2(x - t), \quad \varphi_2(x, t) = f_1(x + t) - f_2(x - t), \tag{4.88b}$$

where $f_m(z), m = 1, 2$, are 2 *arbitrary* functions of the single variable z.
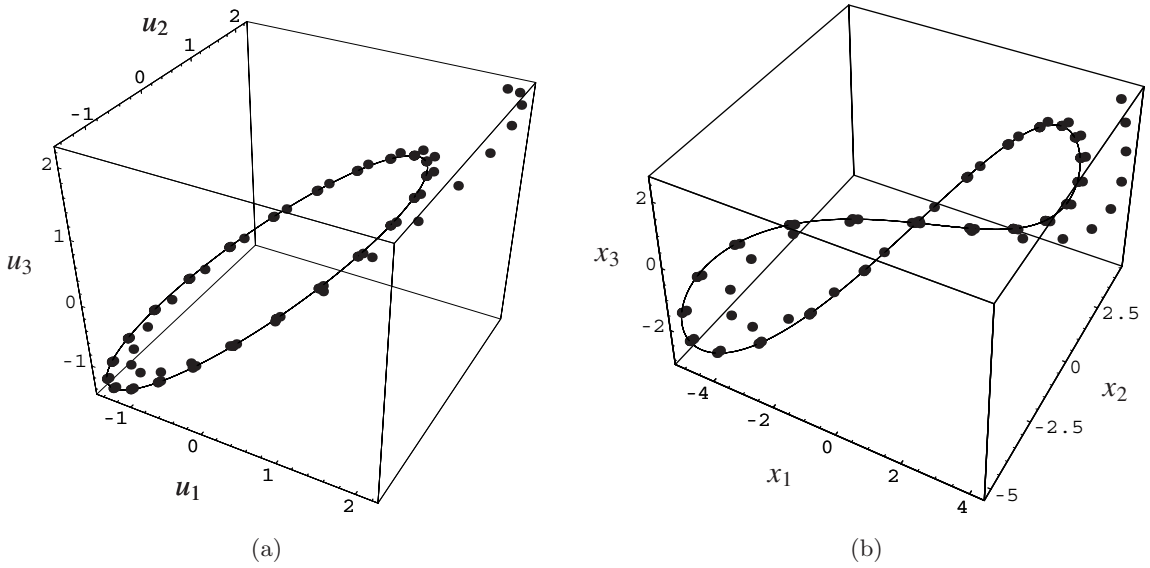
(a)   (b)

Fig. 3. Orbit and limit trajectory in the $u_1u_2u_3$-space for system (4.83) and corresponding orbit and limit trajectory in the $x_1x_2x_3$-space for system (4.85), obtained via transformation (3.5) with $F_1(p,q) = p+q-\frac{pq}{2}$, $F_2(p,q) = p - q$ and $F_3(p,q) = p^2 - q^2$. (a) Orbit (dots) and limit trajectory (solid line) for system (4.83) in the $u_1u_2u_3$-space for $\alpha = \frac{9}{10}$, $\lambda = 23$ and $u_1(0) = 2$, $u_2(0) = 2.13271$ and $u_3(0) = 2.184$ (so that $A = C = S = 1$). (b) Orbit (dots) and limit trajectory (solid line) for system (4.85) in the $x_1x_2x_3$-space when $F_1(p,q) = p + q - \frac{pq}{2}$, $F_2(p,q) = p - q$ and $F_3(p,q) = p^2 - q^2$ emerging from $x_1(0) = 3.98779$, $x_2(0) = 3.9365$ and $x_3(0) = 2.59043$.

We now apply the transformation (2.1) — with the two quantities u_1, u_2 replaced by the two functions $\varphi_1(x,t), \varphi_2(x,t)$ and, likewise, the two quantities x_1, x_2 replaced by two functions $\psi_1(x,t), \psi_2(x,t)$. Thereby we find — via a bit of standard algebra — that the system of two linear PDEs (4.88a) becomes the following system of 2, generally nonlinear, PDEs, for the 2 functions $\psi_1(x,t), \psi_2(x,t)$:

$$\psi_{1,t} = \psi_{2,x} - F_2'(\psi_1)\psi_{1x}$$
$$+ F_1'(\psi_2 - F_2(\psi_1))\{\psi_{1x} - F_1'(\psi_2 - F_2(\psi_1))[\psi_{2,x} - F_2'(\psi_1)\psi_{1x}]\}, \qquad (4.89)$$

$$\psi_{2,t} = \psi_{1,x} - F_1'(\psi_2 - F_2(\psi_1))[\psi_{2,x} - F_2'(\psi_1)\psi_{1x}] + F_2'(\psi_1)\psi_{1t}, \qquad (4.90a)$$

or, equivalently,

$$\psi_{2,t} = \psi_{1,x} - F_1'(\psi_2 - F_2(\psi_1))[\psi_{2,x} - F_2'(\psi_1)\psi_{1x}] + F_2'(\psi_1)[\psi_{2,x} - F_2'(\psi_1)\psi_{1x}$$
$$+ F_1'(\psi_2 - F_2(\psi_1))\{\psi_{1x} - F_1'(\psi_2 - F_2(\psi_1))[\psi_{2,x} - F_2'(\psi_1)\psi_{1x}]\}]. \qquad (4.90b)$$

Here of course $F_1(w), F_2(w)$ are 2 *arbitrary* functions.

And it is as well easily seen that this system features the following *general* solution:

$$\psi_1(x,t) = f_1(x+t) + f_2(x-t) + F_1(f_1(x+t) - f_2(x-t)), \qquad (4.91a)$$

$$\psi_2(x,t) = f_1(x+t) - f_2(x-t)$$
$$+ F_2(f_1(x+t) + f_2(x-t) + F_1(f_1(x+t) - f_2(x-t))), \qquad (4.91b)$$

where again $f_m(z), m = 1, 2$, are 2 *arbitrary* functions of the single variable z.

Readers might amuse themselves by inserting in these formulas — both those displaying the nonlinear system of PDEs (4.89), (4.90), and their solution (4.91) — specific assignments of the 2 functions $F_1(w), F_2(w)$ (for instance the simple assignment (2.2a)).

4.6. *Analytical geometry*

The *invertible transformation* (2.1) is an *area preserving* reparameterization of the Cartesian plane: indeed it is easy to verify that it entails that the Jacobian determinant of this change of variables is unity,

$$
\begin{vmatrix} \dfrac{\partial x_1}{\partial u_1} & \dfrac{\partial x_1}{\partial u_2} \\[2mm] \dfrac{\partial x_2}{\partial u_1} & \dfrac{\partial x_2}{\partial u_2} \end{vmatrix} = 1,
\tag{4.92}
$$

for any arbitrary assignment of the 2 functions $F_1(w), F_2(w)$ (this property corresponds — in the Hamiltonian context, see Subsec. 4.3.1 — to the canonical character of the corresponding change of variables).

Three representative examples of the findings that easily flow from this property are provided by the following three propositions.

Proposition 4.6.1. *Let c_1, c_2 be 2 arbitrary (real) numbers, and draw, in the $x_1 x_2$ Cartesian plane, the following 4 curves: the curve A going from the point $a = (0,0)$ to the point $b = (1, c_2)$ and characterized by the equation (a piece of a parabola)*

$$
A\colon x_2 = c_2 x_1^2;
\tag{4.93a}
$$

the curve B going from the point $b = (1, c_2)$ to the point $c = (c_1, 1 + c_1^2 c_2)$ and characterized by the (quartic) equation

$$
B\colon 1 - x_1 - x_2 + c_2 x_1^2 + c_1 x_2^2 - 2 c_1 c_2 x_1^2 x_2 + c_1 c_2^2 x_1^4 = 0;
\tag{4.93b}
$$

the curve C going from the point $c = (c_1, 1 + c_1^2 c_2)$ to the point $a = (0,0)$ and characterized by the (quartic) equation

$$
C\colon x_1 - c_1 x_2^2 + 2 c_1 c_2 x_1^2 x_2 - c_1 c_2^2 x_1^4 = 0;
\tag{4.93c}
$$

and the curve D going from the point $a = (0,0)$ to the point $(\frac{1}{2} + \frac{c_1}{4}, \frac{1}{2} + c_2(\frac{1}{2} + \frac{c_1}{4})^2)$ (lying on the curve B) and characterized by the (quartic) equation

$$
D\colon x_1 - x_2 + c_2 x_1^2 - c_1 x_2^2 + 2 c_1 c_2 x_1^2 x_2 - c_1 c_2^2 x_1^4 = 0.
\tag{4.93d}
$$

Fact. The region enclosed by the 3 curves A, B, C has area $1/2$, and the curve D divides this region in two parts of *equal* area (see Fig. 4).

The proof of this Proposition 4.6.1 is an immediate consequence of the fact that, via the transformation (2.1) with $F_n(w) = c_n w^2$, see (2.2a), the region enclosed by the 3 curves A, B, C corresponds, in the $x_1 x_2$ Cartesian plane, to the triangle of vertices $(0,0)$, $(1,0)$, $(0,1)$ in the $u_1 u_2$ Cartesian plane, and likewise the curve D corresponds to the segment starting from the vertex $(0,0)$ and bisecting that triangle (see Fig. 4).
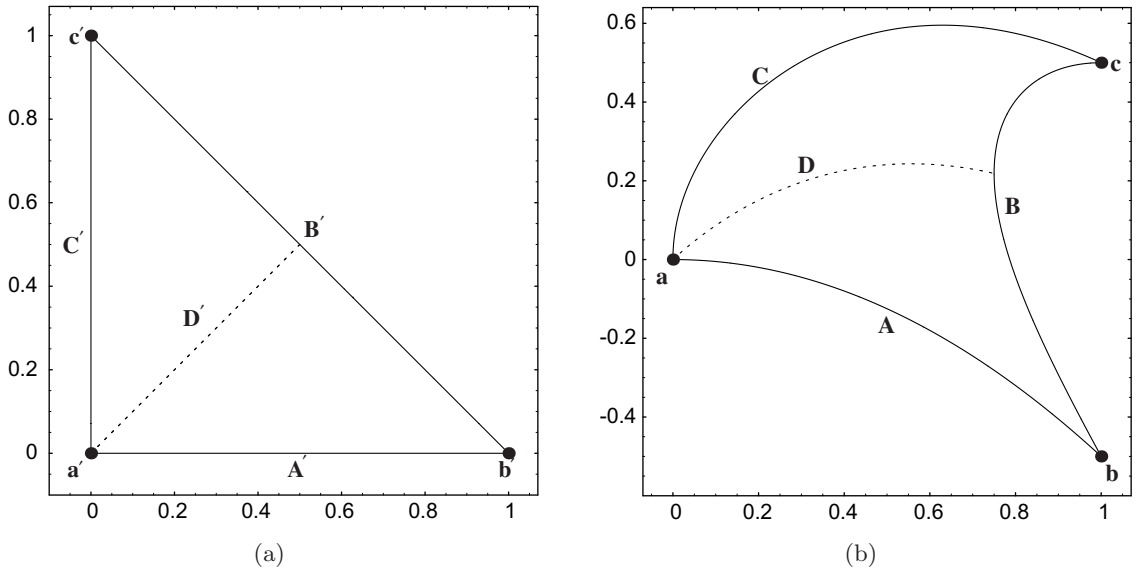
Fig. 4. The triangle of vertices $(0,0)$, $(1,0)$ and $(0,1)$ in the u_1u_2-plane and the corresponding region in the x_1x_2-plane via transformation (2.1) with $F_1(w) = w^2$ and $F_2(w) = -\frac{w^2}{2}$. (a) The triangle of vertices $a' = (0,0)$, $b' = (1,0)$ and $c' = (0,1)$ in the u_1u_2-plane. The segment D' is the bisectrix of the triangle starting from the vertex a'. (b) The four curves A, B, C and D in the x_1x_2-plane, as defined in (4.93) for $c_1 = 1$ and $c_2 = -\frac{1}{2}$, crossing at $a = (0,0)$, $b = (1, -\frac{1}{2})$, $c = (1, \frac{1}{2})$. The curve D divides the enclosed region in two parts of *equal* area.

Proposition 4.6.2. *Draw, in the x_1x_2 Cartesian plane, the 5 curves described parametrically by the following 5 equations*:

$$B: \quad x_1 = 1 + \frac{\alpha}{u}, \quad x_2 = u\left(1 + \frac{\beta u}{u + \alpha}\right), \tag{4.94a}$$

$$C: \quad x_1 = u + \alpha, \quad x_2 = 1 + \frac{\beta}{u + \alpha}, \tag{4.94b}$$

$$D: \quad x_1 = \frac{\alpha}{u}, \quad x_2 = u\left(1 + \frac{\beta}{\alpha}\right), \tag{4.94c}$$

$$E: \quad x_1 = u + \frac{\alpha}{u}, \quad x_2 = u\left(1 + \frac{\beta}{u^2 + \alpha}\right), \tag{4.94d}$$

$$F: \quad x_1 = u + \frac{\alpha}{1 - u}, \quad x_2 = (1 - u)\left(1 + \frac{\beta}{u(1 - u) + \alpha}\right). \tag{4.94e}$$

Here α and β are 2, arbitrarily assigned, (real) numbers ($\alpha \neq 0, \alpha \neq 1$), the running parameter u ranges in all cases from 0 to 1, and we introduced the 5 capital letters B, C, D, E, F to label both the curves and the corresponding equations. Likewise, we denote as A the asymptotic point $x_1 = \text{sign}(\alpha)\infty$, $x_2 = 0$. Note that the first curve, B, has only a finite span, going from the point $b \equiv (1 + \alpha, 1 + \frac{\beta}{1+\alpha})$ to the point $c \equiv (\alpha, 1 + \frac{\beta}{\alpha})$, while the other 4 curves all run all the way to the asymptotic point A, the curves C and E both starting from the point b, the curves D and F both starting from the point c. Fact: the 3 curves B, C, D enclose a
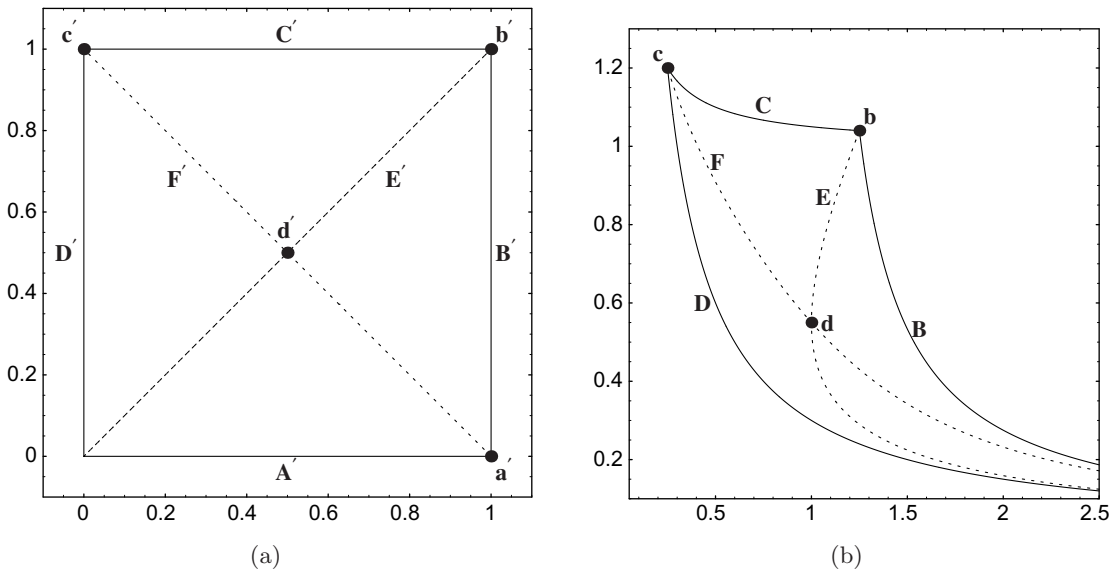
Fig. 5. The square of vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ in the u_1u_2-plane and the corresponding region in the x_1x_2-plane via transformation (2.1) with $F_1(w) = \frac{1}{4\,w}$ and $F_2(w) = \frac{1}{20\,w}$. (a) The square in the u_1u_2-plane having one of its vertices in the origin $(0,0)$ and the other vertices in $a' = (1,0)$, $b' = (1,1)$ and $c' = (0,1)$. (b) The five curves B, C, D, E and F in the x_1x_2-plane, as defined in (4.94) for $\alpha = \frac{1}{4}$ and $\beta = \frac{1}{20}$, crossing at $b = (\frac{5}{4}, \frac{26}{25})$ and $c = (\frac{1}{4}, \frac{6}{5})$.

region (*which of course extends all the way to the asymptotic point* A) *the area of which is unity*; *while the* 2 *curves* E *and* F (*which cross at the point* $d \equiv (\frac{1}{2} + 2\alpha, \frac{1}{2} + \frac{2\beta}{1+4\alpha})$, *inside this region*), *divide this region in* 4 *parts of equal area* (*see Fig.* 5).

The proof of this Proposition 4.6.2 is analogous to that of *Proposition 4.2*, except that now the curves that correspond — again, via (2.1b) with $F_1(w) = \alpha/w$, $F_2(w) = \beta/w$ — in the Cartesian u_1u_2 plane to those characterized in the Cartesian x_1x_2 plane by the 5 expressions (4.94), are the 3 sides B, C, D of the unit square having the 4 vertices $(0,0), (1,0), (1,1), (0,1)$ (labeled counterclockwise, starting from the first side A going, in that u_1u_2 plane, from the origin to the point $(1,0)$, a segment which correspond in the x_1x_2 plane to the asymptotic point A); while the 2 curves E and F correspond to the 2 diagonals of that square, see Fig. 5.

Let us now turn to the third example. Draw, in the x_1x_2-plane, the curve defined by the following *quartic* equation:

$$k_1(x_1 + k_2x_2)^4 + (x_1 + k_2x_2)^2[k_3x_1 + (k_2k_3 + 2\,k_1)x_2] + k_4x_1^2 + (2\,k_2k_4 + k_3)x_1x_2$$
$$+(k_1 + k_2{}^2k_4 + k_2k_3)x_2 + k_5x_1 + k_6x_2 + k_7 = 0, \tag{4.95}$$

where k_i, with $i = 1, 2, \ldots, 7$, are 7 *arbitrary* parameters. It is then possible to associate to this *quartic* curve (4.95) the quantity

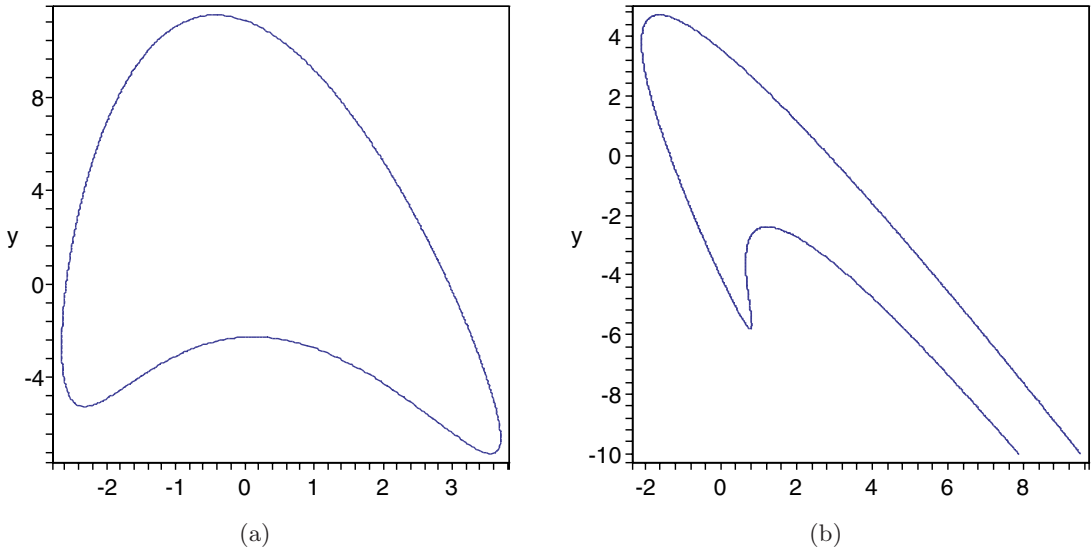$$\Delta = k_3^2 - 4\,k_1(k_4 + k_2k_5 - k_6), \tag{4.96}$$

Fig. 6. The curve (4.95) for two different choices of the 7 arbitrary parameters k_i's. (a) The curve (4.95) for $k_1 = -0.119$, $k_2 = 0.5$, $k_3 = 0.5$, $k_4 = 0.35$, $k_5 = 1.1$, $k_6 = 1.1$ and $k_7 = 3.1$, so that $\Delta = -0.280720 < 0$ and the corresponding conic (4.97) is an ellipse. (b) The curve (4.95) for $k_1 = -0.3125$, $k_2 = 0.5$, $k_3 = 0.5$, $k_4 = 0.35$, $k_5 = 1.1$, $k_6 = 1.1$ and $k_7 = 3.1$, so that $\Delta = 0$ and the corresponding conic (4.97) is a parabola.

playing for this *quartic* curve a role analogous to that of the *discriminant* for a *conic*. Indeed there holds then the following

Proposition 4.6.3. *If $\Delta \geq 0$, then the quartic curve* (4.95) *is open; if $\Delta < 0$, it is instead closed and its area coincides with that of the ellipse* (*in the $u_1 u_2$-plane*) *characterized by the equation*

$$(k_4 + k_2 k_5 - k_6)u_1^2 + k_3 u_1 u_2 + k_1 u_2^2 + k_5 u_1 + (-k_2 k_5 + k_6)u_2 + k_7 = 0 \qquad (4.97)$$

(*see Fig.* 6).

The proof of this Proposition 4.6.3 is again analogous to that of the preceding two propositions, being implied by the fact that (4.95) is the image of (4.97) via the transformation (2.1a) with

$$F_1(w) = k_2 w, \quad F_2(w) = w^2. \qquad (4.98)$$

We leave to the interested reader the geometrical formulation of the additional findings entailed by the fact that, because the area of the disk enclosed by the circle in the $u_1 u_2$ Cartesian plane defined by (4.53a) is $K\pi$, this is also the area of each of the regions enclosed, in the $x_1 x_2$ Cartesian plane, by the curves defined by each of the following equations: (4.63), (4.66), (4.70), (4.74), (4.81) (note that these findings hold for arbitrary values of the parameters characterizing these equations).

And we end this Subsec. 4.6 with the following

Remark 4.6.4. The property of the transformation (2.1) to be area-preserving — as discussed above, see (4.92) — does not necessarily extend to the more general transformations discussed in Sec. 3.

4.7. *Functional equations*

In this Subsec. 4.7 we report, quite tersely, a single example of functional equation. It reads as follows:

$$
\begin{aligned}
[U_1(x) + a]&[U_1(y) + a][U_1(xy)U_2(xy) + aU_2(xy) + bU_1(xy) + c] \\
&= [U_1(x + y) + a]\{U_1(x)U_2(x) + U_1(y)U_2(y) \\
&\quad + a[U_2(x) + U_2(y)] + b[U_1(x) + U_1(y)] + 2c\},
\end{aligned} \tag{4.99}
$$

where a, b are two arbitrary numbers.

Proposition 4.7.1. *A solution of this functional equation reads as follows*:

$$
U_1(z) = \exp(\gamma z) - a, \tag{4.100a}
$$

$$
U_2(z) = [ab - c - \eta \ln(z)] \exp(-\gamma z) - b. \tag{4.100b}
$$

Here and below c, γ and η are 3 arbitrary parameters.

The proof of this finding can be performed by direct substitution, or by verifying that, via the transformation (2.1b) with

$$
\begin{aligned}
u_1 = u_1(z), \quad u_2 &= u_2(z), \quad x_1 = U_1(z), \quad x_2 = U_2(z), \\
F_1(w) = -a, \quad &F_2(w) = -b + \frac{ab - c}{w + a},
\end{aligned} \tag{4.101}
$$

the functional equation (4.99), becomes

$$
u_1(x)u_1(y)u_1(xy)u_2(xy) = u_1(x + y)[u_1(x)u_2(x) + u_1(y)u_2(y)]. \tag{4.102}
$$

One then notes that this functional equation, (4.102), by setting

$$
u_1(z) = f(z), \quad u_2(z) = \frac{g(z)}{f(z)}, \tag{4.103}
$$

is implied by the two standard functional equations

$$
f(x + y) = f(x)f(y), \quad g(xy) = g(x) + g(y), \tag{4.104a}
$$

the unique analytic solutions of which are well known to read

$$
f(z) = \exp(\gamma z), \quad g(z) = \eta \ln(z). \tag{4.104b}
$$

And clearly these formulas imply (4.100) via (2.1a) with (4.101) and (4.103) with (4.104b).

References

[1] C. Jacobi, *Vorlesungen über Dynamik* (Reimer, Berlin, 1848).
[2] A. Reghini, *Dei numeri Pitagorici* (*Parte I, Libro I*), eds. S. Loretoni and C. Scimiterna (Arché, Milano, Italy, 2006, ISBN: 88-7252-275-7 and Edizioni Pizeta, San Donato (Mi), Italy, 2006, ISBN: 88-87625-16-6).
[3] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th edn. (Clarendon Press, Oxford, 1979).

[4] A. Erdélyi (ed.), *Higher Transcendental Functions*, Vol. 2 (McGraw-Hill, New York, 1954).

[5] F. Calogero and S. Graffi, On the quantization of a nonlinear Hamiltonian oscillator, *Phys. Lett. A* **313** (2003) 356–362.

[6] F. Calogero, On the quantization of two other nonlinear harmonic oscillators, *Phys. Lett. A* **319** (2003) 240–245.

[7] F. Calogero, On the quantization of yet another two nonlinear harmonic oscillators, *J. Nonlinear Math. Phys.* **11** (2004) 1–6.

[8] F. Calogero, *Isochronous Systems* (Oxford University Press, Oxford, 2008).

[9] F. Calogero, Isochronous dynamical systems, *Phil. Trans. R. Soc. A* **369** (2011) 1118–1136.

[10] F. Calogero and D. Gómez-Ullate, Asymptotically isochronous systems, *J. Nonlinear Math. Phys.* **15** (2008) 410–426.

[11] F. Calogero and F. Leyvraz, Solvable systems of isochronous, multi-periodic or asymptotically isochronous nonlinear oscillators, *J. Nonlinear Math. Phys.* **17** (2010) 111–120.

[12] F. Calogero and F. Leyvraz, Isochronous oscillators, *J. Nonlinear Math. Phys.* **17** (2010) 103–110.

[13] F. Calogero and F. Leyvraz, Oscillatory and isochronous rate equations possibly describing chemical reactions, *J. Phys. A* **42** (2009) 265208.

[14] F. Calogero and F. Leyvraz, How to embed an arbitrary Hamiltonian dynamics in a superintegrable (or just integrable) Hamiltonian dynamics, *J. Phys. A* **42** (2009) 145202.

[15] F. Calogero and F. Leyvraz, How to extend any dynamical system so that it becomes isochronous, asymptotically isochronous or multi-periodic, *J. Nonlinear Math. Phys.* **16** (2009) 311–338.

[16] F. Calogero, F. Leyvraz, Isochronous systems, the arrow of time, and the definition of deterministic chaos, *Lett. Math. Phys.* **96** (2011) 37–52.

[17] F. Calogero, F. Leyvraz. and M. Sommacal, Isochronous rate equations describing chemical reactions, *J. Phys. A* **43** (2010) 434010.

[18] Novel rare equations describing T Sochronous chemical reactions, *J. Math. Chem.* **48** (2011) 870–879.

[19] J. Chavarriga and M. Sabatini, A survey of isochronous centers, *Qual. Theory Dyn. Syst.* **1** (1999) 1–70.

[20] F. Leyvraz and F. Calogero, Short-time Poincaré recurrence in a broad class of many-body systems, *J. Stat. Mech.: Theory Exper.* (2009) P02022. doi:10.1088/1742-5468/2009/02/P02022.

[21] F. Calogero, The discrete-time goldfishing, SIGMA (submitted to, April 2011).