

## GENERALIZED DIMENSION ESTIMATES FOR IMAGES OF POROUS SETS UNDER MONOTONE SOBOLEV MAPPINGS

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ABSTRACT. We give an essentially sharp estimate in terms of generalized Hausdorff measures for images of porous sets under monotone Sobolev mappings, satisfying suitable Orlicz-Sobolev conditions.

### 1. INTRODUCTION

In this paper, we study dimension distortion under monotone Sobolev mappings, defined in the Euclidean space  $\mathbb{R}^n$  and with images in  $\mathbb{R}^m$ , where  $n, m \in \mathbb{N}$  are such that  $m \geq n \geq 2$ . We assume additional integrability for the distributional differential of the mappings in question in the form of (1). Some dimension distortion estimates under this integrability condition were obtained in [7–9, 13, 14]. We prove the following theorem (see the next section for the definitions).

**Theorem 1.1.** *Let  $f \in W^{1,1}(\Omega, \mathbb{R}^m)$ , where  $\Omega$  is a domain in  $\mathbb{R}^n$ , be a monotone mapping, satisfying*

$$(1) \quad |Df|^n \log^\lambda(e + |Df|) \in L^1(\Omega)$$

for some  $\lambda \geq -1$ . Then  $\mathcal{H}^{h_{\lambda+1}}(f(E)) = 0$  for each porous set  $E \subset \Omega$ , when  $h_{\lambda+1}(t) = t^n \log^{\lambda+1}(1/t)$ .

The papers [8, 9, 13, 14] mentioned above use similar kinds of scales to measure the size of the image sets and consider mappings  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $n \geq 2$ . In those papers, the pre-image sets are assumed to have dimensions strictly less than  $n$ , which is known to be a weaker assumption than porosity [15, 17]. As a consequence of our restrictions, we obtain an essentially optimal gauge under the integrability condition (1). More precisely, the gauge in [9, Theorem 2] and [14, Theorem 1(ii)], where monotone mappings were considered, is  $h_\lambda(t) = t^n \log^\lambda(1/t)$ , which corresponds to larger image sets than the ones given by Theorem 1.1. The gauge of Theorem 1.1 was obtained in the plane in [13] for the image sets under mappings that are additionally assumed to be homeomorphisms. The generalization of this result to higher dimensions required a further regularity assumption for the inverse mapping (see [14, Theorem 1(iii)]).

Results on Lusin's property N for a Sobolev mapping  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ , satisfying (1) for some  $\lambda$ , were obtained in [4–6, 11]. It is enough to require (1) with  $\lambda = -1$ , to

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obtain property N for a sense-preserving mapping [5]. However, even  $\lambda = n - 1$  is not enough, if a mapping is assumed to be continuous only [4]. As for monotone mappings, so far, it is known that such mappings satisfy condition N whenever they belong to the class  $W^{1,n}$  [11], that is, once (1) is satisfied with  $\lambda = 0$ . One can see that we obtain a better gauge ( $t^n \log(1/t)$ ) for the images of porous sets in that case. It remains open whether or not condition N holds for monotone mappings that satisfy (1) for some  $\lambda \in [-1, 0[$ .

Theorem 1.1 is sharp in the following sense. In [2, Section 5], given a positive  $\lambda$ , a monotone mapping  $f \in W^{1,1}(\mathbb{R}^n, \mathbb{R}^n)$  was constructed which maps a porous set  $C \subset ]0, 1[^n$  onto a set  $C' \subset ]0, 1[^n$  of positive generalized Hausdorff measure with gauge  $t^n \log^\lambda(1/t)$ . The integrability of the differential of  $f$  was examined in [8, Section 2] in the case  $n = 2$  and  $|Df|^n \log^s(e + |Df|)$  was proved to be locally integrable for any  $s < \lambda - 1$ . The same integrability bounds may be easily verified for any larger  $n \in \mathbb{N}$ . On the other hand, [5] gives a construction of a monotone mapping  $f \in W^{1,1}(]0, 1[^n, \mathbb{R}^n)$ , which satisfies (1) for each  $\lambda < -1$  and maps a porous set onto a set of positive  $n$ -dimensional Lebesgue measure.

The proof of Theorem 1.1 uses ideas rather similar to the ones applied in the proof of Theorem 2 of [9]. The novelty in our argument comes from the multiplicity of the covering layers that we are able to provide for porous sets. This theorem remains true if we assume local integrability only in (1) and  $f \in W_{\text{loc}}^{1,1}$ . Moreover, it suffices to assume that the mapping  $f$  be pseudomonotone (see [11] for the definition).

## 2. PRELIMINARIES

We assume that our mapping  $f: \Omega \rightarrow \mathbb{R}^m$  is defined in a domain  $\Omega \subset \mathbb{R}^n$ ,  $n, m \in \mathbb{N}$ ,  $m \geq n \geq 2$ , and it is at least of class  $W^{1,1}(\Omega, \mathbb{R}^m)$ , that is, the component functions of  $f$  have integrable distributional first-order derivatives. By monotonicity we mean that  $f$  is continuous and the equalities

$$(2) \quad \sup_{\partial B} f_i = \sup_B f_i \quad \text{and} \quad \inf_{\partial B} f_i = \inf_B f_i$$

are true for every ball  $B \subset \Omega$  and each component function  $f_i$ ,  $i = 1, \dots, m$ , of the mapping  $f$ .

Let us further agree on some basic notation. We write  $B(x, r)$  for an open ball in  $\mathbb{R}^n$  centred at  $x \in \mathbb{R}^n$  and having radius  $r > 0$ . If  $B = B(x, r)$  is a ball and  $a$  is a positive number,  $aB$  denotes the ball  $aB = B(x, ar)$ . All cubes mentioned in this paper are  $n$ -dimensional cubes in  $\mathbb{R}^n$ . By  $|A|$ , we mean the  $n$ -dimensional Lebesgue measure of the set  $A \subset \mathbb{R}^n$ . We denote the diameter of a set  $A$  in a Euclidean space by  $\text{diam } A$  and the distance between two sets  $A, B \subset \mathbb{R}^n$  by  $\text{dist}(A, B)$ . The expression  $A + a$ , with  $A \subset \mathbb{R}^n$  and  $a > 0$ , stands for the set  $\{x \in \mathbb{R}^n: \text{dist}(\{x\}, A) < a\}$ . The function  $\chi_A: \mathbb{R}^m \rightarrow \{0, 1\}$  is the characteristic function of the set  $A \subset \mathbb{R}^m$ . When we write  $L = L(\cdot)$ , we mean that the number  $L \in \mathbb{R}$  depends only on the parameters listed in the parentheses. Finally, the constant  $C > 0$  may differ from occurrence to occurrence, but it depends only on the dimensions of the domain and image spaces  $n$  and  $m$ , the porosity parameter  $\delta$  (see Section 2.2) and the integrability constant  $\lambda$ .

**2.1. Measures and dimensions.** We define a dimension gauge as a non-decreasing function  $h: [0, \infty[ \rightarrow [0, \infty[$  with  $h(0) = 0$ . The only type of gauges we are going

to use is the gauge  $h_p$ ,  $p > 0$ , mentioned in the statement of Theorem 1.1. In order to make  $h_p$  a proper gauge, we redefine it as  $h_p(t) = t^n \varphi_p(t)$ , where

$$\varphi_p(t) = \begin{cases} \log^p \frac{1}{t}, & \text{if } t \in ]0, t_p]; \\ \log^p \frac{1}{t_p}, & \text{if } t > t_p; \end{cases}$$

and  $t_p < 1$  is chosen so that the function  $t^n \log^p \frac{1}{t}$  is increasing in  $t$  on  $]0, t_p[$ . Notice that  $\varphi_p$  is decreasing.

Let  $h$  be a dimension gauge. We write  $\mathcal{H}^h(A)$  for the generalized Hausdorff measure of a set  $A \subset \mathbb{R}^m$ , given by

$$\mathcal{H}^h(A) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^h(A),$$

where

$$\mathcal{H}_\delta^h(A) = \inf \left\{ \sum_{i=1}^{\infty} h(\text{diam } U_i) : A \subset \bigcup_{i=1}^{\infty} U_i, \text{diam } U_i \leq \delta \right\}.$$

We also need the generalized weighted Hausdorff content of a set  $A \subset \mathbb{R}^m$ , given by

$$\lambda_\infty^h(A) = \inf \left\{ \sum_{i=1}^{\infty} c_i h(\text{diam } U_i) : \chi_A(x) \leq \sum_{i=1}^{\infty} c_i \chi_{U_i}(x), \forall x \in \mathbb{R}^m \right\}.$$

Here,  $h$  is also a gauge function. The sequence of pairs  $(c_i, U_i)_{i=1}^\infty$ , where  $c_i \geq 0$  and  $U_i \subset \mathbb{R}^m$  for every  $i \in \mathbb{N}$ , satisfying  $\chi_A \leq \sum_{i=1}^\infty c_i \chi_{U_i}$ , is called a weighted covering of the set  $A$ , and the infimum is taken over all weighted coverings of  $A$ .

The relation to the generalized Hausdorff content is given by the following lemma, which follows from Corollary 8.2 and the proof of Theorem 9.7 of [3].

**Lemma 2.1.** *Let  $A \subset \mathbb{R}^m$  be bounded and  $h$  be a continuous gauge function satisfying  $h(t) > 0$  and  $h(2t) \leq ch(t)$  for some  $c > 0$  and each  $t > 0$ . Then  $\mathcal{H}_\infty^h(A) \leq L\lambda_\infty^h(A)$  with  $L = L(c) > 0$ .*

The upper Minkowski dimension  $\dim_{\mathcal{M}}(A)$  of a bounded set  $A \subset \mathbb{R}^n$  is defined by

$$\dim_{\mathcal{M}}(A) = \inf \left\{ s : \limsup_{\varepsilon \rightarrow 0^+} N(A, \varepsilon) \varepsilon^s = 0 \right\},$$

where  $N(A, \varepsilon)$ ,  $\varepsilon > 0$ , denotes the smallest number of balls of radius  $\varepsilon$  needed to cover  $A$ :

$$N(A, \varepsilon) = \min \left\{ k : A \subset \bigcup_{i=1}^k B(x_i, \varepsilon) \text{ for some } x_i \in \mathbb{R}^n \right\}.$$

**2.2. Porous sets and Whitney cubes.** Let  $E$  be a set in  $\mathbb{R}^n$ ,  $x \in E$  and  $r > 0$ . Put

$$\text{Por}(E, x, r) = \sup \{ \delta \geq 0 : B(y, \delta r) \subset B(x, r) \setminus E \text{ for some } y \in B(x, r) \}.$$

We call a set  $E$   $\delta$ -porous for some  $\delta \in ]0, 1/2]$ , if

$$\liminf_{r \rightarrow 0^+} \text{Por}(E, x, r) \geq \delta$$

for all  $x \in E$ . If, on the other hand, there exists a number  $r > 0$  such that  $\text{Por}(E, x, r') \geq \delta$  for all  $x \in E$  and  $r' \in ]0, r[$ , we say that the set  $E$  is  $(\delta, r)$ -porous.

By ‘‘porosity’’ in the statement of Theorem 1.1, we mean that  $E$  may be any  $\delta$ -porous set for some  $\delta > 0$ . However, due to the  $\sigma$ -additivity of the generalized Hausdorff measure, it is enough to verify the theorem for a set  $E \subset \Omega$  which is

$(\delta, r)$ -porous for some  $\delta > 0$  and  $r > 0$ . Indeed, it is obvious that once  $0 < \delta' < \delta$ , each  $\delta$ -porous set  $E \subset \mathbb{R}^2$  may be written as a countable union  $E = \bigcup_{i=1}^{\infty} E_i$  of  $(\delta', r_i)$ -porous sets  $E_i$  for some  $r_i > 0$ ,  $i = 1, 2, \dots$  [15].

It is known that each bounded  $(\delta, r)$ -porous set in  $\mathbb{R}^n$  has upper Minkowski dimension strictly less than  $n$  (see, for instance, the proof of Theorem 2 in [17]).

In order to obtain a multiple covering for a porous set, we utilize the tool of Whitney cubes. Recall that for each open subset  $A$  of  $\mathbb{R}^n$  there exist a decomposition  $A = \bigcup_{i=1}^{\infty} Q_i$ , where  $Q_i$  are closed cubes with mutually parallel sides, pairwise disjoint interiors and of side lengths  $2^k$  for some integer  $k$ , such that the relation

$$(3) \quad \frac{1}{4} \leq \frac{\text{diam } Q_i}{\text{dist}(Q_i, \partial A)} \leq 1$$

holds for all  $i = 1, 2, \dots$ . See [16] for details.

We need the following simple lemma proved in [12, Lemma 4.6], for example.

**Lemma 2.2.** *Let  $Q_0$  be a cube in  $\mathbb{R}^n$  and  $\mathcal{X}$  its Whitney decomposition, whose cubes have sides parallel to the ones of  $Q_0$ . If  $\tilde{Q} \subset Q_0$  is a cube sharing a face with part of a face of  $Q_0$ , then there exists a cube  $Q \in \mathcal{X}$ , such that  $Q \subset \tilde{Q}$  and  $\text{diam}(Q) \geq \text{diam}(\tilde{Q})/c_0$  (with  $c_0 = c_0(n) > 0$ ).*

The definition of porosity implies the following lemma.

**Lemma 2.3.** *Let  $E \subset \mathbb{R}^n$  be a closed  $(\delta, r)$ -porous set. Then there exist a countable collection  $\mathcal{B}$  of pairwise disjoint balls and constants  $C_0 = C_0(n, \delta) > 1$  and  $C_1 = C_1(n, \delta) > 1$ , such that for every  $j_0 \in \mathbb{N}$ , satisfying  $2^{-j_0} < 8r$ , and  $x \in E$ , we are able to find  $j_0$  balls  $B_1, \dots, B_{j_0}$  of  $\mathcal{B}$  with the following properties:*

- (i)  $\text{diam}(B_i) \geq 2^{-2j_0}/C_0$ ;
- (ii)  $B_i \subset B(x, 2^{-j_0})$  and  $\text{diam } B_i \leq 2^{-j_0}$ ;
- (iii)  $x \in C_1 B_i$ ;

for each  $i \in \{1, \dots, j_0\}$ .

*Proof.* Let  $\mathcal{Q}$  be a Whitney decomposition of the set  $\mathbb{R}^n \setminus E$ . For each cube  $Q \in \mathcal{Q}$ , we fix its Whitney decomposition  $\mathcal{X}_Q$ , whose cubes have sides parallel to the ones of  $Q$ . We define an auxiliary collection of cubes by

$$\mathcal{W} = \bigcup_{Q \in \mathcal{Q}} \mathcal{X}_Q.$$

Finally, the collection  $\mathcal{B}$  consists of the largest balls, contained in the cubes of  $\mathcal{W}$ .

Let  $\delta' \in ]0, \delta[$ . Fix  $x \in E$  and  $j_0 \in \mathbb{N}$  such that  $2^{-j_0} < 8r$ . Denote by  $A_k = A_k(x)$ , with  $k = j_0, j_0 + 1, \dots, 2j_0 - 1$ , the annuli  $B(x, 2^{-k}) \setminus B(x, 2^{-k-1})$ . Let us fix  $k \in \{j_0, j_0 + 1, \dots, 2j_0 - 1\}$  and prove that we are able to find a ball from  $\mathcal{B}$ , contained in  $A_k$  and having the required properties. Consider the smaller annulus  $\tilde{A}_k = (A_k \cap B(x, 7 \cdot 2^{-k}/8)) \setminus B(x, 5 \cdot 2^{-k}/8)$ , which is an annulus of width  $2^{-k}/4$ , twice smaller than that of  $A_k$ . There are two possibilities. The first case is that the annulus  $\tilde{A}_k$  contains a point  $y$  of  $E$ :  $y \in E \cap \tilde{A}_k$ . Since  $E$  is porous, there is a ball  $B \subset B(y, 2^{-k}/8) \subset A_k$  of radius  $\delta' \cdot 2^{-k}/8$ , which has empty intersection with  $E$ . If the annulus  $\tilde{A}_k$  contains no points of the set  $E$ ,  $B$  may be chosen to be any large enough ball  $B \subset \tilde{A}_k$ .

Let us take a Whitney cube  $Q \in \mathcal{Q}$  containing the centre of  $B$ . This cube cannot contain  $A_k$ ; therefore, we are able to pick the largest possible cube  $\tilde{Q} \subset (Q \cap A_k)$ ,

sharing at least one face with part of a face of  $Q$ . This cube may be  $Q$  itself, which is the worst case. If  $Q = \tilde{Q}$ , (3) implies

$$5 \operatorname{diam}(\tilde{Q}) \geq \operatorname{diam}(\tilde{Q}) + \operatorname{dist}(\tilde{Q}, E) \geq \frac{\delta' \cdot 2^{-k}}{8} \geq \frac{\delta' \cdot 2^{-2j_0}}{4}.$$

Otherwise, this inequality is even more obvious. Using Lemma 2.2, we obtain a cube  $Q_k \in \mathcal{W}$ ,  $Q_k \subset \tilde{Q}$ , of diameter  $\operatorname{diam}(Q_k) \geq \delta' \cdot 2^{-k}/40c_0 \geq \delta' \cdot 2^{-2j_0}/20c_0$ . Finally, we take the largest ball  $B_k \in \mathcal{B}$ , contained in  $Q_k$ . Thus, the property (i) is valid with  $C_0 = 20\sqrt{n}c_0/\delta'$ , while the constant  $C_1$  may be taken as  $C_1 = 4C_0$ , because  $\operatorname{diam}(4C_0B_k) \geq 2 \cdot 2^{-k}$ .  $\square$

**2.3. Oscillation on balls.** One of the important tools in estimating the diameters of the images of balls under monotone mappings is the maximal operator. Assume that  $\Omega \subset \mathbb{R}^n$  is an open cube and  $h: \Omega \rightarrow \mathbb{R}$  is a non-negative and integrable function. The maximal operator  $\mathcal{M}_\Omega$  is defined by

$$\mathcal{M}_\Omega h(x) = \sup \left\{ \int_Q h \, dx : x \in Q \subset \Omega \right\},$$

where the supremum is taken over all subcubes  $Q$  of  $\Omega$ , containing the given point  $x \in \Omega$ . Recall that the value

$$\int_A h = \frac{1}{|A|} \int_A h,$$

where  $A \subset \mathbb{R}^n$  with  $|A| < \infty$ , is called the mean integral of the function  $h$  over the set  $A$ . We need the following result on the maximal operator, proved in [1, Lemma 5.1].

**Lemma 2.4.** *Let  $\Omega \subset \mathbb{R}^n$  be an open cube and  $h: \Omega \rightarrow \mathbb{R}$  be a non-negative and integrable function. If  $A: [0, \infty[ \rightarrow [0, \infty[$  is increasing and  $\Phi(t) = A(t)t^q$  for some  $q > 1$ , then there exists a constant  $L = L(n, q) > 0$ , such that*

$$\int_\Omega \Phi(\mathcal{M}_\Omega h) \leq L \int_\Omega \Phi(Lh).$$

The second lemma was basically proved in Theorem 4.3 of [10] for the usual Hardy-Littlewood maximal operator. Since our setting is a little bit different, we would like to reprove it for convenience.

**Lemma 2.5.** *Let  $p \in ]n - 1, n[$  and  $C_1 > 1$  be some positive parameters. There exists a number  $L = L(n, m, p, C_1) > 0$ , such that once  $\Omega \subset \mathbb{R}^n$  is an open cube and  $f \in W^{1,p'}(\Omega, \mathbb{R}^m)$  is a monotone mapping with  $p' > p$ , we have*

$$\operatorname{diam}(f(C_1B)) \leq L \operatorname{diam} B \left( \int_B \mathcal{M}_\Omega (|Df|^p) \right)^{\frac{1}{p}}$$

for each ball  $B \in \mathbb{R}^n$ , satisfying  $2C_1B \subset \Omega$ .

*Proof.* In the proof of this lemma, we allow  $C$  to depend on  $p$ . Let  $B$  be a ball such that  $2C_1B \subset \Omega$  and  $t \in [1, 2]$ . Applying the inequalities (2) to each of the component functions of the monotone mapping  $f$ , we easily obtain the estimate

$$\operatorname{diam} f(C_1B) \leq \operatorname{diam} f(tC_1B) \leq \sqrt{m} \operatorname{diam} f(\partial(tC_1B)).$$

By the Sobolev imbedding theorem on spheres, we further observe

$$\begin{aligned} t^{n-p-1} \operatorname{diam}^p f(C_1 B) &\leq C t^{n-p-1} \operatorname{diam}^p f(\partial(tC_1 B)) \\ &\leq C(C_1 \operatorname{diam} B)^{p-n+1} \int_{\partial(tC_1 B)} |Df|^p \end{aligned}$$

for  $C = C(n, m, p) > 0$  and  $\mathcal{L}^1$ -a.e.  $t \in [1, 2]$ . We integrate both parts over  $t$  to obtain

$$\operatorname{diam}^p(f(C_1 B)) \leq C(C_1 \operatorname{diam} B)^{p-n} \int_{2C_1 B} |Df|^p.$$

Pick the smallest cube  $Q_B$ , containing  $2C_1 B$  and having sides parallel to the sides of  $\Omega$ . This cube is contained in  $\Omega$  and has the ball  $B$  as its subset. We proceed, obtaining

$$\begin{aligned} \operatorname{diam}^p(f(C_1 B)) &\leq C(C_1 \operatorname{diam} B)^{p-n} \int_{Q_B} |Df|^p = C(C_1 \operatorname{diam} B)^p \int_{Q_B} |Df|^p \\ &\leq CC_1^p \operatorname{diam}^p B \mathcal{M}_\Omega(|Df|^p)(x) \end{aligned}$$

for each  $x \in B$ . We complete the proof taking the mean integral over  $x \in B$  of both sides of the last inequality.  $\square$

### 3. PROOF OF THEOREM 1.1

We are now ready to start with the proof of Theorem 1.1. The proof differs slightly for negative and positive values of  $\lambda$ . We give most of the details for both cases. Taking a Whitney decomposition of  $\Omega$ , enlarging the cubes suitably to make them open, considering the intersection of  $E$  with the original Whitney cubes and using the  $\sigma$ -additivity of the generalized Hausdorff measure, we may assume that  $\Omega$  is an open cube and that the closure of  $E$  is contained in  $\Omega$ . Moreover, since the closure of a  $(\delta, r)$ -porous set is  $(\delta, r)$ -porous, we may assume that  $E$  is a closed  $(\delta, r)$ -porous set for some  $\delta \in ]0, 1/2]$  and  $r > 0$ . Let us prove that  $\mathcal{H}^{h_{\lambda+1}}(f(E)) = 0$ .

Fix  $p = n - 1/2$ . Obviously,  $f \in W^{1,p'}(\Omega, \mathbb{R}^m)$  for each  $p' \in ]p, n[$ . Let us take a function  $\Psi$  so that  $\Psi$  is continuous, non-negative, strictly increasing and convex on  $[0, \infty[$  and  $\Psi(t) = t^{n/p} \log^\lambda t > 1$  for  $t \geq t_0$ , where the number  $t_0 > 1$ , depending on  $n$  and  $\lambda$ , is fixed suitably. We also require  $\log t \geq 2 \log \log^{|\lambda|p/n} t > 0$  for all  $t \geq t_0$ . Put  $\Phi(y) = y^{p/n} \log^{-\lambda p/n} y$  for each  $y \geq y_0 = \Psi(t_0)$ . When  $t \geq t_0$ , we have

$$\Phi(\Psi(t)) = \left(\frac{p}{n}\right)^{\frac{\lambda p}{n}} \frac{\log^{\frac{\lambda p}{n}} t}{\log^{\frac{\lambda p}{n}}(t \log^{\frac{\lambda p}{n}} t)} t \geq \left(\frac{p}{n}\right)^{\frac{\lambda p}{n}} \left(\frac{1}{2}\right)^{\frac{|\lambda|p}{n}} t,$$

which yields  $\Psi^{-1}(y) \leq C\Phi(y)$  for  $y \geq y_0$ . Clearly, the assumed integrability (1) implies that  $\Psi(|Df|^p) \in L^1(\Omega)$ . Moreover, by Lemma 2.4, we obtain

$$(4) \quad M := \int_{\Omega} \Psi(\mathcal{M}_\Omega |Df|^p) < \infty.$$

Finally, define  $M_1 = \max\{1, M\}$ .

Fix a positive number  $\eta$ . Choose an integer  $j_0 \in \mathbb{N}$  with  $2^{-j_0} < 8r$ , large enough to guarantee the inclusion  $E + (C_1 + 1)2^{-j_0} \subset \Omega$ . This makes it possible to apply

Lemma 2.5 to all the balls, given by Lemma 2.3. In addition,  $2^{-j_0}$  should be less than or equal to  $t_{\lambda+1}$  and the estimate

$$\int_{E+2^{-j_0}} \Psi(\mathcal{M}_\Omega |Df|^p) < \eta$$

should be valid. We refine the choice of  $j_0$  later.

When  $\lambda > 0$ , we use the fact that the upper Minkowski dimension of the set  $E$  is strictly less than  $n$ . This fact gives a constant  $C > 0$ , depending only on  $n$ , a number  $\varepsilon \in ]0, n[$  and a radius  $r_1 > 0$ , such that  $|E + r'| < C(r')^\varepsilon$  for each  $r' \in ]0, r_1[$ . If necessary, we redefine  $j_0$  so that  $2^{-j_0} < r_1$ . We also fix a positive parameter  $\alpha \in ]n - \varepsilon/2, n[$ . Then  $\sigma := 2\alpha + \varepsilon - 2n$  is positive and we are able to make  $j_0$  large enough to ensure the inequality  $j_0^\lambda 2^{-\sigma j_0} < \eta$ . In the case of non-positive  $\lambda$ 's, it is enough to require  $|E + 2^{-j_0}| < \eta$ .

We apply Lemma 2.3 to the set  $E$ , obtaining a collection of balls  $\mathcal{B}$ . In what follows, the constants  $C_0$  and  $C_1$  are the ones given by Lemma 2.3. Let  $\mathcal{B}_0 \subset \mathcal{B}$  be the subcollection of  $\mathcal{B}$ , consisting of those balls only, which are contained in  $E + 2^{-j_0}$  and whose diameters lie in the range  $[2^{-2j_0}/C_0, 2^{-j_0}]$ . Notice that  $(1/j_0, f(C_1 B))_{B \in \mathcal{B}_0}$  is a weighted covering of the set  $f(E)$ . Let us estimate the generalized weighted Hausdorff content of this image set with the help of this covering. Similarly to what is done in [9], we consider two classes of balls:

$$\mathcal{B}_1 = \{B \in \mathcal{B}_0 : \text{diam } f(C_1 B) \leq \text{diam } B\}$$

and

$$\mathcal{B}_2 = \{B \in \mathcal{B}_0 : \text{diam } f(C_1 B) > \text{diam } B\}.$$

Hence, we obtain

$$\begin{aligned} (5) \quad \lambda_\infty^{h_{\lambda+1}}(f(E)) &\leq \frac{1}{j_0} \sum_{B \in \mathcal{B}_0} \text{diam}^n(f(C_1 B)) \varphi_{\lambda+1}(\text{diam}(f(C_1 B))) \\ &\leq \frac{1}{j_0} \sum_{B \in \mathcal{B}_1} \text{diam}^n B \log^{\lambda+1} \frac{1}{\text{diam } B} \\ &\quad + \frac{1}{j_0} \sum_{B \in \mathcal{B}_2} \text{diam}^n(f(C_1 B)) \log^{\lambda+1} \frac{1}{\text{diam } B} \\ &\leq C \sum_{B \in \mathcal{B}_1} \text{diam}^n B \log^\lambda \frac{1}{\text{diam } B} + C \sum_{B \in \mathcal{B}_2} \text{diam}^n(f(C_1 B)) \log^\lambda \frac{1}{\text{diam } B}. \end{aligned}$$

Let us estimate the first sum on the right-hand side in the case that  $\lambda > 0$ . We consider a larger sum for our future purposes:

$$\begin{aligned} (6) \quad \sum_{B \in \mathcal{B}_0} \text{diam}^n B \log^\lambda \frac{1}{\text{diam } B} &\leq \sum_{B \in \mathcal{B}_0} \text{diam}^n B \text{diam}^{\alpha-n} B \log^\lambda \frac{1}{\text{diam } B} \\ &\leq (C_0^n \log^\lambda(2C_0)) 2^\lambda j_0^\lambda 2^{2j_0(n-\alpha)} \sum_{B \in \mathcal{B}_0} \text{diam}^n B \\ &\leq C j_0^\lambda 2^{2j_0(n-\alpha)} |E + 2^{-j_0}| \\ &\leq C j_0^\lambda 2^{-\sigma j_0} < C\eta. \end{aligned}$$

When  $-1 \leq \lambda \leq 0$ , we do not have to consider the larger sum and the estimations become much simpler, because the logarithmic factor is bounded from above by  $\log^\lambda 2$ . Then the whole sum becomes less than or equal to  $C|E + 2^{-j_0}| < C\eta$ .

In order to estimate the second sum in (5), we apply Lemma 2.5 to each ball  $B \in \mathcal{B}_2$  and use Jensen's inequality. We obtain

$$\begin{aligned} \text{diam } f(C_1 B) &\leq C \text{diam } B \left( \int_B \mathcal{M}_\Omega(|Df|^p) \right)^{\frac{1}{p}} \\ &\leq C \text{diam } B \left( \Psi^{-1} \left( \int_B \Psi(\mathcal{M}_\Omega(|Df|^p)) \right) \right)^{\frac{1}{p}} \\ &\leq C \text{diam } B \left( \int_B \Psi(\mathcal{M}_\Omega(|Df|^p)) \right)^{\frac{1}{n}} \log^{-\frac{\lambda}{n}} \left( \int_B \Psi(\mathcal{M}_\Omega(|Df|^p)) \right), \end{aligned}$$

once  $M_B := \int_B \Psi(\mathcal{M}_\Omega(|Df|^p)) \geq y_0$ ; otherwise,

$$\text{diam } f(C_1 B) \leq C (\Psi^{-1}(y_0))^{1/p} \text{diam } B.$$

Therefore,

$$\begin{aligned} (7) \quad &\sum_{B \in \mathcal{B}_2} \text{diam}^n(f(C_1 B)) \log^\lambda \frac{1}{\text{diam } B} \\ &\leq C (\Psi^{-1}(y_0))^{\frac{n}{p}} \sum_{\substack{B \in \mathcal{B}_2 \\ M_B < y_0}} \text{diam}^n B \log^\lambda \frac{1}{\text{diam } B} \\ &\quad + C \sum_{\substack{B \in \mathcal{B}_2 \\ M_B \geq y_0}} \int_B \Psi(\mathcal{M}_\Omega(|Df|^p)) \log^{-\lambda}(M_B) \log^\lambda \frac{1}{\text{diam } B}. \end{aligned}$$

The first sum appearing here was estimated in (6). The estimations for the remaining sum differ for positive and negative values of  $\lambda$  again.

*Case  $\lambda \in [-1, 0]$ .* In the first case, the estimations are rather straightforward. Denoting by  $\omega_n$  the  $n$ -dimensional Lebesgue measure of the unit ball in  $\mathbb{R}^n$ , we obtain

$$\begin{aligned} (8) \quad &\sum_{\substack{B \in \mathcal{B}_2 \\ M_B \geq y_0}} \int_B \Psi(\mathcal{M}_\Omega(|Df|^p)) \log^{-\lambda}(M_B) \log^\lambda \frac{1}{\text{diam } B} \\ &\leq \sum_{\substack{B \in \mathcal{B}_2 \\ M_B \geq y_0}} \int_B \Psi(\mathcal{M}_\Omega(|Df|^p)) \log^{-\lambda} \frac{2^n M_1}{\omega_n \text{diam}^n B} \log^\lambda \frac{1}{\text{diam } B} \\ &\leq \frac{(2n j_0)^{-\lambda} \log^{-\lambda} \left( \frac{2^{n+1} M_1 C_0}{\omega_n} \right)}{j_0^{-\lambda} \log^{-\lambda} 2} \int_{E+2^{-j_0}} \Psi(\mathcal{M}_\Omega(|Df|^p)) < T\eta \end{aligned}$$

with  $T = T(n, \delta, \lambda, M_1) > 0$ .



Case  $\lambda > 0$ . In order to deal with this case, we again consider two possibilities. If  $B \in \mathcal{B}_2$  is such that  $M_B \in [y_0, \text{diam}^{\alpha-n} B[$ , then  $\int_B \Psi(\mathcal{M}_\Omega(|Df|^p)) \leq C \text{diam}^\alpha B$ ; otherwise, when  $M_B \geq \max\{y_0, \text{diam}^{\alpha-n} B\}$ , we have  $\log^{-\lambda}(M_B) \leq \frac{1}{(n-\alpha)^\lambda} \log^{-\lambda} \frac{1}{\text{diam} B}$ . Hence, we observe

$$(9) \quad \sum_{\substack{B \in \mathcal{B}_2 \\ M_B \geq y_0}} \int_B \Psi(\mathcal{M}_\Omega(|Df|^p)) \log^{-\lambda}(M_B) \log^\lambda \frac{1}{\text{diam} B} \\ \leq C \log^{-\lambda}(y_0) \sum_{\substack{B \in \mathcal{B}_2 \\ y_0 \leq M_B < \text{diam}^{\alpha-2} B}} \text{diam}^\alpha B \log^\lambda \frac{1}{\text{diam} B} \\ + \frac{1}{(n-\alpha)^\lambda} \int_{E+2^{-j_0}} \Psi(\mathcal{M}_\Omega(|Df|^p)) < \left( C + \frac{1}{(n-\alpha)^\lambda} \right) \eta$$

by (6).

Finally, collecting together (5), (6), (7), (8) and (9), we obtain the estimate  $\lambda_\infty^{h_\lambda+1}(f(E)) \leq T_1 \eta$ , where the constant  $T_1 > 0$  depends only on  $n, m, \delta, \lambda$ , the upper Minkowski dimension of the set  $E$  and the value of the integral (4). Since  $\eta$  was an arbitrary number, we have  $\lambda_\infty^{h_\lambda+1}(f(E)) = 0$ , which yields  $\mathcal{H}_\infty^{h_\lambda+1}(f(E)) = 0$  (see Lemma 2.1). This in turn implies  $\mathcal{H}^{h_\lambda+1}(f(E)) = 0$ .

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