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RESOLUTIONS OF IDEALS OF FAT POINTS WITH SUPPORT IN A HYPERPLANE

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ABSTRACT. Let Z' be a fat point subscheme of $\mathbb{P}^d$, and let x_0 be a linear form such that some power of x_0 vanishes on Z' (i.e., the support of Z' lies in the hyperplane H defined by $x_0 = 0$, regarded as $\mathbb{P}^{d-1}$). Let $Z(i) = H \cap Z'(i)$, where $Z'(i)$ is the subscheme of $\mathbb{P}^d$ residual to x_0^i; note that $Z(i)$ is a fat points subscheme of $\mathbb{P}^{d-1} = H$. In this paper we give a graded free resolution of the ideal $I(Z')$ over $R' = K[\mathbb{P}^d]$, in terms of the graded minimal free resolutions of the ideals $I(Z(i)) \subset R = K[\mathbb{P}^{d-1}]$. We also give a criterion for when the resolution is minimal, and we show that this criterion always holds if $\mathrm{char}(K) = 0$.

1. INTRODUCTION

Let $R = K[\mathbb{P}^{d-1}] = K[x_1, \ldots, x_d]$ and $R' = K[\mathbb{P}^d] = K[x_0, x_1, \ldots, x_d]$ be the homogeneous coordinate rings of projective space, over an algebraically closed field K of arbitrary characteristic. We regard $\mathbb{P}^{d-1}$ as the hyperplane $x_0 = 0$ in $\mathbb{P}^d$. We will denote homogeneous components by subscripts; thus, for example, R_1 denotes the K-vector space of homogeneous linear forms in $K[\mathbb{P}^{d-1}]$.

Given points $p_1, \ldots, p_r \in \mathbb{P}^{d-1}$ and nonnegative integers m_i, we have the fat point subschemes $Z = m_1p_1 + \cdots + m_rp_r \subset \mathbb{P}^{d-1}$ (so $I(Z) \subset R$) and $Z' = m_1p_1 + \cdots + m_rp_r \subset \mathbb{P}^d$ (so $I(Z') \subset R'$). In particular, the ideal $I(Z)$ is $I_1^{m_1} \cap \cdots \cap I_r^{m_r}$, where I_j is the ideal generated by all forms in R vanishing at p_j, and $I(Z')$ is $(I')_1^{m_1} \cap \cdots \cap (I')_r^{m_r}$, where $(I')_j$ is the ideal generated by all forms in R' vanishing at p_j.

We also have the obvious canonical inclusion $R \subset R'$, so we can regard ideals in R as R-submodules of R'. Now define $Z'_{m-i} = (m_1 - i)_+p_1 + \cdots + (m_r - i)_+p_r \subset \mathbb{P}^d$ and $Z_{m-i} = (m_1 - i)_+p_1 + \cdots + (m_r - i)_+p_r \subset \mathbb{P}^{d-1}$ for each $0 \le i \le m = \max\{m_1, \ldots, m_r\}$, where for any integer n we define $n_+ = \max\{n, 0\}$. Note that $Z' = Z'_m$ and $Z = Z_m$, and that $\emptyset = Z_0 \subset Z_1 \subset \cdots \subset Z_m = Z$. Alternatively, let $Z'(i)$ denote the subscheme of $\mathbb{P}^d$ defined by the ideal $I(Z') : (x_0^i)$ (and thus residual to x_0^i); then $Z'(i) = Z'_{m-i}$ and Z_{m-i} is the subscheme $Z'(i) \cap \mathbb{P}^{d-1}$ of $\mathbb{P}^{d-1}$.

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In this paper, we construct a graded free resolution of $I(Z')$ over R', given graded minimal free resolutions of each $I(Z_i)$ over R. Under the condition that $I(Z_{i+1}) \subset R_1 I(Z_i)$ for each i, we show that the constructed resolution is minimal. We also show that this condition always holds if $\mathrm{char}(K) = 0$. In fact, we do not know any examples where the condition does not hold.

As a corollary we obtain a result about the Poincaré polynomial of $I(Z')$. (Recall that the Poincaré polynomial encodes the Betti numbers of a resolution. Given a subscheme $W \subset \mathbb{P}^n$ and its ideal $I(W) \subset A = K[\mathbb{P}^n]$, we have its minimal free resolution $0 \to F_t \to \cdots \to F_1 \to F_0 \to I(W) \to 0$. The Poincaré polynomial $P(W)$ is defined as follows. Each syzygy module F_i is a free graded A-module of the form $F_j = \bigoplus_i A(-i)^{a_{ij}}$, and we take

$$P(W) = \sum_{ij} a_{ij} T^i X^j \in \mathbf{Z}[X, T].$$

So, for example, the empty subscheme $W = \emptyset$, whose ideal is thus (1), has polynomial $P(W) = 1$, and if $W = p$ is a single reduced point in $\mathbb{P}^3$, then $P(W) = 3T + 3XT^2 + X^2T^3$.)

Theorem 1.1. *Let $Z' = m_1 p_1 + \cdots + m_r p_r \subset \mathbb{P}^d$, where the points p_i lie in a hyperplane of $\mathbb{P}^d$. If Z_i is defined as above and $I(Z_{i+1}) \subset R_1 I(Z_i)$ for each i, then*

$$P(Z') = (1 + XT) \left(\sum_{0 < i \leq m} T^{m-i} P(Z_i) \right) + T^m,$$

where m is the maximum of the multiplicities $m_1, \ldots, m_r$.

We give our resolution construction in Section 2. To most usefully apply our construction, we also need examples in which the condition $I(Z_{i+1}) \subset R_1 I(Z_i)$ holds. More generally, given $p_i \in \mathbb{P}^d$ and fat points $Z = m'_1 p_1 + \ldots + m'_r p_r$ and $Y = m_1 p_1 + \ldots + m_r p_r$ with $m'_i > m_i$ for all i whenever $m'_i > 0$, it seems possible that $I(Z) \subset R_1 I(Y)$ always holds. We do not know of any counterexamples, and we show in Section 3 various situations, such as whenever $\mathrm{char}(K) = 0$, where this condition does hold. We also discuss various examples in Section 3. In particular, our construction allows us to determine the resolution of the ideal of two fat points in any projective space, recovering some results (but not the explicit formulas) of [FL] and [V]. Since it is easy to give the resolution of any number of fat points in $\mathbb{P}^1$, our result also allows us, among other cases, to determine the resolution for any number of fat points in any projective space if the points lie on a line.

One can speculate on possible generalizations. In one direction, giving a resolution of an ideal $I \subset R'$ is equivalent to giving a resolution for R'/I, and in the cases we study here the annihilator of R'/I contains a power of a linear form. Perhaps our approach can be applied in other cases of modules over a polynomial ring annihilated by a power of a linear form. But even for cyclic modules it is unclear what kind of criterion to expect for resolutions to be minimal. Since our immediate interest is minimal resolutions of ideals of fat points, we have not pursued such questions here. In another direction, one can ask to relax the condition of considering points that lie only on a hyperplane. However, this would seem to change the problem in a fundamental way, since we implicitly use the fact that R' modulo a linear form is not only a quotient of R' but also a subring of R'.

2. The construction

We will need to refer to the following two elementary results.

Lemma 2.1. *Let $h : M \to N$ be a (not necessarily graded) homomorphism of graded R-modules. Let F and G be free modules over R with surjective R-homomorphisms $\alpha : F \to M$ and $\beta : G \to N$. Then there is an R-homomorphism $h_0 : F \to G$ such that $\beta h_0 = h\alpha$, where h_0 is homogeneous of degree t if h is homogeneous of degree t. If, furthermore, $h(M) \subset R_1N$, then, in addition, h_0 can be chosen such that $h_0(F) \subset R_1G$.*

Proof. The first part is clear, since F is free, so assume $h(M) \subset R_1N$ and consider $B : G^d \to R_1N \subset N$, where $B : (c_1, \dots, c_d) \mapsto x_1\beta(c_1) + \cdots + x_d\beta(c_d)$. Clearly, B is surjective, so we can lift $h\alpha : F \to M \to R_1N$ to $h' : F \to G^d$. We also have the canonical map $\gamma : G^d \to G$ (in which $(c_1, \dots, c_d) \mapsto x_1c_1 + \cdots + x_dc_d$), and $\beta\gamma = B$, and hence $\beta\gamma h' = Bh'$. If we take $h_0 = \gamma h'$, then $\beta h_0 = Bh' = h\alpha$ as desired, and $h_0(F) \subset \gamma(G^d) = R_1G$. □

By recursively applying the previous lemma, we obtain:

Corollary 2.2. *Let $h : M \to N$ be a (not necessarily graded) homomorphism of graded R-modules, with $F_\bullet$ and $G_\bullet$ free resolutions over R of M and N, respectively. Then there are R-homomorphisms $h_j : F_j \to G_j$, $j \geq 0$, compatible with the differential morphisms of the resolutions, where each h_j is homogeneous of degree t if h is homogeneous of degree t. Moreover, if $h(M) \subset R_1N$, then the maps h_j can be chosen such that $h_j(F_j) \subset R_1G_j$ for every j.*

Now consider $\emptyset = Z_0 \subset \cdots \subset Z_m = Z \subset \mathbb{P}^{d-1}$ and $Z' \subset \mathbb{P}^d$ as in the introduction. Given minimal graded free resolutions (over R) for each $I(Z_i)$, we now construct a graded resolution (over R') for $I(Z')$. We will use the following notation: the graded free modules in the resolution of $I(Z_i)$ will be denoted $F_{i,j}$ (so $F_{i,0}$ is the free R-module on the generators of $I(Z_i)$, with the suitable shifts; $F_{i,1}$ the free R-module on the first syzygies of $I(Z_i)$, with the suitable shifts; etc.). The free generators for $F_{i,j}$ will be denoted $s_{k,i,j}$, indexed by k. The graded resolution differential $F_{i,j+1} \to F_{i,j}$ will be denoted $\phi_{i,j+1}$. We denote the augmentation maps $F_{i,0} \to I(Z_i)$ by $\phi_{i,0}$.

We will also need the maps $f_{i+1,j} : F_{i+1,j} \to F_{i,j}$ guaranteed by Corollary 2.2, where, in the notation of the corollary, $h : M \to N$ is the inclusion $M = I(Z_{i+1}) \subset I(Z_i) = N$, and $f_{i+1,j} = h_j$. By the corollary, each $f_{i+1,j}$ is homogeneous of degree 0 and $f_{i+1,j}\phi_{i+1,j+1} = \phi_{i,j+1}f_{i+1,j+1}$, for $j \geq 0$, and $\phi_{i+1,0} = \phi_{i,0}f_{i+1,0}$.

We now construct a resolution of $I(Z')$ of the form

$$\cdots \overset{\phi'_{j+2}}{\to} F'_{j+1} \overset{\phi'_{j+1}}{\to} F'_j \overset{\phi'_j}{\to} \cdots \overset{\phi'_1}{\to} F'_0 \overset{\phi'_0}{\to} I(Z') \to 0.$$

To avoid repeatedly having to indicate certain shifts explicitly, we denote $F_{i,j}(-(m-i)) \otimes_R R'$ by $F'_{i,j}$. Each map $\phi_{i,j} : F_{i,j} \to F_{i,j-1}$ induces an obvious map $F_{i,j}(-(m-i)) \to F_{i,j-1}(-(m-i))$ which extends to give a map $\phi'_{i,j} : F'_{i,j} \to F'_{i,j-1}$; i.e., after accounting for the shift, $\phi'_{i,j}$ is just $\phi_{i,j} \otimes_R \mathrm{id}_{R'}$. Similarly, $f'_{i+1,j} : F'_{i+1,j}(-1) \to F'_{i,j}$ denotes the map coming from $f_{i+1,j}$. Now take

$$F'_0 = \bigoplus_{i=0}^{m} F'_{i,0}$$

and, for $j \geq 1$,

$$F'_j = F'_{0,j} \oplus \left(\bigoplus_{i=1}^{m} \left(F'_{i,j} \oplus F'_{i,j-1}(-1) \right) \right)$$

(note $F'_{0,j} = 0$ for $j \geq 1$; we include it in F'_j for consistency). Define the augmentation map as

$$\phi'_0(s_{k,i,0} \otimes 1) = \phi'_{i,0}(s_{k,i,0} \otimes x_0^{m-i})$$

and differentials as

$$\phi'_1(s_{k,i,0} \otimes 1) = s_{k,i,0} \otimes x_0 - f'_{i,0}(s_{k,i,0} \otimes 1),$$

and, for $j \geq 1$,

$$\phi'_j(s_{k,i,j} \otimes 1) = \phi'_{i,j}(s_{k,i,j} \otimes 1)$$

and

$$\phi'_{j+1}(s_{k,i,j} \otimes 1) = s_{k,i,j} \otimes x_0 - f'_{i,j}(s_{k,i,j} \otimes 1) - \phi'_{i,j}(s_{k,i,j} \otimes 1).$$

Note the ambiguity of whether an element $s_{k,i,j-1} \otimes 1$ lies in F'_j or F'_{j-1}. We will resolve this ambiguity either by an explicit declaration, such as $s_{k,i,j-1} \otimes 1 \in F'_j$, or implicitly, as in $\phi'_j(s_{k,i,j-1} \otimes 1)$ (keeping in mind that the differential ϕ'_j is a mapping defined on F'_j).

The modules F'_j and maps ϕ'_j can also be described in terms of mapping cones. In fact, we have the mapping of complexes $f'_{i+1} : F'_{i+1,\bullet}(-1) \to F'_{i,\bullet}$, where for each j the map $F'_{i+1,j}(-1) \to F'_{i,j}$ is given by $f'_{i+1,j}$. Thus this mapping of complexes is ultimately induced by the inclusion $Z_i \subset Z_{i+1}$. We also have the mapping of complexes $\mu_i : F'_{i,\bullet}(-1) \to F'_{i,\bullet}$, where $\mu_i(a) = x_0 a$ is given by multiplying by x_0. Then, for $j \geq 0$, the modules F'_j and maps ϕ'_{j+1} can be seen as coming from amalgamating the mapping cones of μ_m, f'_m, $\cdots$, μ_1, f'_1, as shown in Figure 1. Any two consecutive rows give a mapping cone, either for some μ or for some f'. The direct sum of the modules in column j gives F'_j. The differential $\phi'_j : F'_j \to F'_{j-1}$ is the direct sum of the maps between columns.

Lemma 2.3. *The differentials ϕ'_j, $j > 0$, form a complex and $\phi'_0 \phi'_1 = 0$; i.e., $\phi'_j \phi'_{j+1} = 0$ for all $j \geq 0$.*

Proof. We have

$$\phi'_0 \phi'_1(s_{k,i,1} \otimes 1) = \phi'_0(\phi_{i,1}(s_{k,i,1}) \otimes 1) = \phi_{i,0}(\phi_{i,1}(s_{k,i,1})) \otimes x_0^{m-i},$$

which equals 0 since $\phi_{i,j}\phi_{i,j+1} = 0$, and we have

$$\begin{aligned} \phi'_0 \phi'_1(s_{k,i,0} \otimes 1) &= \phi'_0(s_{k,i,0} \otimes x_0 - f_{i,0}(s_{k,i,0}) \otimes 1) \\ &= \phi_{i,0}(s_{k,i,0}) \otimes x_0^{m-i+1} - \phi_{i-1,0}(f_{i,0}(s_{k,i,0})) \otimes x_0^{m-i+1}, \end{aligned}$$

which equals 0 since $\phi_{i,0} = \phi_{i-1,0} f_{i,0}$. Similarly, for $j > 0$, we have

$$\phi'_j \phi'_{j+1}(s_{k,i,j+1} \otimes 1) = \phi'_j(\phi_{i,j+1}(s_{k,i,j+1}) \otimes 1) = \phi_{i,j}(\phi_{i,j+1}(s_{k,i,j+1})) \otimes 1 = 0$$

and

$$\begin{aligned} \phi'_j \phi'_{j+1}(s_{k,i,j} \otimes 1) &= \phi'_j(s_{k,i,j} \otimes x_0 - f_{i,j}(s_{k,i,j}) \otimes 1 - \phi_{i,j}(s_{k,i,j}) \otimes 1) \\ &= \phi_{i,j}(s_{k,i,j}) \otimes x_0 - \phi_{i-1,j}(f_{i,j}(s_{k,i,j})) \otimes 1 \\ &\quad -(\phi_{i,j}(s_{k,i,j}) \otimes x_0 - f_{i,j-1}(\phi_{i,j}(s_{k,i,j})) \otimes 1 - \phi_{i,j-1}\phi_{i,j}(s_{k,i,j}) \otimes 1), \end{aligned}$$

which equals 0 since $\phi_{i-1,j} f_{i,j} = f_{i,j-1} \phi_{i,j}$. □

$$\begin{array}{cccccccc}
\to & F'_{m,3} & \overset{\phi'_{m,3}}{\to} & F'_{m,2} & \overset{\phi'_{m,2}}{\to} & F'_{m,1} & \overset{\phi'_{m,1}}{\to} & F'_{m,0} \\
 & & {\scriptstyle \mu_m} \nearrow & & {\scriptstyle \mu_m} \nearrow & & {\scriptstyle \mu_m} \nearrow & \\
\to & F'_{m,2}(-1) & \overset{-\phi'_{m,2}}{\to} & F'_{m,1}(-1) & \overset{-\phi'_{m,1}}{\to} & F'_{m,0}(-1) & & \\
 & & {\scriptstyle -f'_{m,2}} \searrow & & {\scriptstyle -f'_{m,1}} \searrow & & {\scriptstyle -f'_{m,0}} \searrow & \\
\to & F'_{m-1,3} & \overset{\phi'_{m-1,3}}{\to} & F'_{m-1,2} & \overset{\phi'_{m-1,2}}{\to} & F'_{m-1,1} & \overset{\phi'_{m-1,1}}{\to} & F'_{m-1,0} \\
 & & {\scriptstyle \mu_{m-1}} \nearrow & & {\scriptstyle \mu_{m-1}} \nearrow & & {\scriptstyle \mu_{m-1}} \nearrow & \\
\to & F'_{m-1,2}(-1) & \overset{-\phi'_{m-1,2}}{\to} & F'_{m-1,1}(-1) & \overset{-\phi'_{m-1,1}}{\to} & F'_{m-1,0}(-1) & & \\
 & & {\scriptstyle -f'_{m-1,2}} \searrow & & {\scriptstyle -f'_{m-1,1}} \searrow & & {\scriptstyle -f'_{m-1,0}} \searrow & \\
 & & \cdots & & \cdots & & \cdots & \\
 & & {\scriptstyle \mu_1} \nearrow & & {\scriptstyle \mu_1} \nearrow & & {\scriptstyle \mu_1} \nearrow & \\
\to & F'_{1,2}(-1) & \overset{-\phi'_{1,2}}{\to} & F'_{1,1}(-1) & \overset{-\phi'_{1,1}}{\to} & F'_{1,0}(-1) & & \\
 & & {\scriptstyle -f'_{1,2}} \searrow & & {\scriptstyle -f'_{1,1}} \searrow & & {\scriptstyle -f'_{1,0}} \searrow & \\
\to & F'_{0,3} = 0 & \overset{\phi'_{0,3}}{\to} & F'_{0,2} = 0 & \overset{\phi'_{0,2}}{\to} & F'_{0,1} = 0 & \overset{\phi'_{0,1}}{\to} & F'_{0,0}
\end{array}$$

Figure 1.

The next result is that this complex gives a resolution. This means first that generators for $I(Z')$ are given by taking for each i the generators of $I(Z_i)$ times x_0^{m-i}, and second that syzygies for $I(Z')$ are of two types. One type comprises the obvious syzygies coming from each of the ideals $I(Z_i)$ individually, which give syzygies of $I(Z')$ based on the fact that $x_0^{m-i}I(Z_i)_t \subset I(Z')_{t+m-i}$. The other type comprises syzygies between elements of $I(Z_{i+1})$ and elements of $I(Z_i)$. (These are easy to see too. Not only do we have $x_0^{m-i}I(Z_i)_t \subset I(Z')_{t+m-i}$, but given a generator g of $I(Z_{i+1})$ of degree t, we get an element $a = x_0^{m-i-1}g \in I(Z')_{t+m-i-1}$. Now $g \in I(Z_{i+1})_t \subset I(Z_i)_t$, so we also get an element $b = x_0^{m-i}g \in I(Z_i)_{t+m-i} \subset I(Z')_{t+m-i}$, and of course $x_0a = b$, which gives a syzygy of the second type.)

Lemma 2.4. *The complex $F'_\bullet$ gives a resolution of $I(Z')$.*

Proof. We must check that the image of ϕ'_0 is $I(Z')$ and, for all $j \geq 1$, that the image of ϕ'_j is the kernel of ϕ'_{j-1}. Let $f \in I(Z')$. We may write $f = x_0g + h$, where no term of h is divisible by x_0. By restricting to the hyperplane $x_0 = 0$, we see that $h \in I(Z_m) \subset I(Z')$, hence that $x_0g \in I(Z')$, and so $g \in I(Z'_{m-1})$. If $m = 1$, then $g \in I(Z'_0) = R'$, so $f \in x_0^1I(Z_0) + I(Z_1)$. It follows that $I(Z') = x_0I(Z_0) + I(Z_1)$. If $m > 1$, it follows by induction on m that $I(Z') = x_0^mI(Z_0) + \cdots + x_0^1I(Z_{m-1}) +$

$I(Z_m)$, and since the image of $\phi_{i,0}$ is $I(Z_i)$, it follows from the definition that ϕ'_0 maps onto $I(Z')$, as required.

Now suppose $\phi'_0(f) = 0$. Say $f \in F'_{0,0} = R'$. Then $\phi'_0(f) = x_0^m f$, so $f = 0$. Now induct on i; say $f \in F'_{0,0} \oplus \cdots \oplus F'_{i,0}$. We can write $f = g + a$, where $g \in F'_{0,0} \oplus \cdots \oplus F'_{i-1,0}$ and $a \in F'_{i,0}$. Write $a = c + x_0 d$, where c is the sum of the terms of a not involving x_0. Then $0 = \phi'_0(f) = \phi'_0(g) + x_0^{m-i+1}\phi'_{i,0}(d) + x_0^{m-i}\phi'_{i,0}(c)$. If $\phi'_{i,0}(c) \neq 0$, then x_0^{m-i+1} divides $\phi'_0(g) + x_0^{m-i+1}\phi'_{i,0}(d)$ but not any term of $x_0^{m-i}\phi'_{i,0}(c)$ since c and hence $\phi'_{i,0}(c)$ does not involve x_0, so we see that $\phi'_0(g) + x_0^{m-i+1}\phi'_{i,0}(d) = 0$ and $x_0^{m-i}\phi'_{i,0}(c) = 0$. Thus, in fact, $\phi'_{i,0}(c) = 0$, and hence $c = \phi'_{i,1}(c')$ for some $c' \in F'_{i,1}$. Since we can regard d as being in $F'_{i,0}(-1)$ and $\phi'_1(d) = x_0 d - f'_{i,0}(d)$, it is enough now to show $g + f'_{i,0}(d)$ is in the image of ϕ'_1. This follows by induction, since $g + f'_{i,0}(d) \in F'_{0,0} \oplus \cdots \oplus F'_{i-1,0}$ and, by Lemma 2.3, $\phi'_0\phi'_1(d) = 0$, so $\phi'_0 f'_{i,0}(d) = \phi'_0(\mu_i(d))$, hence

$$\phi'_0(g + f'_{i,0}(d)) = \phi'_0(g + \mu_i(d)) = \phi'_0(f) = 0.$$

Next, suppose $\phi'_j(f) = 0$ for some f and some $j \geq 1$. Inducting on i, we consider $f \in (F'_{0,j}) \oplus (F'_{1,j} \oplus F'_{1,j-1}(-1)) \oplus \cdots \oplus (F'_{i,j} \oplus F'_{i,j-1}(-1))$. The case $i = 0$ is immediate, so say $i \geq 1$. We can write $f = g + a + b$, where $g \in (F'_{0,j}) \oplus (F'_{1,j} \oplus F'_{1,j-1}(-1)) \oplus \cdots \oplus (F'_{i-1,j} \oplus F'_{i-1,j-1}(-1))$, $a \in F'_{i,j}$ and $b \in F'_{i,j-1}(-1)$. Now $0 = \phi'_j(f) = \phi'_j(g) - f'_{i,j-1}(b) - \phi'_{i,j-1}(b) + \phi'_{i,j}(a) + x_0 b$, when $j > 1$, and $0 = \phi'_1(f) = \phi'_1(g) - f'_{i,0}(b) + \phi'_{i,1}(a) + x_0 b$ for $j = 1$. But $\phi'_j(g) - f'_{i,j-1}(b) - \phi'_{i,j-1}(b) \in (F'_{0,j}) \oplus (F'_{1,j} \oplus F'_{1,j-1}(-1)) \oplus \cdots \oplus (F'_{i-1,j} \oplus F'_{i-1,j-1}(-1))$, for $j > 1$, $\phi'_1(g) - f'_{i,0}(b) \in F'_{0,0} \oplus \cdots \oplus F'_{i-1,0}$ for $j = 1$ and $\phi'_{i,j}(a) + x_0 b \in F'_{i,j-1}$, so each is 0.

Denote by c the sum of all terms of a not divisible by x_0. Then $a = c + x_0 d$ for some $d \in F'_{i,j}(-1)$. As above we must have $\phi'_{i,j}(c) = 0$ (hence $c = \phi'_{i,j+1}(e)$ for some $e \in F'_{i,j+1}$) and $\phi'_{j+1}(d) = x_0 d - f'_{i,j}(d) - \phi'_{i,j}(d) = x_0 d - f'_{i,j}(d) + b$ (since $x_0 b = -\phi'_{i,j}(a) = -\phi'_{i,j}(c + x_0 d) = -\phi'_{i,j}(x_0 d)$), so $\phi'_{j+1}(e + d) = a + b - f'_{i,j}(d)$. Thus it is enough to show that $f - \phi'_{j+1}(e+d) = g + f'_{i,j}(d)$ is in the image of ϕ'_{j+1}. But $g + f'_{i,j}(d) \in (F'_{0,j}) \oplus (F'_{1,j} \oplus F'_{1,j-1}(-1)) \oplus \cdots \oplus (F'_{i-1,j} \oplus F'_{i-1,j-1}(-1))$ and $\phi'_j(g + f'_{i,j}(d)) = \phi'_j(g) - f'_{i,j-1}(b) = 0$, so this follows by induction. □

Corollary 2.5. *If $I(Z_{i+1}) \subset R_1 I(Z_i)$ for each i, then the maps $f_{i,j}$ can be chosen so that the resolution $F'_\bullet$ is minimal.*

Proof. By Corollary 2.2, we may assume $f_{i+1,j} : F_{i+1,j} \to R_1 F_{i,j}$, for all i and j. Since the resolutions $F_{i,\bullet}$ are minimal, we know that the matrix for each map $\phi_{i,j}$ has entries in R_1, and we now know the same is true for each $f_{i+1,j}$. The same now follows for each ϕ'_j by an inspection of the definition of ϕ'_j. □

We can now prove Theorem 1.1.

Proof of Theorem 1.1. Since $F'_0 = \bigoplus_{i=0}^m F'_{i,0}$ and $F'_j = \bigoplus_{i=1}^m (F'_{i,j} \oplus F'_{i,j-1}(-1))$ for $j \geq 1$, then, accounting for shifts, we see that the Poincaré polynomial $P(Z')$ is given by the sum $T^m + XT^m P(Z_1) + T^{m-1}P(Z_1) + \cdots + XTP(Z_m) + P(Z_m)$, which simplifies to $T^m + (1 + XT)(T^{m-1}P(Z_1) + \cdots + P(Z_m))$, as claimed. □

3. Applications

To apply our results to obtain minimal resolutions, we need to verify the condition $I(Z_{i+1}) \subset R_1 I(Z_i)$ of Corollary 2.5. We first do this when char(K) is either 0 or is sufficiently large, then in various additional situations, such as the case of monomial ideals.

Proposition 3.1. *Given $p_i \in \mathbb{P}^d$ and fat points $Z = m'_1 p_1 + \ldots + m'_r p_r$ and $Y = m_1 p_1 + \ldots + m_r p_r$ with $m'_i > m_i$ for all i whenever $m'_i > 0$, assume char(K) is either 0 or bigger than the degree of each generator in a minimal set of homogeneous generators of $I(Z)$. Then $I(Z) \subset R_1 I(Y)$.*

Proof. This follows easily using Euler's identity, that $\delta F = \sum_i x_i \partial F/\partial x_i$ for any homogeneous form F with either char$(K) = 0$ or char$(K) > \delta$, where δ is the degree of F. □

Proposition 3.2. *Let Y and Z be as in Proposition 3.1. Assume that the points p_i are located at coordinate vertices of $\mathbb{P}^{d-1}$. Then $I(Z) \subset R_1 I(Y)$.*

Proof. The ideals $I(Y)$ and $I(Z)$ are generated by monomials in this case, and we may assume that the variables are indexed so that x_j vanishes at p_i for all $i \neq j$. Now assume there is a monomial $f = x_1^{n_1} \cdots x_d^{n_d} \in (I(Z) \setminus R_1 I(Y)) \subset I(Y)$. Since $f \in I(Z)$, we know $m'_i \leq (n_1 + \cdots + n_d) - n_i$ for every i, but $f \in I(Y) \setminus R_1 I(Y)$, so $m_i = (n_1 + \cdots + n_d) - n_i$ for some i. However, $m'_i > m_i$, so this is impossible. □

We also have the following bootstrapping result:

Proposition 3.3. *Let $p_1, \ldots, p_r \in L$, where $L \subset \mathbb{P}^d$ is a proper linear subspace of $\mathbb{P}^d$. Let R be the homogeneous coordinate ring for L, and R' that for $\mathbb{P}^d$. Given positive integers $m_1, \ldots, m_r$, let $Z = m_1 p_1 + \cdots + m_r p_r \subset L$ be the fat point subscheme of L, and let $Z' = m_1 p_1 + \cdots + m_r p_r \subset \mathbb{P}^d$ be the fat point subscheme of $\mathbb{P}^d$ specified by the same multiplicities. If $I(Z_{i+1}) \subset R_1 I(Z_i)$ holds for all i, then so does $I(Z'_{i+1}) \subset R_1 I(Z'_i)$.*

Proof. It is enough by induction to prove this in the case that L is a hyperplane. But by Lemma 2.4, we have that $I(Z'_i) = \sum_{0 \leq j \leq i} x_0^{i-j} I(Z_j) R'$, for all i. Thus $I(Z'_{i+1}) = \sum_{0 \leq j \leq i+1} x_0^{i+1-j} I(Z_j) R' = x_0 (\sum_{0 \leq j \leq i} x_0^{i-j} I(Z_j) R') + I(Z_{i+1}) R'$, and this is a subset of $x_0(\sum_{0 \leq j \leq i} x_0^{i-j} I(Z_j) R') + R_1(I(Z_i) R') \subset R'_1 I(Z'_i)$. □

Example 3.4. Resolutions of ideals for fat point subschemes supported at up to $d+1$ general points of $\mathbb{P}^d$ are known in various cases ([F], [FL], [Fr], [V]). Proposition 3.2, and Corollary 2.5 (or Theorem 1.1 for just the Betti numbers), reduce the problem of determining resolutions of fat point subschemes with support at up to $d+1$ general points of $\mathbb{P}^d$ to cases in which the support spans the entire projective space. For example, to determine the resolution for $2p_1 + 2p_2 + p_3$ for general points $p_i \in \mathbb{P}^d$ with $d > 2$, it is enough to determine the resolutions of $2p_1 + 2p_2 + p_3 \subset \mathbb{P}^2$ and $p_1 + p_2 \subset \mathbb{P}^2$, and to do $p_1 + p_2 \subset \mathbb{P}^2$ it suffices to do $p_1 + p_2 \subset \mathbb{P}^1$. Since, in fact, resolutions for ideals of fat points with support at 3 general points of $\mathbb{P}^2$ are known ([C]), our results as a consequence give the resolution and Betti numbers for ideals of fat points supported at any 3 general points in projective space of any dimension. This generalizes the result for two points ([FL], [V]).

Example 3.5. Another way to generalize the known resolution of fat points with support at two points is to consider supports consisting of collinear points. Let $p_1, \dots, p_r \in L$, where $L \subset \mathbb{P}^d$ is a line. Let $Z = m_1p_1 + \cdots + m_rp_r \subset L$. Then $I(Z) = fR$, where f is a polynomial vanishing at each point p_i to order m_i and R is the homogeneous coordinate ring of L. Since the ideals are principal and the degree of the generator is the sum of the multiplicities, it is easy to see that $I(Z_{i+1}) \subset R_1I(Z_i)$ holds for all i. Proposition 3.3 and Corollary 2.5 now give the minimal resolution of $I(Z') \subset \mathbb{P}^d$, and Theorem 1.1 gives the Betti numbers, extending the result of [H] for $d = 2$.

Example 3.6. Various facts are known for resolutions of points in $\mathbb{P}^2$; our results thus extend these to higher dimension, at least in characteristic 0. For example, [C] works out the resolution of the ideal of fat points with support on an irreducible conic in $\mathbb{P}^2$, while [H] determines the Betti numbers for the case of any plane conic, irreducible or not, and [FHH] determines the Betti numbers for any fat point subscheme with support at up to 8 general points of $\mathbb{P}^2$. Thus our results give the resolution for the ideal of fat points whose support lies on an irreducible conic curve in any projective space (since an irreducible degree 2 curve is contained in a plane), and they give the Betti numbers when the support either consists of up to 8 general points in a plane or lies in any conic in a plane, for a plane in any projective space. There are also many additional examples of sets of points $p_1, \dots, p_r$ contained in configurations of lines in the plane for which the graded Betti numbers for both $p_1 + \cdots + p_r$ and $2p_1 + \cdots + 2p_r$ are known (see [GMS]). Our results thus give the graded Betti numbers for these examples regarded as subschemes of $\mathbb{P}^n$, by regarding $\mathbb{P}^2$ as a linear subspace of $\mathbb{P}^n$.

Remark 3.7. We close with a remark about an additional situation in which our criterion for minimality will hold. Consider a fat point subscheme $Z \subset \mathbb{P}^d$. For each i, let D_i (d_i, resp.) be the degree of the generator of maximal (resp., minimal) degree in a minimal set of homogeneous generators for $I(Z_i)$. Since $I(Z_{i+1}) \subset I(Z_i)$ and $R_1I(Z_i)_t = I(Z_i)_{t+1}$ for $t \geq D_i$, it is clear that the condition $I(Z_{i+1}) \subset R_1I(Z_i)$ holds if the degrees of the generators of $I(Z_{i+1})$ are shifted enough with respect to those of $I(Z_i)$ (in particular, if $d_{i+1} > D_i$ for each $i > 0$). This occurs, for example, if, for each i, the fat points in Z_{i+1} of multiplicity 1 are general and if there are enough of them. More explicitly, let Z_1 consist of r_0 simple points. Let Z_2 include the same points as does Z_1, but take these points with multiplicity 2, and add on r_1 additional general simple points. Continue in this way, defining Z_i and r_i. The condition $I(Z_{i+1}) \subset R_1I(Z_i)$ holds for all i, if, for example, $r_i \geq \binom{D_i+d}{d}$ for all i, since $I(Z_{i+1})$ has no elements of degree less than $D_i + 1$.

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