Gravity theory through affine spheres

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Gravity theory through affine spheres

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Abstract. In this work it is argued that in order to improve our understanding of gravity and spacetime our most successful theory, general relativity, must be destructured. That is, some geometrical assumptions must be dropped and recovered just under suitable limits. Along this line of thought we pursue the idea that the roundness of the light cone, and hence the isotropy of the speed of light, must be relaxed and that, in fact, the shape of light cones must be regarded as a dynamical variable. Mathematically, we apply some important results from affine differential geometry to this problem, the idea being that in the transition we should preserve the identification of the spacetime continuum with a manifold endowed with a cone structure and a spacetime volume form. To that end it is suggested that the cotangent indicatrix (dispersion relation) must be described by an equation of Monge-Ampère type determining a hyperbolic affine sphere, at least whenever the matter content is negligible. Non-relativistic spacetimes fall into this description as they are recovered whenever the center of the affine sphere is at infinity. In the more general context of Lorentz-Finsler theories it is shown that the lightlike unparametrized geodesic flow is completely determined by the distribution of light cones. Moreover, the transport of lightlike momenta is well defined though there could be no notion of affine parameter. Finally, we show how the perturbed indicatrix can be obtained from the perturbed light cone.

1. Introduction

In this work we are going to present some recent results on anisotropic gravity theories obtained in [1,2]. Although, ultimately the theory is of Lorentz-Finsler type we wish to introduce it from a different angle, emphasizing the role of affine differential geometry in its development.

The author was motivated by some general ideas and principles on how Physics should be expected to evolve. The history of Physics is intimately connected with that of Geometry and more, broadly, with that of Mathematics. With the work of Albert Einstein we learned that the gravitational phenomena are manifestations of a dynamical metric manifold. The special relativistic spacetime had to be replaced by a more dynamical structure which broke the, at the time undisputed, property of translational invariance.

Similarly, if our not so humble aim is that of trying to anticipate the next physical revolution, it seems a good idea to look for other tacit "rigidity" assumptions and to work to relax them. Some authors have focused on the topology of the manifold. Since one of the main goals remains the unification of the gravitational dynamics with that of quantum fields, and since the latter display peculiar phenomena of quantization in the observables, it has been suggested that, perhaps, the hidden assumption which has to be removed is contained in the very use of differentiable manifolds. Some authors suggested that manifolds should be replaced by graphs,

since graphs, being by their very nature quantized, are philosophically compatible with the hints coming from the quantum work. The most notable program in this direction is that of "Causal Set Theory", for which the spacetime manifold, with its causal structure is replaced by a directed graph [3]. The directed links provide a primitive notion of causality.

Although we shall not be concerned with quantum gravity, we wish to mention Causal Set Theory because it recognizes that two spacetime ingredients of general relativity should be preserved in the transition to the quantum world. They are the notion of *causality* and the notion of *spacetime volume*. There are good reason to do so; reasons that might be better understood by briefly recalling the geometrical assumptions behind general relativity.

We recall that in general relativity the spacetime is modeled by a Lorentzian time-oriented manifold (M, g). At each even $x \in M$ we have a Lorentzian bilinear form on $T_x M$ as in special relativity

$$g_x = -(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2.$$

where $\{y^\mu\}$ are coordinates induced on $T_x M$ by the local coordinates $\{x^\mu\}$ on M. The cone of timelike vectors is, see Fig. 1,

$$\Omega = \{y \colon g_x(y, y) < 0, \ y^0 > 0\}$$

The set of lightlike vectors is

$$\partial\Omega = \{y \colon g_x(y, y) = 0, y \neq 0\}$$

The *velocity space* of massive particles/observers is

$$\mathbb{H} = \{y \colon g_x(y, y) = -1\}.$$

This is the usual *hyperboloid.* As a consequence on spacetime we have a distribution $x \to \Omega_x$ of timelike cones and a distribution $x \to \mathbb{H}_x$ of hyperboloids. A curve $x \colon t \mapsto x(t)$ is

Figure 1. Light cone and velocity space in special and general relativity.

- Timelike: if $g(\dot{x}, \dot{x}) < 0$,   (massive particles),
- Lightlike: if $g(\dot{x}, \dot{x}) = 0$,   (massless particles).

It is *causal* if at every point it is timelike or lightlike. The *proper time* of a massive particle is

$$\tau = \int_{x(t)} \sqrt{-g(\dot{x}, \dot{x})} \mathrm{d}t.$$

A simple algebraic results tells us that any Lorentzian bilinear form is almost completely characterized by its light cone [4, App. D].
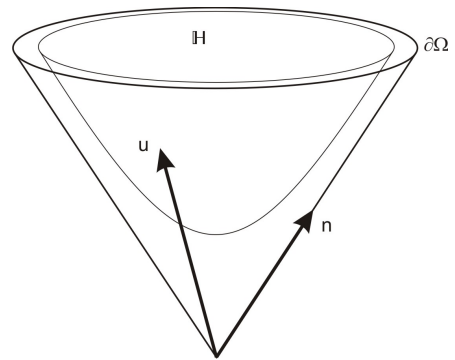
Proposition 1.1. *On a vector space of dimension* $n \geq 3$ *two Lorentzian bilinear forms* η_1, η_2 *are proportional if and only if they have the same timelike cone* Ω.

$$\Omega_1 = \Omega_2 \Leftrightarrow \exists a \in \mathbb{R} : \eta_1 = a^2 \eta_2.$$

Since the volume form induced by the bilinear form is $\sqrt{-\det \eta} \, \mathrm{d}y^0 \wedge \cdots \wedge \mathrm{d}y^n$ we have

Corollary 1.2. *On a vector space of dimension* $n \geq 3$ *two Lorentzian bilinear forms* η_1, η_2 *coincide if and only if they share the same timelike cone and they induce the same volume form.*

So, introducing the x-dependence of general relativity we arrive at

Corollary 1.3. *Two spacetime metrics g_1 and g_2 coincide if and only if they induce the same distribution of timelike cones $x \to \Omega_x$ and the same volume form $\mathrm{d}\mu = \sqrt{-\det g}\,\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n$.*

The distribution of light cones defines the causal relation

$$J = \{(x,y) \in M^2 : \ x = y \text{ or there is a future directed causal curve from } x \text{ to } y\}$$

so as a minimal requirement the transition from the non-quantum to the quantum world must preserve the notions of causal relation and spacetime volume. Causal Set Theory has this feature. The spacetime volume is obtained from the number of vertices of the graph while the causal relation is obtained from the direction of the links. While we share the opinion that the pair

$$\textit{spacetime measure} + \textit{causal order}$$

is central and must be preserved we do not believe that the transition to the quantum world could be accomplished in one big step, by working directly with discretized objects such as graphs. In a sense, it would be like pretending to do Quantum Mechanics by working with the spectra of operators. In Quantum Mechanics the discretized features are a byproduct of the operator theory in Hilbert spaces, so the mathematics must evolve into something richer rather than simpler.
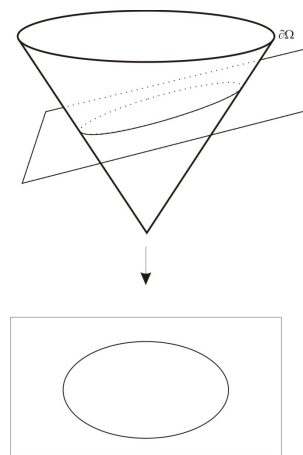
Furthermore, in this work we do not ask: what are the physical quantum theories which accommodate reasonable concepts which can be interpreted as causal relation and spacetime volume. Rather we stay in the non-quantum world and ask: what is the most general non-quantum theory which embodies these notions? So we are really looking for non-quantum modifications of general relativity.

In order to ask this question we have to look for some "rigidity" assumption, with the idea of dropping it. We shall focus on the rigidity assumption that light cones are round, namely have ellipsoidal sections, see Fig. 2. So the new dynamical variable will be the *shape of the light cones*. Since they are no more round, the speed of light in our theory could be (slightly) anisotropic.

We have seen that the distribution of light cones in general relativity determines the metric up to a conformal factor. Moreover, unparametrized lightlike geodesics are conformally invariant so the specification of the light cone distribution fixes the motion of massless particles. There is reasonable hope that a distribution of non-isotropic light cones (not to count the specification of a volume form) could fix the motion of massless particles,

Figure 2. Usual light cones are round.

namely a notion of lightlike geodesics (we shall see that this is indeed the case, cf. Sec. 3.3).

However, on spacetime we have also the massive particles. Since in general relativity the metric gets determined from the cone distribution and the spacetime volume, so does the hyperboloid $\mathbb{H}$ inside the timelike cone. In the anisotropic theory from the same ingredients we don't get a bilinear form on T_xM and so we don't get a velocity space for massive particles.

Here comes the main idea of our work on this problem: to use affine differential geometry to show that there is, in fact, a privileged hypersurface inside the timelike cone which, in the end, can be successfully interpreted as velocity space. So, let us recall some elements of affine differential geometry.

2. Affine differential geometry

Affine differential geometry was developed by Wilhelm Blaschke and his school in the first half of the XX century when he realized that non-degenerate (e.g. convex) hypersurfaces admit a natural notion of transverse. So let E be an affine space modeled on a $n+1$-dimensional vector space V endowed with a volume form ω. We stress that we don't have any scalar product on V and so no notion of angle or orthogonality. In order to fix the ideas let us consider a convex hypersurface S, namely a hypersurface bounding a strongly convex set.

Let us pick a point $p \in S$ and let us consider the plane P tangent to p and its translates P_t, $P_0 = P$, where t is a parameter. For every t the plane P_t cuts the hypersurface and selects a compact sector Q_t of finite volume inside S, see Fig. 3. The notion of barycenter (centroid) is affine, namely it is well defined in any affine space. So let G_t be the centroid of Q_t. As $t \to 0$ the point G_t draws a curve which ends at $G_0 = p$. The tangent to the curve at p selects a special direction, which is the direction of the affine normal. Of course, should V be endowed with a scalar product, the Euclidean normal need not coincide with the affine normal, though this turns out to be the case for spheres.
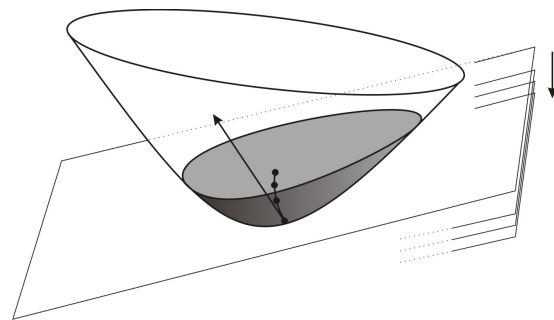
Figure 3. Geometrical construction of the affine normal.

Formally, the immersion is denoted $f \colon N \to E$, where $S = f(N)$. Next, though one does not know the privileged transverse one chooses one $\xi \colon N \to V$, so that $\xi(x)$ is transverse to S at $f(x)$. This choice allows one to split the derivatives, so for $X, Y \in TN$ we have, denoting with D the natural derivative one E

$$D_{f_*(X)} f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi, \qquad \text{(Gauss equation)}$$
$$D_{f_*(X)} \xi = -f_*(S(X)) + \tau(X)\xi. \qquad \text{(Weingarten equation)}$$

which are used to define a torsionless connection ∇, a metric h, an endomorphism S and a one-form τ on N. Here the transverse field is determined by $n+1$ arbitrary parameters, so we shall need $n+1$ conditions to fix it.

Observe that the affine space has two natural structures, namely the torsionless (flat) connection D and the volume form ω which share the compatibility condition $D\omega = 0$ (the volume form is translationally invariant). So, one wants the same properties to hold on N where on this manifold we have already identified a torsionless connection ∇, while the volume form can be defined through $\theta = f^*(i_\xi \omega)$. By imposing the (equiaffine) condition $\nabla\theta = 0$ we remove some of the arbitrariness which resides in the choice of transverse ξ (we are imposing n equations, since the derivative takes n-directions). It turns out that this condition is equivalent to $\tau = 0$. A manifold endowed with a pair (∇, θ) satisfying the condition $\nabla\theta = 0$ is called *equiaffine space*. Its connection ∇ has necessarily a symmetric Ricci tensor. The notion of equiaffine space generalizes that of affine space with translationally invariant volume form.

Actually, here we can really define the volume form following a different route, namely, for any θ-positively oriented n-tuple, set

$$\theta'(X_1, \cdots, X_n) = |\det h(X_i, X_j)|^{1/2}.$$

So the remaining condition demands that there should be no ambiguity in the definition of volume, that is $\theta' = \theta$. In this way ξ is fixed unambiguously to the affine (Blaschke) normal,

and so we obtain, as a consequence, the affine volume form θ, the affine connection ∇, the affine metric h, and the affine shape operator S.

From a privileged transverse direction we can introduce an important concept. Indeed, suppose that the transverse directions meet at the same point, see Fig. 4. Then the hypersurface is called *affine sphere* and the point of convergence is the *center* of the sphere.

It turns out that the are just three types of affine spheres: (a) the improper spheres, for which the the center is at infinity, namely all the privileged transverse directions are parallel, (b) the (proper) elliptic spheres, for which the center is inside the convex set bounded by the hypersurface, and (c) the (proper) hyperbolic spheres, for which the center is outside the convex set bounded by the hypersurface. Some deep theorems from affine differential geometry assure that the improper affine spheres are just the paraboloids while the elliptic affine spheres are just the ellipsoids.

Figure 4. The definition of affine sphere.

What about the hyperbolic affine spheres? Eugenio Calabi [5] in 1971 conjectured that they are much more abundant, and that, in fact, up to homotheties there is a one-to-one correspondence between open sharp convex cones and hyperbolic affine spheres. Namely, every hyperbolic affine sphere is asymptotic to an open sharp convex cone and every such cone contains, up to homotheties, a unique hyperbolic affine sphere asymptotic to it, see Fig. 5. Calabi's conjecture was proved by Cheng and Yau in 1977, some steps being clarified by other mathematicians.
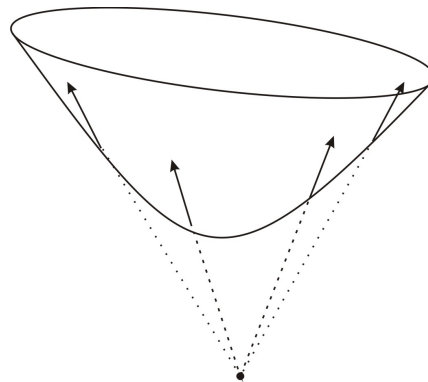
Suppose that we have been given on E a centroaffine hypersurface, namely a hypersurface transverse to the half-lines starting from some point $o \in E$. It is convenient to regard E as a vector space V with origin in o, to introduce linear coordinates $\{y^\alpha\}$ over V in such a way that the hyperplane $y^0 = 0$ does not intersect the closure of the cone generated by the hypersurface and o, saved for the origin. Moreover, it is convenient to parametrize the hypersurface using Klein's type coordinates $(v^i = y^i/y^0)$ and regard the hypersurface as the embedding

Figure 5. Calabi's conjecture.

$$f\colon \boldsymbol{v} \mapsto y(\boldsymbol{v}) = -\frac{1}{u(\boldsymbol{v})}(1, \boldsymbol{v})$$

so that $N = D \subset R^n$. Then the hypersurface is an affine sphere if u is convex, vanishes at the boundary of the convex set D and

$$\det u_{ij} = \left(-\frac{1}{u}\right)^{n+2}, \quad u|_{\partial D} = 0, \qquad \text{(affine sphere equation)}. \qquad (1)$$

Here n is the affine sphere dimension (we shall be interested in $n = 3$). The mathematical problem was precisely that of showing that this equations admits one and only one solution.

3. Physical interpretation

Now, we return to our physical problem. We needed a special hypersurface inside the non-round light cone so as to assign to it the role of velocity space of the observer. Our idea is now to let it coincide with the affine sphere of the light cone. In this way the distribution of light cones determines the velocity space of the observer, and hence the dispersion relation.

Observe that the coordinates $\{y^\alpha\}$ can be fixed as follows: first we choose a timelike direction and choose the y^0-axis along it, then the other axes are chosen so that the hyperplane $y^0 = 1$ is tangent to the hypersurface and the affine metric h is the identity at $y = (1, 0, \cdots, 0)$. The coordinates, up to rotation of the space coordinates $\{y^i\}$ are uniquely determined by the choice of timelike direction and are called *observer coordinates* relative to the observer selected by that direction. The coordinates $\boldsymbol{v}$ defined through $v^i = y^i/y^0$ are nothing else, in our physical interpretation, than the velocity of the particle as seen from the observer, while the domain D is nothing but the (observer dependent) shape of the domain of allowed velocities for massive particles. The remarkable fact is that the variables which most simplify the affine sphere PDE are precisely the variables which have physical meaning.

In our theory u is nothing but the classical Lagrangian for the particle (per unit mass). For instance, whenever the domain D is isotropic the solution of the Monge-Ampére equation is $u = -\sqrt{1 - \boldsymbol{v}^2}$, so that we recover the general relativistic kinematical space from the assumption of light speed isotropy. It turns out that in observer coordinates the function u has always the expansion

$$u = -1 + \tfrac{1}{2}\boldsymbol{v}^2 + o(v^2),$$

which is an equivalent form of expressing the fact that we are in observer coordinates. Furthermore, the Legendre transform $u^*(\boldsymbol{p})$ represents the classical Hamiltonian (per unit mass) for the theory and in observer coordinates it admits the expansion

$$E(\boldsymbol{p}) = u^*(\boldsymbol{p}) = 1 + \tfrac{1}{2}\boldsymbol{p}^2 + o(p^2),$$

it can be shown that $\boldsymbol{p} \mapsto (-u^*(\boldsymbol{p}), \boldsymbol{p})$ is an affine sphere dual to that on velocity space with which we worked so far.

It can be observed that we have obtained the velocity space on $E = T_x M$ by regarding it as an affine space endowed with a translationally invariant volume form and by assuming to have been given a timelike cone Ω_x on $T_x M$ centered at 0. In order to have a translationally invariant volume form at each event we need a volume form on spacetime. So our spacetimes which we term *affine sphere spacetimes* can be equivalently characterized by two properties, namely as (a) a distribution of affine spheres with center in the zero section of TM or as (b) a pair given by a spacetime volume and a cone distribution. Thanks to the latter characterization the affine sphere spacetimes reflect the notions of measure and order, precisely the sort of objects we were looking for. Actually, there is a third characterization which will be mentioned in the next section.

3.1. Lorentz-Finsler geometry

In this section we clarify that our theory is a special type of Lorentz-Finsler theory.

In Finslerian generalizations of general relativity the spacetime is a $n+1$-dimensional manifold endowed with a Finsler Lagrangian $\mathscr{L}: \Omega \to \mathbb{R}$, $\Omega \subset TM\backslash\{0\}$, where Ω is an open sharp convex cone subbundle of the slit tangent bundle, $\mathscr{L}$ is positive homogeneous of degree two, that is $\forall s > 0, y \in \Omega_x$, $\mathscr{L}(x, sy) = s^2 \mathscr{L}(x, y)$, $\mathscr{L}$ is negative on Ω and converges to zero at the boundary $\partial\Omega$, and finally, the fiber Hessian $g_{\mu\nu} = \partial^2 \mathscr{L}/\partial y^\mu \partial y^\nu$ is Lorentzian. We shall not demand $\mathscr{L}$ to be differentiable at the boundary $\partial\Omega$, namely we adopt the *rough model* discussed in [6]. The set Ω_x represents the set of future directed timelike vectors at $x \in M$.

The indicatrix $\mathscr{I}_x \subset \Omega_x$ is the locus where $2\mathscr{L} = -1$ and it represents the velocity space of observers (this is the usual hyperboloid in general relativity). By positive homogeneity the Finsler Lagrangian can be recovered from the indicatrix as follows, for $y \in \Omega_x$

$$\mathscr{L}(x, y) = -s^2/2, \quad \text{where } s \text{ is such that } y/s \in \mathscr{I}_x. \tag{2}$$

By positive homogeneity the formulas $\mathscr{L} = \frac{1}{2}g_{\mu\nu}y^\mu y^\nu$, $\frac{\partial \mathscr{L}}{\partial y^\mu} = g_{\mu\nu}y^\nu$, hold true, where the metric might depend on y. If it is independent of y then we are in the quadratic case which corresponds to Lorentzian geometry and general relativity. The Cartan torsion is $C_{\mu\nu\alpha} = \frac{1}{2}\frac{\partial}{\partial y^\alpha}g_{\mu\nu}$. It is symmetric and annihilated by y^μ. The mean Cartan torsion is its contraction

$$I_\alpha := g^{\mu\nu}C_{\mu\nu\alpha} = \frac{1}{2}\frac{\partial}{\partial y^\alpha}\log|\det g_{\mu\nu}|. \tag{3}$$

In a series of recent works we have stressed the importance of the Lorentz-Finsler spaces for which $I_\alpha = 0$, see [1, 7, 8]. They are precisely the affine sphere spacetime which we met before. Observe that the condition $I_\alpha = 0$ implies that we have a well defined volume form on M given by the standard expression $\mathrm{d}\mu = |\det g_{\mu\nu}|^{1/2}\mathrm{d}x^0 \wedge \cdots \mathrm{d}x^n$. In general Lorentz-Finsler spaces there is no such volume form.

For a long time it was believed that Finslerian spacetimes which satisfy $I_\alpha = 0$ are not interesting. In fact a theorem by Deicke establishes that Finsler spaces which satisfy this property are Riemannian and so trivial. Previous researchers did not pay much attention to the signature of the metric. In the spacetime case Deicke's theorem does not really hold, in fact we can prove that the condition $I_\alpha = 0$ is really equivalent to the fact that the indicatrix is a hyperbolic affine sphere centered in the zero section.

The proof passes through the following symmetric tensor on N (Pick cubic form, the $\frac{1}{2}$ factor is due to a better correspondence with Finslerian objects)

$$c(X, Y, Z) = \frac{1}{2}[(\nabla_X h)(Y, Z) + \tau(X)h(Y, Z)]. \tag{4}$$

which in the context of Finsler geometry one evaluates on the indicatrix with respect to the centroaffine normal y (which is equiaffine, namely $\tau = 0$). It turns out that under the pullback of the immersion f, C corresponds to c and its trace I corresponds to $\mathrm{tr}_h c$, so $I = 0$ if and only if $\mathrm{tr}_h c = 0$. But the latter condition together with $\tau = 0$ really implies that y is proportional to the Blaschke normal, so that the affine normal directions meet at o and hence the indicatrix $\mathscr{I}$ is an affine sphere.

So $I = 0$ if and only if the indicatrix is an affine sphere centered at the origin of the tangent space $T_x M$. Once we have the indicatrix we can recover the Finsler Lagrangian from (2) which means that all the metrical properties follow from the indicatrix and hence from the light cone. The relationship between the classical Lagrangian u and the Finsler Lagrangian $\mathscr{L}$ is given by

$$\mathscr{L}((y^0, \mathbf{y})) = -\frac{1}{2}(y^0)^2 u^2(\mathbf{y}/y^0), \tag{5}$$

$$u(\mathbf{v}) = -\sqrt{-2\mathscr{L}((1, \mathbf{v}))}. \tag{6}$$

A remarkable fact concerning the concept of affine sphere spacetime is that it embodies in a nice way the non-relativistic limit [1]. In fact, non-relativistic spacetimes admit the same description of (relativistic) affine sphere spacetimes: the cotangent indicatrix is determined by a distribution of affine spheres which, in the non-relativistic case, have center at infinity. This result is connected with a theorem due to Jörgens, Pogorelov, Calabi, Cheng and Yau that establishes that these improper affine spheres are paraboloids. The algebraic ingredients entering the definition of paraboloid define a one-form field $\mathrm{d}t$, the classical time, and a space metric on its kernel, precisely the geometric ingredients needed to express the geometry of non-relativistic spacetimes [9].

3.2. Relativistic invariance

In [2] we have argued that the relativity principle holds at an event x if a linear subgroups of the endomorphisms of $T_x M$ acts transitively on the indicatrix $\mathscr{I}_x$. In other words, the velocity space is homogeneous and so for an observer it becomes impossible to infer its position on velocity space from local measurements. One might ask what are the affine sphere spacetimes which satisfy this form of relativity principle. I have shown that there are just three possibilities connected with a classification of homogeneous cones due to Vinberg [10]. In these models, due to homogeneity, the shape of D is really independent of the observer.

The first model is isotropic, i.e. D is a ball, and has the usual Finsler Lagrangian

$$\mathscr{L} = \frac{1}{2}\big(-(y^0)^2 + \mathbf{y}^2\big). \tag{7}$$

The second model has a velocity domain D of tetrahedral shape, with Finsler Lagrangian

$$\mathscr{L}_C = -\frac{1}{2}\big[(y^0 - y^1 - y^2 - y^3)^{1/2}(y^0 - y^1 + y^2 + y^3)^{1/2} \\ (y^0 + y^1 - y^2 + y^3)^{1/2}(y^0 + y^1 + y^2 - y^3)^{1/2}\big], \tag{8}$$

This model falls into a family investigated by Bogoslovsky and Goenner [11, 12] but from the geometric point of view was also considered by many other authors including Berwald and Moór [13, 14], Matsumoto [15] and Calabi [5].

The third model was not previously noticed. The Finsler Lagrangian is

$$\mathscr{L} = -\frac{2}{3^{3/4}}\Big(\frac{1}{2}y^0 + \frac{\sqrt{3}}{2}y^3\Big)^{1/2}\Big(\Big(\frac{\sqrt{3}}{2}y^0 - \frac{1}{2}y^3\Big)^2 - (y^1)^2 - (y^2)^2\Big)^{3/4}. \tag{9}$$

and the domain D is a cone in which the height is equal to the diameter of the base.

As can be seen the isotropic model is the only one with C^1 dependence of the speed of light on direction, so in the classical derivation of the Lorentz transformations the isotropy assumption could be replaced by a C^1 light velocity assumption.

For small velocities, namely for $y^i/y^0 \ll 1$ all the previous expressions reduce themeselves to (7), the special relativistic case. This is so because we wrote these models in observer coordinates.

I also considered general relativistic Lorentz-Finsler affine sphere spacetimes which in the analogous low-velocity limit reduce themselves to notable spacetimes of GR. For instance, with $r = \sqrt{z^2 + \rho^2}$ the next Lagrangian

$$\mathscr{L} = -\frac{2(1 - \frac{2m}{r})}{3^{3/4}}\left(\left(\frac{1}{2}\,\mathrm{d}t_S + \frac{\sqrt{3}}{2}\Big(1 - \frac{2m}{r}\Big)^{-1}\frac{(1 - 2m\rho^2/r^3)\mathrm{d}z + 2mz\rho\,\mathrm{d}\rho}{\sqrt{1 - 2m\rho^2/r^3}}\right)^2\right)^{1/4}$$

$$\left(\left(\frac{\sqrt{3}}{2}\,\mathrm{d}t_S - \frac{1}{2}\Big(1 - \frac{2m}{r}\Big)^{-1}\frac{(1 - 2m\rho^2/r^3)\mathrm{d}z + 2mz\rho\,\mathrm{d}\rho}{\sqrt{1 - 2m\rho^2/r^3}}\right)^2 - \Big(1 - \frac{2m}{r}\Big)^{-1}\Big(\frac{\mathrm{d}\rho^2}{1 - 2m\rho^2/r^3} + \rho^2\mathrm{d}\varphi^2\Big)\right)^{3/4}.$$

is conic anisotropic at each event and reduces itself to the Schwarzschild metric in the said limit. Similarly, we obtained conic anisotropic versions of the Kerr-Schild, Kerr-de Sitter, Kerr-Newman, Taub, and FLRW spacetimes. These metrics have been deduced from their low-velocity limit (with respect to some conformal stationary observer) and from the requirement of relativistic invariance at every spacetime event, not from the imposition of dynamical field equation. However, the latter can perhaps be better identified if we regard these metrics as reasonable spacetime solutions.
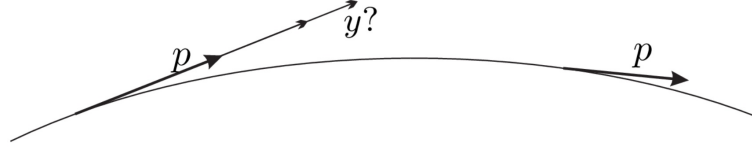
Figure 6. In Lorentz-Finsler theories the unparametrized lightlike geodesics and the transport of momenta make sense while the notion of affine parameter might not make sense.

3.3. Geodesic flow on the lightlike cotangent bundle
We have seen that saved for the isotropic case, the Lorentz-Finsler spaces which satisfy the relativity principle have a non smooth light cone. In general, we might regard these spacetimes as idealizations since they are considerably anisotropic. We are more interested in small deviations from isotropy and so in light cones which could be smooth. Unfortunately, even if the cone is smooth the function $\mathscr{L}$ determined from it has been proved [1] to be $C^{1,1/2}(\bar{\Omega}_x)$ and with non vanishing differential at the cone. This degree of differentiability is insufficient, for the geodesics solve Lagrange's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial y^\mu} - \frac{\partial \mathscr{L}}{\partial x^\mu} = 0, \qquad y^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}$$

so that lightlike geodesics are well defined only if $\mathscr{L}(x,y) \in C^2$ for y belonging to the light cone. So it seem that we have a problem in the definition of lightlike geodesic.

This problem has a nice solution which shows that the unparametrized lightlike geodesics really depend only on the distribution of light cones, namely on $\mathscr{L}^{-1}(0)$, and not on the whole Finsler Lagrangian $\mathscr{L}$. Since we can always find another C^2 (subsidiary) Finsler Lagrangian $\hat{\mathscr{L}}$ with the same light cones, the unparametrized lightlike geodesics are indeed well defined provided we can show that they are independent of $\hat{\mathscr{L}}$. This is precisely what we proved in [1].

Indeed, since $\hat{\mathscr{L}}$ and $\hat{\mathscr{L}}'$ vanish on the light cone bundle $N = \cup_x \partial\Omega_x$, which is a codimension one hypersurface on $TM\backslash 0$, and are negative on Ω we must have $\mathrm{d}\hat{\mathscr{L}} = \phi\,\mathrm{d}\hat{\mathscr{L}}'$ for some positive function ϕ on N. This means that for $(x,y) \in N$,

$$\frac{\partial \hat{\mathscr{L}}}{\partial x^\mu} = \phi\frac{\partial \hat{\mathscr{L}}'}{\partial x^\mu}, \qquad \frac{\partial \hat{\mathscr{L}}}{\partial y^\mu} = \phi\frac{\partial \hat{\mathscr{L}}'}{\partial y^\mu}. \tag{10}$$

Let $x(t) = x'(t')$ be a curve in two different parametrization. Thanks to the identity

$$\left(\frac{\mathrm{d}t}{\mathrm{d}t'}\right)^2 \phi^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\hat{\mathscr{L}}}{\partial y^\mu} - \frac{\partial\hat{\mathscr{L}}}{\partial x^\mu}\right) = \left(\frac{\mathrm{d}t'}{\mathrm{d}t}\right)^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\phi\frac{\mathrm{d}t'}{\mathrm{d}t}\right)\right)\frac{\partial\hat{\mathscr{L}}'}{\partial y^\mu} + \frac{\mathrm{d}}{\mathrm{d}t'}\frac{\partial\hat{\mathscr{L}}'}{\partial y^\mu} - \frac{\partial\hat{\mathscr{L}}'}{\partial x^\mu}$$

we have that every lightlike $\hat{\mathscr{L}}$-geodesic is a lightlike $\hat{\mathscr{L}}'$-geodesic for a parameter t' such that $\phi\frac{\mathrm{d}t'}{\mathrm{d}t} = cnst$. Thus unparametrized lightlike geodesics do not depend on the choice of $\hat{\mathscr{L}}$.

But there is much more to it. Notice that if $x(t) = x'(t')$ with $x(t)$ a $\hat{\mathscr{L}}$-lightlike geodesic and $x'(t')$ a $\hat{\mathscr{L}}'$-lightlike geodesic, then $\phi\frac{\mathrm{d}t'}{\mathrm{d}t} = cnst$. Given an unparametrized geodesic we say that a $\hat{\mathscr{L}}$-affine parameter t and a $\hat{\mathscr{L}}'$-affine parameter t' are syntonized if $\phi\frac{\mathrm{d}t'}{\mathrm{d}t} = 1$. So if the two parameters are syntonized at one event then they are syntonized at every event. But this fact implies that the transport of momenta is well defined since we have the identity

$$\frac{\partial\hat{\mathscr{L}}}{\partial y^\mu}(x(t),\tfrac{\mathrm{d}x}{\mathrm{d}t}(t)) = \phi\frac{\partial\hat{\mathscr{L}}'}{\partial y^\mu}(x(t),\tfrac{\mathrm{d}x}{\mathrm{d}t}(t)) = \phi\frac{\mathrm{d}t'}{\mathrm{d}t}\frac{\partial\hat{\mathscr{L}}'}{\partial y^\mu}(x'(t'),\tfrac{\mathrm{d}x'}{\mathrm{d}t'}(t')). \tag{11}$$

Thus on the unparametrized lightlike geodesic $x(t) = x'(t')$ if $p_\mu(t_1) = p'_\mu(t'_1)$ at one event then $\phi \frac{\mathrm{d}t'}{\mathrm{d}t} = 1$ at the same event, namely t and t' are syntonized, which means $\phi \frac{\mathrm{d}t'}{\mathrm{d}t} = 1$ at any later event and so $p_\mu(t) = p'_\mu(t')$ at any later event. That is, the transport of lightlike momenta is well defined as independent of the subsidiary Lagrangian.

There is a similar Hamiltonian argument which allows us to reach the same conclusion. Let $\hat{\mathscr{L}} \colon \bar{\Omega} \to \mathbb{R}$ be a subsidiary Lagrangian and let $\hat{\mathscr{H}} \colon \bar{\Omega}^* \to \mathbb{R}$ be its Legendre transform. Here Ω^* is the polar cone of Ω, cf. [16, Sec. 2.5], $\hat{\mathscr{H}}$ vanishes on $N^* = \partial\Omega^*$, it has non-vanishing differential there and is negative in Ω^*.

The dynamics induced by a subsidiary Lagrangian $\hat{\mathscr{L}}$ can also be obtained from Hamilton's equations

$$i_X\omega = -\mathrm{d}\hat{\mathscr{H}},$$

where $\omega = \mathrm{d}p_\alpha \wedge \mathrm{d}x^\alpha$ is the symplectic form and X is a vector field on the slit cotangent bundle $T^*M\backslash 0$ defining the dynamical system on momentum space. Due to

$$L_X\omega = (\mathrm{d}i_X + i_X\mathrm{d})\omega = -\mathrm{d}\mathrm{d}\hat{\mathscr{H}} = 0,$$

the flow preserves the Hamiltonian and so it is tangent to $N^* = \hat{\mathscr{H}}^{-1}(0)$.

Any other subsidiary Hamiltonian $\hat{\mathscr{H}}'$ vanishes on N^*, thus there is a function $\varphi > 0$ such that $\hat{\mathscr{H}}' = \varphi\hat{\mathscr{H}}$ on N^*. Let X' be the field determined by $\hat{\mathscr{H}}'$, then since the integral curves stay in N^*

$$i_{X'}\omega = -\mathrm{d}\hat{\mathscr{H}}' = -\hat{\mathscr{H}}\mathrm{d}\varphi - \varphi\mathrm{d}\hat{\mathscr{H}} = -\varphi\mathrm{d}\hat{\mathscr{H}},$$

which implies, by the non degeneracy of the symplectic form, $X = \varphi X'$. In other words the integral lines coincide and so there is a well defined unparametrized flow on N^* determined solely by this codimension one hypersurface. Due to the positive homogeneity of the Hamiltonians, this flow is invariant under homotheties, a fact which for any given event and direction on the base M allows us to find a unique projected integral line passing through that event in that direction. This is the unparametrized lightlike geodesic on M.

The possibility of defining the transport of momenta without the need for an affine parameter is a remarkable feature, indeed the notion of affine parameter in general relativity is, for most part, an unnecessary concept quite detached from observations.

3.4. Perturbation of the isotropic theory

Since in practice the light cone is expected to be almost isotropic, we are interested in the perturbation of the isotropic affine sphere solution $u = -\sqrt{1 - v^2}$ as induced by a small deformation of the light cone. It is convenient to introduce the variable

$$w(\boldsymbol{v}) = -\frac{1}{2}\,u^2 = \hat{\mathscr{L}}((1, \mathbf{v})). \tag{12}$$

where the coordinates $v^i = y^i/y^0$ correspond to the Klein's parametrization of $\mathbb{H}^n$. The metric of the hyperboloid is

$$h = h_{ij}\mathrm{d}v^i\mathrm{d}v^j = \frac{1}{(1 - v^2)^2}\{(1 - v^2)\delta_{ij} + v_iv_j\}\mathrm{d}v^i \otimes \mathrm{d}v^j.$$

Perturbing (1) we obtain

$$(\delta^{ij} - v^iv^j)(\delta w)_{ij} + 2v^j(\delta w)_j - 2\delta w = 0, \tag{13}$$

where the new domain D is obtained from the unit ball B through a displacement $-\delta w|_{\partial B}\hat{\mathbf{v}}$ of the boundary at $\hat{\mathbf{v}} \in \partial B = S^{n-1}$. If we define $\delta w = (1 - v^2)f$ we can write it in the form

$$\Delta^{\mathbb{H}} f - 2(n+1)f = 0, \qquad \lim_{v \to 1}[(1 - v^2)f(\boldsymbol{v})] = \delta w|_{S^{n-1}} \qquad (14)$$

where there appears the Laplacian of the hyperbolic metric. The advantage of this equation stays in its coordinate invariance. Expanding the perturbation in spherical harmonics

$$\delta w = \sum_{\ell,\mu} a_{\ell\mu} U_\ell(v) Y_\ell^\mu(\theta, \phi), \qquad (15)$$

where $a_{\ell\mu}$ are constants and $U_\ell(1) = 1$, and replacing in (13) we arrive at a hypergeometric differential equation whose solution is

$$U_\ell = n_\ell \, v^\ell \, \mathrm{Re}\left(F\left(\tfrac{\ell}{2} - 1, \tfrac{\ell}{2} - \tfrac{1}{2}, \ell + \tfrac{n}{2}; v^2 + i0^+\right)\right), \qquad (16)$$

where n_ℓ is a normalization constant (not to be confused with the space dimension; of course we are mostly interested in the physical dimension $n = 3$).
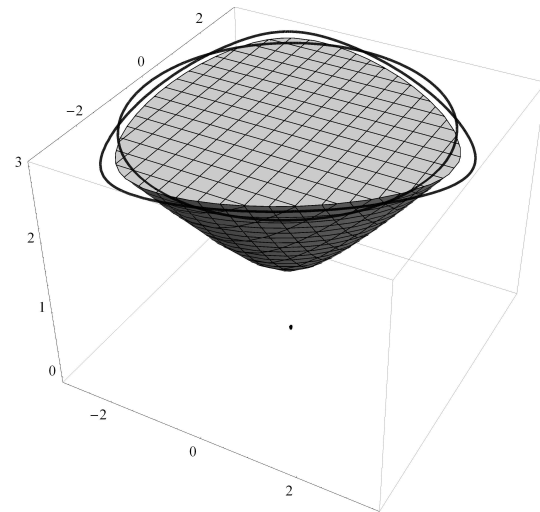
Here we have taken the real part of the value obtained as the limit $z \to v^2$ for z converging to the real line from the upper half plane. We have done this because the hypergeometric function has a singularity at $z = 1$. Both Re and $+i0^+$ can be removed from the expression in the region $v \le 1$.

Here we have taken the real part of the value obtained as the limit $z \to v^2$ for z converging to the real line from the upper half plane. We have done this because the hypergeometric function has a singularity at $z = 1$. Finally, the perturbation of the classical Lagrangian is obtained from $u + \delta u = \sqrt{-2(w + \delta w)} = u(1 + \frac{1}{2}\frac{\delta w}{w}) + ...,$ that is $\delta u = \delta w/\sqrt{1 - v^2}$. Using this expansion we can relate the deformation of the light cone and the deformation of the indicatrix. We give an example in Figure 7.

As it is well known, the Cosmic Microwave Background radiation presents some puzzling anisotropy feature at large angles (small ℓ). We believe that affine sphere relativity might provide the correct theoretical framework for

Figure 7. Perturbation of the 3-dimensional isotropic cone with the third spherical harmonic (black ring), and consequent deformation of the hyperboloid.

the interpretation and explanation of such anomalies. The idea is simply that they are due to a deformation of the light cone in our spacetime neighborhood or in a spacetime neighborhood of the last scattering hypersurface. These considerations will be expanded in a future work.

4. Conclusions
We have presented a novel kinematical framework for a gravitational theory of Lorentz-Finsler type. This theory has several interesting features. It preserves the correspondence between the notion of spacetime and the pair given by (a) a cone distribution (i.e. a causal order) and (b)

a spacetime measure. It also provides a dynamics for massive particles since these ingredients determine univocally a Finsler Lagrangian, and hence all the required metrical concepts of the theory. We have investigated the notion of relativity principle showing that the theories which satisfy it either coincide with GR or are highly anisotropic and so are not expected to be physically relevant in our spacetime neighborhood. Therefore, we have moved to a perturbative approach. Here we could only introduce some of the results we obtained in this direction. We believe that they are really promising since they seem to have the potential to explain some anomalies observed in the CMB radiation.

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