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FROM LÉVY WALKS TO SUPERDIFFUSIVE SHOCK ACCELERATION

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ABSTRACT

In this paper, we present a general scenario for nondiffusive transport and we investigate the influence of anomalous, superdiffusive transport on Fermi acceleration processes at shocks. We explain why energetic particle superdiffusion can be described within the Lévy walk framework, which is based on a power-law distribution of free path lengths and on a coupling between free path length and free path duration. A self-contained derivation of the particle mean square displacement, which grows as $\langle \Delta x^2 \rangle = 2D_\alpha t^\alpha$ with $\alpha > 1$, and the particle propagator, is presented for Lévy walks, making use of a generalized version of the Montroll–Weiss equation. We also derive for the first time an explicit expression for the anomalous diffusion coefficient D_α and we discuss how to obtain these quantities from energetic particle observations in space. The results are applied to the case of particle acceleration at an infinite planar shock front. Using the scaling properties of the Lévy walk propagator, the energy spectral indices are found to have values smaller than the ones predicted by the diffusive shock acceleration theory. Furthermore, when applying the results to ions with energies of a few MeV accelerated at the solar wind termination shock, the estimation of the anomalous diffusion coefficient associated with the superdiffusive motion gives acceleration times much smaller than the ones related to normal diffusion.

Key words: acceleration of particles – cosmic rays – diffusion – shock waves – Sun: heliosphere – turbulence

Online-only material: color figures

1. INTRODUCTION

Galactic cosmic rays, as well as heliospheric energetic particles, are usually thought to be accelerated at shock waves by a first order Fermi process called diffusive shock acceleration (DSA; Bell 1978; Fisk & Lee 1980; Drury 1983; Blandford & Eichler 1987). Recently, X-ray and γ-ray observations of supernova remnants (SNRs) have confirmed the presence of highly energetic particles at SNR shocks (e.g., Aharonian et al. 2004; Ackermann et al. 2013). DSA is based on the acceleration of particles crossing the shock many times because of their random, diffusive motion: at each shock crossing, particles gain an amount of energy and momentum corresponding to Fermi head-on collisions. The typical scenario consists of an infinite planar shock and leads to a power-law energy spectrum, which, for ultrarelativistic particles, has a spectral index

$$\gamma = \frac{r+2}{r-1}, \tag{1}$$

where $r = V_u/V_d$ is the shock compression ratio and V_u and V_d are the upstream and downstream plasma velocities in the shock frame, respectively. Also, an acceleration time t_{acc} can be estimated as (e.g., Drury 1983; Gaisser 1990)

$$t_{\mathrm{acc}} = \frac{3}{V_u - V_d}\left(\frac{D_u}{V_u} + \frac{D_d}{V_d}\right), \tag{2}$$

where D_u (D_d) is the upstream (downstream) particle diffusion coefficient. Although DSA is currently the standard model for the acceleration of high energy particles, many details are not clear (see, for instance, Balogh et al. 2013; Giacalone 2013) and several observations remain unexplained. For instance, assuming the maximal compression ratio $r = 4$, Equation (1) yields $\gamma = 2$, which matches nicely the observed spectrum of Galactic cosmic rays up to $\sim 10^{15}$ eV. However, harder spectral indices are deduced for relativistic electrons from radio observations of shell type SNRs (e.g., Bogdan et al. 1985; Whiteoak & Green 1996). In interplanetary space, it is possible to check Equation (1) by measuring the compression ratio r and the spectral index γ in situ. In spite of the large experimental uncertainties on r and γ, a broad agreement with the predictions of DSA is obtained (Giacalone 2012). Nevertheless, spectral indices harder than expected are also found (van Nes et al. 1984; Lee et al. 2012). Furthermore, the recent crossing of the solar wind termination shock by the *Voyager 2* spacecraft has shown that the observed spectral index of the termination shock particles (i.e., ions up to a few MeV, not to be confused with anomalous cosmic rays) is harder than that corresponding to the observed compression ratio $r \simeq 2$ (Decker et al. 2008). This discrepancy persists even when the spatially extended slowing down of the solar wind, which occurs well ahead of the shock and which increases the effective compression ratio to $r \simeq 2.4$, is taken into account (Florinski et al. 2009; Perrone et al. 2013). On the other hand, questions about the acceleration time, Equation (2), also arise. For instance, Lagage & Cesarsky (1983) have shown that such a time can be too long to explain the acceleration of cosmic rays up to energies of 10^{15} eV at SNRs (see also Kirk & Dendy 2001; Ptuskin et al. 2010).

These and other observations have stimulated a number of studies to bring DSA predictions closer to the observed properties; such studies go from the inclusion of second order Fermi acceleration in the cosmic ray transport equation (Droge et al. 1987) to the inclusion of upstream and downstream free escape boundaries (Ostrowski & Schlickeiser 1996) to the motion of the "scattering centers," e.g., Alfvén waves, with respect to the bulk plasma flow (Vainio & Schlickeiser 1999; Sokolov et al. 2006), to the amplification of the upstream magnetic field because of the streaming cosmic rays (Bell 2004; Ptuskin et al. 2010; Blasi et al. 2012). Although those corrections correspond to important physical effects that have to be taken into account by a fully developed theory, here we consider a more radical extension of the basic model of DSA to the case of anomalous, superdiffusive transport, i.e., the case when the mean square displacement of an energetic particle grows superlinearly with time. We consider such an extension to be more radical because the fundamental statistical mechanism of interaction between particles and the shock is modified. Indeed, the superdiffusive transport regimes are based on Lévy statistics, characterized by probability distributions with power-law tails, rather than by Gaussian statistics. In the last few years, anomalous transport has been found in a large variety of physical systems (Metzler & Klafter 2000, 2004; Klafter & Sokolov 2011) that are characterized by a mean square displacement growing as

$$\langle \Delta x^2 \rangle = 2 D_\alpha t^\alpha, \tag{3}$$

with $\alpha < 1$ in the case of subdiffusion, $1 < \alpha < 2$ in the case of superdiffusion, and $\alpha = 2$ representing the ballistic regime. Such anomalous regimes are a generalization of the normal diffusion regime, where $\langle \Delta x^2 \rangle \propto t$, and are obtained when long range correlations and memory effects in space and/or time are involved (for a quick overview, see Shlesinger et al. 1993 and Klafter et al. 1996). Numerical simulations of charged particle transport in the presence of magnetic turbulence show that superdiffusion along the average magnetic field is possible: in particular, Zimbardo et al. (2006) and Pommois et al. (2007) found that superdiffusion parallel to the magnetic field is obtained for low magnetic turbulence levels and when the turbulence is either isotropic or anisotropic with a quasi-slab spectrum (i.e., when the turbulence wavevectors are mostly aligned with the background magnetic field). Conversely, for the same turbulence level but quasi-two-dimensional (2D) anisotropy (i.e., when the turbulence wavevectors are mostly perpendicular to the background magnetic field), normal diffusion is obtained because of an increased level of pitch angle diffusion. On the other hand, Shalchi & Kourakis (2007) found parallel superdiffusion also for a composite 20% slab and an 80% 2D turbulence model. Further, Tautz (2010) found that including the time dependence of turbulence induces parallel superdiffusion (see also Zimbardo et al. 2012). Nondiffusive transport also attracts significant attention in the fusion plasma field (e.g., Mier et al. 2008; Gustafson et al. 2012, and many others). Looking at experimental measurements in the solar wind, evidence of electron transport corresponding to "a wide variety of propagation modes [...] ranging from diffusive to essentially scatter free" was already found by Lin (1974, p. 189) by analyzing solar nonrelativistic electron time profiles. More recently, evidence of electron and ion superdiffusion upstream of interplanetary shocks was reported by Perri & Zimbardo (2007, 2008, 2009a, 2009b) and Sugiyama & Shiota (2011).

The influence of subdiffusion on DSA was considered by Duffy et al. (1995) and Kirk et al. (1996), while a first account of the influence of superdiffusion on DSA has been given by Perri & Zimbardo (2012a). Considering Fermi acceleration in the presence of shocks, but assuming superdiffusion rather than normal diffusion, and following the technique used in Duffy et al. (1995) and Kirk et al. (1996), superdiffusion modifies the main predictions of DSA in a substantial way, i.e., the energy spectral index (see Equation (1)) and the acceleration time (Equation (2)). Therefore, the new theory, which we termed superdiffusive shock acceleration (SSA), has the potential to explain a number of observations of hard spectral indices and maximal cosmic ray energies (Perri & Zimbardo 2012a).

The theoretical description of anomalous transport involves the use of a variety of tools like non-Gaussian statistics, Lévy flights and Lévy walks, long range correlations, Hurst exponents, and fractional derivatives (e.g., Eule et al. 2012; Perrone et al. 2013), but it is not always clear which model is the most appropriate to describe a specific physical system. In this paper, we give arguments to show that the appropriate framework for the transport of cosmic rays and energetic particles is that of Lévy walks and we give a more comprehensive and detailed account of SSA than earlier reported. To this end, we shall give just the theoretical background that is actually needed for deducing the spectral index and the acceleration time envisaged by SSA; we refer the reader to the original works for a more thorough treatment. The basic properties of the Lévy random walk will be described and the resulting superdiffusive transport and non-Gaussian propagator will be derived. The scaling properties of the propagator, which are crucial for obtaining the new expression of the spectral index (Kirk et al. 1996), are discussed in detail. Then, we show how this yields the spectral indices, both in the ultrarelativistic and in the nonrelativistic cases, and a new profile for the accelerated particle density across the shock. We give for the first time a detailed expression for the anomalous diffusion coefficient D_α. We note in particular that D_α enters the value of the acceleration time in the superdiffusive case; furthermore, knowledge of its value is required to determine the actual transport properties of, e.g., solar energetic particles. We also show how the values of α and D_α, which has physical dimensions ℓ^2/t^α, can be determined from the energetic particle profiles observed in situ at heliospheric shocks.

2. COSMIC RAY PROPAGATION AND LÉVY WALKS

Before discussing superdiffusion in terms of Lévy walks, we consider here the minimal elements necessary for cosmic rays or energetic particles to attain a transport regime different from the normal one. We can describe a random transport process like the sum of random displacements x_i (in 1D), each step requiring a time t_i. Assuming equal probability ψ for positive and negative displacements, $\psi(x_i) = \psi(-x_i)$, the expected value (the mean) is zero. Then, if the variance of x_i is finite, $\sigma^2 = \int x^2 \psi(x) dx < \infty$, the central limit theorem (CLT) requires that the limit distribution of the random walker probability density be a Gaussian of width $2Dt$, with $D = \sigma^2/\tau$ and τ being the (finite) average of t_i. Therefore, the finite value of σ^2 means that there is a well defined transport scale and normal diffusion with $\langle \Delta x^2 \rangle = 2Dt$ is obtained. As a consequence, in order to be able to obtain superdiffusion, the assumption of finite σ^2 has to be relaxed (e.g., Klafter et al. 1987; Metzler & Klafter 2000, 2004). This immediately implies that there is no typical transport scale and that the particle mean free path and the standard diffusion coefficient are diverging. From the definition of σ^2, and assuming that $\psi(x)$ is well behaved for small $|x|$, the divergency of σ^2 is obtained when the probability of displacement lengths has power-law tails of the form $\psi(x) \sim |x|^{-\mu}$ with $\mu \leqslant 3$. The generalized CLT then requires that the limit distribution be a symmetric Lévy distribution of index $\mu - 1$, whose Fourier transform can be expressed as $\hat{L}_{\mu-1}(k,t) = \exp(-Ct|k|^{\mu-1})$, where C is a scale parameter (e.g., Metzler & Klafter 2000; Zaslavsky 2002). The explicit Fourier inversion of the Lévy distribution can only be performed in some cases; even so, for $1 < \mu < 3$, it has power-law tails of the form $L_{\mu-1}(x,t) \simeq Ct|x|^{-\mu}$ (e.g., Zaslavsky 2002; Metzler & Klafter 2004). Considering the symmetric, zero-average case so that $\langle \Delta x^2 \rangle = \langle x^2 \rangle$, this result has the implication that the mean square displacement $\langle x^2(t) \rangle = \int x^2 L_{\mu-1}(x,t)dx$ is diverging, too, for any finite time t. In order to avoid such an unphysical result, it is necessary to consider a process called the Lévy walk, which is characterized by a coupling between the displacement length x_i and the displacement duration t_i (Geisel et al. 1985; Shlesinger & Klafter 1985; Shlesinger et al. 1987; Klafter et al. 1987). In such a way, it is not possible to have arbitrarily long displacements in a finite time (something which would be unphysical) and this eliminates the divergence of $\langle x^2(t) \rangle$, while superdiffusive transport can still be obtained, as shown below. In summary, the essential ingredients to describe superdiffusion in terms of a stochastic process are (1) a probability of free path lengths $\psi(x)$ with power-law tails such that the variance σ^2 is diverging (so that a Gaussian process is no longer required by the CLT) and (2) a space–time coupling of the free path lengths x and the free path durations t, e.g., $|x| = vt$, so that $\langle x^2(t) \rangle$ is not diverging at finite times. Such a space–time coupling corresponds well to energetic particles moving with a given speed v. We remark that decoupled random walks, like for instance Lévy flights where the free path length is independent of the corresponding time, are not appropriate to describe the transport of particles having mass, since Lévy flights may imply infinite velocities.

For energetic particles of speed v in a magnetic field, we are concerned mostly with parallel transport, since this usually prevails (see the discussion in Perri & Zimbardo 2012b), even though perpendicular motion can be anomalous, too (e.g., Zimbardo 2005). For the velocity parallel to the magnetic field, we have $v_\parallel = v \cos\theta$, where θ is the pitch angle. Making the assumption of a nearly isotropic velocity distribution function—a standard assumption of DSA that is also made for SSA—one has $\langle v_\parallel^2 \rangle = v^2/3$. Therefore, there exists a typical, well-defined speed for parallel motion and the constant velocity model is appropriate to describe the parallel transport of energetic particles with an isotropic velocity distribution function. This situation can be contrasted with other descriptions of anomalous transport that are based on a power-law distribution of velocities (e.g., Mier et al. 2008). About notation, although in practice we are considering the rms value of $v_\parallel$, in the following we will write v for brevity.

3. THE DERIVATION OF SUPERDIFFUSION IN TERMS OF THE LÉVY RANDOM WALK

In order to show how superdiffusion modifies the properties of DSA, we consider the simplest case, making the standard assumption of an infinite planar shock, so that the system only depends on the coordinate x perpendicular to the shock. Also, the average magnetic field is assumed to be parallel to the shock normal, so that we will mostly discuss transport parallel to the magnetic field (i.e., along x), keeping in mind that the case of oblique shocks can be described in the same way (e.g., Drury 1983), except for shock normal angles θ_{Bn} very close to 90°.

Here, we make use of the microscopic, probabilistic approach introduced by Montroll & Weiss (1965), which is called continuous time random walks, and extended to Lévy walks by Shlesinger et al. (1987), Klafter et al. (1987), and Zumofen & Klafter (1993); in this case, a particle moves at constant speed until it reaches a point at which it changes direction of motion randomly (the so-called velocity model). For this purpose, let us introduce the probability, which defines Lévy walks, of performing a jump of length $|x|$ in a time interval t:

$$\psi(x,t) = \tfrac{1}{2}\delta(|x| - vt)\psi(t), \tag{4}$$

where v is the particle speed and, for acceleration at a planar shock, we assume 1D motion along the x direction; nevertheless, this methodology can be extended to the fully 3D motion.

The delta function in Equation (4) plays the role of coupling space and time, which is the qualifying property of Lévy walks, so that very long displacements require more time (Shlesinger et al. 1987; Klafter et al. 1987; Metzler & Klafter 2004). Furthermore, $\psi(x,t)$ is normalized to one: $\int \psi(x,t)dxdt = 1$. Thus, from Equation (4), we can define the probability $W(x,t)$ for particles to pass through x at time t in a single motion step from the origin, even without stopping or changing direction in x (Shlesinger et al. 1987):

$$W(x,t) = \delta(|x| - vt) \int_t^\infty dt' \int_{|x|}^\infty dx' \psi(x',t'). \tag{5}$$

This expression clarifies also that particles going farther than $|x|$ in a single step contribute to the probability density of being at x at time t, thus emphasizing the importance of the time spent in the walk as compared with models that envisage instantaneous jumps between the random walk sites. A complementary quantity is the probability density $Q(x,t)$ of just arriving at a point x at time t, coming from "anywhere," and then changing the direction of motion randomly to perform the next step. Assuming homogeneity in space and time, one has

$$Q(x,t) = \int_{-\infty}^\infty dx' \int_0^t dt' Q(x-x', t-t')\psi(x',t') + \delta(x)\delta(t), \tag{6}$$

where the initial condition ($x = 0, t = 0$) gives rise to the delta functions. Thus, only those particle displacements of length $(x - x', t - t')$ contribute to the integral in Equation (6); in the case that a particle does not move at all, Equation (6) reduces to $Q(x,t) = \delta(x)\delta(t)$. In the wording of Shlesinger et al. (1987), the points where the direction of motion changes are called "turning points," therefore $Q(x,t)$ is the probability of being a turning point. The probability density for particles to either stop or pass through the location x at time t, namely the propagator, is therefore obtained as (Zumofen & Klafter 1993)

$$P(x,t) = \int dx' \int_0^t dt' Q(x-x', t-t')W(x',t'). \tag{7}$$

Let us now obtain the propagator in Fourier–Laplace space. As usual, we have

$$P(k,s) = \int_0^\infty dt \exp(-st) \int_{-\infty}^\infty dx \exp(-ikx) P(x,t), \tag{8}$$

where we adopt the convention that the argument k indicates the Fourier transform and the argument s the Laplace transform. Substituting Equation (7) in Equation (8) and adopting $\xi = x - x'$, we can rewrite

$$\begin{aligned} P(k,s) = &\int_0^\infty dt \exp(-st) \int_{-\infty}^\infty d\xi \int_{-\infty}^\infty dx' \int_0^t dt' \exp(-ik\xi) \\ &\times \exp(-ikx') Q(\xi, t-t') W(x',t') \\ = &\int_0^\infty dt \int_0^t dt' \exp(-st) Q(k, t-t') W(k,t'). \end{aligned}$$

Considering the property of the Laplace transform for the convolution product of two functions, the above expression reduces to

$$P(k,s) = Q(k,s)W(k,s). \tag{9}$$

If the same procedure is applied to $Q(x,t)$ as given by Equation (6), it is easy to show that $Q(k,s) = Q(k,s)\psi(k,s)+1$, so that

$$Q(k,s) = \frac{1}{1-\psi(k,s)}. \tag{10}$$

Replacing Equation (10) in Equation (9), the Fourier–Laplace transform of the propagator is obtained:

$$P(k,s) = \frac{W(k,s)}{1-\psi(k,s)}. \tag{11}$$

This expression can be considered to be the extension of the Montroll–Weiss equation (Montroll & Weiss 1965; Ragot & Kirk 1997; del-Castillo-Negrete et al. 2004) to the case of Lévy walks (Zumofen & Klafter 1993), the main differences with the classical case being $W(k,s)$ and the coupled form of $\psi(k,s)$. It is interesting to note that the Montroll–Weiss equation was already considered in the astrophysical literature in connection with the transport of high energy electrons (Ragot & Kirk 1997); however, the use of a decoupled jump probability $\psi(x,t) = f(x)\psi(t)$ leads to problematic results, such as the divergence of $\langle x^2(t)\rangle$ at finite times (Klafter et al. 1987), when the variance σ^2 is diverging. In the case that the mean scattering time and the variance of $\psi(x,t)$ are finite, normal diffusion and the Gaussian propagator are readily recovered from the Montroll–Weiss equation (in the astrophysical literature, see, e.g., Webb et al. 2006 and Bian & Browning 2008).

The first quantity we want to obtain is the mean square displacement, namely the second order moment of the probability distribution function $P(x,t)$, $\langle x^2(t)\rangle = \int dx\, x^2 P(x,t)$. After Fourier transforming and straightforward calculations, the expression to study is

$$\langle x^2(t)\rangle = -\int_{-\infty}^{\infty} \delta(k)\frac{\partial^2 P(k,t)}{\partial k^2}dk = -\left.\frac{\partial^2 P(k,t)}{\partial k^2}\right|_{k=0}.$$

In order to calculate the mean square displacement, it is necessary to find an explicit expression for the propagator in Equation (11), so that we have to obtain $W(k,s)$ and $\psi(k,s)$ in Fourier–Laplace space. Proper forms of $W(k,s)$ and $\psi(k,s)$ for the Lévy walk jump probability, Equation (4), are derived in the Appendix. Applying the Laplace transform to the equation for $\langle x^2(t)\rangle$, we have

$$\begin{aligned}\langle x^2(s)\rangle &= -\left.\frac{\partial^2 P(k,s)}{\partial k^2}\right|_{k=0} = -\frac{\partial^2}{\partial k^2}\left[\frac{W(k,s)}{1-\psi(k,s)}\right]_{k=0}\\ &= \left[-\frac{\partial^2 W(k,s)}{\partial k^2}\frac{1}{1-\psi(k,s)}\right.\\ &\quad \left.-\frac{W(k,s)}{(1-\psi(k,s))^2}\frac{\partial^2\psi(k,s)}{\partial k^2}\right]_{k=0},\end{aligned} \tag{12}$$

where we have taken into account that $\partial W(k,s)/\partial k = 0$ and $\partial\psi(k,s)/\partial k = 0$ in the limit of $k \to 0$, as shown in the Appendix. We note that the $\delta(k)$ in the above equations requires the computation of the derivatives for $k = 0$: this means that in the Fourier–Laplace transform of W and ψ, we can make an expansion of $\exp(ikx)$ for small k, do the derivative, and then take the limit $k \to 0$. Exploiting all the terms in Equation (12) in the limit of $k \to 0$ and making use of the expressions for $W(k,s)$, $\psi(k,s)$, $W''(k,s)$, and $\psi''(k,s)$ found in the Appendix, it is easy to show that

$$\langle x^2(s)\rangle = \frac{2Av^2t_0^{\mu}}{\tau}\frac{\Gamma(3-\mu)}{\mu-1}s^{\mu-5} \equiv 2D_\mu s^{\mu-5}, \tag{13}$$

where A is a normalization constant, v is the particle speed, t_0 and τ are defined below, Γ is Euler's gamma function, and D_μ is the anomalous diffusion coefficient in Fourier–Laplace space. After inverse Laplace transforming $\langle x^2(s)\rangle$, we end up with

$$\langle x^2(t)\rangle \sim t^{4-\mu} \equiv t^\alpha, \tag{14}$$

where $\alpha = 4-\mu$, which describes a superdiffusive process for $2 < \mu < 3$ (Geisel et al. 1985; Klafter et al. 1987; Zumofen & Klafter 1993). We note that while the limits $\mu = 3$ and $\mu = 2$ correspond to normal diffusion and ballistic transport, respectively, some logarithmic corrections in the mean square displacement and different forms of the propagator are obtained for those limits (e.g., Zumofen & Klafter 1993); anyway, we do not describe those cases here.

It is interesting to study the trend of the anomalous diffusion coefficient as a function of the anomalous diffusion exponent α that describes the superdiffusive process. From Equation (13), it is possible to re-write $D_\alpha = (Av^2/\tau)t_0^{4-\alpha}\Gamma(\alpha-1)/(3-\alpha)$. Thus, the value of D_α does depend on the mean scattering time of particles, namely τ. The latter is defined as $\tau = \int t\psi(x,t)dx\, dt = (1/2)\int \delta(|x|-vt)\psi(t)\, t\, dx\, dt = \int_0^\infty t\psi(t)dt$, after using Equation (4) and exploiting the delta function. Therefore, it is necessary to have an explicit functional form for the distribution of the particle jump times $\psi(t)$ to determine τ and then D_α. We make the following choice:

$$\psi(t) = \begin{cases} A & t \leqslant t_0 \\ A\left(\dfrac{t}{t_0}\right)^{-\mu} & t > t_0 \end{cases}. \tag{15}$$

In other words, t_0 is a scale parameter such that for times $t > t_0$, ψ corresponds to a power-law distribution of the particle jump times. Here, we choose $\psi(t)$ to be constant for $t < t_0$, although other choices are possible. Thus,

$$\begin{aligned}\tau &= A\left[\int_0^{t_0} t dt + \int_{t_0}^{\infty} t\left(\frac{t}{t_0}\right)^{-\mu} dt\right]\\ &= A\left[\frac{t_0^2}{2} + t_0^{\mu}\frac{t_0^{2-\mu}}{\mu-2}\right]\\ &= At_0^2\frac{\mu}{2(\mu-2)} \equiv At_0^2\frac{4-\alpha}{2(2-\alpha)}.\end{aligned} \tag{16}$$

Now the normalization constant A can be determined by the requirement $\int\psi(x,t)dxdt = 1$, which gives $A = (\mu-1)/\mu t_0$, and finally

$$\tau = \frac{\mu-1}{2(\mu-2)}t_0 = \frac{3-\alpha}{2(2-\alpha)}t_0. \tag{17}$$

Inserting Equation (16) in the expression for the anomalous diffusion coefficient D_α, we get

$$D_\alpha = \frac{2(2-\alpha)}{(3-\alpha)(4-\alpha)}\Gamma(\alpha-1)v^2t_0^{2-\alpha}. \tag{18}$$

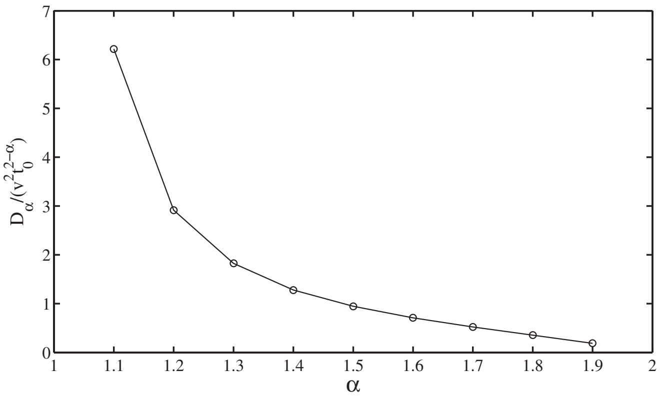

Figure 1. Anomalous diffusion coefficient D_α as a function of the anomalous diffusion exponent α. D_α has been normalized to the physical dimensional value of $v^2 t_0^{2-\alpha}$.

It is worth noting that, to our knowledge, this anomalous diffusion coefficient has never explicitly been given in the literature: on the one hand, the interest in understanding the origin of anomalous transport has been so strong that most attention has gone into deriving Equation (14) for the time evolution of $\langle x^2\rangle$. On the other hand, D_α is considered to be a "nonuniversal prefactor" because it depends on the form of $\psi(t)$ for small t: provided that $\psi(t)$ is well behaved for $t \to 0$, its specific form is not important for obtaining the anomalous diffusion exponent $\alpha = 4 - \mu$, to the point that $\psi(t)$ for small t is not even discussed in most papers on anomalous transport. However, we are interested in having an explicit expression of D_α because this influences the acceleration time (e.g., Duffy et al. 1995; Perri & Zimbardo 2012a) and finally the maximum achievable cosmic ray energy. We also note that, apart from factors of order 1, $D_\alpha \sim v^2 t_0^{2-\alpha}$, corresponding to the appropriate physical dimensions required by Equation (3). Furthermore, we can introduce a typical length $\ell_0 = vt_0$, which is uniquely defined by the delta function in Equation (4) as the length beyond which the distribution of free path lengths is a power law. Then, we can also write $D_\alpha \sim \ell_0^{2-\alpha} v^\alpha$. This expression can be "compared" with the normal diffusion coefficient $D \simeq \frac{1}{3}\lambda v$ (although the physical dimensions are clearly different): for particles of a given energy, the speed v is a known quantity, so that in the case of normal diffusion, the knowledge of the mean free path λ is sufficient to determine the diffusion coefficient. Conversely, in the case of anomalous diffusion, one needs to know both the scale length ℓ_0 and the anomalous diffusion exponent α. As shown later, the analysis of the energetic particle profile upstream of interplanetary shocks can yield both these quantities. The trend of D_α as a function of α in the case of superdiffusion is shown in Figure 1. One can see that D_α decreases as α increases.

4. THE PROPAGATOR FOR LÉVY WALKS

The next quantity we want to obtain is the particle propagator for a Lévy walk. The general form of the propagator in Fourier–Laplace space is given by Equation (11). To obtain an analytical expression, we follow Zumofen et al. (1990) and Zumofen & Klafter (1993) and make an expansion for small k and small s, which corresponds in physical space to the large $|x|$ and large t regime, i.e., to the time asymptotic regime. We note, however, that the condition $kv \ll s$, implicitly used above to obtain the mean square displacement, is not required here (see Zumofen et al. 1990 and Blumen et al. 1990 for a discussion of this point). In what follows, we only retain the leading (lowest order) terms in k and s. It is easy to see that the leading term of $W(k,s)$ in k is proportional to k^2 (see the Appendix); since a lower order term in k is coming from $\psi(k,s)$, we can take $W(k=0,s)$ in Equation (11). Then, at the lowest order in k, $P(k,s)$ can be further simplified, since

$$W(0,s) = \int_0^\infty \exp(-st)\,dt \int_t^\infty \psi(t')\,dt'$$
$$= \int_0^\infty dt \exp(-st)\left[1 - \int_0^t \psi(t')\,dt'\right] = \frac{1-\psi(s)}{s}$$

(also see the Appendix). Therefore, Equation (11) reduces to the classical form of the Montroll–Weiss equation, i.e., $P(k,s) \sim (1-\psi(s))/[s(1-\psi(k,s))]$ (Klafter et al. 1987; Ragot & Kirk 1997). In Fourier–Laplace space, we have for the coupled jump probability $\psi(x,t)$

$$\psi(k,s) = \frac{1}{2}\int_{-\infty}^{\infty} dx \exp(-ikx)\int_0^\infty dt \exp(-st)\delta(|x|-vt)\psi(t)$$
$$= \int_0^\infty dt \exp(-st)\frac{\exp(ikvt)+\exp(-ikvt)}{2}\psi(t)$$
$$= \int_0^\infty dt \frac{\exp(-qt)+\exp(-\bar{q}t)}{2}\psi(t). \tag{19}$$

In Equation (19), a compact notation has been adopted by introducing the complex variables $q = s + ikv$ and $\bar{q} = s - ikv$. Let us now manipulate Equation (19) by adding and subtracting the terms qt and $\bar{q}t$ and adding and subtracting two times the factor 1 in the integrand. In the case of a Lévy process where the probability distribution of the particle jump times is a power-law decay, $\psi(t) = A(t/t_0)^{-\mu}$ with $2 < \mu < 3$, this leads to

$$\psi(k,s) = \frac{1}{2}\int_0^\infty dt[\exp(-qt) - 1 + qt]A\left(\frac{t}{t_0}\right)^{-\mu}$$
$$+ \frac{1}{2}\int_0^\infty dt[1-qt]A\left(\frac{t}{t_0}\right)^{-\mu}$$
$$+ \frac{1}{2}\int_0^\infty dt[\exp(-\bar{q}t) - 1 + \bar{q}t]A\left(\frac{t}{t_0}\right)^{-\mu}$$
$$+ \frac{1}{2}\int_0^\infty dt[1-\bar{q}t]A\left(\frac{t}{t_0}\right)^{-\mu}.$$

The terms without exponentials yield

$$\frac{1}{2}\int_0^\infty dt[1-qt]A\left(\frac{t}{t_0}\right)^{-\mu} \equiv \frac{1}{2}\int_0^\infty dt[1-qt]\psi(t)$$
$$= \frac{1}{2}(1-q\tau),$$

where, as discussed above, $\tau = \int_0^\infty dt\ t\psi(t)$ is the mean scattering time. Also, the other terms are now converging even for $t \to 0$ and they can be manipulated as

$$\frac{1}{2}\int_0^\infty dt[\exp(-qt) - 1 + qt]A\left(\frac{t}{t_0}\right)^{-\mu}$$
$$= \frac{A}{2t_0^{-\mu}}q^{\mu-1}\int_0^\infty y^{-\mu}[\exp(-y) - 1 + y]\,dy$$
$$= \frac{A}{2t_0^{-\mu}}q^{\mu-1}\Gamma(1-\mu),$$

having applied the change of variable $y = qt$ and taken into account the Cauchy–Saalschütz's formula for the Euler gamma function of a negative argument up to order $n = 1$, $\Gamma(z) = \int_0^\infty x^{z-1}[\exp(-x) - \sum_{m=0}^{n}(-1)^m(x^m/m!)]\,dx$ (Whittaker & Watson 1927). Since similar relationships can be easily found for the terms with $\bar{q}$, the probability distributions of the jump lengths in Fourier–Laplace space eventually give

$$\psi(k,s) = 1 - \frac{1}{2}(q+\bar{q})\tau + \frac{A}{2t_0^{-\mu}}\Gamma(1-\mu)(q^{\mu-1}+\bar{q}^{\mu-1})$$
$$= 1 - s\tau + \frac{A}{2t_0^{-\mu}}\Gamma(1-\mu)[(s+ikv)^{\mu-1}+(s-ikv)^{\mu-1}].$$

If we retain the lowest orders both in s and k, considering that $1 < \mu - 1 < 2$, we get

$$\psi(k,s) \simeq 1 - \tau s + C_2|k|^{\mu-1}, \tag{20}$$

with

$$C_2 = \frac{A\Gamma(1-\mu)}{2t_0^{-\mu}}[(\exp(i\pi/2))^{\mu-1} + (\exp(-i\pi/2))^{\mu-1}]v^{\mu-1}$$
$$\equiv \frac{A\Gamma(1-\mu)}{t_0^{-\mu}}\cos\left[\frac{\pi}{2}(\mu-1)\right]v^{\mu-1}.$$

Finally, it is possible to derive the particle propagator in Fourier–Laplace space

$$P(k,s) \sim \frac{1-\psi(s)}{s}\frac{1}{1-\psi(k,s)} = \frac{1-(1-\tau s)}{s(1-(1-\tau s + C_2|k|^{\mu-1}))}$$
$$= \frac{\tau}{\tau s - C_2|k|^{\mu-1}} = \left(s - \frac{C_2|k|^{\mu-1}}{\tau}\right)^{-1}, \tag{21}$$

which corresponds to Equation (86) in Zumofen & Klafter (1993). By applying the inverse Laplace transform to Equation (21), we get

$$P(k,t) \simeq \exp\left(\frac{C_2|k|^{\mu-1}}{\tau}t\right). \tag{22}$$

Note that the constant C_2 given above is always less than zero for $2 < \mu < 3$. The scaling properties that follow from this expression are discussed in the next section. The Fourier transform of the propagator just obtained has the same form as the Fourier transform of the Lévy distribution, given in Section 2. However, $P(k,t)$ above is just the lowest order expansion for small k and s and other terms can be obtained, which make the Lévy walk $P(k,t)$ different from $\hat{L}_{\mu-1}(k,t)$. In particular, higher order terms in k, within the exponential in Equation (22), would lead to a steeper power-law decay of the tails of the propagator in physical space. Therefore, the propagator for Lévy walks differs from the Lévy stable laws, especially for large $|x|$. The space–time coupling of Lévy walks has two effects on the propagator: (1) $P(x,t) = 0$ for $|x| > vt$, since no particle can travel farther than vt in a time t and (2) the power-law tails of $P(x,t)$ are somewhat steeper than $L_{\mu-1}(x,t) \sim t/\tau C_2|x|^{-\mu}$. Indeed, Zumofen et al. (1990) and Blumen et al. (1990) obtained the propagator for different values of μ by a numerical inversion of Equation (11) and found that the propagator tails actually decay slightly faster than $|x|^{-\mu}$ and then smoothly go to zero for $|x| \simeq vt$. The fact that $P(x,t) = 0$ for $|x| > vt$ is important to avoid the divergency of $\langle x^2(t)\rangle = \int x^2 P(x,t)dx$, which affects Lévy stable laws, making them unsuitable for describing the superdiffusion of particles with a finite velocity. Furthermore, the faster decay of the tails implies that a correction has to be made to the technique to extract the transport properties from the slope of the energetic particle profile upstream of interplanetary shocks, developed by Perri & Zimbardo (2007, 2008). Inspection of Figure 2 of Zumofen et al. (1990) suggests that this correction is on the order of -0.2 for the exponent μ in Equation (35). However, this correction may depend on the value of μ and we reserve a more detailed analysis of this point for future work.

4.1. Scaling Properties of the Particle Propagator

We will now derive the scaling properties of the particle propagator shown in Equation (22) for an arbitrary scale transformation of space and time. The stability of Lévy distributions for general scale transformations has been discussed, for instance, in Consolini et al. (2005). In order to obtain the scaling properties of $P(x,t)$, i.e., of the particle propagator in real space, let us consider the following general transformations (Consolini et al. 2005)

$$x \to \beta^a x, \quad t \to \beta^b t, \tag{23}$$

where $a, b \in \Re$, and β are positive, dimensionless transformation parameters. Thus,

$$P(\beta^a x, \beta^b t) = \int dk \exp(ikx\beta^a)\exp\left(\frac{C_2|k|^{\mu-1}}{\tau}t\beta^b\right).$$

Considering the change of variable $k = \beta k'$, it is easy to obtain

$$P(\beta^a x, \beta^b t) = \beta\int dk' \exp(ik'x\beta^{a+1})\exp\left(\frac{C_2|k'|^{\mu-1}}{\tau}t\beta^{b+\mu-1}\right).$$

Assuming $a = -1$ and $b = 1-\mu$, the propagator simply becomes

$$P\left(\frac{x}{\beta}, \beta^{1-\mu}t\right) = \beta\int dk' \exp(ik'x)\exp\left(\frac{C_2|k'|^{\mu-1}}{\tau}t\right)$$
$$= \beta P(x,t).$$

Considering the time t_0 and the length ℓ_0 defined in Section 3, we introduce the dimensionless variables $\hat{t} = t/t_0$ and $\hat{x} = x/\ell_0$. Then, if we set $\beta = \hat{t}^{1/(\mu-1)}$, we obtain

$$P\left(\frac{\hat{x}\ell_0}{\hat{t}^{1/(\mu-1)}}, t_0\right) = \beta P(x,t) = \hat{t}^{1/(\mu-1)}P(x,t),$$

that is, we obtain a scaling variable $\xi = \hat{x}/\hat{t}^{1/(\mu-1)}$ and, consequently, a scaling function $f(\xi) = P(\xi\ell_0, t_0)$. Finally, we get a particle propagator for a superdiffusive process that has the scaling property (Zumofen & Klafter 1993)

$$P(x,t) = \frac{f(\xi)}{\hat{t}^{1/(\mu-1)}}. \tag{24}$$

We point out that such a scaling depends solely on the properties of the Fourier transform of $P(x,t)$, Equation (22).

4.2. Spectral Index for Superdiffusive Shock Acceleration

Kirk et al. (1996) have shown that the change in the energy spectral index of particles accelerated at shocks depends precisely on the scaling properties of the particle propagator.

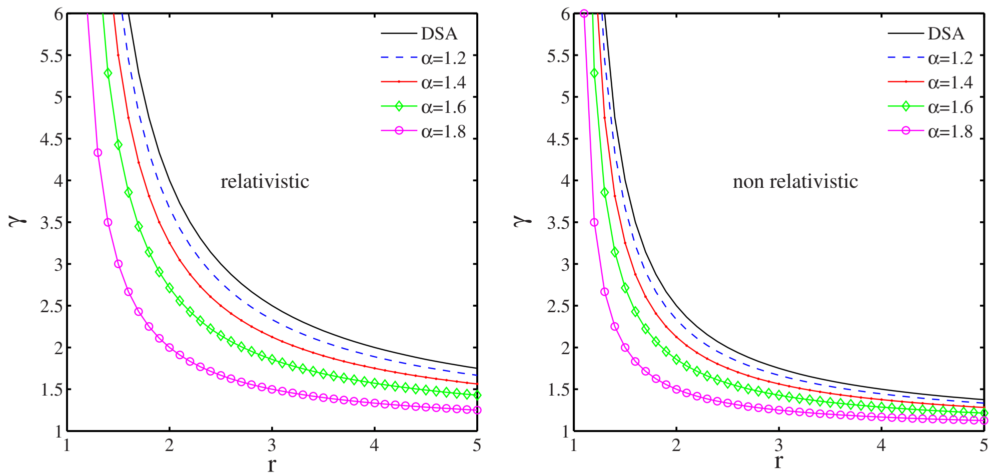

Figure 2. Energy spectral index as a function of the compression ratio of the shock. The spectral index predicted by SSA for different values of the exponent α of the mean square displacement are compared with the DSA prediction (black solid lines) both for the relativistic (left panel) and for the nonrelativistic (right panel) cases.
(A color version of this figure is available in the online journal.)

Following Kirk et al. (1996), Perri & Zimbardo (2012a) explicitly derived the ratio of the particle densities at the shock (n_0) and far downstream (n_d) from the shock front via the expression in Equation (24). Indeed, the propagator allows us to compute the particle density as

$$n(x,t) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{t} dt'\, P(x-x', t-t') Q_{\rm sh}(x', t'), \quad (25)$$

where $Q_{\rm sh} = \Phi_0 \delta(x' - V_{\rm sh} t')$ indicates the particle injection at the shock, which is moving with speed $V_{\rm sh}$. Under steady state conditions, the injection flux density Φ_0 equals the flux density advected far downstream, $\Phi_0 = n_d V_d$; also, in the downstream frame, $V_{\rm sh} = V_d$. The density at the shock can be obtained by computing $n_0 = n(x = V_{\rm sh} t, t)$ and, integrating over x', one obtains

$$n_0 = n_d V_d \int_0^{\infty} P(V_{\rm sh} z, z)\, dz, \quad (26)$$

where we introduced $z = t - t'$. Using Equation (24) and changing to the integration variable $\xi = (V_{\rm sh} z/\ell_0)(z/t_0)^{1/(1-\mu)}$, we have

$$n_0 = n_d \frac{V_d}{V_{\rm sh}} \left(\frac{\mu - 1}{\mu - 2}\right) \int_0^{\infty} \ell_0 f(\xi)\, d\xi. \quad (27)$$

Considering that the normalization condition for the propagator can be expressed as $\int_{-\infty}^{\infty} P(x,t)dx = \ell_0 \int_{-\infty}^{\infty} f(\xi) d\xi = 1$, we finally obtain (Perri & Zimbardo 2012a):

$$\frac{n_d}{n_0} = 2\,\frac{\mu - 2}{\mu - 1} = 2\,\frac{2-\alpha}{3-\alpha}. \quad (28)$$

We point out that for $1 < \alpha < 2$ the density far downstream n_d is lower than the density just downstream of the shock, which equals n_0. This result is at variance with the scenario of DSA, which envisages a constant density downstream. Having obtained the ratio of particle densities n_d/n_0, it is easy to derive the probability of leaving the acceleration region (e.g., Gaisser 1990): $P_{\rm esc} = \Phi_d/\Phi_{u\to d}$, where $\Phi_d = n_d V_d$ the particle flux exiting far downstream and $\Phi_{u\to d} = n_0 v/4$ is the incoming particle flux crossing the shock from upstream to downstream (in the shock frame). As shown in, e.g., Drury (1983) and Gaisser (1990), one has $P_{\rm esc} = 4 n_d V_d / n_0 v$, so that the probability for each particle to escape the acceleration region is $P_{\rm esc} = 8(V_d/v)(2-\alpha)/(3-\alpha)$. The first order Fermi acceleration predicts a relative momentum gain for particles making a complete upstream–downstream cycle across the shock given by $\Delta p/p = (4/3)(V_u - V_d)/v$ (e.g., Drury 1983) and, for ultrarelativistic particles where $\Delta E/E = \Delta p/p$, an integral energy spectrum having a slope $\tilde{\gamma} = P_{\rm esc}/(\Delta p/p)$; therefore, the slope of the differential energy spectrum to be compared with that in Equation (1) for DSA is

$$\gamma = \tilde{\gamma} + 1 = \frac{6}{r-1}\,\frac{\mu-2}{\mu-1} + 1 \equiv \frac{6}{r-1}\,\frac{2-\alpha}{3-\alpha} + 1, \quad (29)$$

where we used again the relation between the anomalous diffusion exponent α of superdiffusive motion and μ, $\alpha = 4-\mu$. Basically, the new expression of the spectral index depends on the ratio of densities, Equation (28), which stems from the non-Gaussian properties of the propagator in Equation (25).

The expression in Equation (29) changes if nonrelativistic particles are considered to undergo a first order Fermi acceleration at the shock. Indeed, for nonrelativistic particles $\Delta E/E = 2\Delta p/p$ and, introducing the integral spectral index for momentum $\tilde{\gamma}_p$, it is easy to show that $\tilde{\gamma}_p = 2(\gamma - 1)$, so that

$$\gamma = \frac{3}{r-1}\,\frac{\mu-2}{\mu-1} + 1 \equiv \frac{3}{r-1}\,\frac{2-\alpha}{3-\alpha} + 1. \quad (30)$$

The spectral slopes in Equations (29) and (30) are plotted in Figure 2 as a function of the compression ratio r and for different values of the anomalous diffusion exponent α (see the legend in the panels). A comparison with the slope expected for a normal, Gaussian process, i.e., for DSA, is also given both for the relativistic (left panel) and for the nonrelativistic (right panel) cases. It can be noted that a shock acceleration process based on superdiffusive transport of particles is able to give us values of γ lower than DSA for a fixed value of the compression ratio of the shock. We can understand this result considering that for $2 < \mu < 3$ the escape probability is smaller than in the case of normal diffusion, which is recovered for $\mu = 3$. A smaller escape probability implies a larger return probability to the shock, whose crossings provide acceleration, so that this leads to harder spectral indices than those obtained by DSA. In practice, in Equation (25) the power-law tails of the propagator give, to those particles that are moving upwind from downstream, a larger chance to return to the shock compared with the Gaussian propagator. In a similar way, we can think that the long free paths allowed by $\psi(x, t)$ permit some downstream particles to move effectively against the plasma flow and to meet the shock again.

We can compare our results with those of Kirk et al. (1996). They considered subdiffusive propagation, $\langle x^2(t)\rangle \propto t^\alpha$ with $\alpha < 1$, and propagators depending on a scaling variable $\xi = \hat{x}/\hat{t}^{\alpha/2}$. The corresponding ratio of densities is $n_d/n_0 = 2 - \alpha$, which is different from Equation (28) and which implies that, for subdiffusion, the density far downstream n_d is larger than the density n_0 at the shock. Also, the subdiffusive spectral index turns out to be, for relativistic particles,

$$\gamma_{\rm sub} = \frac{3}{r-1}(2-\alpha) + 1, \tag{31}$$

which yields softer spectral indices than DSA for $\alpha < 1$. Clearly, the above expression gives harder spectral indices for $\alpha > 1$ but, as we have shown, the propagator scaling properties for Lévy walks are different from those for subdiffusion. We also note that a number of different propagators for different transport regimes have been given by Webb et al. (2006).

4.3. *Explicit Forms of the Propagator*

In the previous subsection, we highlighted the scaling properties of the Lévy walk propagator $P(x,t)$ using its Fourier transform. The reason for this way of derivation is that the Fourier back transform can be obtained only for limiting cases (see below), while the scaling properties, being obtained from the pristine Fourier transform, hold for all values of x and t. Here, we give the limiting forms of the propagator in coordinate space, which are obtained by Zumofen et al. (1990) and Zumofen & Klafter (1993). For small values of the scaling variable $\xi = \hat{x}/\hat{t}^{1/(\mu-1)} \ll 1$, one has a modified Gaussian:

$$P(x,t) \simeq \frac{a_0}{t^{1/(\mu-1)}} \exp\left[-a_1\left(\frac{x}{t^{1/(\mu-1)}}\right)^2\right], \tag{32}$$

where

$$a_0 = \Gamma\left(\frac{\mu}{\mu-1}\right) \Big/ \pi\left(\frac{|C_2|}{\tau}\right)^{1/(\mu-1)} \tag{33}$$

and

$$a_1 = \Gamma\left(\frac{3}{\mu-1}\right) \Big/ 2\Gamma\left(\frac{1}{\mu-1}\right)\left(\frac{|C_2|}{\tau}\right)^{2/(\mu-1)}. \tag{34}$$

Conversely, for large distances, $\xi \gg 1$ but $|x| < vt$, one obtains a power law,

$$P(x,t) \simeq \frac{\Gamma(\mu)}{\pi} \sin\left[\frac{\pi}{2}(\mu-1)\right]\frac{|C_2|}{\tau}\frac{t}{|x|^\mu} \tag{35}$$

(see also Zaslavsky 2002). We can see that the value of ξ determines the limiting forms to be used for the propagator.

We note that normal diffusion leads, in the case of a constant diffusion coefficient, to an exponential energetic particle profile upstream of the shock and to a constant density downstream

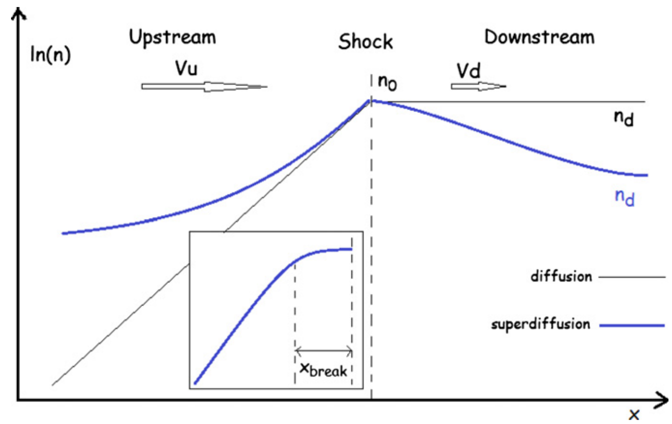

Figure 3. Energetic particle density profile along the shock normal coordinate. The thin black lines represent normal diffusion. The thick blue lines represent superdiffusion. In log–lin axes, the upstream density profile is a straight line for normal diffusion and a bent line for superdiffusion. While the downstream density is constant for normal diffusion, it decreases with distance from the shock for superdiffusion. The inset shown the upstream density profile in log–log axes for superdiffusion: the power-law profile breaks in the vicinity of the shock and this allows us to determine the distance $x_{\rm break}$ (see the text).

(A color version of this figure is available in the online journal.)

(e.g., Drury 1983). Conversely, superdiffusion leads, at some distance upstream of the shock, to a power-law profile as well as to a nonconstant density downstream; see Figure 3. We can see that superdiffusion allows particles to move more effectively against the advective motion of the plasma, so that the energetic particle density is larger than the one for normal diffusion when going upwind from the shock as well as when going from far downstream toward the shock. The above power-law propagator was used by Perri & Zimbardo (2007, 2008) to derive the slope $a = \mu - 2$ of the power-law profile of energetic particles well upstream of the shock. Clearly, the upstream power-law profile is obtained from the power-law form of the propagator inserted into Equation (25). When $\xi \sim 1$, the power-law form of $P(x,t)$ does not hold any more and indeed a break in the energetic particle power-law profile is observed close to the shock; see Figure 3 and experimental data in Perri & Zimbardo (2008, 2009a, 2009b). The distance in space and time of the break from the shock can be used to estimate the length $\ell_0 = vt_0$, as well as the time t_0. Indeed, from $\xi = 1$, we obtain

$$\frac{x}{\ell_0} = \left(\frac{t}{t_0}\right)^{1/(\mu-1)}, \tag{36}$$

so that we can determine ℓ_0 and t_0 for particles of speed v from the observed break distance $x_{\rm break} = V_R t_{\rm break}$ and the observed break time $t_{\rm break}$ (here, V_R is the relative velocity between the shock and the spacecraft). For instance, solving for t_0, we have

$$t_0 = (V_R/v)^{\mu/(\mu-1)} t_{\rm break}. \tag{37}$$

As discussed in Perri & Zimbardo (2009a, 2012a), for nonrelativistic protons accelerated at the solar wind termination shock, the time $t_{\rm break}$ at which the particle time profile upstream of the shock changes to a power-law decay is about 10 days; since $t_0 = \ell_0/v$ and considering protons of 2×10^3 keV (i.e., $v \sim 10^4$ km s^{-1}), the length ℓ_0 in the spacecraft frame has been estimated to be $\ell_0 \sim 1.5 \times 10^5$ km. Note that when one assumes normal particle transport, the diffusion coefficient $D \sim \lambda v$, where λ is the particle mean free path (in the case of superdiffusion, that quantity is diverging). Common estimates of the mean free path for MeV particles in the heliosphere are on the order 1 AU, that is, much larger than ℓ_0 at the termination shock.

5. ANOMALOUS DIFFUSION COEFFICIENT AND PARTICLE ACCELERATION TIME AT A PLANAR SHOCK

In Section 3, we derived the form of the anomalous diffusion coefficient D_α as a function of the exponent α (see Equation (18)). Having obtained D_α, it is possible to compute an expression for the mean acceleration time of particles accelerated at a shock front. In the case of DSA, as discussed in Drury (1983) and Gaisser (1990), the acceleration time can be extrapolated by simple microscopic arguments. More precisely, a permanence time can be defined by considering the ratio between the total number of particles per unit surface in the upstream region N_u and the particle incident flux coming from upstream $\Phi_{u\to d} = nv/4$ (under the assumption of particle isotropy), namely $t_u^* = N_u/\Phi_{u\to d} = [\int_{-\infty}^0 n \exp(V_u x/D_u)dx]/(nv/4) \equiv 4D_u/(vV_u)$. In a similar way, a downstream permanence time can be obtained by equating the distance perpendicular to the shock front overcome by the advecting motion of the plasma, $\Delta x = V_d \Delta t$, to the mean square displacement performed by particles during their diffusive motion, $\langle \Delta x^2 \rangle \sim D_d \Delta t$; thus, we obtain $t_d^* = N_d/\Phi_{u\to d} = n\Delta x/(nv/4) = 4D_d/(vV_d)$. The total permanence time (also called the cycle time) depends on both the diffusion coefficients upstream and downstream, $t^* = t_u^* + t_d^* = (4/v)[(D_u/V_u) + (D_d/V_d)]$. Consequently, the acceleration time can be defined as the ratio between the permanence time within the acceleration region and the fraction of momentum gained by particles $\Delta p/p = (4/3)(V_u - V_d)/v$ by Fermi acceleration in a cycle:

$$t_{\rm acc}^D = \frac{t^*}{\Delta p/p} = \frac{3}{V_u - V_d}\left(\frac{D_u}{V_u} + \frac{D_d}{V_d}\right). \qquad (38)$$

In the case of SSA, the same considerations can been used, that is, the permanence time is obtained by equating the distance due to advective motion, $\Delta x = V_d \Delta t$, to the distance covered because of superdiffusive propagation, $\langle \Delta x^2 \rangle \sim D_{d\alpha} \Delta t^\alpha$. Eliminating Δx, one finds a superdiffusive acceleration time

$$t_{\rm acc}^S = \frac{3}{V_u - V_d}\left[\left(\frac{D_{u\alpha}}{V_u^\alpha}\right)^{1/(2-\alpha)} + \left(\frac{D_{d\alpha}}{V_d^\alpha}\right)^{1/(2-\alpha)}\right]. \qquad (39)$$

Equation (39) allows us to estimate the acceleration time within the framework of SSA, with the result of the standard theory recovered for $\alpha = 1$. We can estimate the ratio between the superdiffusive acceleration time and the normal diffusive acceleration time by taking the ratio of Equation (39) and Equation (38), making the assumption that $D_d = D_u = D \simeq \lambda v$ and that $D_{d\alpha} = D_{u\alpha} = D_\alpha$ (see Equation (18)). Considering that $t_0 = \ell_0/v$, we end up with

$$\frac{t_{\rm acc}^S}{t_{\rm acc}^D} = K_\alpha v^{\frac{2\alpha-2}{2-\alpha}} \frac{\ell_0}{\lambda},$$

where

$$K_\alpha = \left[\frac{2(2-\alpha)\Gamma(\alpha-1)}{(3-\alpha)(4-\alpha)}\right]^{1/(2-\alpha)} \times \left[\left(\frac{1}{V_u}\right)^{\alpha/(2-\alpha)} + \left(\frac{1}{V_d}\right)^{\alpha/(2-\alpha)}\right] \frac{V_u V_d}{V_u + V_d}.$$

From the above equations, it appears that the ratio between the acceleration times coming from the two theories depends on the ratio between the length ℓ_0 at which the particle time profiles exhibit a power-law decay, typical of a superdiffusive regime, and the particle mean free path λ defined for normal diffusion. Of course, this kind of ratio varies with the system considered; for example, for nonrelativistic ions accelerated at the termination shock $\ell_0/\lambda \sim 10^{-5}$ (Perri & Zimbardo 2012a), implying acceleration times much faster in the case of superdiffusion. This property is probably a general one: indeed, besides the decrease of D_α with α shown in Figure 1, there is numerical evidence that when transport is superdiffusive, $\alpha > 1$, the dimensionless anomalous diffusion constant D_α is orders of magnitude smaller than the dimensionless normal diffusion coefficient (e.g., Gkioulidou et al. 2007; Pommois et al. 2007).

6. DISCUSSION AND CONCLUSIONS

In this paper, we have presented a self-contained derivation of the basic transport theory underlying SSA. We have given arguments to show that the superdiffusion of particles having finite velocity, e.g., the speed of light c, has to be described by a Lévy random walk, that is, by a microscopic free path (or jump) probability $\psi(x, t)$ that is characterized by a δ-coupling between the free path length and the free path duration, as well as by power-law tails for the probability of long displacements, $\psi(x, t) \simeq (A/2)\delta(|x| - vt)(t/t_0)^{-\mu}$. In this connection, we note that Perri & Zimbardo (2012b) found, by analyzing the magnetic variances near the resonant time scale for energetic electrons upstream of interplanetary shocks detected by the *Ulysses* spacecraft, that the corresponding scattering times have a power-law probability distribution with slopes $2.5 < \mu < 3.5$; these are, at least partly, consistent with Lévy walks leading to superdiffusion. This finding can give a microscopic explanation for the electron superdiffusion observed at several shocks in the solar wind.

Within the framework of Lévy walks, we have derived the anomalous diffusion exponent α and, for the first time, an explicit expression for the anomalous diffusion coefficient D_α. We also derived the non-Gaussian, Lévy-like propagator to leading order: the derivation of the propagator from first principles allows us to envisage how next order corrections can be obtained with the aim of improving the diagnostic tools to analyze the observations. The scaling properties of the propagator, which are derived in detail, are used to obtain the new expression of the energy spectral indices, which are plotted as a function of the compression ratio r and α in Figure 2. It is important to note that SSA predicts harder spectral indices than DSA and that this can explain a number of observations like the hard spectra of SNR synchrotron emitting electrons (e.g., Bogdan et al. 1985; Whiteoak & Green 1996) and the hard spectra of MeV ions at the solar wind termination shock (Perri & Zimbardo 2009a, 2012a; Florinski et al. 2009). Also, in situ observations of shock crossing in the solar wind give evidence of some very hard spectra that are not easily explained by DSA (van Nes et al. 1984; Lee et al. 2012). Furthermore, the Lévy walk propagator leads to energetic particle downstream densities lower than their density at the shock; see Equation (28) and Figure 3. We point out that such density profiles are routinely observed at heliospheric shocks; see, e.g., Figure 2 of Giacalone (2012).

On the other hand, the value of the anomalous diffusion coefficient D_α is fundamental for determining the acceleration time at shocks: we have derived D_α making a specific assumption about the shape of $\psi(t)$ for $t < t_0$, i.e., a constant ψ. Other choices, like a bell shaped $\psi(t)$ for $t < t_0$, would simply change the value of the normalization constant A in Equation (18) by a factor of order unity. It is interesting to note that superdiffusion leads to a new scaling of the acceleration time at shocks for the ratio of particle speed over plasma bulk speed: assuming, for the purpose of an estimate, that V_u and V_d are of the same order, one has

$$\frac{t^S_{\rm acc}}{t^D_{\rm acc}} \sim \left(\frac{v}{V_u}\right)^{\frac{2\alpha-2}{2-\alpha}} \frac{\ell_0}{\lambda}. \quad (40)$$

Since both numerical (e.g., Gkioulidou et al. 2007; Pommois et al. 2007) and experimental (Perri & Zimbardo 2012a) estimates show that $\ell_0/\lambda \ll 1$, we find that SSA may lead to shorter acceleration times than DSA.

We have shown how α and ℓ_0, and hence D_α, can be determined by the analysis of the energetic particle profiles upstream of the shock, i.e., by the slope of the energetic particle intensity and by the break in the power-law profile. We note that knowledge of both α and D_α is necessary to quantify the pace of transport; see Equation (3). This knowledge is crucial for determining the propagation of, e.g., solar energetic particles, which are one of the main threats of space weather. In summary, we can say that a number of theoretical predictions to be tested and to be used as diagnostic tools are now developed up to a stage appropriate to address data analysis, both in situ and remotely.

Finally, DSA is often studied by means of the cosmic ray transport equation called the Parker equation. It is desirable to describe SSA by a differential transport equation, too, and it is well known that anomalous transport can be effectively studied by fractional derivatives (Metzler & Klafter 2000, 2004; del-Castillo-Negrete et al. 2004; Zaslavsky 2002; Ragot & Kirk 1997; Webb et al. 2006; Bian & Browning 2008). We point out, however, that consideration of the space–time coupling that characterizes Lévy walks requires the use of the so-called fractional material derivative (Sokolov & Metzler 2003; Metzler & Klafter 2004), which indeed involves both the space and time derivatives and which can be written, in Fourier–Laplace space, in terms of the complex variables $q = s + ikv$ and $\bar{q} = s - ikv$ introduced in Section 3 (cf. with Equation (56) of Metzler & Klafter 2004). Future theoretical work will address the formulation of a generalized Parker equation in terms of fractional material derivatives.

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APPENDIX

DERIVATION OF THE PROBABILITY OF JUMP LENGTH $\psi(k, s)$ AND WALK DENSITY $W(k,s)$ IN THE FOURIER–LAPLACE SPACE

In order to derive an expression for the propagator in Fourier–Laplace space (see Equation(11)), we need to compute the probabilities $W(k,s)$ and $\psi(k, s)$ that describe the microscopic motion of particles. Inserting Equation (4) in Equation (5), we get $W(x, t) = \frac{1}{2}(\delta(|x| - vt) \int_t^\infty dt' \psi(t'))$, which, in Fourier–Laplace space, under symmetry conditions, becomes

$$\begin{aligned} W(k, s) &= \frac{1}{2} \int_{-\infty}^{\infty} dx \int_0^\infty dt \exp(-ikx) \\ &\quad \times \exp(-st)\delta(|x| - vt) \int_t^\infty dt'\psi(t') \\ &= \frac{1}{2} \int_0^\infty dt \exp(-ikvt) \exp(-st) \int_t^\infty dt'\psi(t') \\ &\quad + \frac{1}{2} \int_0^\infty dt \exp(ikvt) \exp(-st) \int_t^\infty dt'\psi(t') \\ &= \frac{1}{2} \int_0^\infty dt[\exp(-ikvt) + \exp(ikvt)] \\ &\quad \times \exp(-st) \int_t^\infty dt'\psi(t') \end{aligned}$$

so that we finally obtain

$$W(k, s) = \int_0^\infty dt \exp(-st) \cos(kvt) \int_t^\infty dt'\psi(t'). \quad (A1)$$

The first order derivative with respect to k is simply

$$W'(k, s) = -\int_0^\infty \exp(-st) \sin(kvt) vt \int_t^\infty dt'\psi(t'), \quad (A2)$$

which goes to zero in the limit of $k \to 0$. The second order derivative with respect to k, in the limit of $k \to 0$, has the expression $W''(k, s) = -\int_0^\infty dt \exp(-st) v^2 t^2 \int_t^\infty dt'\psi(t')$. In the framework of a Lévy walk model, the probability distribution function of the times associated with the jumps of particles follows a power-law decay for times $t > t_0$, namely $\psi(t) = A(t/t_0)^{-\mu}$, where A is the normalization constant. Note that this power-law behavior holds for asymptotic times, more specifically far from any transient phase of the system evolution. This condition is fundamental when deriving the mean square displacement of particles, since the physical time dependence is the one for times much greater than the characteristic times of the system. Thus, the second order derivative becomes

$$\begin{aligned} \left[\frac{\partial^2 W(k, s)}{\partial k^2}\right]_{k=0} &= -\int_0^\infty dt \exp(-st) v^2 t^2 \int_t^\infty dt' A \left(\frac{t'}{t_0}\right)^{-\mu} \\ &= -\int_0^\infty dt \exp(-st) \frac{v^2 t^2}{t_0^{-\mu}} \frac{A}{1-\mu} t^{1-\mu} \\ &= \frac{A}{\mu - 1} v^2 t_0^\mu \int_0^\infty dt \exp(-st) t^{3-\mu} \\ &= \frac{A}{\mu - 1} \frac{v^2 t_0^\mu}{s^{4-\mu}} \Gamma(4 - \mu). \end{aligned} \quad (A3)$$

The probability distribution of the jump lengths in Fourier–Laplace space and in the limit of $k \to 0$ can be written as

$$\psi(k, s) \sim \int_0^\infty dt \int_{-\infty}^{\infty} dx\ \psi(x, t)\left[1 - ikx - \frac{k^2 x^2}{2}\right] \exp(-st), \quad (A4)$$

where the term $\exp(-ikx)$ has been expanded up to the second order. The second term in brackets in Equation (A4) is odd in the variable x, therefore it goes to zero when integrated over $[-\infty, \infty]$. If we also make the assumption $st \ll 1$, Equation (A4) can be further approximated as

$$\psi(k,s) \sim \int_0^\infty dt \int_{-\infty}^{\infty} dx\ \psi(x,t)[1-st] - \int_0^\infty dt \int_{-\infty}^{\infty} dx\ \psi(x,t)\frac{k^2x^2}{2}\exp(-st), \quad \text{(A5)}$$

having expanded the factor $\exp(-st)$ in the first term on the right-hand side. Note that the term $\int_0^\infty dt \int_{-\infty}^{\infty} dx\ \psi(x,t)t \equiv \tau$ in Equation (A5) gives the definition of the mean scattering time of particles, so that

$$\psi(k,s) \sim 1 - s\tau - \int_0^\infty dt \int_{-\infty}^{\infty} dx\ \psi(x,t)\frac{k^2x^2}{2}\exp(-st). \quad \text{(A6)}$$

Exploiting the probability for the jump lengths $\psi(x,t)$ in Equation (A6), by using Equation (4) for a Lévy walk, and, after some algebra, we obtain

$$\psi(k,s) \sim 1 - s\tau - \frac{A}{2}k^2v^2t_0^\mu\frac{\Gamma(3-\mu)}{s^{3-\mu}} = 1 - s\tau - C_1k^2s^{\mu-3}, \quad \text{(A7)}$$

where the constant $C_1 = Av^2t_0^\mu\Gamma(3-\mu)/2$. Equation (A7) corresponds to Equation (37) in Klafter et al. (1987) with their parameter $\nu = 1$. The first order derivative $\partial\psi(k,s)/\partial k \rightarrow 0$ for $k \rightarrow 0$, while the second order derivative gives

$$\frac{\partial^2\psi(k,s)}{\partial k^2} = -2C_1s^{\mu-3}. \quad \text{(A8)}$$

REFERENCES

Ackermann, M., Ajello, M., Allafort, A., et al. 2013, Sci, 339, 807
Aharonian, F. A., Akhperjanian, A. G., Aye, K.-M., et al. 2004, Natur, 432, 75
Balogh, A., Bykov, A., Lin, R. P., Raymond, J., & Scholer, M. (eds.) 2013, Particle Acceleration in Cosmic Plasmas, Space Science Series of ISSI (New York: Springer)
Bell, A. R. 1978, MNRAS, 182, 147
Bell, A. R. 2004, MNRAS, 353, 550
Bian, N. H., & Browning, P. K. 2008, ApJ, 687, L111
Blandford, R., & Eichler, D. 1987, PhR, 154, 1
Blasi, P., Amato, E., & Serpico, P. D. 2012, PhRvL, 109, 061101
Blumen, A., Klafter, J., & Zumofen, G. 1990, EL, 13, 223
Bogdan, T. J., Lee, M. A., Lerche, I., & Webb, G. M. 1985, in Proc. 19th Int. Cosmic Ray Conf., Vol. 9, 543
Consolini, G., Kretzschmar, M., Lui, A. T. Y., Zimbardo, G., & Macek, W. M. 2005, JGR, 110, A07202
Decker, R.-B., Krimigis, S. M., Roelof, E. C., et al. 2008, Natur, 454, 67
del-Castillo-Negrete, D., Carreras, B. A., & Lynch, V. E. 2004, PhPl, 11, 3854
Droge, W., Lerche, I., & Schlickeiser, R. 1987, A&A, 178, 252
Drury, L. O'C. 1983, RPPh, 46, 973
Duffy, P., Kirk, J.-G., Gallant, Y.-A., et al. 1985, A&A, 302, L21
Eule, S., Zaburdaev, V., Friedrich, R., & Geisel, T. 2012, PhRvE, 86, 041134
Fisk, L. A., & Lee, M. A. 1980, ApJ, 237, 620
Florinski, V., Decker, R. B., le Roux, J. A., & Zank, G. P. 2009, GeoRL, 36, L12101
Gaisser, T. K. 1990, Cosmic Rays and Particle Physics (Cambridge: Cambridge Univ. Press)
Geisel, T., Nierwetberg, J., & Zacherl, A. 1985, PhRvL, 54, 616
Giacalone, J. 2012, ApJ, 761, 28
Giacalone, J. 2013, SSRv, 176, 73
Gkioulidou, M., Zimbardo, G., Pommois, P., Veltri, P., & Vlahos, L. 2007, A&A, 462, 1113
Gustafson, K., Ricci, P., Furno, I., et al. 2012, PhRvL, 108, 035006
Kirk, J. G., & Dendy, R. O. 2001, JPhG, 27, 1589
Kirk, J. G., Duffy, P., & Gallant, Y. A. 1996, A&A, 314, 1010
Klafter, J., Blumen, A., & Shlesinger, M. F. 1987, PhRvA, 35, 3081
Klafter, J., Shlesinger, M. F., & Zumofen, G. 1996, PhT, 49, 33
Klafter, J., & Sokolov, I. M. 2011, First Steps in Random Walks: From Tools to Applications (Oxford: Oxford Univ. Press)
Lagage, P. O., & Cesarsky, C. J. 1983, A&A, 125, 249
Lee, M. A., Mewaldt, R. A., & Giacalone, J. 2012, SSRv, 173, 247
Lin, R. P. 1974, SSRv, 16, 189
Metzler, R., & Klafter, J. 2000, PhR, 339, 1
Metzler, R., & Klafter, J. 2004, JPhA, 37, R161
Mier, J. A., Sanchez, R., Garcia, L., et al. 2008, PhRvL, 101, 165001
Montroll, E. W., & Weiss, G. H. 1965, JMP, 6, 167
Ostrowski, M., & Schlickeiser, R. 1996, SoPh, 167, 391
Perri, S., & Zimbardo, G. 2007, ApJL, 671, L177
Perri, S., & Zimbardo, G. 2008, JGR, 113, A03107
Perri, S., & Zimbardo, G. 2009a, ApJL, 693, L118
Perri, S., & Zimbardo, G. 2009b, AdSpR, 44, 465
Perri, S., & Zimbardo, G. 2012a, ApJ, 750, 87
Perri, S., & Zimbardo, G. 2012b, ApJ, 754, 8
Perrone, D., Dendy, R. O., Furno, I., et al. 2013, SSRv
Pommois, P., Zimbardo, G., & Veltri, P. 2007, PhPl, 14, 012311
Ptuskin, V., Zirakashvili, V., & Seo, E. 2010, ApJ, 718, 31
Ragot, B. R., & Kirk, J. G. 1997, A&A, 327, 432
Shalchi, A., & Kourakis, I. 2007, A&A, 470, 405
Shlesinger, M. F., & Klafter, J. 1985, PhRvL, 54, 2551
Shlesinger, M. F., West, B., & Klafter, J. 1987, PhRvL, 58, 1100
Shlesinger, M. F., Zaslavsky, G. M., & Klafter, J. 1993, Natur, 363, 31
Sokolov, I. M., & Metzler, R. 2003, PhRvE, 67, 010101
Sokolov, I. V., Roussev, I. I., Fisk, L. A., et al. 2006, ApJL, 642, L81
Sugiyama, T., & Shiota, D. 2011, ApJL, 731, L34
Tautz, R. C. 2010, PPCF, 52, 045016
Vainio, R., & Schlickeiser, R. 1999, A&A, 343, 303
van Nes, P., Reinhard, R., Sanderson, T. R., Wenzel, K.-P., & Zwickl, R. D. 1984, JGR, 89, 2122
Webb, G. M., Zank, G. P., Kaghashvili, E. Kh., & le Roux, J. A. 2006, ApJ, 651, 211
Whiteoak, J. B. Z., & Green, A. J. 1996, A&AS, 118, 329
Whittaker, E. T., & Watson, G. N. 1927, A Course of Modern Analysis II (Cambridge: Cambridge Univ. Press)
Zaslavsky, G. M. 2002, PhR, 371, 461
Zimbardo, G. 2005, PPCF, 47, B755
Zimbardo, G., Perri, S., Pommois, P., & Veltri, P. 2012, AdSpR, 49, 1633
Zimbardo, G., Pommois, P., & Veltri, P. 2006, ApJL, 109, L91
Zumofen, G., & Klafter, J. 1993, PhRvE, 47, 2
Zumofen, G., Klafter, J., & Blumen, A. 1990, CP, 146, 433