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Black strings in AdS_5

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Abstract: We present non-extremal magnetic black string solutions in five-dimensional gauged supergravity. The conformal infinity is the product of time and $\mathrm{S}^1 \times \mathcal{S}_h$, where $\mathcal{S}_h$ denotes a compact Riemann surface of genus h. The construction is based on both analytical and numerical techniques. We compute the holographic stress tensor, the Euclidean action and the conserved charges of the solutions and show that the latter satisfy a Smarr-type formula. The phase structure is determined in the canonical ensemble, and it is shown that there is a first order phase transition from small to large black strings, which disappears above a certain critical magnetic charge that is obtained numerically. For another particular value of the magnetic charge, that corresponds to a twisting of the dual super Yang-Mills theory, the conformal anomalies coming from the background curvature and those arising from the coupling to external gauge fields exactly cancel. We also obtain supersymmetric solutions describing waves propagating on extremal BPS magnetic black strings, and show that they possess a Siklos-Virasoro reparametrization invariance.

Keywords: Black Holes, p-branes, AdS-CFT Correspondence, Classical Theories of Gravity.

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Contents

1. Introduction

Four dimensional black holes are well understood. The uniqueness theorems ensure that for a given set of asymptotic charges, a unique black hole phase exists and belongs to the Kerr-Newman family of solutions. Things change drastically in higher dimensions. The discovery of a five dimensional black ring solution [1] has shown that uniqueness is violated. This leads to interesting phase diagrams, where a phase transition between Myers-Perry black holes [2] and black rings occurs as one increases the angular momentum of the system. The phase diagram becomes much richer if one spacelike dimension is compactified. The uniform black string solution, that can be constructed as a direct product of the Schwarzschild solution times the compact dimension, suffers from a long wavelength gravitational instability, the Gregory-Laflamme instability [3], but becomes stable above a critical mass. At this critical mass new static non-uniform strings emerge, with the same horizon topology of the black string, but without the translational symmetry along the circle. Another phase is given by Kaluza-Klein black holes, black holes with S^3 horizon topology localized on the circle. This localized black hole phase meets the non-uniform string phase in a topology transition point, the merger point. The dynamics of the decay of these strings is still unclear and opens interesting questions related to the cosmic censorship. Other phases coexist in the full diagram, and phases where Kaluza-Klein bubbles are attached to black holes and multi-black hole configurations have been studied in the literature, showing a continuous non-uniqueness of static classical black hole solutions for given asymptotic charges (see [4, 5] for reviews on the subject).

The study of black holes in presence of a negative cosmological constant is of particular interest in the context of the AdS/CFT correspondence [6] since their thermodynamics opens the opportunity to shed some light on the non-perturbative aspects of certain field theories and on their thermodynamical phases. For example, the Hawking-Page transition [7] between thermal AdS_5 and the Schwarzschild-AdS_5 black hole corresponds to a thermal phase transition from a confining to a deconfining phase of $D = 4$, $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ [8], while the phase structure of Reissner-Nordström-AdS black holes, similar to that of the van der Waals-Maxwell liquid-gas system, is connected to the study of the SYM theory coupled to some background R-symmetry current [9, 10]. Moreover, if one compactifies the SYM theory on a Scherk-Schwarz S^1, the resulting low energy dynamics is that of a $2+1$ dimensional Yang-Mills theory that undergoes a deconfining phase transition at finite temperature. Lumps of deconfined plasma have then an effective description in terms of fluid dynamics. The phases of this fluid, studied in [11, 12], are directly related to the black hole phases of the gravitational description.[1] The existence of plasmarings suggests the existence of asymptotically AdS_5 black rings, yet to be discovered. Further independent evidence has been obtained by the authors of [14], that found supersymmetric black rings in AdS_5 presenting a conical singularity, which plausibly will disappear out of extremality by balancing the forces.

If one allows for more general topology of the boundary, locally asymptotically AdS black holes with different horizon topology are permitted [15, 16]. The horizon of these so-called topological black holes can be any Einstein space with positive, negative or vanishing curvature [17]. In this article we will focus on minimal gauged supergravity in five dimensions, on locally asymptotically AdS_5 spacetimes with one non-trivial S^1 cycle at infinity. The boundary where the CFT lives has then the topology $\mathbb{R}_t \times \mathcal{S} \times S^1$ where $\mathcal{S}$ can be a two-sphere S^2, the Lobatchevski plane $\mathbb{H}^2$ or the Euclidean space E^2. Not much is known on the black hole phases in this case; uniform, neutral black strings with $S^2 \times S^1$ topology have been found by Copsey and Horowitz in [18] and then generalized to higher dimensions and arbitrary $\mathcal{S}$ by Mann, Radu and Stelea in [19]. Electric charge and angular momentum were included in [20], while non-abelian black string solution (but also abelian U(1)) were obtained in [21]. These solutions are typically numerical, since in presence of a cosmological constant the black strings cannot be constructed by taking a direct product with a circle, and the differential equations they verify cannot be solved analytically. There are exact supersymmetric magnetically charged solutions [22, 23] with

[1]See [13] for recent work on the subject.

horizon topology $S^2 \times S^1$ or $\mathbb{H}^2 \times S^1$, but their magnetic charge is quantized in terms of the AdS radius and there is no smooth limit connecting them to the uncharged strings.

In this article we will obtain the general family of magnetically charged black strings in AdS_5, connecting the supersymmetric to the uncharged ones. We construct them by matching numerically the near-horizon expansion of the metric to their asymptotic Fefferman-Graham expansion [24]. Furthermore, for some special cases, we are able to find their exact analytical expressions. We compute their masses and tensions, using the counterterm prescription [25], and fix the vacuum energy with the Kounterterm procedure [26, 27]. If the magnetic charge assumes a certain value given in terms of the inverse gauge coupling constant (corresponding to a twisting of the dual super Yang-Mills theory), the conformal anomalies coming from the background curvature and those arising from the coupling to external gauge fields exactly cancel.[2] The study of their thermodynamics shows in the canonical ensemble (fixed magnetic charge) a van der Waals-Maxwell phase structure similar to the one of the Reissner-Nordström-AdS black holes [9, 10] and electrically charged black strings in AdS_5 [20]: for small magnetic charges, we find two coexisting black string phases separated by a first order phase transition, which disappears at a critical point as we increase the charge.

Performing a double analytic continuation, these solutions describe static, magnetically charged bubbles of nothing in AdS_5. For vanishing cosmological constant, the Kaluza-Klein bubbles can have arbitrarily negative energy, implying an instability of this vacuum sector of Einstein's equations. For asymptotically AdS solutions, the existence of a stable ground state in the dual field theory description implies a lower bound on the mass of these solutions. This has led to the formulation of the positive energy conjecture for locally asymptotically AdS spacetimes [28], that has been discussed in [18] in presence of one compact direction on the boundary. We show that the static magnetically charged bubbles exist for any size of the S^1 at infinity in contrast to the uncharged case where bubbles exist only below a critical size of the S^1. Moreover, the quantum phase transition occuring in the strongly coupled gauge theory as one varies the size of the S^1 [18] becomes a quantum phase transition between the vacua dual to the small and large bubbles of nothing, and disappears for magnetic charges above a critical charge. We expect the lowest energy bubble to have the lowest energy among all solutions sharing the same asymptotic structure, as was conjectured for the uncharged one. It would be interesting to check this by some explicit calculation.

Finally, we turn the attention to supersymmetric black strings and generalize the exact supersymmetric black string solution of [22, 23] in two ways. First, we include electric charge, and find new exact solutions representing supersymmetric dyonic black strings with $\mathbb{H}^2 \times S^1$ topology of the horizon. Then, we construct BPS solutions corresponding to magnetic black strings with waves propagating along them. We show that the latter, enjoy a large Siklos-Virasoro reparametrization invariance.

The outline of this article is as follows. In section 2 we derive the new non-extremal magnetic black string solutions in five-dimensional minimal gauged supergravity, and

[2] This was already noticed in [21].

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present along with the numerical results some new exact solutions. Section 3 is devoted to the properties of the solutions. We compute the conserved quantities, the conformal anomaly of the dual CFT, and we study their thermodynamics, showing the emergence of a van der Waals-Maxwell phase structure. In section 4 we find new exact supersymmetric solutions with $\mathbb{H}^2 \times \mathrm{S}^1$ horizon topology, generalizing the known BPS magnetic string to non-vanishing electric charge and momentum waves in addition to the quantized magnetic charge. We also show that it enjoys a large Siklos-Virasoro reparameterization invariance. We conclude in section 5 with some final remarks. The first terms of the Fefferman-Graham expansion for the magnetic black strings are given in appendix A.

2. Non-extremal magnetic black string solutions

2.1 Action principle and field equations

The theory we shall be considering is minimal gauged supergravity in five dimensions, with bosonic action

$$I_0 = \frac{1}{4\pi G}\int_{\mathcal{M}}\left[\left(\frac{R}{4}+3g^2\right)\star 1 - \frac{1}{2}F\wedge\star F - \frac{2}{3\sqrt{3}}F\wedge F\wedge A\right] - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} \mathrm{d}^4x\sqrt{-\gamma}K\,, \tag{2.1}$$

where R is the scalar curvature and $F = \mathrm{d}A$ is the field strength of the U(1) gauge field. K, appearing in the Gibbons-Hawking term, represents the trace of the extrinsic curvature of the boundary defined as

$$K_{ij} = -\frac{1}{2}(\nabla_i n_j + \nabla_j n_i)\,, \tag{2.2}$$

with n^j denoting the outward pointing normal vector to the boundary and γ_{ij} the induced metric. The equations of motion following from (2.1) are

$$R_{\mu\nu} = 2F_{\mu\rho}F_\nu{}^\rho - \frac{1}{3}g_{\mu\nu}(F^2+12g^2)\,, \tag{2.3}$$

$$\mathrm{d}\star F + \frac{2}{\sqrt{3}}F\wedge F = 0\,, \tag{2.4}$$

where $F^2 = F_{\mu\nu}F^{\mu\nu}$.

In order to find non-extremal string solutions, we choose the ansatz

$$\mathrm{d}s^2 = -\,\mathrm{e}^{2V}\,\mathrm{d}t^2 + \mathrm{e}^{2T}\,\mathrm{d}z^2 + \mathrm{e}^{2U}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_k^2\,, \tag{2.5}$$

where V, T, U are functions of r only and $\mathrm{d}\Omega_k^2$ denotes the metric of a two-manifold $\mathcal{S}$ of constant Gaussian curvature k. Without loss of generality we can restrict ourselves to the cases $k = 0, \pm 1$; $\mathcal{S}$ is a quotient space of the universal coverings S^2 $(k=1)$, $\mathbb{H}^2$ $(k=-1)$ or E^2 $(k=0)$. Explicitly, we choose

$$\mathrm{d}\Omega_k^2 = \mathrm{d}\theta^2 + S(\theta)^2\,\mathrm{d}\varphi^2\,, \tag{2.6}$$

with

$$S(\theta) = \begin{cases} \sin\theta\,, & k=1\,,\\ \theta\,, & k=0\,,\\ \sinh\theta\,, & k=-1\,. \end{cases}$$

We also assume that the direction z is periodic with period L. In five dimensions, strings can carry magnetic charge. So we take the magnetic ansatz

$$F_{\theta\varphi} = kqS(\theta), \qquad A_\varphi = kq \int S(\theta)\,\mathrm{d}\theta \tag{2.7}$$

for the U(1) gauge field. With this choice, the Maxwell equations (2.4) are trivially satisfied. Plugging the line element (2.5) into the Einstein equations (2.3), yields the following set of coupled ordinary differential equations:

$$\mathrm{e}^{-2U}\left[V'T' - V'U' + V'' + (V')^2 + \frac{2V'}{r}\right] = \frac{2}{3}\left(\frac{kq}{r^2}\right)^2 + 4g^2\,, \tag{2.8}$$

$$\mathrm{e}^{-2U}\left[V'T' - T'U' + T'' + (T')^2 + \frac{2T'}{r}\right] = \frac{2}{3}\left(\frac{kq}{r^2}\right)^2 + 4g^2\,, \tag{2.9}$$

$$\mathrm{e}^{-2U}\left[V'' + (V')^2 - V'U' + T'' + (T')^2 - T'U' - \frac{2U'}{r}\right] = \frac{2}{3}\left(\frac{kq}{r^2}\right)^2 + 4g^2\,, \tag{2.10}$$

$$-\,\mathrm{e}^{-2U}\left[r(V' + T' - U') + 1\right] + k = \frac{4}{3}\left(\frac{kq}{r}\right)^2 - 4g^2r^2\,. \tag{2.11}$$

We now define $F = V + T$ and $G = V - T$ and consider the difference between (2.8) and (2.9), which, after integration, gives

$$G' = \frac{\mu}{r^2}\,\mathrm{e}^{U-F}\,, \tag{2.12}$$

where μ is an integration constant. Note that for extremal solutions one has $T = V$, which implies $G = 0$ and hence $\mu = 0$. Thus μ can be interpreted as a non-extremality parameter. Equation (2.11) can be rewritten in the form

$$F' = \mathrm{e}^{2U} f + U' - \frac{1}{r}\,, \tag{2.13}$$

with

$$f(r) = 4g^2r + \frac{k}{r} - \frac{4(kq)^2}{3r^3}\,. \tag{2.14}$$

Using (2.13) in the sum of (2.8) and (2.9) yields an equation for U only,

$$\mathrm{e}^{2U} f^2 + 3U'f + f' + U''\,\mathrm{e}^{-2U} + \frac{U'}{r}\,\mathrm{e}^{-2U} = 8g^2 + \frac{4}{3}\left(\frac{kq}{r^2}\right)^2\,, \tag{2.15}$$

which can be written in the more convenient form

$$y''y + 3y'f + \frac{1}{r}y'y - (y')^2 - 2y\left[f' - 8g^2 - \frac{4}{3}\left(\frac{kq}{r^2}\right)^2\right] = 2f^2 \tag{2.16}$$

by defining $y = \mathrm{e}^{-2U}$. The only equation that we did not use up to now is (2.10). Subtracting the sum of (2.8) and (2.9) from (2.10), and using (2.12) and (2.13), one obtains

$$\frac{\mu^2}{2r^4}\,\mathrm{e}^{-2F} = \mathrm{e}^{-2U}\left[\frac{3U'}{r} + \frac{1}{2}(U')^2 - \frac{3}{2r^2}\right] + \frac{1}{2}\,\mathrm{e}^{2U} f^2 + fU' + \frac{k}{r^2} - 2\left(\frac{kq}{r^2}\right)^2\,. \tag{2.17}$$

Notice that in the extremal case $T = V$, $\mu = 0$, this implies a first order differential equation for U.[3] It is straightforward to show that solving (2.17) for F, deriving with respect to r, and using (2.15), leads to (2.13). Thus, in order to find the complete solution, one first solves (2.16) to get y. Plugging this into (2.17) gives then F, and finally (2.12) yields G. Then all the field equations are satisfied. We have thus decoupled completely the field equations and reduced the problem to solving a non-linear ordinary second order differential equation. Unfortunately, solving (2.16) is a quite formidable task, and we did not succeed in finding the most general solution, so that in the general case we have to resort to numerical techniques. Nevertheless, it is possible to obtain some particular exact solutions, which we will discuss in section 2.3.

2.2 Asymptotics

At large r, the functions appearing in the metric admit the Fefferman-Graham expansions [24]

$$\begin{aligned}
y &= (gr)^2 + f_0 + \frac{\xi\ln(gr)}{(gr)^2} + \frac{c_z + c_t + c_0}{(gr)^2} + O\left(\frac{\ln r}{r^4}\right), \\
\mathrm{e}^{2T} &= (gr)^2 + a_0 + \frac{\rho\ln(gr)}{(gr)^2} + \frac{c_z}{(gr)^2} + O\left(\frac{\ln r}{r^4}\right), \\
\mathrm{e}^{2V} &= (gr)^2 + b_0 + \frac{\chi\ln(gr)}{(gr)^2} + \frac{c_t}{(gr)^2} + O\left(\frac{\ln r}{r^4}\right).
\end{aligned} \tag{2.18}$$

Substituting these into the equations (2.16), (2.17) and (2.12), yields the coefficients

$$\begin{aligned}
f_0 &= \frac{2k}{3}, \qquad a_0 = b_0 = \frac{k}{2}, \qquad c_0 = \frac{(kgq)^2}{3}, \\
\xi &= \frac{k^2}{6}[1 - 12(gq)^2], \qquad \rho = \chi = \frac{\xi}{2},
\end{aligned} \tag{2.19}$$

plus the relation

$$c_z - c_t = \frac{\mu g}{2}. \tag{2.20}$$

Therefore the expansion at large r depends only on the two constants c_t and c_z.
We assume in addition that there is an event horizon at $r = r_\mathrm{h}$, and that the metric functions can be expanded into a Taylor series near r_h.[4] Then the Einstein equations

[3] This first order equation implies (2.15).

[4] Note that the latter assumption, while reasonable for non-extremal solutions, is violated for the extremal supersymmetric black strings found in [23]. We will see below how the near-horizon expansion looks like in the extremal case.

imply

$$
\begin{aligned}
y &= \frac{12g^2r_{\rm h}^4 + 3kr_{\rm h}^2 - 4(kq)^2}{3r_{\rm h}^3}(r - r_{\rm h}) \\
&\quad - \frac{6g^2r_{\rm h}^4 + 3kr_{\rm h}^2 - 8(kq)^2}{3r_{\rm h}^4}(r - r_{\rm h})^2 + O((r - r_{\rm h})^3)\,, \\
\mathrm{e}^{2T} &= a_{\rm h} + \frac{4a_{\rm h}\left[6g^2r_{\rm h}^4 + (kq)^2\right]}{r_{\rm h}[12g^2r_{\rm h}^4 + 3kr_{\rm h}^2 - 4(kq)^2]}(r - r_{\rm h}) \\
&\quad + \frac{4a_{\rm h}[36g^4r_{\rm h}^8 + 4(kq)^4 - 3k^3(qr_{\rm h})^2 - 6r_{\rm h}^4(kgq)^2]}{r_{\rm h}^2[12g^2r_{\rm h}^4 + 3kr_{\rm h}^2 - 4(kq)^2]^2}(r - r_{\rm h})^2 + O((r - r_{\rm h})^3)\,, \\
\mathrm{e}^{2V} &= b_{\rm h}(r - r_{\rm h}) - \frac{b_{\rm h}[6g^2r_{\rm h}^4 + 3kr_{\rm h}^2 - 4(kq)^2]}{r_{\rm h}[12g^2r_{\rm h}^4 + 3kr_{\rm h}^2 - 4(kq)^2]}(r - r_{\rm h})^2 + O((r - r_{\rm h})^3)\,,
\end{aligned}
\tag{2.21}
$$

where

$$
b_{\rm h} = \frac{12\mu^2}{a_{\rm h}r_{\rm h}\left[12g^2r_{\rm h}^4 + 3kr_{\rm h}^2 - 4(kq)^2\right]}\,. \tag{2.22}
$$

Equation (2.16) actually admits two further expansions for y with different coefficients. One of them is not compatible with the desired behaviour of V and T close to the horizon, while the other is given by

$$
y = \alpha_2(r - r_{\rm h})^2 + O((r - r_{\rm h})^3)\,, \tag{2.23}
$$

with

$$
\alpha_2 = 22g^2 + \frac{5k}{2r_{\rm h}^2} \pm \sqrt{228g^4 + \frac{9k^2}{4r_{\rm h}^4} + \frac{46g^2k}{r_{\rm h}^2}}\,. \tag{2.24}
$$

Notice that the position of the event horizon $r_{\rm h}$ is not fixed by the equations of motion for the expansion (2.21), while for the second possibility (2.23) one finds $f(r_{\rm h}) = 0$, so $r_{\rm h}$ is given in terms of k and q. (2.23) is related to the extremal case $\mu = 0$ that we shall consider below.

For $q = 0$, (2.18) and (2.21) reduce correctly to the expansions at infinity and near the horizon for uncharged black strings [19].

The conditions for a regular event horizon are $y'(r_{\rm h}) > 0$ and $(\mathrm{e}^{2V})'(r_{\rm h}) > 0$. Using (2.21), these imply $a_{\rm h} > 0$ and, in the $k = \pm 1$ case, the further condition

$$
r_{\rm h} > \frac{\sqrt{-18k + 6\sqrt{9 + 192(gq)^2}}}{12g}\,, \tag{2.25}
$$

which gives a minimal value for $r_{\rm h}$.

We note that globally regular solutions with $r_{\rm h} = 0$, which exist for $q = 0$ [19], are not allowed for non-vanishing magnetic charge q.

From eqs. (2.17) and (2.12) it is clear that changing the sign of μ leaves F invariant, while $G \to -G$. This means that V and T are interchanged under $\mu \to -\mu$. Looking at (2.21), we see that, if a solution with a given value of μ describes a black string, then the corresponding solution with $-\mu$ is a bubble of nothing, which could have been obtained also by a double analytic continuation of the black string. For $q = 0$, such bubble geometries were considered in [18].

Let us finally determine the near-horizon geometry in the extremal case $\mu = 0$. Expanding (2.17) near $r = r_{\rm h}$ implies then $f(r_{\rm h}) = 0$. Using (2.21), the terms of order $(r-r_{\rm h})^2$ in (2.17) give the condition $14g^2r_{\rm h}^2 + k = 0$, that, combined with $f(r_{\rm h}) = 0$, yields $q^2 < 0$, which is impossible. Thus, for $\mu = 0$, the correct near-horizon expansion for y must be given by (2.23). Using this in (2.13) gives after integration

$$\mathrm{e}^F = \mathrm{e}^{2V} = \mathrm{e}^{2T} = C(r - r_{\rm h})^\gamma(1 + O(r - r_{\rm h}))\,, \tag{2.26}$$

with the exponent

$$\gamma = \frac{16g^2r_{\rm h}^2 - r_{\rm h}^2\alpha_2 + 2k}{r_{\rm h}^2\alpha_2}\,, \tag{2.27}$$

and C is an integration constant. Note that γ is positive if and only if we choose the lower sign in (2.24). In the supersymmetric case $q^2 = 1/12g^2$, $k = -1$ [23] one has $r_{\rm h}^2 = 1/3g^2$ (cf. the following subsection) and thus $\alpha_2 = 4g^2$ and $\gamma = 3/2$, which is indeed the correct exponent [23]. Introducing the new coordinate $\rho = (r - r_{\rm h})^\gamma$, it is easy to see that the near-horizon geometry of the extremal solutions is $\mathrm{AdS}_3 \times \mathcal{S}$.

2.3 Exact solutions

Despite the complexity of the differential equation (2.16) it is possible to find some exact solutions that we list in the following.

- $k = 0$, $q = 0$:

  In this case the equations of motion are solved by

  $$\mathrm{e}^{2T} = (gr)^2, \qquad \mathrm{e}^{2V} = y = (gr)^2 - \frac{\mu}{2gr^2}\,. \tag{2.28}$$

  Considering z as a coordinate of the transverse space, this can also be viewed as a black hole, it is the metric found in [17].

- k arbitrary, $q^2 = 1/12g^2$:

  In the case of quantized magnetic charge, one finds two exact solutions. The first is the supersymmetric magnetic string [22, 23] given by

  $$\mathrm{e}^{2T} = \mathrm{e}^{2V} = (gr)^{\frac{1}{2}}\left(gr + \frac{k}{3gr}\right)^{\frac{3}{2}}, \qquad y = \left(gr + \frac{k}{3gr}\right)^2. \tag{2.29}$$

  For $k = -1$ this has an event horizon at $r = r_{\rm h} = 1/\sqrt{3}g$, which is the limiting value of (2.25).

  The second is a non-extremal black string with finite FG expansion for y,

  $$\begin{aligned} y &= (gr)^2 + \frac{2k}{3} - \frac{2k^2}{9(gr)^2} + \frac{k}{9(gr)^4}\,, \\ \mathrm{e}^{2T} &= \mathrm{e}^{-G}\, gr\sqrt{(gr)^2 + k}\,, \\ \mathrm{e}^{2V} &= \mathrm{e}^{G}\, gr\sqrt{(gr)^2 + k}\,, \end{aligned} \tag{2.30}$$

where

$$\mathrm{e}^G = \left[\frac{3(gr)^2(k-2)}{-2+3k(gr)^2 - 2\sqrt{9(gr)^4 - 3k(gr)^2+1}}\right]^{\frac{3g\mu k}{2}}$$
$$\cdot\left[\frac{5-21k(gr)^2+2\sqrt{13}\sqrt{9(gr)^4-3k(gr)^2+1}}{((gr)^2+k)(6\sqrt{13}-21k)}\right]^{\frac{3g\mu k}{2\sqrt{13}}} ,$$

and

$$\mu = \pm\frac{\sqrt{13}k}{3g} . \tag{2.31}$$

Notice that the two signs in (2.31) are related by an interchange of T and V; $\mu \to -\mu$ implies $G \to -G$ and thus $T \to V$, $V \to T$.

For $k = 0$ the above metric gives simply AdS_5. In the non-trivial cases $k = \pm 1$ the Kretschmann scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ blows up as r goes to zero. This curvature singularity is naked if $k = 1$, while for $k = -1$ it is hidden by an event horizon at $r = r_\mathrm{h} = 1/g$ if we choose the lower sign in (2.31). At the horizon e^{2V} vanishes linearly while e^{2T} goes to a constant. Comparing with the Fefferman-Graham expansion (2.18), we see that this particular black string solution corresponds to $c_t = -1/8 - \sqrt{13}/12$, $c_z = -1/8 + \sqrt{13}/12$. It has non-vanishing Hawking temperature and Bekenstein-Hawking entropy. For the upper sign in (2.31) one obtains a bubble solution similar in spirit to those considered in [18].

2.4 Numerical computation

So far we have not been able to solve the differential equations (2.8) to (2.11) analytically for general r_h and q. A numerical evaluation requires some care, since the constants c_t and c_z appear as subleading terms in the Fefferman-Graham expansion (2.18). In subsubsection 3.1.1 we will show that these constants determine the mass M and the tension $\mathcal{T}$ of the corresponding solution. In addition, we have to determine the a priori unknown constants a_h and b_h in the near-horizon expansions (2.21). In particular, as we will show in subsection 3.2, a_h is required to compute the area of the event horizon, and hence the entropy S and Hawking temperature T_H of the corresponding solution. With boundary conditions given at $r \gtrsim r_\mathrm{h}$, the numerical evolution of the solutions to large r therefore has to be accurate enough to extract c_t and c_z from the subleading terms in the asymptotic expansions (2.18), and also to fix a_h and b_h a posteriori.

The boundary conditions taken from the near-horizon expansions in (2.21) depend on the a priori unknown constants a_h and b_h. These constants have to be chosen such that the asymptotic expansions are of the Fefferman-Graham form as in (2.18), i.e. the coefficients in front of $(gr)^2$ should be exactly one. This is always the case for $y = \mathrm{e}^{-2U}$, but not for e^{2T} and e^{2V}. Since T and V enter the differential equations (2.8) to (2.11) only via their first and second derivatives, the respective solutions are only determined up to an additive constant, i.e. the corresponding warp factors e^{2T} and e^{2V} can be rescaled by appropriate constants. Such rescalings correspond to rescalings of respectively z and t in the ansatz for the metric (2.5). The freedom of the rescalings is uniquely fixed by the condition that the asymptotic expansion has to be of the Fefferman-Graham form, which determines $a_{\rm h}$ and $b_{\rm h}$ in the respective near-horizon expansions (2.21) of e^{2T} and e^{2V}. We can therefore start from arbitrary non-vanishing initial values for $a_{\rm h}$ and $b_{\rm h}$, and determine the solution of the differential equations. The correct values for $a_{\rm h}$ and $b_{\rm h}$ are then found afterwards by dividing our initially chosen values by the corresponding coefficients in front of the $(gr)^2$ terms in the asymptotic expansions of our found numerical solutions.

We use (2.8), the linear combination (2.10) $-$ (2.8) $-$ (2.9) + $\frac{2}{(gr)^2}$(2.11) and (2.11) for the numerical evaluation which we perform with `mathematica`. It turns out that a direct integration of the three equations for the three functions $y_1 = y = \mathrm{e}^{-2U}$, $y_2 = \mathrm{e}^{2T}$, $y_3 = \mathrm{e}^{2V}$ does not provide the required accuracy: the values for c_t and c_z strongly depend on the values of $r \gg r_{\rm h}$ which we use to fix the asymptotic expansions. We therefore first separate the leading near-horizon and leading asymptotic behaviours from the unknown subleading contributions by introducing new functions for which we derive differential equations that can be integrated with higher numerical accuracy. The original functions y_i are therefore split into a product of two functions as[5]

$$y_i(u) = \mu_i(u)\left(1 + \frac{w_i(u)}{1+u^6}\right), \qquad i = 1,2,3\,, \tag{2.32}$$

where the μ_i are chosen as follows

$$\mu_i(u) = u^2 + f_i - (f_i - g_i + u_{\rm h}^2)\frac{1}{1+(u-u_{\rm h})^4} + \left(\xi_i \frac{\log u}{u^2} + \frac{c_i}{u^2}\right)\frac{(u-u_{\rm h})^4}{1+(u-u_{\rm h})^4}\,. \tag{2.33}$$

We have thereby introduced the dimensionless variables $u = gr$ and $u_{\rm h} = gr_{\rm h}$. The constants f_i, ξ_i, c_i follow from (2.18) and inherit their names from the expansion of $y_1 = y$. The constants g_i consider that the near-horizon expansion of y_2 in (2.21) starts with a constant. We identify

$$f_i = (f_0, a_0, b_0)\,, \qquad \xi_i = (\xi, \rho, \chi)\,, \qquad c_i = (c_0, 0, 0)\,, \qquad g_i = (0, 1, 0)\,. \tag{2.34}$$

The functions μ_i, which expand as

$$\mu_{{\rm nh},i}(u) = g_i + 2u_{\rm h}(u-u_{\rm h}) + (u-u_{\rm h})^2 + O((u-u_{\rm h})^3) \tag{2.35}$$

near the horizon then have the asymptotic behaviours

$$\mu_{{\rm asy},i}(u) = u^2 + f_i + \xi_i\frac{\log u}{u^2} + \frac{c_i}{u^2} + O\left(\frac{1}{u^4}\right), \tag{2.36}$$

which are the known parts of the asymptotic expansion of y_i in (2.18) without c_t and c_z.

[5] A similar ansatz has been used in [18].

To match the near-horizon expansion of (2.32) to the near-horizon expansions in (2.21), using that μ_i expands as given in (2.35), the w_i themselves have to expand as

$$
\begin{aligned}
w_1(u) &= (1+u_{\mathrm{h}}^6)\left[1+\frac{1}{2u_{\mathrm{h}}^4}\left(u_{\mathrm{h}}^2 k-\frac{4}{3}(kgq)^2\right)\right] \\
&\quad+\left[\frac{1+13u_{\mathrm{h}}^6}{4u_{\mathrm{h}}^5}\left(u_{\mathrm{h}}^2 k-\frac{4}{3}(kgq)^2\right)+6u_{\mathrm{h}}^5\right](u-u_{\mathrm{h}})+O((u-u_{\mathrm{h}})^2)\,, \\
w_2(u) &= (1+u_{\mathrm{h}}^6)(a_{\mathrm{h}}-1) \\
&\quad-\left[\frac{a_{\mathrm{h}}}{u_{\mathrm{h}}}\left(1+2u_{\mathrm{h}}^2-5u_{\mathrm{h}}^6+2u_{\mathrm{h}}^8-\frac{3u_{\mathrm{h}}^2(1+u_{\mathrm{h}}^6)(12u_{\mathrm{h}}^2+k)}{12u_{\mathrm{h}}^4+3u_{\mathrm{h}}^2 k-4(kgq)^2}\right)+6u_{\mathrm{h}}^5\right](u-u_{\mathrm{h}}) \\
&\quad+O((u-u_{\mathrm{h}})^2)\,, \\
w_3(u) &= (1+u_{\mathrm{h}}^6)\left[\frac{b_{\mathrm{h}}}{2u_{\mathrm{h}}}-1\right] \\
&\quad+\left[\frac{3b_{\mathrm{h}}}{4u_{\mathrm{h}}}\frac{1}{12u_{\mathrm{h}}^4+3u_{\mathrm{h}}^2 k-4(kgq)^2}\left(\frac{1+13u_{\mathrm{h}}^6}{u_{\mathrm{h}}}\left(u_{\mathrm{h}}^2 k-\frac{4}{3}(kgq)^2\right)+48u_{\mathrm{h}}^9\right)\right. \\
&\quad\left.-\,6u_{\mathrm{h}}^5\right](u-u_{\mathrm{h}})+O((u-u_{\mathrm{h}})^2)\,.
\end{aligned}
\tag{2.37}
$$

This result is used to specify the boundary conditions for integrating the differential equations for the w_i. In our numerical evaluation this cannot be done exactly at $u=u_{\mathrm{h}}$. We have to specify $w_1(u_0)$, $w_2(u_0)$, $w_3(u_0)$, $w_3'(u_0)$ at $u_0=u_{\mathrm{h}}(1+\varepsilon)$, where the minimal possible value for $\varepsilon \ll 1$ depends on the numerical integration routine. The results presented here are all obtained with $\varepsilon < 10^{-4}$. We initially fix $a_{\mathrm{h}}=b_{\mathrm{h}}=1$. We then fit to the found solutions the two functions $\mu_{\mathrm{asy},i}$ and $\frac{1}{u^2}$. As explained above, the correct values for a_{h} and b_{h} are then found by dividing their initially chosen values by the coefficients of respectively $\mu_{\mathrm{asy},2}$ and $\mu_{\mathrm{asy},3}$ in the corresponding fit-function. To increase the accuracy, it is advantageous to repeat this procedure with the new values for a_{h} and b_{h} until the coefficients in front of $\mu_{\mathrm{nh},2}$, $\mu_{\mathrm{nh},3}$ are compatible with one. In this way we can achieve an accuracy better than 10^{-8} for these coefficients. The final fit then yields c_t and c_z as the respective coefficient of the second fit function $\frac{1}{u^2}$. A fit of the functions $\mu_{\mathrm{asy},1}$ and $\frac{1}{u^2}$ to the result for y_1 independently also determines the sum c_t+c_z.

The information on c_t and c_z is directly encoded in the asymptotic expansion of the second factor in (2.32). It is designed in such a way that the second derivatives $\frac{1}{2}w_i''$ at large u directly give c_t+c_z, c_t, c_z. However, in practice only $\frac{1}{2}w_1''$ becomes constant at large u such that one can directly read off c_t+c_z. In the other two cases, a fit of the three functions u^2, u^4, u^6 to the solution based on the correct values for a_{h} and b_{h} yields c_t and c_z as the coefficients of u^2. We also find a very small admixture of u^4 and u^6 terms spoils the constancy of the derivatives $\frac{1}{2}w_2''$, $\frac{1}{2}w_3''$ at large u. To obtain accurate values for c_t and c_z, the fits to w_i require a determination of the solutions to much larger values of u than required for a fit to the y_i.

The numerical results for the functions w_i are nevertheless useful to find the regimes in which we can trust the numerical results and to determine which interval of large u should be used for the fits to y_i to achieve the highest accuracy. We find that our ansatz (2.32)

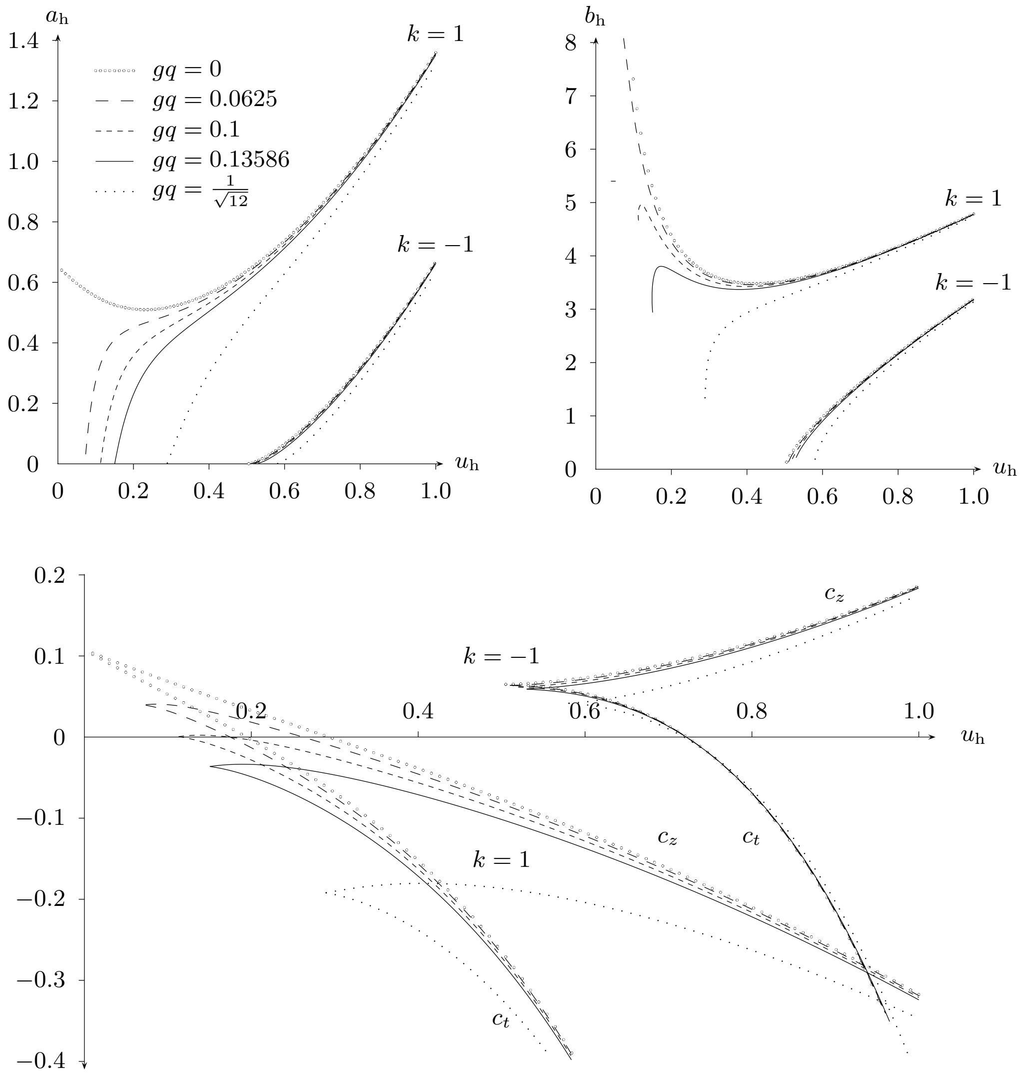

Figure 1: a_{h}, b_{h}, c_t, c_z as functions of u_{h} for $k = 1$ and $k = -1$ at five values of the magnetic charge q.

works fine at least for $0 < u_{\mathrm{h}} \lesssim 2$, which is sufficient for our purposes. For larger u_{h} it is enough to slightly modify (2.32) by replacing $u \to \frac{u}{u_{\mathrm{h}}}$ and $(u - h) \to (\frac{u}{u_{\mathrm{h}}} - 1)$ in the interpolating factors. Furthermore, in the regime $0 < u_{\mathrm{h}} \lesssim 2$ a restriction to $u_{\mathrm{h}} < u \lesssim 100$ avoids the regime of increasing noise above $u \approx 120$, and it suffices to fit the asymptotics with high precision. This we have checked by reproducing the exact results of subsection 2.3. For fits in the interval $0.8\, u_{\mathrm{max}} \leq u \leq u_{\mathrm{max}}$ with $u_{\mathrm{max}} = 100$ we obtain the highest relative accuracy, which in any case is better than 10^{-5} for c_t and c_z.

For given $g = 1$, $k = \pm 1$ and charge q we compute $a_{\mathrm{h}}(u_{\mathrm{h}})$, $b_{\mathrm{h}}(u_{\mathrm{h}})$, $c_t(u_{\mathrm{h}})$, $c_z(u_{\mathrm{h}})$ for sufficiently many values of u_{h}. The corresponding results for $u_{\mathrm{min}} \leq u_{\mathrm{h}} \leq 1$ are presented in figure 1 for $k = 1$ and $k = -1$ at five values of the magnetic charge q. These results allow us to determine the inverse Hawking temperature T_{H} as a function of the entropy S, and the free energy F as a function of $T_{\rm H}$. To determine the critical charge $q_{\rm crit}$, we fit a linear function to data points which we concentrate around the estimated turning point and vary q until the slope of the fit function is compatible with zero. This allows us to determine $gq_{\rm crit} = 0.13586(1)$. We have determined the error, which only affects the last digit, by the lower and upper bound for $q_{\rm crit}$ which undoubtedly are below and above $q_{\rm crit}$, respectively. At $q_{\rm crit}$ we then fit a cubic polynomial to the data points around the estimated turning point. This then fixes the turning point more precisely to $(S\frac{Gg^2}{LV_k}, \frac{g}{T_{\rm H}}) = (0.017278(^{-0.000006}_{+0.000010}), 7.12673(^{-0.00013}_{+0.00007}))$, where we estimate the errors from the corresponding fits in which q assumes the value of the lower or upper bound of $q_{\rm crit}$. Finally, we should remark that the corresponding data-files for all the plots are available as parts of the source files of this paper. They are called `[xquantity][yquantity]q[value of q as integer][neg/pos]k.dat`.

3. Properties of magnetic black strings

3.1 Conserved quantities

3.1.1 Standard counterterm method

To compute the mass and tension, which we expect to be encoded in the constants c_t and c_z appearing in (2.18), we use the counterterm procedure for spacetimes with negative cosmological constant proposed in [25]. One obtains a finite quasilocal stress tensor

$$T^{ij} = \frac{2}{\sqrt{-\gamma}} \frac{\delta I}{\delta \gamma_{ij}} \tag{3.1}$$

by adding to the action (2.1) a counterterm I_{ct},

$$I_{\rm ren} = I_0 + I_{\rm ct}(\gamma_{ij}) \,, \tag{3.2}$$

where γ_{ij} denotes the induced metric on the boundary $\partial\mathcal{M}$, which we take to be a hypersurface at constant radial coordinate r. By requiring cancellation of divergences in the limit $r \to \infty$, one finds the explicit expression for $I_{\rm ct}$ [25],

$$\begin{aligned} I_{\rm ct} = -\frac{1}{8\pi G}\int_{\partial\mathcal{M}} {\rm d}^4x\,\sqrt{-\gamma}\Bigg\{ -3g\left(1+\frac{\mathcal{R}}{12g^2}\right) \\ + \frac{\ln(gr)}{2g}\left[\frac{1}{4g^2}\left(\frac{1}{3}\mathcal{R}^2 - \mathcal{R}_{ij}\mathcal{R}^{ij}\right) + F_{ij}F^{ij}\right]\Bigg\} \,, \end{aligned} \tag{3.3}$$

where $\mathcal{R}$ and $\mathcal{R}^{ij}$ are the curvature and the Ricci tensor associated with the metric γ. The first term in the second line of (3.3) is the usual expression required to cancel logarithmic divergences [29] that appear in odd dimensions, while the second one is due to the presence of additional matter fields (in our case a U(1) gauge field) in the bulk [30]. A logarithmic contribution to the counterterms also appears naturally in a reformulation of the holographic renormalization procedure in terms of the extrinsic curvature [31].

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Varying the renormalized action (3.2) with respect to the boundary metric γ_{ij} leads to the holographic energy-momentum tensor

$$\begin{aligned} T^{ij} &= \frac{2}{\sqrt{-\gamma}} \frac{\delta I_{\rm ren}}{\delta \gamma_{ij}} \\ &= \frac{1}{8\pi G} \Bigg\{ K^{ij} - K\gamma^{ij} + \frac{1}{2g} G^{ij} \\ &\qquad + \frac{\ln(gr)}{2g} \Bigg[\frac{1}{4g^2} \Bigg(\frac{1}{3}\gamma^{ij}\mathcal{R}^2 - \gamma^{ij}\mathcal{R}_{kl}\mathcal{R}^{kl} - \frac{4}{3}\mathcal{R}\mathcal{R}^{ij} + 4\mathcal{R}^{ikjl}\mathcal{R}_{kl} \\ &\qquad\qquad + 2\Box\left(\mathcal{R}^{ij} - \frac{1}{2}\gamma^{ij}\mathcal{R}\right) + \frac{2}{3}\left(\gamma^{ij}\Box - \nabla^i\nabla^j\right)\mathcal{R}\Bigg) \\ &\qquad\qquad + \gamma^{ij}F_{kl}F^{kl} - 4F^{ik}F^{jl}\gamma_{kl}\Bigg]\Bigg\} \,. \end{aligned} \tag{3.4}$$

Let ${}^4\mathcal{S}_t$ be a spacelike hypersurface at constant t, with unit normal n, and $\Sigma = {}^4\mathcal{S}_t \cap \partial\mathcal{M}$, with induced metric σ. Then, for any Killing vector field ξ associated with an isometry of the boundary four-metric, one defines the conserved charge

$$Q_\xi = \int_\Sigma \mathrm{d}^3x \sqrt{\sigma}\, n^i\, T_{ij}\, \xi^j \,. \tag{3.5}$$

The charge associated to time translation invariance ($\xi = \partial_t$) is the mass M of the spacetime. Evaluating (3.5) for $r \to \infty$ we obtain

$$M = \frac{LV_k}{16\pi G g}\left[c_z - 3c_t + \frac{k^2}{12}\right] , \tag{3.6}$$

where L denotes the period of the z direction, and V_k is the area of the angular sector,[6] $V_k = \int \mathrm{d}\theta\, \mathrm{d}\varphi\, S(\theta)$.

A naive application of (3.5) to compute the conserved charge associated to $\xi = \partial_z$ yields zero, because there is no momentum along the string.[7] Nevertheless, there exists a non-vanishing charge corresponding to translations in z, namely the string tension $\mathcal{T}$ [33–36], which can also be computed using (3.5), but now n is the unit normal to a surface of constant z. In our case we get for the tension per unit time

$$\mathcal{T} = \frac{V_k}{16\pi G g}\left[3c_z - c_t - \frac{k^2}{12}\right] . \tag{3.7}$$

Note that (3.6) and (3.7) coincide with the results obtained in [19] for the uncharged case.

The vacuum is given by the supersymmetric solution (2.29), which corresponds to $c_t = c_z = k^2/24$ and has thus vanishing mass and tension. Therefore, the standard regularization does not produce any vacuum energy. Notice, however, that this procedure suffers from an ambiguity. In fact, to the minimal counterterm action (3.3) one can always add terms

---

[6] V_k is finite if one compactifies the two-surface $\mathcal{S}$ to a torus ($k = 0$) or to a Riemann surface of genus $h > 1$ ($k = -1$). One has then $V_0 = |\mathrm{Im}\tau|$, with τ the Teichmüller parameter of the torus, and, using Gauss-Bonnet, $V_{-1} = 4\pi(h-1)$. In the case of non-compact $\mathcal{S}$ one can define a mass per unit volume.

[7] Solutions with momentum along z will be discussed in section 4.

quadratic in the Riemann tensor, Ricci tensor and Ricci scalar of the boundary. As in four dimensions the variation of the Euler term

$$\mathcal{E}_4 = \sqrt{-\gamma}\left(\mathcal{R}_{ijkl}\mathcal{R}^{ijkl} - 4\mathcal{R}_{ij}\mathcal{R}^{ij} + \mathcal{R}^2\right) \tag{3.8}$$

vanishes, the general quadratic term that produces the ambiguity simplifies to

$$\Delta I_{\rm ct} = -\frac{1}{8\pi G g^3}\int_{\partial\mathcal{M}} \mathrm{d}^4x\,\sqrt{-\gamma}\left(\alpha\mathcal{R}_{ij}\mathcal{R}^{ij} + \beta\mathcal{R}^2\right)\,, \tag{3.9}$$

with α and β denoting arbitrary constants. This yields an additional contribution to the stress tensor,

$$\Delta T^{ij} = \frac{1}{8\pi G g^3}\left[\gamma^{ij}\left(\alpha\mathcal{R}_{kl}\mathcal{R}^{kl} + \beta\mathcal{R}^2\right) - 4\left(\alpha\mathcal{R}^{ikjl}\mathcal{R}_{kl} + \beta\mathcal{R}\mathcal{R}^{ij}\right)\right.$$
$$\left. -2\alpha\Box\left(\mathcal{R}^{ij} - \frac{1}{2}\gamma^{ij}\mathcal{R}\right) - (\alpha+2\beta)\left(\gamma^{ij}\Box - \nabla^i\nabla^j\right)\mathcal{R}\right]\,;$$

in particular, the variations of the energy (3.6) and the tension (3.7) are

$$\Delta M = -\frac{k^2 L V_k}{4\pi G g}\,(\alpha+2\beta) \qquad \text{and} \quad \Delta\mathcal{T} = \frac{k^2 V_k}{4\pi G g}\,(\alpha+2\beta)\,. \tag{3.10}$$

Therefore, conserved quantities are well determined only with the further specification of the prescription that is assumed regarding quadratic terms in the counterterm action.

3.1.2 Holographic stress tensor and conformal anomaly

An important information on the CFT dual to the black string solutions (2.5) is encoded in the expectation value of its energy-momentum tensor, that we wish to compute now. The metric of the background upon which the dual field theory resides is found by rescaling

$$h_{ij} = \lim_{r\to\infty}\frac{1}{(gr)^2}\,\gamma_{ij}\,, \tag{3.11}$$

which yields

$$h_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = -\,\mathrm{d}t^2 + \mathrm{d}z^2 + \frac{1}{g^2}\,\mathrm{d}\Omega_k^2\,, \tag{3.12}$$

and so the conformal boundary, where the $\mathcal{N}=4$ SU(N) SYM theory lives, is $\mathbb{R}\times\mathrm{S}^1\times\mathcal{S}$.

The stress tensor expectation value $\langle\hat{T}_{jk}\rangle$ can be computed using the relation [37]

$$\sqrt{-h}\,h^{ij}\langle\hat{T}_{jk}\rangle = \lim_{r\to\infty}\sqrt{-\gamma}\,\gamma^{ij}\,T_{jk}\,, \tag{3.13}$$

which gives

$$\begin{aligned}\langle\hat{T}^t{}_t\rangle &= \frac{g}{16\pi G}\left[3c_t - c_z - \frac{k^2}{12}\right]\,, \qquad \langle\hat{T}^z{}_z\rangle = \frac{g}{16\pi G}\left[3c_z - c_t - \frac{k^2}{12}\right]\,,\\ \langle\hat{T}^\theta{}_\theta\rangle &= \langle\hat{T}^\varphi{}_\varphi\rangle = -\frac{g}{16\pi G}\left[c_t + c_z + (kgq)^2 - \frac{k^2}{6}\right]\,,\end{aligned} \tag{3.14}$$

i.e. an anisotropic perfect fluid form. As expected, this stress tensor is not traceless,

$$\langle \hat{T}^i_{\ i} \rangle = \frac{gk^2}{96\pi G}\left[1 - 12\,(gq)^2\right]\,. \tag{3.15}$$

The first part of (3.15) matches exactly the conformal anomaly of the boundary CFT coming from the background curvature

$$\mathcal{A} = \frac{N^2}{32\pi^2}\left(\mathcal{R}_{ij}\mathcal{R}^{ij} - \frac{1}{3}\mathcal{R}^2\right)\,, \tag{3.16}$$

if we use the AdS/CFT dictionary $N^2 = \pi/(2Gg^3)$. The second part of (3.15), proportional to q^2, results from the coupling of the CFT to a background gauge field. The two contributions exactly cancel when the magnetic charge assumes the value

$$q^2 = \frac{1}{12g^2}\,. \tag{3.17}$$

This is precisely the behaviour we expect; in fact choosing (3.17) in (2.18) has the effect of cancelling the logarithmic terms in the Fefferman-Graham expansion, which produce the anomaly. Notice that the generalized Dirac quantization condition (3.17) was found in [22, 23] by requiring supersymmetry. Maldacena and Nuñez showed that it corresponds to a twisting of the dual SYM theory [40]: Putting a supersymmetric field theory on a curved manifold generally breaks supersymmetry, because one will not have a Killing spinor obeying $(\partial_i + \omega_i)\,\epsilon = 0$, where ω_i denotes the spin connection. If, however, the field theory has a global R-symmetry, it can be coupled to an external gauge field that couples to the R-symmetry current. If we choose this external gauge field to be equal to the spin connection, $A_i = \omega_i$, we can find a covariantly constant spinor since $(\partial_i + \omega_i - A_i)\,\epsilon = \partial_i\epsilon$, which vanishes for constant ϵ. The resulting theory is called 'twisted', because the coupling to the external gauge field effectively changes the spins of all fields. The requirement $A_i = \omega_i$ yields precisely the charge quantization condition (3.17).

Notice finally that the ambiguity due to the quadratic contributions (3.9) to the action does not affect the Weyl anomaly, because

$$\langle \Delta\hat{T}^i_{\ i} \rangle = 0\,, \tag{3.18}$$

as it can be seen from (3.10).

3.1.3 Kounterterm procedure

A different approach to regularize both the conserved quantities and the Euclidean action for asymptotically AdS spacetimes is given by the Kounterterm proposal [26, 27], considering covariant boundary terms depending on both extrinsic and intrinsic quantities, instead of the Gibbons-Hawking term plus intrinsic counterterms,

$$I_{\text{ren}} = \frac{1}{16\pi G}\int_{\mathcal{M}} \mathrm{d}^{d+1}x\sqrt{-g}\,(R - 2\Lambda) - c_d\int_{\partial\mathcal{M}} \mathrm{d}^d x B_d(\gamma, \mathcal{R}(\gamma), K)\,. \tag{3.19}$$

B_d is a polynomial in the induced metric, the boundary Riemann tensor and the extrinsic curvature. With a suitable choice of B_d one is able to achieve a well-posed variational principle and to solve the regularization problem at the same time. The main advantage of this procedure is that it provides a closed formula for the charges in all dimensions. This is a consequence of the use of geometrical boundary terms related to topological invariants and Chern-Simons forms which are not obtained through the algorithm given by holographic renormalization. Therefore, to obtain their explicit form, we do not face the technical difficulties of standard AdS gravity regularization.

We now focus on the main features of the procedure and specialize to the five-dimensional case. The corresponding coupling constant c_4 is fixed demanding the total action to be stationary under arbitrary on-shell variations of the fields that respect the asymptotic form of the metric in asymptotically AdS spacetimes, i.e.,

$$R^{\alpha\beta}_{\mu\nu} + g^2\delta^{[\alpha\beta]}_{[\mu\nu]} = 0 \tag{3.20}$$

and

$$K^i_j = g\delta^i_j\,. \tag{3.21}$$

As it has been argued in [31], a Dirichlet boundary condition on the metric γ_{ij} does not really make sense in spacetimes with conformal boundary, as it is the case of asymptotically AdS spaces. Indeed, the boundary metric blows up as the boundary is reached. This can be seen as a motivation to introduce the regular asymptotic condition (3.21), because it does not induce additional divergences in the variation of the action, and yet it is compatible with the idea of holographic reconstruction of the spacetime.

The boundary term in five dimensions is given by

$$B_4 = \sqrt{-\gamma}\delta^{[i_1i_2i_3]}_{[j_1j_2j_3]}K^{j_1}_{i_1}\left(\mathcal{R}^{j_2j_3}_{i_2i_3} - K^{j_2}_{i_2}K^{j_3}_{i_3} + \frac{g^2}{3}\delta^{j_2}_{i_2}\delta^{j_3}_{i_3}\right)\,,$$

and is multiplied by a coupling constant $c_4 = 1/(128\pi G g^2)$.

Conserved quantities are defined as Noether charges associated to asymptotic Killing vectors ξ. Their expression appears naturally split into two parts,

$$Q(\xi) = q(\xi) + q_0(\xi)\,, \tag{3.22}$$

where the quantity

$$\begin{aligned} q(\xi) &= \int_\Sigma \mathrm{d}^3x\sqrt{\sigma}\,\left(u_j\, q^j_i\,\xi^i\right)\,, \\ q^j_i &= \frac{1}{64\pi G g^2}\delta^{[jj_2\ldots j_4]}_{[i_1i_2\ldots i_4]}K^{i_1}_i\delta^{i_2}_{j_2}\left(R^{i_3i_4}_{j_3j_4} + g^2\delta^{[i_3i_4]}_{[j_3j_4]}\right)\,, \end{aligned} \tag{3.23}$$

provides, in general, the mass and the angular momentum for point-like solutions, but also for topological black holes. It can be noticed that the above formula is identically vanishing for the AdS vacuum. Therefore, the second contribution in (3.22)

$$\begin{aligned} q_0(\xi) &= \int_\Sigma \mathrm{d}^3x\,\sqrt{\sigma}\,\left(u_j\, q^j_{(0)i}\,\xi^i\right)\,, \\ q^j_{(0)i} &= -\frac{1}{128\pi G g^2}\,\delta^{[jj_2\ldots j_4]}_{[i_1i_2\ldots i_4]}\,\left(\delta^{i_2}_{j_2}K^{i_1}_i + \delta^{i_2}_i K^{i_1}_{j_2}\right)\left(\mathcal{R}^{i_3i_4}_{j_3j_4} - K^{i_3}_{j_3}K^{i_4}_{j_4} + g^2\delta^{i_3}_{j_3}\delta^{i_4}_{j_4}\right)\,, \end{aligned} \tag{3.24}$$

is truly a covariant formula for the vacuum energy for any asymptotically AdS spacetime (see [32] for the vacuum energy results obtained in the standard regularization approach).

For the quantized charge case (3.17), evaluating the first piece $q(\xi)$ in the conserved charges formula (3.22), we obtain the same values as in the Dirichlet regularization shown above, for the energy (3.6) if $\xi = \partial_t$ and the tension (3.7) if $\xi = \partial_z$. For the second part, $q_0(\xi)$, one gets

$$q_0(\partial_t) = -\frac{1}{16\pi G g}\frac{k^2}{24}LV_k\,, \qquad q_0(\partial_z) \;=\frac{1}{16\pi G g}\frac{k^2}{24}V_k\,, \tag{3.25}$$

such that there is an additional contribution to the total mass and tension in the Kounterterm formalism. One could eventually match these results with the ones obtained using intrinsic counterterms through the addition of quadratic terms in the curvature (3.9). In fact, the ambiguity pointed out in (3.10) is able to reproduce the same vacuum energy and tension for supersymmetric magnetic strings if α and β satisfy

$$\alpha + 2\beta = \frac{1}{96}\,,$$

because there is no an a priori reasoning to rule out the existence of a vacuum energy for a supersymmetric solution in asymptotically AdS gravity.

One might also supplement the Kounterterms with the logarithmic terms in (3.3), which would allow the regularization of the conserved charges in the general case.

3.2 Thermodynamics

The Hawking temperature of the black string solutions is obtained by requiring the absence of conical singularities in the Euclidean section of the metric (2.5). Setting $t = i\tau$ and using (2.21), the (τ, r)-part of the near-horizon metric becomes

$$\mathrm{d}\sigma^2 = \frac{b_{\mathrm{h}}\,\alpha}{4}\,\rho^2\,\mathrm{d}\tau^2 + \mathrm{d}\rho^2, \tag{3.26}$$

where

$$\alpha = y'(r_{\mathrm{h}}) = \frac{12g^2r_{\mathrm{h}}^4 + 3kr_{\mathrm{h}}^2 - 4(kq)^2}{3r_{\mathrm{h}}^3}\,, \tag{3.27}$$

and the new radial coordinate ρ is defined by

$$\mathrm{d}\rho = \frac{\mathrm{d}r}{\sqrt{\alpha}\sqrt{r - r_{\mathrm{h}}}}\,. \tag{3.28}$$

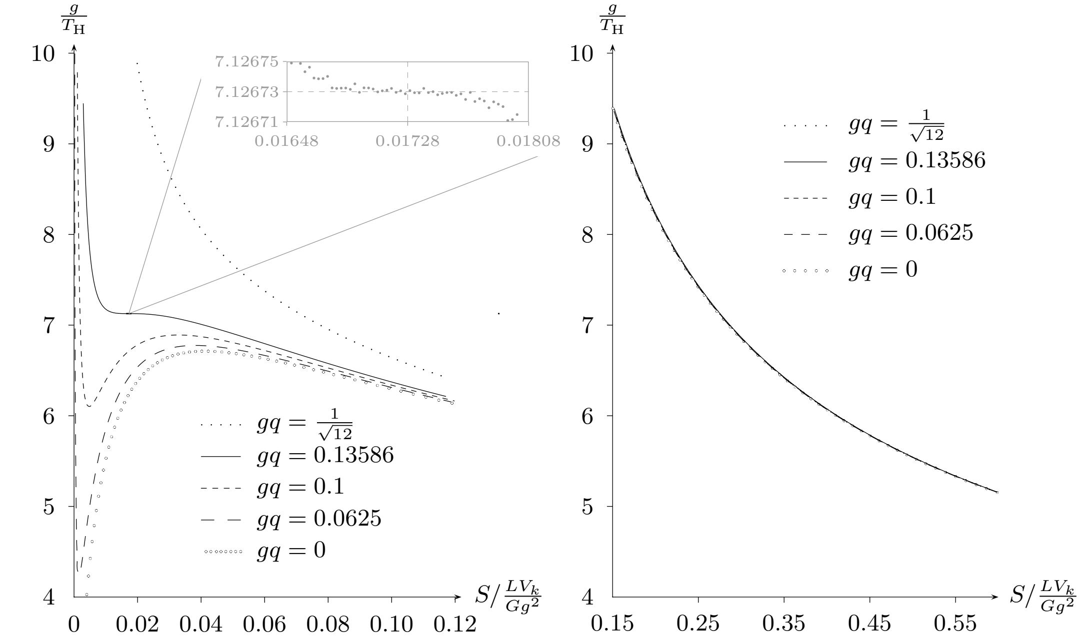
$4\pi/\sqrt{b_{\mathrm{h}}\alpha}$. This gives the Hawking temperature

$$T_{\mathrm{H}} = \frac{1}{4\pi}\sqrt{b_{\mathrm{h}}\alpha} = \frac{|\mu|}{2\pi\sqrt{a_{\mathrm{h}}}\,r_{\mathrm{h}}^2}\,. \tag{3.29}$$

In the last step, we used (2.22).

Computation of the area of the event horizon

$$A = \int_{r=r_{\mathrm{h}}} \mathrm{e}^T\, r_{\mathrm{h}}^2\, S(\theta)\,\mathrm{d}z\,\mathrm{d}\theta\,\mathrm{d}\varphi \tag{3.30}$$



(a) $k = 1$ (b) $k = -1$

Figure 2: The inverse Hawking temperature $\frac{1}{T_{\mathrm{H}}}$ as a function of the entropy S 2a for $k = 1$ and 2b for $k = -1$ at five values of the magnetic charge q. The critical charge for a first order phase transition present in 2a is given by $q_{\mathrm{crit}} = 0.13586(1)$. In the first two graphs of 2a with $q < q_{\mathrm{crit}}$ one finds two local extrema and a turning point. At $q = q_{\mathrm{crit}}$ the extrema and turning point merge into a saddle point, which vanishes at $q > q_{\mathrm{crit}}$. The graphs of 2b nearly coincide for the five values q in the plotted range.

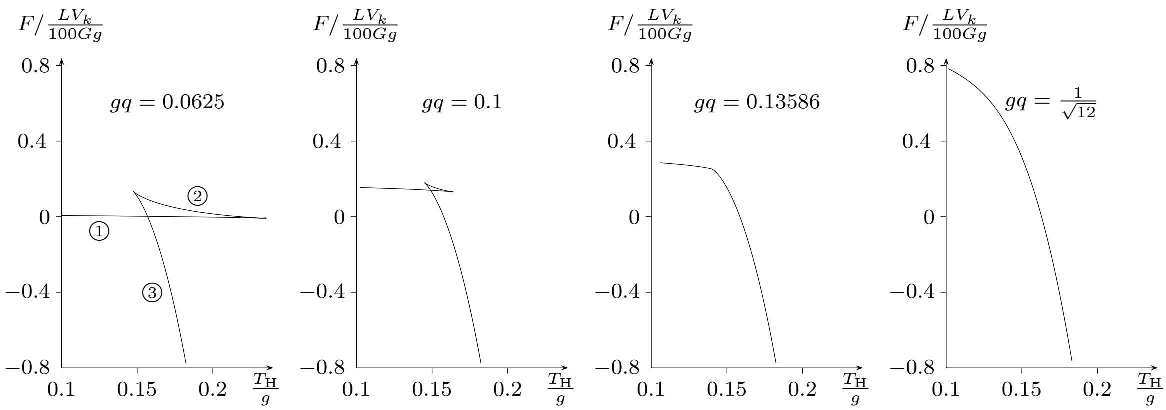


Figure 3: The free energy F as a function of the Hawking temperature T_{H} for $k = 1$ at four values of the magnetic charge q. The critical charge is given by $q_{\mathrm{crit}} = 0.13586(1)$. In the first two diagrams with $q < q_{\mathrm{crit}}$ one finds three branches. At $q = q_{\mathrm{crit}}$ the branches merge into a single one, which remains also at $q > q_{\mathrm{crit}}$.

yields for the entropy

$$S = \frac{A}{4G} = \frac{\sqrt{a_{\rm h}}\, r_{\rm h}^2\, L\, V_k}{4G}\,. \tag{3.31}$$

Note that for $\mu \geq 0$ the thermodynamic quantities (3.6), (3.7), (3.29) and (3.31) obey the Smarr-type formula[8]

$$M + \mathcal{T} L = T_{\rm H} S\,. \tag{3.32}$$

As a confirmation of the results obtained so far, we compute the Euclidean action

$$I_{\rm E} = -\frac{1}{4\pi G}\int \sqrt{g}\left[\frac{R}{4} + 3g^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{6\sqrt{3}}\epsilon^{\mu\nu\alpha\beta\gamma}F_{\mu\nu}F_{\alpha\beta}A_\gamma\right]{\rm d}^5x + I_{\rm GH} + I_{\rm ct}\,, \tag{3.33}$$

where

$$I_{\rm GH} = \frac{1}{8\pi G}\int {\rm d}^4x\sqrt{\gamma}K\,, \tag{3.34}$$

and $I_{\rm ct}$ is given by (3.3), with γ instead of $-\gamma$, but with the same overall sign. In our case, the Chern-Simons term does not contribute, and $I_{\rm E}$ reduces on-shell to

$$I_{\rm E} = \frac{1}{4\pi G}\int \sqrt{g}\left[2g^2 + \frac{1}{6}F_{\mu\nu}F^{\mu\nu}\right]{\rm d}^5x + I_{\rm GH} + I_{\rm ct}\,. \tag{3.35}$$

Plugging in our expressions for $\sqrt{g}$ and F^2, the bulk term reads

$$I_{\rm E,\,bulk} = \frac{\beta L V_k}{4\pi G}\int_{r_{\rm h}}^{R} {\rm e}^{F+U}\, r^2\left[2g^2 + \frac{k^2q^2}{3r^4}\right]{\rm d}r\,, \tag{3.36}$$

where β denotes the inverse temperature and R is some cutoff to be sent to infinity at the end of the calculation. Using the equation of motion (2.8), this can be integrated to give

$$I_{\rm E,\,bulk} = \frac{\beta L V_k}{8\pi G}\left[V'\,{\rm e}^{F-U}\, r^2\right]_{r_{\rm h}}^{R}\,. \tag{3.37}$$

Evaluation of the boundary terms yields

$$I_{\rm GH} + I_{\rm ct} = -\frac{\beta L V_k}{8\pi G}\,{\rm e}^F\left[{\rm e}^{-U}\, r^2\left(F' + \frac{2}{r}\right) - 3gr^2 - \frac{k}{2g} + \frac{k^2\ln(gr)}{gr^2}\left(q^2 - \frac{1}{12g^2}\right)\right]_{r=R}\,. \tag{3.38}$$

Adding this to (3.37) and taking the limit $R \to \infty$, the final result takes the form

$$I_{\rm E} = \beta(M - T_{\rm H}S)\,, \tag{3.39}$$

with M, $T_{\rm H}$ and S given by (3.6), (3.29) and (3.31) respectively. The Helmholtz free energy is thus

$$F = \frac{I_{\rm E}}{\beta} = M - T_{\rm H}S\,, \tag{3.40}$$

which correctly coincides with the result we would have obtained by simply Legendre transforming the mass. Using (3.32), one gets

$$F = -\mathcal{T} L\,. \tag{3.41}$$

[8] Note that a Smarr-type formula has been shown to hold in full generality [41].

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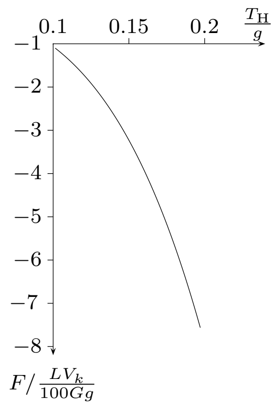

Figure 4: The free energy F as a function of the Hawking temperature $T_{\rm H}$ for $k = -1$ at $gq = 0.13586(1)$.

Note that the Smarr formula (3.32) is a simple consequence of the scaling behaviour

$$F(T_{\rm H}, \lambda L, q) = \lambda F(T_{\rm H}, L, q)\,. \tag{3.42}$$

Deriving this with respect to λ, using $\mathcal{T} = -(\partial F/\partial L)_{T_{\rm H},q}$ and setting $\lambda = 1$ gives (3.41) and thus (3.32).

Figures 2a and 2b show the inverse temperature $\beta = 1/T_{\rm H}$ as a function of the entropy S (in units of LV_k/G) for $k = 1$ and $k = -1$ respectively. In both cases, the curves, which were obtained numerically, are drawn for five different values of the magnetic charge q. In the $k = -1$ case the graphs nearly coincide for the chosen values of q. Note that the functional relationship $\beta = \beta(S)$ represents one of the equations of state. We see that for $k = -1$ the entropy decreases monotonically with β for all values of q, so the heat capacity

$$c_{L,q} = T_{\rm H}\left(\frac{\partial S}{\partial T_{\rm H}}\right)_{L,q} \tag{3.43}$$

is always positive, which is a necessary condition for local thermodynamic stability. This situation changes drastically for $k = 1$: If the magnetic charge is smaller than the critical value $gq_{\rm crit} = 0.13586(1)$, there are three branches of black string solutions, with the middle branch being thermodynamically unstable. The left one is present only for non-vanishing q, because otherwise no extremal $k = 1$ black strings exist. At $q = q_{\rm crit}$ the turning points of $\beta(S)$ merge, and disappear for $q > q_{\rm crit}$, so that we are left with one single branch of stable black strings. Notice that the limit $S \to 0$ corresponds to the extremal solutions that have zero Hawking temperature, and horizon coordinate $r_{\rm h}$ given by the limiting value of (2.25). The $\beta(S)$ curve reminds us of the $P(V)$ van der Waals equation of state, where the pressure P is replaced here by β and the volume V by S. This analogy was noticed for the first time and explored in detail in [9, 10] for the case of electrically charged AdS black holes.[9] We will have to say more about this later.

[9]Rotation was included in [42].

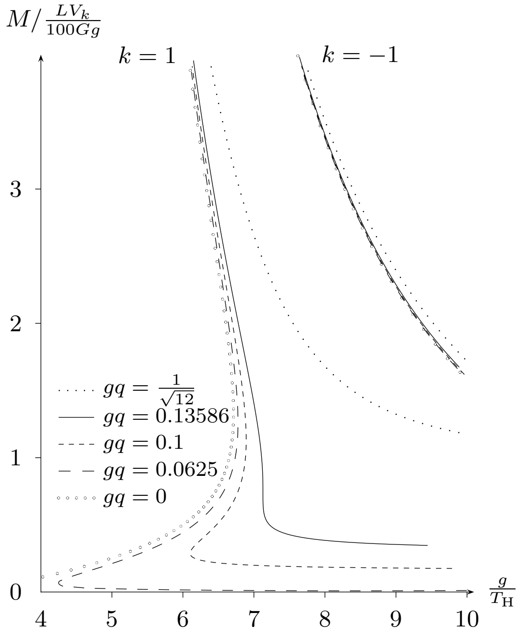

Figure 5: The mass M in dependence of the inverse Hawking temperature T_{H} for $k=1$ and $k=-1$ at five values of the magnetic charge q.

The free energy (in units of $LV_k/100Gg$) as a function of temperature is shown in figure 3 ($k=1$) for four different values of the magnetic charge q and in figure 4 ($k=-1$), for a single value of q.

In the hyperbolic case $k=-1$, there is always one single branch. For $k=1$ and small charge $q<q_{\mathrm{crit}}$, starting at the left of the plot (low temperature), there is a single branch of free energy, corresponding to stable small black strings, which we shall refer to as branch 1. At a certain temperature, branches 2 (unstable black strings) and 3 (stable large black strings) appear and separate from each other at higher temperatures. At some still larger temperature, branches 1 and 2 meet and disappear, whereas branch 3 continues to the right.

If we raise the magnetic charge, the swallowtail (i.e. the triangle confined by the three branches) becomes smaller and finally disappears at $q=q_{\mathrm{crit}}$. At this critical point, the first order phase transition degenerates and becomes of higher order. Finally, for $q>q_{\mathrm{crit}}$, only a single stable branch remains.

Notice that the free energy does not go to zero for $T_{\mathrm{H}}\to 0$, which means that in general the extremal black string has non-vanishing F. It is easy to see that $F=0$ for the supersymmetric solution (2.29), which therefore represents some sort of background.

We are however free to add finite counterterms to (3.3), e.g. a term proportional to

$$\int \mathrm{d}^4x\sqrt{-\gamma}F_{ij}F^{ij}\,. \tag{3.44}$$

This would give an additional contribution to the Euclidean action that depends on the

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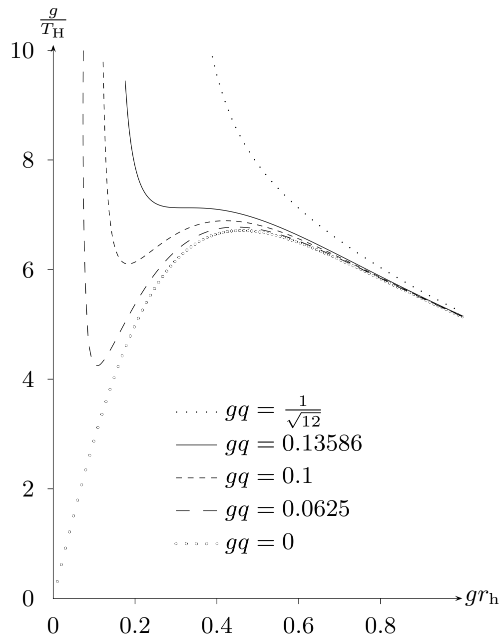

Figure 6: The inverse Hawking temperature $\frac{1}{T_\mathrm{H}}$ in dependence of the horizon r_h for $k = 1$ at five values of the magnetic charge q.

magnetic charge. By adjusting the prefactor of (3.44) one can probably obtain a free energy that vanishes for the extremal solution, although we did not check this explicitly.

Finally, by performing the double analytic continuation $\chi = it$, $\tau = iz$, corresponding to a change of the sign of μ, the black string solutions become static magnetically charged bubble of nothing solutions. The S^1 factor of the metric pinches of at the radius r_h, and the solution is regular if the spatial coordinate χ is identified modulo $s = 1/T_\mathrm{H} = 4\pi/\sqrt{b_\mathrm{h}\alpha}$. These are zero-temperature solutions, in the $L \to \infty$ limit, with the same spatial asymptotic structure $\mathcal{S} \times \mathrm{S}^1$ as the black strings. As shown in figure 6, the length of the S^1 is fixed by the size r_h of the bubble. Note that in presence of a magnetic charge there is a minimal size of the bubble, unlike in the uncharged $k = 1$ case studied in [18]. Also, these uncharged $k = 1$ bubbles of nothing where shown to exist only under some critical length of the S^1. If, however, one adds magnetic charge, bubble solutions exist for any length of the S^1. The ground state is given by the lowest total energy bubble, which is given (in terms of the black string quantities) by

$$E_\mathrm{b} = -\frac{\mathcal{T}}{T_\mathrm{H}} = \frac{F}{LT_\mathrm{H}}. \tag{3.45}$$

The energy of the bubble is plotted in figure 7 as a function of the size s of the asymptotic S^1. For non-vanishing magnetic charge below the critical value there are three branches of bubbles, that merge in a single branch above the critical charge. Therefore, there is a first order phase transition between a small size bubble and a large size one for charges lower than the critical charge q_crit, that disappears for higher charge. Therefore, the quantum phase transition that occurs in the strongly coupled dual gauge theory as one varies the

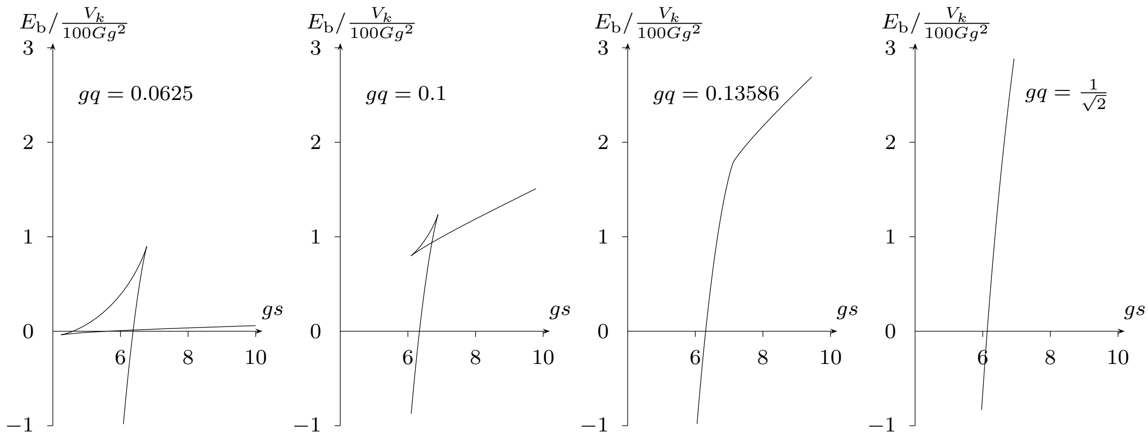

Figure 7: The energy E_b as a function of the of the size $s = \frac{1}{T_\mathrm{H}}$ of the asymptotic S^1 for $k = 1$ at four values of the magnetic charge q. The critical charge is given by $q_\mathrm{crit} = 0.13586(1)$. In the first two diagrams with $q < q_\mathrm{crit}$ one finds three branches. At $q = q_\mathrm{crit}$ the branches merge into a single one, which remains also at $q > q_\mathrm{crit}$.

size of the S^1 found in [18] for the uncharged $k = 1$ bubbles becomes a quantum phase transition between the states dual to the small/large bubbles, and then disappears for $q > q_\mathrm{crit}$.

4. Supersymmetric waves on strings

4.1 Construction of the solution

We now would like to construct supersymmetric generalizations of the magnetic string solutions of [22, 23], that carry momentum along the string. To this end, we recall that supersymmetric solutions of minimal gauged supergravity in five dimensions are divided into timelike and null classes, according to the nature of the Killing vector constructed as a bilinear from the Killing spinor. The general null solution has been obtained in [43] and reads[10]

$$\begin{aligned} \mathrm{d}s^2 &= -H^{-1}(\mathcal{F}\,\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}v) + H^2[(\mathrm{d}x^1 + a_1\,\mathrm{d}u)^2 + \mathrm{e}^{3\phi}(\mathrm{d}x^\alpha + \mathrm{e}^{-3\phi}\,a_\alpha\,\mathrm{d}u)^2]\,, \\ A &= A_u\,\mathrm{d}u + \frac{\sqrt{3}}{4g}\varepsilon_{\alpha\beta}\phi_{,\alpha}\,\mathrm{d}x^\beta\,. \end{aligned} \tag{4.1}$$

The function $\phi(u, x^i)$ is determined by the equation

$$\mathrm{e}^{2\phi}\,\partial_z^2\,\mathrm{e}^\phi + \Delta^{(2)}\phi = 0\,. \tag{4.2}$$

[10] In this section, the five-dimensional geometries are described by the coordinates $\{u, v, x^1, x^2, x^3\}$, where $x^1 = z$, $x^2 = x$ and $x^3 = y$. Latin letters $i, j, \ldots$ are indices on the three-dimensional flat space parameterized by $\{x^1, x^2, x^3\}$. Early greek letters $\alpha, \beta, \ldots$ are indices of the two-dimensional space $\{x^2, x^3\}$, again with flat metric. The antisymmetric tensor $\varepsilon_{\alpha\beta}$ on this space is defined such that $\varepsilon_{23} = 1$, and $\Delta^{(2)} = \partial_\alpha\partial_\alpha$ is the flat Laplacian in two dimensions.

Given a solution of (4.2), $H(u, x^i)$ is obtained from

$$H = -\frac{1}{2g}\phi_{,z}\,, \tag{4.3}$$

and $A_u(u, x^i)$ is found by solving the Maxwell equation

$$\partial_z[H^2\,\mathrm{e}^{2\phi}\,\partial_z(\mathrm{e}^{\phi}\,A_u)] + \partial_\alpha(H^2 A_{u,\alpha}) = \frac{\sqrt{3}}{2g}H\varepsilon_{\alpha\beta}\phi_{,\alpha u}H_{,\beta}\,. \tag{4.4}$$

Then, the functions $a_i(u, x^j)$ are determined by the system

$$\begin{aligned}\frac{1}{2\sqrt{3}}\varepsilon_{\alpha\beta}\partial_\alpha(H^3 a_\beta) &= -H^2\,\mathrm{e}^{2\phi}\,\partial_z(\mathrm{e}^{\phi}\,A_u)\,,\\ \frac{1}{2\sqrt{3}}[\partial_\alpha(H^3 a_1) - \partial_z(H^3 a_\alpha)] &= H^2\varepsilon_{\alpha\beta}A_{u,\beta} - \frac{\sqrt{3}}{4g}H^2\phi_{,\alpha u}\,,\end{aligned} \tag{4.5}$$

whose integrability condition is (4.4). Finally, the function $\mathcal{F}(u, x^i)$ follows from the uu-component of the Einstein equations,

$$R_{uu} = 2F_{u\sigma}F_u{}^{\sigma} - \frac{1}{3}g_{uu}(F^2 + 12g^2)\,. \tag{4.6}$$

To find the subset of supersymmetric null solutions which describe waves on strings, we suppose ϕ to be separable,

$$\phi(u, x, y, z) = \phi_1(u, z) + \phi_2(u, x, y)\,. \tag{4.7}$$

Substituting this expression of ϕ in the equation (4.2), we find that ϕ_1 and ϕ_2 have to satisfy the equations

$$\partial_z^2\,\mathrm{e}^{\phi_1} = \frac{k}{24g}\,\mathrm{e}^{-2\phi_1}\,, \tag{4.8}$$

$$\Delta^{(2)}\phi_2 = -\frac{k}{24g}\,\mathrm{e}^{3\phi_2}\,, \tag{4.9}$$

where $k(u)$ is an arbitrary function. (4.8) implies

$$\mathrm{e}^{3\phi_1}(\phi_{1,z})^2 = \mu\,\mathrm{e}^{\phi_1} - \frac{k}{12g}\,, \tag{4.10}$$

where $\mu(u)$ denotes again an arbitrary function. Equation (4.9) is the Liouville equation. As a particular solution we choose

$$\mathrm{e}^{3\phi_2} = \frac{64g}{\Upsilon^2}\,, \tag{4.11}$$

where $\Upsilon(u, x, y) = 1 + k(x^2 + y^2)$. To proceed we suppose that A_u is a function of u and z only. Then (4.4) implies that

$$A_u = \mathrm{e}^{-\phi_1}\left[\alpha\int \mathrm{d}z\;(\mathrm{e}^{\phi_1}\,\phi_{1,z})^{-2} + \beta\right]\,, \tag{4.12}$$

with $\alpha(u)$ and $\beta(u)$ arbitrary functions. The system (4.5) is solved by

$$Ha_1 = \frac{k'}{gk}\frac{\Upsilon - 1}{\Upsilon} + \Gamma\,, \qquad H^3 a_2 = \frac{16\sqrt{3}\alpha y}{g\Upsilon}\,, \qquad H^3 a_3 = -\frac{16\sqrt{3}\alpha x}{g\Upsilon}\,, \tag{4.13}$$

where $k' = \partial_u k$ and $\Gamma(u,z)$ is an arbitrary function. We introduce the new coordinate ρ defined by

$$\rho = \frac{1}{2\phi_{1,z}(g\,\mathrm{e}^{\phi_1})^{3/2}}\,, \tag{4.14}$$

and choose

$$\Gamma = -H\partial_u z\,, \tag{4.15}$$

where z has to be considered as a function of u and ρ, and the derivative has to be taken considering ρ as fixed. Then, if we rescale $\mathrm{d}u \to \mu^{3/2}\,\mathrm{d}u$, $\mathcal{F} \to \mu^{-3/2}\mathcal{F}$ and $\beta \to \mu^{-5/2}\beta$ to eliminate μ, in the coordinate system $\{u, v, \rho, x, y\}$, the general solution (4.1) takes the form

$$\begin{aligned}
\mathrm{d}s^2 &= \frac{h^{\frac{3}{2}}}{g^2\rho^2}\left(\frac{1}{2}\mathcal{F}\,\mathrm{d}u^2 + \mathrm{d}u\,\mathrm{d}v\right) + \left(\frac{1}{g\rho h}\,\mathrm{d}\rho + \frac{k'}{gk}\frac{\Upsilon-1}{\Upsilon}\,\mathrm{d}u\right)^2 \\
&\quad + \frac{4}{g^4\rho^2\Upsilon^2}\left[\left(\mathrm{d}x - 2\sqrt{3}g\alpha y\Upsilon h^{\frac{3}{2}}\,\mathrm{d}u\right)^2 + \left(\mathrm{d}y + 2\sqrt{3}g\alpha x\Upsilon h^{\frac{3}{2}}\,\mathrm{d}u\right)^2\right]\,, \\
A_u &= \alpha\left\{-2h^{\frac{1}{2}} + 3h^{-\frac{1}{2}} + \frac{g^2k\rho^2}{h}\ln\left[\frac{1}{g\rho}\left(1 + h^{\frac{1}{2}}\right)\right]\right\} + \frac{4\beta g^3\rho^2}{h}\,, \\
A_x &= \frac{ky}{g\sqrt{3}\Upsilon}\,, \qquad A_y = -\frac{kx}{g\sqrt{3}\Upsilon}\,,
\end{aligned} \tag{4.16}$$

where

$$h = 1 + \frac{g^2k\rho^2}{3}\,. \tag{4.17}$$

In the case in which $\alpha = 0$ and k is constant, the solution (4.16) becomes

$$\begin{aligned}
\mathrm{d}s^2 &= \frac{1}{g^2\rho^2h^2}\left[\mathrm{d}\rho^2 + h^{7/2}\left(\frac{\mathcal{F}}{2}\,\mathrm{d}u^2 + \mathrm{d}u\,\mathrm{d}v\right) + \frac{4h^2}{g^2}\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{\Upsilon^2}\right]\,, \\
A_u &= \frac{4\beta g^3\rho^2}{h}\,, \qquad A_x = \frac{ky}{g\sqrt{3}\Upsilon}\,, \qquad A_y = -\frac{kx}{g\sqrt{3}\Upsilon}\,.
\end{aligned} \tag{4.18}$$

This solution describes a string with non-vanishing electric charge density. The uu-component of the Einstein equations gives

$$\Upsilon^2\Delta^{(2)}\mathcal{F} + \frac{8h^2 - 20h}{g^2\rho}\mathcal{F}_{,\rho} + \frac{4h^2}{g^2}\mathcal{F}_{,\rho\rho} + 2048\beta^2g^6\frac{\rho^4}{h^{7/2}} = 0\,. \tag{4.19}$$

A particular solution of this equation is

$$\mathcal{F} = \frac{3456\beta^2g^2}{k^3}\left(\frac{1}{7h^{7/2}} - \frac{1}{5h^{5/2}}\right)\,. \tag{4.20}$$

In the case in which $\alpha = \beta = 0$ the solution (4.16) becomes

$$\begin{aligned} \mathrm{d}s^2 &= \frac{h^{\frac{3}{2}}}{g^2\rho^2}\left(\frac{1}{2}\mathcal{F}\,\mathrm{d}u^2 + \mathrm{d}u\,\mathrm{d}v\right) + \left(\frac{1}{g\rho h}\,\mathrm{d}\rho + \frac{k'}{gk}\frac{\Upsilon - 1}{\Upsilon}\,\mathrm{d}u\right)^2 + \frac{4}{g^4\rho^2}\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{\Upsilon^2}\,, \\ A_u &= 0\,, \qquad A_x = \frac{ky}{g\sqrt{3}\Upsilon}\,, \qquad A_y = -\frac{kx}{g\sqrt{3}\Upsilon}\,. \end{aligned} \tag{4.21}$$

This solution describes a wave (with profile given by $\mathcal{F}$) that propagates on a string in an asymptotically AdS_5 space time. In this case the uu-component of the Einstein equations is

$$\begin{aligned} &\Upsilon^2\Delta^{(2)}\mathcal{F} + \frac{8h^2 - 20h}{g^2\rho}\mathcal{F}_{,\rho} + \frac{4h^2}{g^2}\mathcal{F}_{,\rho\rho} \\ &+ \frac{4\rho^2k'^2}{h^{7/2}k}\left(-\frac{14}{3} + \frac{46h}{3} - 16h^2 - \frac{32h}{3\Upsilon} + \frac{32h^2}{\Upsilon} - \frac{16h^2}{\Upsilon^2}\right) - \frac{16\rho^2k''}{3h^{5/2}}\left(1 - 2h + \frac{2h}{\Upsilon}\right) = 0\,. \end{aligned} \tag{4.22}$$

A particular solution of this equation is

$$\mathcal{F} = \mathcal{H} + \mathcal{P}\,, \tag{4.23}$$

where $\mathcal{H}$ is a solution of the homogeneous part of the equation (4.22),

$$\mathcal{H}(u, \rho, x, y) = \mathcal{H}_1(u, \rho)\mathcal{H}_2(u, x, y)\,, \tag{4.24}$$

where $\mathcal{H}_1$ has to satisfy Heun's equation[11]

$$\ddot{\mathcal{H}}_1 + \left(\frac{5/2}{r - 1} - \frac{1}{r}\right)\dot{\mathcal{H}}_1 - \frac{3c/(16k)}{r(r-1)^2}\mathcal{H}_1 = 0\,, \tag{4.25}$$

and $\mathcal{H}_2$ has to obey the Laplace equation

$$\widehat{\triangle}^{(2)}\mathcal{H}_2 = c\mathcal{H}_2\,, \tag{4.26}$$

where $\widehat{\triangle}^{(2)}$ is the Laplacian related to the metric $\mathrm{d}s^2 = (\mathrm{d}x^2 + \mathrm{d}y^2)/\Upsilon^2$ and c is a function of u only.

$\mathcal{P}$ is a particular solution of the equation (4.22),

$$\mathcal{P} = \mathcal{A} + \frac{\mathcal{B}}{\Upsilon} + \frac{\mathcal{C}}{\Upsilon^2}\,, \tag{4.27}$$

where

$$\begin{aligned} \mathcal{A}(u, \rho) &= \frac{k'^2}{g^2h^{7/2}k^3}\left(\frac{3}{2} - \frac{99h}{10} + 18h^2\right) + \frac{k''}{g^2h^{5/2}k^2}\left(\frac{6}{5} - 12h^2\right)\,, \\ \mathcal{B}(u, \rho) &= \frac{k'^2}{g^2h^{5/2}k^3}(6 - 28h + 12h^2) + \frac{4k''}{g^2h^{3/2}k^2}\,, \\ \mathcal{C}(u, \rho) &= \frac{k'^2}{g^2h^{3/2}k^3}(8 - 6h)\,. \end{aligned} \tag{4.28}$$

[11] We have introduced the new radial coordinate $r = -\frac{1}{3}g^2k\rho^2$ and defined $\dot{\mathcal{H}}_1 = \partial_r\mathcal{H}_1$.

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The sections of the geometry (4.21) with constant u, v and ρ are two-dimensional spaces that have curvature proportional to k. It is amusing to note that, by choosing e.g. $k(u) = \tanh u$, these sections can change continuously from a hyperbolic space $\mathbb{H}^2$ to a sphere S^2 as the coordinate u varies from $-\infty$ to $+\infty$.

In the case $k = -1$ and $\mathcal{F} = 0$ the spacetime is described by the metric

$$\mathrm{d}s^2 = \frac{1}{g^2\rho^2h^2}\left(\mathrm{d}\rho^2 + h^{7/2}\,\mathrm{d}u\,\mathrm{d}v + \frac{4h^2}{g^2}\frac{\mathrm{d}x^2+\mathrm{d}y^2}{\Upsilon^2}\right). \tag{4.29}$$

This geometry describes a magnetic black string that asymptotes to AdS_5 [23]. This solution preserves one quarter of supersymmetry and approaches the half-supersymmetric product space $\mathrm{AdS}_3 \times \mathbb{H}^2$ near the event horizon at $\rho = \sqrt{3}/g$.

4.2 Siklos-Virasoro invariance

An interesting property of the full family of solutions (4.21) is that they enjoy a large reparametrization invariance, called Siklos-Virasoro invariance. Indeed if we perform the diffeomorphism

$$\overline{u} = \chi(u)\,, \qquad \overline{v} = v - \lambda(u,\rho,x^\alpha)\,, \qquad \overline{\rho} = \sqrt{\chi'}\rho\,, \qquad \overline{x}^\alpha = \sqrt{\chi'}x^\alpha\,, \tag{4.30}$$

defined by the arbitrary function $\chi(u)$ and

$$\lambda(u,\rho,x^\alpha) = \frac{\chi''}{2\chi'}\left[-\frac{6}{5g^2kh^{5/2}} + \frac{4}{g^2kh^{3/2}}\frac{\Upsilon-1}{\Upsilon}\right] + \sigma(u)\,, \tag{4.31}$$

where $\sigma(u)$ is an arbitrary function, the metric and the field equations remain invariant in form if the functions k, h, Υ and $\mathcal{F}$ transform according to

$$\overline{k} = \frac{k}{\chi'}\,, \qquad \overline{h} = 1 + \frac{g^2\overline{\rho}^2\overline{k}}{3}\,, \qquad \overline{\Upsilon} = 1 + \overline{k}(\overline{x}^2+\overline{y}^2)\,, \tag{4.32}$$

$$\begin{aligned}\overline{\mathcal{F}} = \frac{1}{\chi'}\Bigg\{&\mathcal{F} + 2\sigma' + \left[\frac{6k'}{5g^2k^2h^{5/2}} + \frac{\rho^2k'}{kh^{7/2}}\right]\frac{\chi''}{\chi'} + \left[\frac{6}{5g^2kh^{5/2}} - \frac{\rho^2}{2h^{7/2}}\right]\left(\frac{\chi''}{\chi'}\right)^2 \\ &- \frac{6}{5g^2kh^{5/2}}\frac{\chi'''}{\chi'} + \left(\frac{\Upsilon-1}{\Upsilon}\right)^2\left[\frac{4\rho^2k'}{kh^{3/2}}\frac{\chi''}{\chi'} - \frac{2\rho^2}{h^{3/2}}\left(\frac{\chi''}{\chi'}\right)^2\right] \\ &+ \frac{\Upsilon-1}{\Upsilon}\Bigg[\left(-\frac{4\rho^2k'}{kh^{5/2}} - \frac{4k'}{g^2k^2h^{3/2}} + \frac{4k'}{g^2k^2h^{3/2}\Upsilon}\right)\frac{\chi''}{\chi'} \\ &\qquad + \left(-\frac{4}{g^2kh^{3/2}} + \frac{2\rho^2}{h^{5/2}} - \frac{2}{g^2kh^{3/2}\Upsilon}\right)\left(\frac{\chi''}{\chi'}\right)^2 + \frac{4}{g^2kh^{3/2}}\frac{\chi'''}{\chi'}\Bigg]\Bigg\}\,.\end{aligned} \tag{4.33}$$

In particular, the equation (4.22) remains invariant in form under this transformation. In the case in which $k \to 0$ this invariance was first obtained by Siklos [44], and if we choose

$$\begin{aligned}\sigma(u) &= \frac{3}{5g^2k}\frac{\chi''}{\chi'} + \tau(u)\,, \\ \tau(u) &\overset{k\to 0}{\longmapsto} 0\,,\end{aligned} \tag{4.34}$$

then $\mathcal{F}$ transforms in a simple way,

$$\overline{\mathcal{F}} \overset{k\to 0}{\longmapsto} \frac{1}{\chi'}\left[\mathcal{F} + \{\chi, u\}\left(\rho^2 + \frac{4}{g^2}(x^2+y^2)\right)\right] , \tag{4.35}$$

where

$$\{\chi(u); u\} = \frac{\chi'''(u)}{\chi'(u)} - \frac{3}{2}\left(\frac{\chi''(u)}{\chi'(u)}\right)^2 , \tag{4.36}$$

denotes the Schwarzian derivative.

It would be interesting to see if this Virasoro symmetry is related to the microstate counting of black strings in AdS (see [45] for related work).

5. Final remarks

While much effort has been dedicated to the study of the phases of black holes in Minkowski spacetime times a circle, the study of the black hole phases in locally asymptotically AdS_5 spacetimes with a non-trivial asymptotic circle is still in its embryonic stages. At the moment, only the uniform black string phase has been investigated, and we generalized this phase to the inclusion of the magnetic charge. Many interesting questions are still open. First of all, it would be interesting to study the stability of these black strings and to see whether a Gregory-Laflamme instability is present. It is also relevant to study whether more general phases exist, as for example non-uniform strings and localized black holes as in the Kaluza-Klein case. In the presence of a negative cosmological constant we have the additional tool of the dual field theory. By interpreting the asymptotic S^1 as a Scherk-Schwarz circle, the study of lumps of deconfined plasma, using the effective fluid dynamics description, can give us some new insight on different possible coexisting phases. This is currently under investigation.

Also, as we saw, the extremal string solutions with $\mu = 0$ interpolate between AdS_5 at infinity and $\mathrm{AdS}_3 \times \mathcal{S}$ near the horizon. The field theory dual of this supergravity solution has the interpretation of an RG flow across dimensions: The $\mathcal{N} = 4$ SYM theory on $\mathbb{R} \times \mathrm{S}^1 \times \mathcal{S}$ flows to a two-dimensional conformal field theory in the infrared. We can compute the central charge of this CFT as follows. According to [46], one has

$$c = \frac{3R_{\mathrm{AdS}_3}}{2G_3} ,$$

where G_3 denotes the effective three-dimensional Newton constant related to G_5 by

$$\frac{1}{G_3} = \frac{r_{\mathrm{h}}^2 V_k}{G_5} = \frac{2g^3 r_{\mathrm{h}}^2 V_k N^2}{\pi} ,$$

and we used the AdS/CFT dictionary in the last step. The curvature radius R_{AdS_3} can be read off from the near-horizon solution (2.26), with the result $R_{\mathrm{AdS}_3} = 2/\gamma\sqrt{\alpha_2}$. This yields the central charge

$$c = \frac{6g^3 r_{\mathrm{h}}^2 V_k N^2}{\pi\gamma\sqrt{\alpha_2}} . \tag{5.1}$$

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In the case where $\mathcal{S}$ is a compact Riemann surface of genus h ($k = -1$, $V_{-1} = 4\pi(h-1)$), and quantized magnetic charge, $q^2 = 1/12g^2$, (5.1) reduces correctly to the result of [40], namely $c = 8N^2(h-1)/3$ (cf. equation (75) of [40] for $a = 1/3$, which is the case of minimal gauged supergravity).

It would be very interesting to see whether the thermodynamic entropy of near-extremal black strings in AdS_5 can be reproduced by using the central charge (5.1) in Cardy's formula [45].

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A. Fefferman-Graham expansion

We give here the first few terms of the Fefferman-Graham expansion of the black strings. For convenience, we define $u = gr$ and $\tilde{q} = gq$. Note that e^{2T} and e^{2V} have the same expansion, up to the exchange of c_z and c_t, which simply corresponds to the double analytic continuation swapping black strings with bubbles of nothing. Also the $\ln u$ terms always come with a ξ factor.

$$
\begin{aligned}
y = {} & u^2 + \frac{2}{3}k + \frac{\xi \ln u}{u^2} + \frac{c_z + c_t + \frac{1}{3}k^2\tilde{q}^2}{u^2} - \frac{k\xi \ln u}{3u^4} - \left(c_z + c_t - \frac{k^2}{24}(1 + 12\tilde{q}^2)\right)\frac{k}{3u^4} \\
& - \frac{\xi^2}{6}\frac{(\ln u)^2}{u^6} - \left(c_z + c_t - \frac{11}{24}k^2\right)\frac{\xi \ln u}{3u^6} \\
& + \left(\frac{11}{12}k^2(c_z + c_t) - 4c_z c_t - \frac{23}{576}k^4 - \frac{1}{3}k^4\tilde{q}^2 - \frac{1}{4}k^4\tilde{q}^4\right)\frac{1}{6u^6} \\
& + \frac{2k\xi^2 (\ln u)^2}{9u^8} + \left(c_z + c_t - \frac{13}{72}k^2 - \frac{1}{12}k^2\tilde{q}^2\right)\frac{4k\xi \ln u}{9u^8} \\
& + \left(c_z^2 + c_t^2 + 6c_z c_t - \frac{k^2}{3}\left(\frac{13}{6} + \tilde{q}^2\right)(c_z + c_t) + \frac{77}{2592}k^4 + \frac{91}{432}k^4\tilde{q}^2 + \frac{7}{36}k^4\tilde{q}^4\right)\frac{k}{9u^8} \\
& + \frac{\xi^3 (\ln u)^3}{12u^{10}} + O\left(\frac{(\ln u)^2}{u^{10}}\right),
\end{aligned} \tag{A.1}
$$

$$
\begin{aligned}
e^{2T} = {} & u^2 + \frac{k}{2} + \frac{\xi \ln u}{2u^2} + \frac{c_z}{u^2} - \frac{7k\xi \ln u}{36u^4} - \left(c_z + \frac{3}{4}c_t - \frac{11}{288}k^2 - \frac{7}{24}k^2\tilde{q}^2\right)\frac{2k}{9u^4} \\
& - \xi^2\frac{(\ln u)^2}{8u^6} - \left(c_z + c_t - \frac{29}{72}k^2 + \frac{1}{6}k^2\tilde{q}^2\right)\frac{\xi \ln u}{4u^6} + \left(\frac{59}{576}k^2 c_z + \frac{19}{192}k^2 c_t - \frac{1}{2}c_z c_t \right. \\
& \left. - \frac{199}{41472}k^4 - \frac{1}{48}k^2 c_z\tilde{q}^2 - \frac{1}{16}k^2 c_t\tilde{q}^2 - \frac{5}{216}k^4\tilde{q}^2 - \frac{1}{32}k^4\tilde{q}^4\right)\frac{1}{u^6} \\
& + \frac{127}{720}k\frac{\xi^2 (\ln u)^2}{u^8} + \left(\frac{25}{72}c_z + \frac{43}{120}c_t - \frac{2591}{43200}k^2 + \frac{17}{1200}k^2\tilde{q}^2\right)\frac{k\xi \ln u}{u^8}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{4}{45}c_z^2+\frac{1}{10}c_t^2+\frac{31}{60}c_zc_t-\frac{1027}{17280}k^2c_z-\frac{581}{9600}k^2c_t\right.\\
&\qquad\left.+\frac{1}{160}k^2c_z\tilde{q}^2+\frac{53}{2400}k^2c_t\tilde{q}^2+\frac{250849}{93312000}k^4+\frac{9989}{972000}k^4\tilde{q}^2\right)\frac{k}{u^8}\\
&+\frac{11\xi^3(\ln u)^3}{144u^{10}}+O\left(\frac{(\ln u)^2}{u^{10}}\right),
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
e^{2V}&=u^2+\frac{k}{2}+\frac{\xi\ln u}{2u^2}+\frac{c_t}{u^2}-\frac{7k\xi\ln u}{36u^4}-\left(c_t+\frac{3}{4}c_z-\frac{11}{288}k^2-\frac{7}{24}k^2\tilde{q}^2\right)\frac{2k}{9u^4}\\
&-\xi^2\frac{(\ln u)^2}{8u^6}-\left(c_z+c_t-\frac{29}{72}k^2+\frac{1}{6}k^2\tilde{q}^2\right)\frac{\xi\ln u}{4u^6}+\left(\frac{59}{576}k^2c_t+\frac{19}{192}k^2c_z-\frac{1}{2}c_zc_t\right.\\
&\qquad\left.-\frac{199}{41472}k^4-\frac{1}{48}k^2c_t\tilde{q}^2-\frac{1}{16}k^2c_z\tilde{q}^2-\frac{5}{216}k^4\tilde{q}^2-\frac{1}{32}k^4\tilde{q}^4\right)\frac{1}{u^6}\\
&+\frac{127}{720}k\frac{\xi^2(\ln u)^2}{u^8}+\left(\frac{25}{72}c_t+\frac{43}{120}c_z-\frac{2591}{43200}k^2+\frac{17}{1200}k^2\tilde{q}^2\right)\frac{k\xi\ln u}{u^8}\\
&+\left(\frac{4}{45}c_t^2+\frac{1}{10}c_z^2+\frac{31}{60}c_zc_t-\frac{1027}{17280}k^2c_t-\frac{581}{9600}k^2c_z\right.\\
&\qquad\left.+\frac{1}{160}k^2c_t\tilde{q}^2+\frac{53}{2400}k^2c_z\tilde{q}^2+\frac{250849}{93312000}k^4+\frac{9989}{972000}k^4\tilde{q}^2\right)\frac{k}{u^8}\\
&+\frac{11\xi^3(\ln u)^3}{144u^{10}}+O\left(\frac{(\ln u)^2}{u^{10}}\right).
\end{aligned}
\tag{A.3}
$$

References

[1] R. Emparan and H.S. Reall, *A rotating black ring in five dimensions*, *Phys. Rev. Lett.* **88** (2002) 101101 [hep-th/0110260].

[2] R.C. Myers and M.J. Perry, *Black holes in higher dimensional space-times*, *Ann. Phys. (NY)* **172** (1986) 304.

[3] R. Gregory and R. Laflamme, *Black strings and p-branes are unstable*, *Phys. Rev. Lett.* **70** (1993) 2837 [hep-th/9301052].

[4] T. Harmark, V. Niarchos and N.A. Obers, *Instabilities of black strings and branes*, *Class. and Quant. Grav.* **24** (2007) R1 [hep-th/0701022].

[5] B. Kol, *The phase transition between caged black holes and black strings: a review*, *Phys. Rept.* **422** (2006) 119 [hep-th/0411240].

[6] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [*Int. J. Theor. Phys.* **38** (1999) 1113] [hep-th/9711200].

[7] S.W. Hawking and D.N. Page, *Thermodynamics of black holes in Anti-de Sitter space*, *Commun. Math. Phys.* **87** (1983) 577.

[8] E. Witten, *Anti-de Sitter space, thermal phase transition and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [hep-th/9803131].

[9] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, *Charged AdS black holes and catastrophic holography*, *Phys. Rev.* **D 60** (1999) 064018 [hep-th/9902170].

[10] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, *Holography, thermodynamics and fluctuations of charged AdS black holes*, *Phys. Rev.* **D 60** (1999) 104026 [hep-th/9904197].

[11] O. Aharony, S. Minwalla and T. Wiseman, *Plasma-balls in large-N gauge theories and localized black holes*, *Class. and Quant. Grav.* **23** (2006) 2171 [hep-th/0507219].

[12] S. Lahiri and S. Minwalla, *Plasmarings as dual black rings*, arXiv:0705.3404.

[13] S. Bhattacharyya, S. Lahiri, R. Loganayagam and S. Minwalla, *Large rotating AdS black holes from fluid mechanics*, arXiv:0708.1770.

[14] H.K. Kunduri, J. Lucietti and H.S. Reall, *Do supersymmetric Anti-de Sitter black rings exist?*, *JHEP* **02** (2007) 026 [hep-th/0611351].

[15] R.B. Mann, *Topological black holes: outside looking in*, gr-qc/9709039.

[16] L. Vanzo, *Black holes with unusual topology*, *Phys. Rev.* **D 56** (1997) 6475 [gr-qc/9705004].

[17] D. Birmingham, *Topological black holes in Anti-de Sitter space*, *Class. and Quant. Grav.* **16** (1999) 1197 [hep-th/9808032].

[18] K. Copsey and G.T. Horowitz, *Gravity dual of gauge theory on $S^2 \times S^1 \times R$*, *JHEP* **06** (2006) 021 [hep-th/0602003].

[19] R.B. Mann, E. Radu and C. Stelea, *Black string solutions with negative cosmological constant*, *JHEP* **09** (2006) 073 [hep-th/0604205].

[20] Y. Brihaye, E. Radu and C. Stelea, *Black strings with negative cosmological constant: inclusion of electric charge and rotation*, *Class. and Quant. Grav.* **24** (2007) 4839 [hep-th/0703046].

[21] Y. Brihaye and E. Radu, *Magnetic solutions in AdS_5 and trace anomalies*, *Phys. Lett.* **B 658** (2008) 164 [arXiv:0706.4378].

[22] A.H. Chamseddine and W.A. Sabra, *Magnetic strings in five dimensional gauged supergravity theories*, *Phys. Lett.* **B 477** (2000) 329 [hep-th/9911195].

[23] D. Klemm and W.A. Sabra, *Supersymmetry of black strings in $D = 5$ gauged supergravities*, *Phys. Rev.* **D 62** (2000) 024003 [hep-th/0001131].

[24] C. Fefferman and C.R. Graham, *Conformal invariants*, in *Élie Cartan et les Mathématiques d'Aujourd'hui*, Astérisque (1985) 95.

[25] V. Balasubramanian and P. Kraus, *A stress tensor for Anti-de Sitter gravity*, *Commun. Math. Phys.* **208** (1999) 413 [hep-th/9902121].

[26] R. Olea, *Mass, angular momentum and thermodynamics in four-dimensional Kerr-AdS black holes*, *JHEP* **06** (2005) 023 [hep-th/0504233].

[27] R. Olea, *Regularization of odd-dimensional AdS gravity: kounterterms*, *JHEP* **04** (2007) 073 [hep-th/0610230].

[28] G.T. Horowitz and R.C. Myers, *The AdS/CFT correspondence and a new positive energy conjecture for general relativity*, *Phys. Rev.* **D 59** (1999) 026005 [hep-th/9808079].

[29] K. Skenderis, *Asymptotically Anti-de Sitter spacetimes and their stress energy tensor*, *Int. J. Mod. Phys.* **A 16** (2001) 740 [hep-th/0010138].

[30] M. Taylor, *More on counterterms in the gravitational action and anomalies*, hep-th/0002125.

[31] I. Papadimitriou and K. Skenderis, *AdS/CFT correspondence and geometry*, hep-th/0404176.

[32] M.C.N. Cheng and K. Skenderis, *Positivity of energy for asymptotically locally AdS spacetimes*, *JHEP* **08** (2005) 107 [hep-th/0506123].

[33] J.H. Traschen and D. Fox, *Tension perturbations of black brane spacetimes*, *Class. and Quant. Grav.* **21** (2004) 289 [gr-qc/0103106].

[34] P.K. Townsend and M. Zamaklar, *The first law of black brane mechanics*, *Class. and Quant. Grav.* **18** (2001) 5269 [hep-th/0107228].

[35] J.H. Traschen, *A positivity theorem for gravitational tension in brane spacetimes*, *Class. and Quant. Grav.* **21** (2004) 1343 [hep-th/0308173].

[36] T. Harmark and N.A. Obers, *New phase diagram for black holes and strings on cylinders*, *Class. and Quant. Grav.* **21** (2004) 1709 [hep-th/0309116].

[37] R.C. Myers, *Stress tensors and Casimir energies in the AdS/CFT correspondence*, *Phys. Rev.* **D 60** (1999) 046002 [hep-th/9903203].

[38] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, *JHEP* **07** (1998) 023 [hep-th/9806087].

[39] S. de Haro, S.N. Solodukhin and K. Skenderis, *Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence*, *Commun. Math. Phys.* **217** (2001) 595 [hep-th/0002230].

[40] J.M. Maldacena and C. Núñez, *Supergravity description of field theories on curved manifolds and a no-go theorem*, *Int. J. Mod. Phys.* **A 16** (2001) 822 [hep-th/0007018].

[41] I. Papadimitriou and K. Skenderis, *Thermodynamics of asymptotically locally AdS spacetimes*, *JHEP* **08** (2005) 004 [hep-th/0505190].

[42] M.M. Caldarelli, G. Cognola and D. Klemm, *Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories*, *Class. and Quant. Grav.* **17** (2000) 399 [hep-th/9908022].

[43] J.P. Gauntlett and J.B. Gutowski, *All supersymmetric solutions of minimal gauged supergravity in five dimensions*, *Phys. Rev.* **D 68** (2003) 105009 [hep-th/0304064].

[44] S.T.C. Siklos, *Lobatchevski plane gravitational waves*, in *Galaxies, axisymmetric systems and relativity*, M.A.H. MacCallum ed., Cambridge University Press, Cambridge U.K. (1985).

[45] M. Bañados, A. Chamblin and G.W. Gibbons, *Branes, AdS gravitons and virasoro symmetry*, *Phys. Rev.* **D 61** (2000) 081901 [hep-th/9911101].

[46] J.D. Brown and M. Henneaux, *Central charges in the canonical realization of asymptotic symmetries: an example from three-dimensional gravity*, *Commun. Math. Phys.* **104** (1986) 207.

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