$(k,+)$–Distance-Hereditary Graphs*

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Abstract

In this work we introduce, characterize, and provide algorithmic results for $(k,+)$–distance-hereditary graphs, $k \geq 0$. These graphs can be used to model interconnection networks with desirable connectivity properties; a network modeled as a $(k,+)$–distance-hereditary graph can be characterized as follows: *if some nodes have failed, as long as two nodes remain connected, the distance between these nodes in the faulty graph is bounded by the distance in the non-faulty graph* plus *an integer constant* k. The class of all these graphs is denoted by DH$(k,+)$. By varying the parameter k, classes DH$(k,+)$ include all graphs and form a hierarchy that represents a parametric extension of the well-known class of distance-hereditary graphs.

Keywords: Interconnection networks, dilation number, distance-hereditary graphs, characterization of graph classes, forbidden subgraphs, recognition algorithms.

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1 Introduction

A fundamental problem in any parallel or distributed system is the efficient communication of data between processors. Such efficiency depends on the *routing scheme* defined over the system, that is the set of paths used to route data, one path for each possible pair of processors. The efficiency of a routing scheme is meanly measured in terms of its *stretch factor* and *dilation*. The stretch factor (dilation) is the maximum ratio (difference) between the length of a path defined by the scheme and the shortest path between the same pair of processors.

In this work we are interested in networks in which routing schemes coincide with shortest paths and node failures may occur. Distances are always computed by means of shortest paths in the subnetwork that is induced by the non-faulty components. In this context, the decrease of the efficiency of the communication only depends on the topology of the networks. To measure this efficiency degradation, some parameters about the topology can be defined. In [8] the authors defined the notion of *stretch number*, while in this paper we introduce the *dilation number* of a graph G. It is defined as the smallest k such that $G \in \mathrm{DH}(k,+)$, where a network modeled as a graph belonging to the class $\mathrm{DH}(k,+)$ can be characterized as follows: *if some nodes have failed, as long as two nodes remain connected, the distance between these nodes in the faulty graph is bounded by the distance in the non-faulty graph* plus *an integer constant* k. Elements of $\mathrm{DH}(k,+)$ are called $(k,+)$–distance-hereditary graphs. The name is motivated by the fact that the well-known class of distance-hereditary graphs [18, 19] corresponds to the class $\mathrm{DH}(0,+)$. So, by varying the parameter k, classes $\mathrm{DH}(k,+)$: (1) include all graphs, (2) form a hierarchy that represents a parametric extension of distance-hereditary graphs. Given the relevance of $(k,+)$–distance-hereditary graphs in the area of communication networks, our purpose is to provide characterization and algorithmic results about the introduced graphs.

Related works. In literature there are several papers devoted to fault-tolerant network design, mainly starting from a given desired topology and introducing fault-tolerance to it (e.g., see [4, 16, 20]).

Papers [8, 9] present several results about $(k,*)$–distance-hereditary graphs, i.e., graphs whose induced distance is bounded by a *multiplicative* factor k. In [14], a study about similar concepts is performed: they give characterizations for graphs in which *no delay* occurs in the case that a *single* node fails. These graphs are called *self-repairing*. In [10], authors introduce and characterize new classes of graphs that guarantee constant stretch factors k even when a multiple number of *edges* have failed. In a first step, they do not limit the number of edge faults at all, allowing for *unlimited* edge faults. Secondly, they examine the more realistic case where the number of edge faults is *bounded* by a value ℓ. The corresponding graphs are called k–self-spanners and (k,ℓ)–self-spanners, respectively. In both cases, the names are motivated by strong relationships to the concept of k–*spanners* [21]. Related works are also those concerning distance-hereditary graphs [18, 19]: they have been investigated to design interconnection network topologies [7, 12, 13], and several papers have been devoted to them (see [3] and references therein).

Results. First, we formally introduce $(k,+)$–distance-hereditary graphs and provide some preliminary results. An initial characterization is given in terms of the dilation number. Then, we remark relationships between $(k,*)$–distance-hereditary graphs and $(k,+)$–distance-hereditary graphs. Starting from these observations, we introduce the notion of *twin graph* G^* of an arbitrary graph G. This graph has the remarkable property that $G \in \mathrm{DH}(k,+)$ if and only if $G^* \in \mathrm{DH}(k,+)$. Thanks to this notion, we are able to provide a characterization of graphs G in $\mathrm{DH}(k,+)$ based on cycle-chord conditions of its twin graph G^*. Since we also show that the recognition problem for the new graph classes is Co-NP-complete (for k not fixed), then we investigate in more detail the smallest class among the new ones, i.e., class $\mathrm{DH}(1,+)$. In this context, our main result consists of listing all the forbidden induced subgraphs of every $G \in \mathrm{DH}(1,+)$. A theoretical consequence of this characterization is that the recognition problem of class $\mathrm{DH}(1,+)$ can be solved in polynomial time.

This paper is organized as follows. Notation and basic concepts used in this work are given in Section 2, while Section 3 formally introduces $(k,+)$–distance-hereditary graphs and provides some preliminary results. Sections 4 and 5 study graphs in $\mathrm{DH}(k,+)$: the former introduces and uses the notion of twin graph to characterize graphs in $\mathrm{DH}(k,+)$, and the latter states the Co-NP-completeness result. Sections 6 and 7 study graphs in $\mathrm{DH}(1,+)$: the former characterizes graphs in $\mathrm{DH}(1,+)$ by listing forbidden induced subgraphs, while the latter uses this characterization to provide a polynomial time recognition. Finally, Section 8 concludes the paper by listing some open problems.

2 Notation and basic concepts

In this work we consider finite, simple, loopless, undirected and unweighted graphs $G = (V,E)$ with node set V and edge set E. We use standard terminologies from [3, 17], some of which are briefly reviewed here.

$|G|$ denotes the cardinality of V. A *subgraph* of G is a graph having all its nodes and edges in G. Given a subset S of V, the *induced subgraph* $\langle S \rangle$ of G is the maximal subgraph of G with node set S. $G - S$ is the subgraph of G induced by $V \setminus S$; when $S = \{x\}$, we write $G - x$ instead of $G - \{x\}$.

If x is a node of G, by $N_G(x)$ we denote the *neighbors* of x in G, that is, the set of nodes in G that are adjacent to x, and by $N_G[x]$ we denote the *closed neighborhood* of x, that is $N_G(x) \cup \{x\}$. Moreover, $N_G(S) = \bigcup_{u \in S} N_G(u)$ and

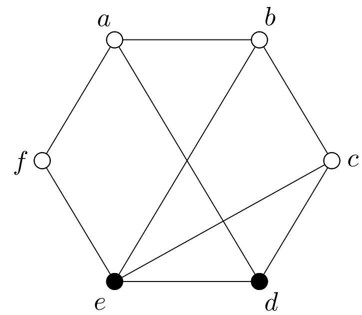

Figure 1: *The chord distance of this cycle* C_6 *is* 2 *because nodes* d *and* e *are consecutive and every chord is incident to one of them. Moreover, there is no other set with less then* 3 *nodes with the same properties.*

$N_G[S] = \bigcup_{u \in S} N_G[u]$. The degree of x is denoted by $deg_G(x)$ and it is equal to $|N_G(x)|$. $S \subseteq V$ is an *independent set* in G if $(x, y) \notin E$, for all $x, y \in S$.

Two nodes v and v' of G are *twins* in G if they have the same neighborhood in G; we distinguish between *false twins* when $N_G(v) = N_G(v')$ and *true twins* when $N_G[v] = N_G[v']$. If $u \in V$, operation $\gamma(G, u)$ (see [3]) extends G by adding a false twin of u; the resulting graph is $G' = (V \cup \{u'\}, E \cup \{(u', v) \mid v \in N_G(u)\})$.

A sequence of pairwise distinct nodes $(x_0, x_1, \ldots, x_n)$ is a *path* in G if $(x_i, x_{i+1}) \in E$ for $0 \leq i < n$. The *length* of a path $p = (x_0, \ldots, x_n)$ is n, whereas $|p|$ denotes the number of its nodes. A path $(x_0, \ldots, x_n)$ is an *induced path* if $\langle \{x_0, \ldots, x_n\} \rangle$ has n edges. Two nodes x and y of G are *connected* if there is a path from x to y in G. A graph G is *connected* if, for each pair of nodes x and y of G, x and y are connected. A *biconnected component* of G is a subgraph of G which remains connected even if we delete any of its nodes.

The length of a shortest path between two nodes x and y in G is called *distance* and is denoted by $d_G(x, y)$; moreover, the length of a longest induced path between the same nodes is denoted by $D_G(x, y)$. We use symbols $P_G(x, y)$ and $p_G(x, y)$ to denote a longest and a shortest induced path between x and y, respectively. Sometimes, when no ambiguity occurs, we use $P_G(x, y)$ and $p_G(x, y)$ to denote the sets of nodes belonging to the corresponding paths.

A *cycle* C_n in G is a path $(x_0, \ldots, x_{n-1})$ where also $(x_0, x_{n-1}) \in E$. Two nodes x_i and x_j are *consecutive* in C_n if $j \equiv (i+1) \bmod n$ or $i \equiv (j+1) \bmod n$. A *chord* of a cycle is an edge joining two non-consecutive nodes in the cycle. H_n denotes an *hole*, i.e., a cycle with n nodes and without chords. The *chord distance* of a cycle C_n is denoted by $cd(C_n)$, and it is defined as the minimum number of consecutive nodes in C_n such that every chord of C_n is incident to some of such nodes (see Fig. 1). We define $cd(H_n) = 0$.

If x and y are two nodes of G such that $d_G(x, y) \geq 2$, then $\{x, y\}$ is a *cycle-pair* if there exist a path $p_G(x, y)$ and a path $P_G(x, y)$ such that $p_G(x, y) \cap P_G(x, y) = \{x, y\}$. In other words, if $\{x, y\}$ is a cycle-pair, then the set $p_G(x, y) \cup P_G(x, y)$

induces a cycle in G.

3 Basic definitions and preliminary results

In this section we formally define $(k,+)$–distance-hereditary graphs and provide some preliminary results.

Definition 3.1 *Let k be a real number. A graph G is a $(k,+)$*–distance-hereditary graph *if, for each connected induced subgraph G' of G:*

$$d_{G'}(x,y) \leq d_G(x,y) + k, \quad \textit{for each } x,y \in G'. \tag{1}$$

The class of all the $(k,+)$–distance-hereditary graphs is denoted by $\mathrm{DH}(k,+)$.

Notice that the above definition holds for both connected and disconnected graphs. Given a rational number $k \geq 1$, the $(k,*)$–distance-hereditary graphs [8, 9] have been defined in a similar way: it is sufficient to replace Eq. 1 by the following one:

$$d_{G'}(x,y) \leq d_G(x,y) \cdot k, \quad \text{for each } x,y \in G'. \tag{2}$$

$\mathrm{DH}(k,*)$ denotes the class of all $(k,*)$–distance-hereditary graphs. By setting $k=0$ in Eq. 1 or $k=1$ in Eq. 2, we get the definition of distance-hereditary graphs [18, 19].

Lemma 3.2 *The class* $\mathrm{DH}(k,+)$ *is closed under taking induced subgraphs.*

Proof. Let G be a graph in $\mathrm{DH}(k,+)$ and G' be an induced subgraph of G. According to Definition 3.1, we have to show that, for each connected induced subgraph G'' of G', $d_{G''}(x,y) \leq d_{G'}(x,y) + k$, for each $x,y \in G''$.

Let x and y be two nodes in G''. Since G'' is a connected induced subgraph of G, then, by Definition 3.1, $d_{G''}(x,y) \leq d_G(x,y) + k$. Relationship $d_G(x,y) \leq d_{G'}(x,y)$ is straightforward. Combining these inequalities, we get

$$d_{G''}(x,y) \;\leq\; d_G(x,y) + k \;\leq\; d_{G'}(x,y) + k$$

□

Definition 3.3 *Let G be a graph, and $\{x,y\}$ be a pair of connected nodes in G. Then:*

1. *the* dilation number $\partial_G(x,y)$ of the pair $\{x,y\}$ *is given by* $\partial_G(x,y) = D_G(x,y) - d_G(x,y)$*;*
2. *the* dilation number $\partial(G)$ of G *is the maximum dilation number over all possible pairs of connected nodes, that is,* $\partial(G) = \max_{\{x,y\}} \partial_G(x,y)$*;*

3. *$\mathcal{D}(G)$ is the set containing all the pairs of nodes inducing the dilation number of G, that is, $\mathcal{D}(G) = \{\{x,y\} \mid \partial_G(x,y) = \partial(G)\}$.*

In the context of $(k,*)$–distance-hereditary graphs, we introduced the corresponding notion of *stretch number*. Shortly, if G is a graph and x and y two connected nodes in G, then : (1) the stretch number $s_G(x,y)$ of $\{x,y\}$ is given by $s_G(x,y) = D_G(x,y)/d_G(x,y)$, and (2) the stretch number $s(G)$ of G is given by $s(G) = \max_{\{x,y\}} s_G(x,y)$.

The following two lemmas list some basic properties of $(k,+)$–distance-hereditary graphs.

Lemma 3.4 *The following facts hold:*

1. $\mathrm{DH}(0,+)$ *coincides with the class of distance-hereditary graphs;*
2. $\mathrm{DH}(k,+) = \mathrm{DH}(\lfloor k \rfloor,+)$*, for each real number $k \geq 0$;*
3. $\mathrm{DH}(k_1,+) \subseteq \mathrm{DH}(k_2,+)$*, for each pair of integers k_1 and k_2, $k_1 \leq k_2$;*
4. *If $(x,y) \in E$ then $\partial_G(x,y) = 0$. As a consequence:*
   - if $\partial(G) = 0$ then $\mathcal{D}(G)$ contains every pairs of connected nodes of G;
   - if $\partial(G) > 0$ then $d_G(x,y) \geq 2$ for each pair $\{x,y\} \in \mathcal{D}(G)$;
5. *if G contains n nodes, then $\partial(G) \leq \max\{0, n-4\}$; moreover, for each $n \in \mathbb{N}$ there exists a graph G' such that $\partial(G') = n$;*

Proof. Facts 1, 2, 3, and 4 directly follow from Definition 3.1. The remainder proves Fact 5.

If $n \leq 4$, then G is a distance-hereditary graph and hence $\partial(G) = 0$. If $n > 4$ and $G \notin \mathrm{DH}(0,+)$, then:

a) $d_G(x,y) \geq 2$, for each $\{x,y\} \in \mathcal{D}(G)$ (see Fact 4);

b) $D_G(x,y) \leq n-2$, for each pair $\{x,y\}$ of connected nodes in G.

Then, if $\{x,y\} \in \mathcal{D}(G)$, the following holds:

$$\partial(G) \;=\; D_G(x,y) \;-\; d_G(x,y) \;\leq\; (n-2)-2 \;=\; n-4$$

To complete the proof we show that $\partial(H_n) = n-4$, for $n \geq 4$. H_4 is a distance-hereditary graph, and hence $\partial(H_4) = 0$. When $n > 4$, for each pair $\{x,y\}$ of nodes in H_n such that $d_{H_n}(x,y) \geq 2$, we have $D_{H_n}(x,y) = n - d_{H_n}(x,y)$. Then, $\partial_{H_n}(x,y) = D_{H_n}(x,y) - d_{H_n}(x,y) = n - 2 \cdot d_{H_n}(x,y)$, which is maximum for $d_{H_n}(x,y) = 2$. Hence $\partial(H_n) = n-4$. □

The dilation number can be used to provide a first characterization of $(k,+)$–distance-hereditary graphs.

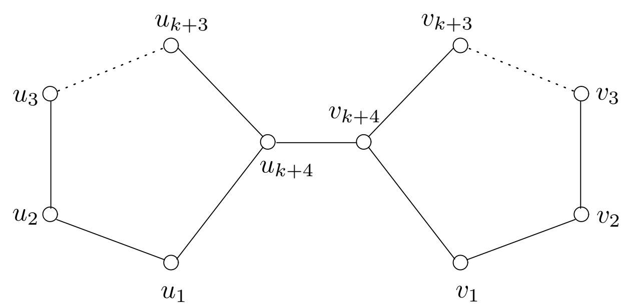

Figure 2: *The graph G_k, $k \geq 1$, used by Lemma 3.6. It consists of two holes H_{k+4} joined by an edge.*

Theorem 3.5 *Let G be a graph. $G \in \mathrm{DH}(k,+)$ if and only if $\partial(G) \leq k$.*

Proof. We first show that the following relationship holds:

$$\partial(G) = \min\{t : G \in \mathrm{DH}(t,+)\} \tag{3}$$

By Definition 3.3, $\partial(G) = \max_{\{x,y\}}\{D_G(x,y) - d_G(x,y)\}$, that is,

$$\partial(G) \geq D_G(x,y) - d_G(x,y)$$

for each pair of connected nodes $x, y \in V$. If $G' = (V', E')$ is a connected induced subgraph of G, then $\partial(G) \geq d_{G'}(x,y) - d_G(x,y)$ for each $x, y \in V'$. Hence $d_{G'}(x,y) \leq d_G(x,y) + \partial(G)$ for each $x, y \in V'$. By the generality of G', it follows that $G \in \mathrm{DH}(\partial(G),+)$.

By contradiction, let us suppose that there exists an integer $t < \partial(G)$ such that $G \in \mathrm{DH}(t,+)$. Let $\{x,y\} \in \mathcal{D}(G)$, and G' be the subgraph induced by $P_G(x,y)$. In this case we have that $d_{G'}(x,y) = D_G(x,y)$, and hence the relation $D_G(x,y) - d_G(x,y) = \partial(G) > t$ implies that

$$d_{G'}(x,y) = D_G(x,y) > t + d_G(x,y).$$

Then $G \notin \mathrm{DH}(t,+)$, a contradiction. The theorem follows by Eq. 3 and Fact 3 of Lemma 3.4. □

In [8], it is shown that $G \in \mathrm{DH}(k,*)$ if and only if $s(G) \leq k$. We conclude this section by providing a relationship between the classes $\mathrm{DH}(k,+)$ and $\mathrm{DH}(k,*)$, useful in the remainder of the paper.

Lemma 3.6 *For each $k \geq 1$, $\mathrm{DH}(k,+) \subset \mathrm{DH}(1 + \frac{k}{2}, *)$.*

Proof. Let $G \in \mathrm{DH}(k,+)$, $k \geq 1$, and $x, y \in G$. We show that $s_G(x,y) \leq 1 + \frac{k}{2}$. By the generality of x and y, this implies $G \in \mathrm{DH}(1 + \frac{k}{2}, *)$, and, in turn, $\mathrm{DH}(k,+) \subseteq \mathrm{DH}(1 + \frac{k}{2}, *)$.

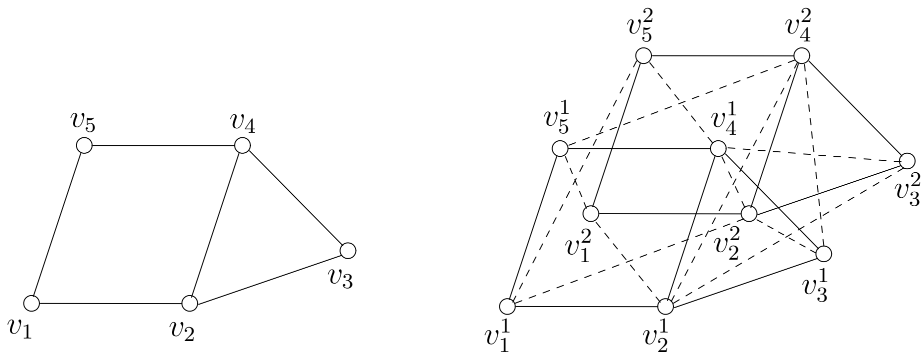

Figure 3: *A graph G and its twin graph G^* (see Definition 4.1). Dashed lines represent edges in E^3. Nodes $v_1^1, v_2^1, \ldots, v_5^1$ and $v_1^2, v_2^2, \ldots, v_5^2$ induce the subgraphs G^1 and G^2, respectively. Both G^1 and G^2 are isomorphic to G.*

By Theorem 3.5, $\partial(G) \leq k$. Hence, $D_G(x,y) \leq d_G(x,y) + k$. If $d_G(x,y) = 1$, then $s_G(x,y) = 1$ and relation $s_G(x,y) \leq 1 + \frac{k}{2}$ trivially holds. If $d_G(x,y) \geq 2$, then:

$$s_G(x,y) = \frac{D_G(x,y)}{d_G(x,y)} \leq \frac{d_G(x,y) + k}{d_G(x,y)} = 1 + \frac{k}{d_G(x,y)} \leq 1 + \frac{k}{2}$$

To complete the proof we show that the graph G_k of Fig. 2 is such that $G_k \in \mathrm{DH}(1 + \frac{k}{2}, *)$ and $G_k \notin \mathrm{DH}(k, +)$, for each $k \geq 1$. In fact, $\partial_{G_k}(u_2, v_2) = (k+2) + 1 + (k+2) - 5 = 2k$, while $s(G_k) = s_{G_k}(u_2, u_{k+4}) = \frac{k+2}{2} = 1 + \frac{k}{2}$ (in [8], it is shown that the stretch number of a graph H coincides with the stretch number of one of the maximal biconnected components of H). □

4 Characterization of graphs in $\mathrm{DH}(k, +)$

In [8], it is shown that the $(k,*)$–distance-hereditary graphs enjoy a nice "locality property": the stretch number of G coincides with the stretch number of an induced subgraph of G that forms a cycle. This property does not hold for $(k,+)$–distance-hereditary graphs. For instance, consider again the graph G_k of Fig. 2: the longest and the shortest induced path between every pair of nodes in $\mathcal{D}(G_k)$ does not induce a cycle (indeed, the pairs induce the whole G_k).

In this section, we introduce the notion of *twin graph*. We show that the locality property recalled above holds for twin graphs of $(k,+)$–distance-hereditary graphs. In this way, we are able to provide a characterization of $(k,+)$–distance-hereditary graphs based on cycle-chord conditions (Theorem 4.6).

Definition 4.1 *Let $G = (V, E)$ be a graph. The* twin graph *of G is a graph $G^* = (V^*, E^*)$ such that $V^* = V^1 \cup V^2$ and $E^* = E^1 \cup E^2 \cup E^3$, where:*

- *$V^1 = \{v^1 \mid v \in V\}$;*

- $V^2 = \{v^2 \mid v \in V\}$;

- $E^1 = \{(u^1, v^1) \mid (u, v) \in E\}$;

- $E^2 = \{(u^2, v^2) \mid (u, v) \in E\}$;

- $E^3 = \{(u^1, v^2), (u^2, v^1), \mid (u, v) \in E\}$.

The subgraphs of G^* given by (V^1, E^1) and (V^2, E^2) are denoted by G^1 and G^2, respectively.

Then $|V^*| = 2 \cdot |V|$ and $|E^*| = 4 \cdot |E|$. The name "twin graph" is due to the fact that, for each pair (v^1, v^2) of nodes in G^* such that $v_1 \in V^1$ and $v_2 \in V^2$, v^2 is a false twin of v^1 in G^*. Hence, the twin graph G^* can be obtained from G by applying operation $\gamma(G, v)$ to each node v of G. Moreover, notice that both G^1 and G^2 are induced subgraphs of G^* and isomorphic to G (see Fig. 3).

Definition 4.2 *Let S be the subgraph of G^* induced by the nodes $v_1^{i_1}, v_2^{i_2}, \ldots, v_n^{i_n}$, $i_j \in \{1, 2\}$ and $1 \leq j \leq n$. The* projection *of S* on *G^1, denoted by S^1, is the subgraph of G^1 induced by the nodes $v_1^1, v_2^1, \ldots, v_n^1$.*

Notice that, the projection of G^2 on G^1 corresponds to G^1. The following lemma deals with projections of generic induced subgraphs of G^*.

Lemma 4.3 *Let S be an induced subgraph of G^*. If there are no false twins in S, then S and its projection S^1 are isomorphic.*

Proof. Let $S = (V_S, E_S)$, $S^1 = (V_{S^1}, E_{S^1})$, and assume that there are no false twins in $V_S = \{v_1^{i_1}, v_2^{i_2}, \ldots, v_n^{i_n}\}$. We show that S and S^1 are isomorphic, that is: (1) $|V_{S^1}| = |V_S|$, and (2) $(v_j^{i_j}, v_k^{i_k}) \in E_S$ if and only if $(v_j^1, v_k^1) \in E_{S^1}$.

1. By construction of V_{S^1}, it follows that $|V_{S^1}| \leq |V_S|$. According to Definition 4.1, $|V_{S^1}|$ contains less elements than $|V_S|$ if and only if two different nodes in V_S have the same corresponding node in V_{S^1}, i.e, there exist two distinct nodes $v_j^{i_j}$ and $v_k^{i_k}$ in V_S, $i_j \neq i_k$, such that $v_j^1 \equiv v_k^1$. If such two nodes exist, by Definition 4.1, $v_j^{i_j}$ and $v_k^{i_k}$ are false twins in G^*. As a consequence, since S is induced in G^* then $v_j^{i_j}$ and $v_k^{i_k}$ are false twins in S, a contradiction. Hence, $|V_{S^1}| = |V_S|$.

2. The property that $(v_j^{i_j}, v_k^{i_k}) \in E_S$ if and only if $(v_j^1, v_k^1) \in E_{S^1}$ directly follows from definition of G^*.

$\square$

Lemma 4.4 *For $k \geq 0$, $G \in \mathrm{DH}(k, +)$ if and only if $G^* \in \mathrm{DH}(k, +)$.*

Proof. $\Longrightarrow$: Assuming $G^* \notin \mathrm{DH}(k,+)$, there exist two nodes $u,v \in V^*$ such that $D_{G^*}(u,v) - d_{G^*}(u,v) > k$. Let $P_{G^*}(u,v) = (u \equiv v_0^{i_0}, v_1^{i_1}, \ldots, v_n^{i_n} \equiv v)$, $i_j \in \{1,2\}$ and $0 \leq j \leq n$, and $p_{G^*}(u,v) = (u \equiv u_0^{\ell_0}, u_1^{\ell_1}, \ldots, u_m^{\ell_m} \equiv v)$, $\ell_j \in \{1,2\}$ and $0 \leq j \leq m$, be a longest and a shortest induced path connecting u and v, respectively. According to the fact that there are false twins in an induced path if and only if the path has three nodes, we analyze two different cases:

1. $m = 2$:
   In this case, $p_{G^*}(u,v)$ has three nodes. Moreover, $n > 2$ otherwise $G^* \in \mathrm{DH}(0,+)$, a contradiction for the hypothesis $G^* \notin \mathrm{DH}(k,+)$. The subgraph S induced by $P_{G^*}(u,v) \cup p_{G^*}(u,v)$ is a cycle with at least 5 nodes and chord distance at most 1. It can be easily observed that such a cycle does not contains false twins. Then, by Lemma 4.3, the projection S^1 of S is isomorphic to S.

   By observing that $\partial(S) > k$, it follows that $\partial(S^1) > k$. Hence, by Lemma 3.2, $G^1 \notin \mathrm{DH}(k,+)$. Finally, since G^1 is isomorphic to G, then $G \notin \mathrm{DH}(k,+)$.

2. $m > 2$:
   In this case, neither the subgraph induced by $P_{G^*}(u,v)$ nor the subgraph induced by $p_{G^*}(u,v)$ has false twins. Let $P^1(v_0^1, v_n^1)$ and $p^1(u_0^1, u_m^1)$ be the projections of $P_{G^*}(u,v)$ and $p_{G^*}(u,v)$ on G^1, respectively. By Lemma 4.3, $P^1(v_0^1, v_n^1)$ and $p^1(u_0^1, u_m^1)$ are isomorphic to $P_{G^*}(u,v)$ and $p_{G^*}(u,v)$, respectively.

   Since $D_{G^1}(v_0^1, v_n^1) - d_{G^1}(u_0^1, u_m^1) \geq |P^1(v_0^1, v_n^1)| - |p^1(u_0^1, u_m^1)| = |P_{G^*}(u,v)| - |p_{G^*}(u,v)| > k$, then $G^1 \notin \mathrm{DH}(k,+)$. Finally, since G^1 is isomorphic to G, then $G \notin \mathrm{DH}(k,+)$.

$\Longleftarrow$: Assume $G^* \in \mathrm{DH}(k,+)$. Since G is an induced subgraph of G^*, then, by Lemma 3.2, $G \in \mathrm{DH}(k,+)$. □

Lemma 4.5 *Let G be a graph such that $\partial(G) > 0$. Then, $\mathcal{D}(G^*)$ contains a cycle-pair of G^*.*

Proof. Let $\{x,y\} \in \mathcal{D}(G)$. Let $P_G(x,y) = (x \equiv v_0, v_1, \ldots, v_n \equiv y)$ and $p_G(x,y) = (x \equiv u_0, u_1, \ldots, u_m \equiv y)$ be a longest and a shortest induced path connecting x and y, respectively. Now, two cases may arise:

1. $P_G(x,y) \cap p_G(x,y) = \{x,y\}$:
   In this case, $P_{G^*}(x,y) = (x \equiv v_0^1, v_1^1, \ldots, v_n^1 \equiv y)$ and $p_{G^*}(x,y) = (x \equiv u_0^1, u_1^1, \ldots, u_m^1 \equiv y)$ form the requested cycle-pair in G^* (remember that, By Lemma 4.4, $\partial(G) = \partial(G^*)$).

2. $P_G(x,y) \cap p_G(x,y) \neq \{x,y\}$:
In this case, select $p'_{G^*}(v_0^1, v_n^1) = (x \equiv v_0^1, v_1^2, v_2^2 \ldots, v_{n-1}^2, v_n^1 \equiv y)$ and $p''_{G^*}(u_0^1, u_m^1) = (x \equiv u_0^1, u_1^1, \ldots, u_m^1 \equiv y)$. To show that $\{x,y\} \in \mathcal{D}(G^*)$, it is sufficient to observe that: $p'_{G^*}(v_0^1, v_n^1)$ and $p''_{G^*}(u_0^1, u_m^1)$ are induced in G^*, $|P_G(x,y)| = |p'_{G^*}(v_0^1, v_n^1)|$, and $|p_G(x,y)| = |p''_{G^*}(u_0^1, u_m^1)|$ (by construction of G^*); $\{x,y\} \in \mathcal{D}(G)$ (by hypothesis); $\partial(G) = \partial(G^*)$ (by Lemma 4.4).

□

The following theorem provides a cycle-chord characterization for graphs in DH$(k,+)$, $k \geq 0$.

Theorem 4.6 *Let G be a graph and $k \geq 0$ be an integer. Then, $G \in$ DH$(k,+)$ if and only if $cd(C_n) \geq \frac{n-k}{2} - 1$ for each cycle C_n, $n > k+4$, of G^*.*

Proof. $\Longrightarrow$: Assume $G \in$ DH$(k,+)$, $k \geq 0$. By contradiction, suppose there exists a cycle C_n, $n > k+4$, in G^* such that $cd(C_n) < \frac{n-k}{2} - 1$. Let $C_n = (x, v_1, v_2, \ldots, v_q, y, u_p, u_{p-1}, \ldots u_1)$, $p + q + 2 = n$, and $\{v_1, v_2, \ldots, v_q\}$ the set of nodes giving the chord distance of C_n (hence, $cd(C_n) = q$). Since $(x, v_1, v_2, \ldots, v_q, y)$ is a path in G^*, then $d_{G^*}(x,y) \leq q+1$; moreover, since $(x, u_1, u_2, \ldots u_p, y)$ is an induced path in G^*, then $D_{G^*}(x,y) \geq p+1$. It follows that:

$$\begin{aligned} D_{G^*}(x,y) - d_{G^*}(x,y) &\geq (p+1) - (q+1) \\ &= p - q \\ &= (n-q-2) - q \\ &= n - 2q - 2 \\ &> n - 2(\frac{n-k}{2} - 1) - 2 \\ &= k \end{aligned}$$

Hence, $D_{G^*}(x,y) - d_{G^*}(x,y) > k$, that is $\partial(G^*) > k$. This is a contradiction, because, by Lemma 4.4, $G \in$ DH$(k,+)$ implies $G^* \in$ DH$(k,+)$.

$\Longleftarrow$: Assume $cd(C_n) \geq \frac{n-k}{2} - 1$ for each cycle C_n, $n > k+4$, of G^*. By contradiction, suppose $G \notin$ DH$(k,+)$. In this case, by Lemma 4.4, $G^* \notin$ DH$(k,+)$, and hence $\partial(G^*) > 0$. Now, by Lemma 4.5 and Fact 4 of Lemma 3.4 there exists a cycle-pair $\{x,y\} \in \mathcal{D}(G^*)$ inducing a cycle C_n with n nodes and such that $d_{G^*}(x,y) \geq 2$. Since $G^* \notin$ DH$(k,+)$, then $D_{G^*}(x,y) > k + d_{G^*}(x,y)$. Moreover,

$$\begin{aligned} n &= D_{G^*}(x,y) + d_{G^*}(x,y) \\ &> k + d_{G^*}(x,y) + d_{G^*}(x,y) \\ &= k + 2 \cdot d_{G^*}(x,y) \end{aligned}$$

implies that $d_{G^*}(x,y) < \frac{n-k}{2}$. Finally, since $cd(C_n) = d_{G^*}(x,y) - 1$, then

$$cd(C_n) < \frac{n-k}{2} - 1,$$

a contradiction. □

5 Recognition problem for DH$(k,+)$

In this section we study the recognition problem for the class DH$(k,+)$ when k is not fixed. We start by defining the following decision problem:

Definition 5.1 *Dilation Number Problem:*
INSTANCE*: A graph $G = (V, E)$, an integer $q \geq 0$.*
QUESTION*: $\partial(G) > q$?*

The NP-completeness of this problem can be shown by providing a polynomial transformation from the NP-complete problem *Induced Path* (cf. [15], GT23), that can be formally defined as follows:

INSTANCE: A graph $G = (V, E)$, a positive integer $k \leq |V|$.
QUESTION: Is there a subset $P \subseteq V$ with $|P| \geq k$ such that the subgraph induced by P is an induced path on $|P|$ nodes ?

In the following result, we use the version of *Induced Path* in which $1 < k \leq |V|$. Obviously, this problem is still NP-complete.

Theorem 5.2 *Dilation Number is NP-complete.*

Proof. It is easy to see that the Dilation Number problem belongs to NP, as given a pair of paths joining two nodes in V it is possible to check in polynomial time whether the difference of their lengths is greater than q.

Given a graph $G = (V, E)$ and a positive integer k representing an instance of Induced Path, in polynomial time we construct a graph G' and define an integer q such that there exists the required induced path in G if and only if $\partial(G')$ is greater than q.

The reduction graph $G' = (V', E')$ is obtained as follows: for each node $v \in V$, add a pendant node $\overline{v}$ to v. These new nodes form the independent set $W = \{\overline{v} \mid v \in V\}$. Then, connect all the nodes in $V \cup W$ to a new node u (see Fig. 4). Formally:

- $V' = V \cup W \cup \{u\}$;
- V, W and $\{u\}$ are pairwise disjoint sets with $|W| = |V|$;
- $E' = E \cup \{(v, \overline{v}) \mid v \in V\} \cup \{(u, v), (u, \overline{v}) \mid v \in V\}$.

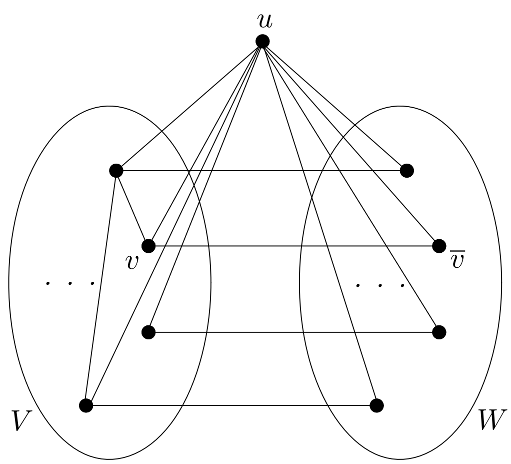

Figure 4: *The graph G' built using the instance $G = (V, E)$ of the Induced Path problem. W is an independent set containing a node $\overline{v}$ for each node $v \in V$.*

Concerning the rational number q, it is given by $q = k - 2$.

Now we prove that the instance of Induced Path has a positive answer if and only if $\partial(G') > q$.

$\Longrightarrow$*:* Assume that the instance of Induced Path has a positive answer. This implies that an induced path $p = (v_1, v_2, \ldots, v_n)$ exists in $\langle V \rangle$ such that $|p| \geq k$. Then the path $\overline{p} = (\overline{v}_1, v_1, \ldots, v_n, \overline{v}_n)$ is also an induced path in G' and $|\overline{p}| \geq k+2$.

By definition of G', nodes $\overline{v}_1$ and $\overline{v}_n$ are not adjacent, and since they are both adjacent to u, then $d_{G'}(\overline{v}_1, \overline{v}_n) = 2$. Hence, the following relation holds:

$$\begin{aligned} \partial(G') &\geq D_{G'}(\overline{v}_1, \overline{v}_n) - d_{G'}(\overline{v}_1, \overline{v}_n) \\ &\geq (k+1) - 2 \\ &= k - 1 \\ &> q \end{aligned}$$

This implies that the instance of *Stretch Number* has a positive answer.

$\Longleftarrow$*:* Let us assume that *Dilation Number* has a positive answer, that is $\partial(G') > q$. By definition of dilation number there exist two nodes $x, y \in G'$ such that $\partial_{G'}(x, y) > q$. Nodes x and y cannot be adjacent otherwise $\partial_{G'}(x, y) = 0$ (a contradiction for $\partial(G') > q \geq 0$). For the same reason, neither x nor y can coincide with u, being u adjacent to each other node in G'. Then, $d_{G'}(x, y) = 2$. This implies that the relation $D_{G'}(x, y) - d_{G'}(x, y) > q$ can be rewritten as $D_{G'}(x, y) > q + 2$. Then,

$$D_{G'}(x, y) \; > \; q + 2 \; = \; (k - 2) + 2 \; = \; k$$

Let $p = (x, v_1, \ldots, v_n, y)$ be an induced path between x and y whose length is equal to $D_{G'}(x, y)$. If p contains u, then $P_{G'}(x, y) = (x, u, y)$, contradicting the relation $D_{G'}(x, y) > q + 2 > 2$. Hence, x, y and v_i, $1 \leq i \leq n$, are elements of $V \cup W$. Moreover, since p does not contain u and since the elements of W are pendant nodes in $\langle V \cup W \rangle$, then $v_i \notin W$, $1 \leq i \leq n$.

Now, three different cases arise, according to the membership of x and y to W. Notice that $|p| > k + 1$ because $|p| = D_{G'}(x, y) + 1 > k + 1$.

1. Both x and y are in V. In this case p is an induced path in G, and since $|p| > k + 1$, then p itself is a solution for the instance of the *Induced Path* problem.

2. $x \in V$ and $y \in W$. In this case $p' = (x, v_1, \ldots, v_n)$ is an induced path in G, and since $|p| > k + 1$, then $|p'| > k$ and p' is a solution for the instance of the *Induced Path* problem.

3. Both x and y are in W. In this case $p'' = (v_1, \ldots, v_n)$ is an induced path in G, and since $|p| > k + 1$, then $|p''| \geq k$ and p'' is a solution for the instance of the *Induced Path* problem.

This implies that the instance of *Induced Path* has a positive answer. □

If we fix $k = 0$, then the recognition problem for the class DH$(k, +)$ can be solved in linear time [19]. If we consider k not fixed, then the recognition problem for the class DH$(k, +)$ is exactly the complementary problem of *Dilation Number*. As a consequence, the following complexity result can be stated.

Corollary 5.3 *If k is not fixed, the recognition problem for the class* DH$(k, +)$ *is Co-NP-complete.*

6 Characterization of graphs in DH$(1, +)$

In this section we provide a characterization for the smallest class among the new ones, i.e., class containing $(1, +)$–distance-hereditary graphs. Theorem 6.6 lists all the forbidden induced subgraphs of every graph in DH$(1, +)$.

Lemma 6.1 *Let G be a graph containing, as induced subgraphs, a cycle C_n with $n \geq 6$ and $cd(C_n) \leq 1$. Then, G contains one of the cycles of Fig. 5 as induced subgraphs, that is:*

1. *H_n, for each $n \geq 6$;*

2. *cycles C_6 with $cd(C_6) = 1$;*

3. *cycles C_7 with $cd(C_7) = 1$;*

4. *cycles C_8 with $cd(C_8) = 1$.*

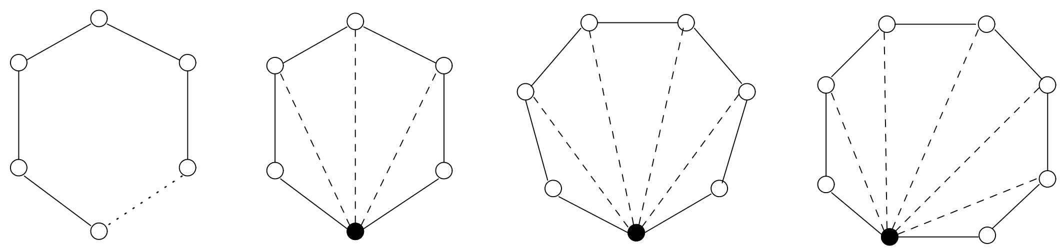

Figure 5: *Cycles used in Lemma 6.1. In each of the last three cycles, at least one chord must exist. The dotted line represents a path; a dashed line represents an edge that may or may not exist.*

Proof. Let C_n be the cycle contained, as induced subgraphs, in G:

1. $n \geq 6$ and $cd(C_n) = 0$:
   In this case the statement is trivially true (since G contains H_n, $n \geq 6$, as induced subgraph);

2. $6 \leq n \leq 8$ and $cd(C_n) = 1$:
   Also in this case the statement is trivially true (since G contains one of the cycles of Fig. 5 having chord distance 1, as induced subgraph);

3. $n \geq 9$ and $cd(C_n) = 1$:
   Let $C_n = (u_1, u_2, \ldots, u_n)$, and assume that all the chords of C are incident to u_1. Denote by l the smallest index j such that u_j and u_1 are connected by a chord of C, i.e. $l = \min\{j \mid (u_j, u_1) \text{ is a chord of } C\}$. If $l \geq 6$, then the cycle $(u_1, u_2, \ldots, u_l)$ is a hole with at least 6 nodes, and then the lemma holds. If $l < 6$, then the cycle $(u_1, u_l, u_{l+1}, \ldots, u_n)$ contains $n' \geq n - 3 \geq 6$ nodes and has chord distance at most 1. Now, if the latter cycle is one of the cycles of Fig. 5 we are done, otherwise we can recursively apply to this cycle the arguments above.

The analysis of these three cases concludes the proof. □

Lemma 6.2 *Let G be a graph and let G^* its twin graph. $G \in \mathrm{DH}(1, +)$ if and only if the following graphs are not induced subgraphs of G^*:*

1. *H_n, for each $n \geq 6$;*

2. *cycles C_6 with $cd(C_6) = 1$;*

3. *cycles C_7 with $cd(C_7) = 1$;*

4. *cycles C_8 with $cd(C_8) = 1$;*

5. *cycles C_{2i+4} with $cd(C_{2i+4}) = i$, for each $i \geq 2$.*

Proof. $\Longrightarrow$: Holes H_n, $n \geq 6$, have dilation number at least 2. Cycles with 6, 7, or 8 nodes and chord distance 1 have dilation number equal to 2, 3, and 4, respectively. Cycles C_{2i+4} with chord distance equal to i have dilation number at least $2i+4-2\cdot(cd(C_{2i+4})+1)=2$. Then, they are forbidden induced subgraphs for G and, by Lemma 4.4, also for G^*.

$\Longleftarrow$: Assuming $G \notin \mathrm{DH}(1,+)$, we show that G^* contains one of the forbidden subgraphs. If $G \notin \mathrm{DH}(1,+)$ then, by Theorem 4.6, G^* contains a cycle C_n, $n \geq 6$, as induced subgraph such that $0 \leq cd(C_n) < \frac{n-3}{2}$. In what follows we show that either C_n contains one of the cycles in the statement of the lemma or C_n contains a cycle $C_{n'} \notin \mathrm{DH}(1,+)$ as induced subgraph, $n' < n$. In the latter case we can recursively apply to $C_{n'}$ this proof.

Letting $q \geq 0$ and $n \geq \max\{6, 2q+4\}$, consider the cycle C_n with chord distance q. The analysis of C_n is performed by cases:

1. $0 \leq q \leq 1$ and $n \geq \max\{6, 2q+4\} = 6$:
   In this case, by Lemma 6.1, C_n contains one of the cycles of Fig. 5 as induced subgraph.

2. $q \geq 2$ and $n = \max\{6, 2q+4\} = 2q+4$:
   In this case, C_n corresponds to the last cycle in statement of the lemma.

3. $q \geq 2$ and $n > \max\{6, 2q+4\} = 2q+4$:
   In this case, assume that the cycle C_n is induced by the nodes of the two node-disjoint paths $P_{G^*}(x,y) = (x, u_1, u_2, \ldots, u_p, y)$ and $p_{G^*}(x,y) = (x, v_1, v_2, \ldots, v_q, y)$, $p+q+2 = n$, such that nodes $v_1, v_2, \ldots, v_q$ give the chord distance of C_n. In this cycle, we denote by r_j the largest index j' such that v_j and $u_{j'}$ are connected by a chord of C_n, i.e. $r_j = \max\{j' \mid (v_j, u_{j'}) \text{ is a chord of } C_n\}$; we assume r_j undefined when v_j is not incident to a chord of C_n. Informally, r_j gives the *rightmost* chord incident to v_j. Notice that, since $q \geq 1$, r_1 is defined.

   If $r_1 > 3$ then the subgraph of C_n induced by the nodes $v_1, x, u_1, \ldots, u_{r_1}$ is a cycle with at least 6 nodes and chord distance at most 1. According to Lemma 6.1, this subgraph contains one of the cycles of Fig. 5 as induced subgraph.

   Assume $r_1 \leq 3$ and let $C_{n'}$ be the subgraph of C_n induced by the nodes $v_1, v_2, \ldots, v_q, y, u_p, u_{p-1}, \ldots, u_{r_1}$ (informally, $C_{n'}$ is one of the two cycles obtained by "cutting" C_n by means of chord (v_1, u_{r_1})). Cycle $C_{n'}$ has $n' \geq n-3 \geq 6$ nodes (because $r_1 \leq 3$) and chord distance at most $q-1$.

   According to Theorem 4.6, by proving $cd(C_{n'}) < \frac{n'-3}{2}$ we get $C_{n'} \notin \mathrm{DH}(1,+)$. Since $cd(C_{n'}) \leq q-1$, $cd(C_{n'}) < \frac{n'-3}{2}$ holds when the following inequality holds:

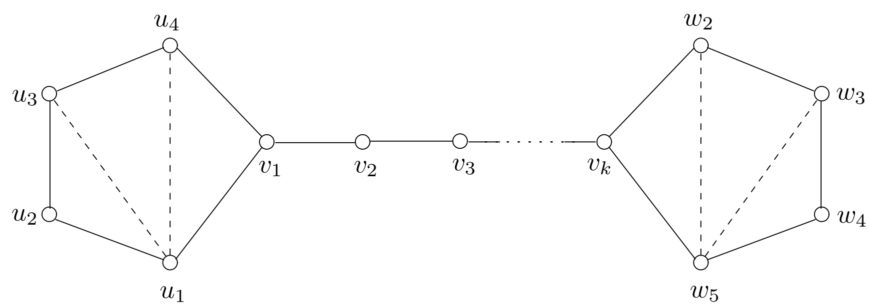

Figure 6: *The clepsydra graph $cl(k)$ (see Definition 6.3). The dashed edges may or may not exist.*

$$\frac{n'-3}{2} > q-1,$$
$$\frac{(n-3)-3}{2} > q-1,$$
$$n > 2q+4. \tag{4}$$

Since $n > 2q+4$ holds by hypothesis, then $cd(C_{n'}) < \frac{n'-3}{2}$. This means that $C_{n'} \notin \mathrm{DH}(1,+)$, and hence we can recursively apply to $C_{n'}$ this proof.

The analysis of the cases above concludes the proof. □

The following definition introduces the notion of *clepsydra* graph (see Fig. 6), useful to characterize graphs in $\mathrm{DH}(1,+)$.

Definition 6.3 *Let $C_5 = (u_1,u_2,u_3,u_4,u_5)$ be a cycle such that $cd(C_5) \leq 1$ and $deg_{C_5}(u_5) = 2$, $P_k = (v_1,v_2,\ldots,v_k)$ be a path with $k \geq 1$, and $C'_5 = (w_1,w_2,w_3,w_4,w_5)$ be a cycle such that $cd(C'_5) \leq 1$ and $deg_{C'_5}(w_1) = 2$. A* clepsydra of order *k is a graph $cl(k) = (V,E)$ such that:*

- *$V = \{u_1,u_2,u_3,u_4,v_1,v_2,\ldots,v_k,w_2,w_3,w_4,w_5\}$;*
- *if $x,y \in V$ then $(x,y) \in E$ if and only if one of the following condition holds:*
  1. *$(x,y) \in \{(u_1,v_1),(u_4,v_1),(v_k,w_2),(v_k,w_5)\}$*
  2. *x and y are adjacent in C_5, P_k, or C'_5*

Lemma 6.4 *Let $G \in \mathrm{DH}(\frac{3}{2},*)$ containing a cycle C_n as induced subgraph such that $n \geq 4+2i$, $i \geq 1$, and $cd(C_n) \leq i$. Then, G contains a clepsydra as induced subgraph.*

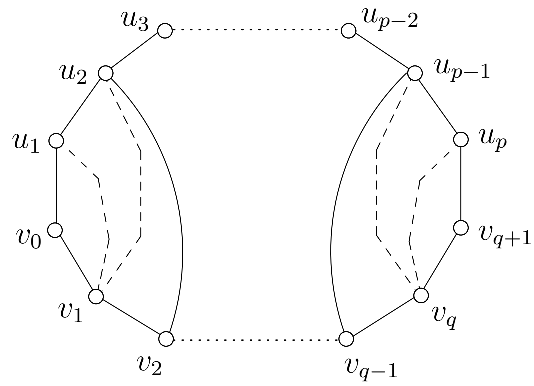

Figure 7: *The cycle C_n considered in Lemma 6.4. Dashed edges represent chords that may or may not exist. Nodes in cycles $(u_2, u_1, v_0, v_1, v_2)$ and $(v_{q-1}, v_q, v_{q+1}, u_p, u_{p-1})$, along with nodes in path $(v_3, v_4, \ldots, v_{q-2})$, induce a clepsydra cl$(k)$, $k \geq 1$ ($k = 1$ when $q = 3$), as subgraph of C_n.*

Proof. Let $C_n = (v_0, v_1, v_2, \ldots, v_q, v_{q+1}, u_p, u_{p-1}, \ldots, u_1)$ and let $(v_1, v_2, \ldots, v_q)$ be the nodes giving the chord distance of C_n (see Figure 7). Without loss of generality, we assume that C_n is a cycle in G that satisfies the hypotheses and has the minimum number of nodes.

We first show that a chord connecting two nodes in $\{v_0, v_1, \ldots, v_{q+1}\}$ does not exist in C_n. This is done by analyzing three cases:

- Consider a chord connecting two nodes in $\{v_0, v_1, \ldots, v_q\}$. Such a chord can be denoted by (v_j, v_{j+t+1}), where j and t are integers such that $j \geq 0$, $t \geq 1$, and $j + t + 1 \leq q$. Chord (v_j, v_{j+t+1}) forms the cycle $C_m = (v_0, v_1, \ldots, v_j, v_{j+t+1}, \ldots v_{q+1}, u_p, u_{p-1}, \ldots, u_1)$. It follows that C_m has $m = n - t$ nodes and chord distance $cd(C_m) \leq q - t$. Since

$$m = n - t \geq 4 + 2i - t \geq 4 + 2q - t \geq 4 + 2q - 2t = 4 + 2(q - t)$$

  and $q - t \geq 1$, then C_m is a cycle which fulfills the hypotheses as in the statement but with less nodes than C_n. This contradicts the minimality of C_n.

- Consider a chord connecting two nodes in $\{v_1, v_2 \ldots, v_{q+1}\}$. This case is symmetric to the previous one.

- Chord (v_0, v_{q+1}) does not exist by definition of chord distance.

Then, every chord (x, y) of C_n is such that $x \in \{v_1, v_2, \ldots, v_q\}$ (by definition of chord distance) and $y \in \{u_1, u_2, \ldots, u_p\}$.

Now, we show some lower bounds for p, q and i. By hypotheses, it follows that $n = p + q + 2 \geq 4 + 2i$ and $q \leq i$, and hence:

$$p \;\geq\; 2i + 2 - q \;\geq\; 2i + 2 - i \;=\; i + 2.$$

Paths $(v_0, v_1, \ldots, v_{q+1})$ and $(v_0, u_1, u_2 \ldots, u_p, v_{q+1})$ can be used to get a lower bound for the stretch number $s_G(v_0, v_{q+1})$ (the concept of stretch number is recalled after Definition 3.3):

$$s_G(v_0, v_{q+1}) \;\geq\; \frac{p+1}{q+1}. \tag{5}$$

Moreover, since $G \in \mathrm{DH}(\frac{3}{2}, *)$, the following upper bound holds:

$$s_G(v_0, v_{q+1}) \;\leq\; \frac{3}{2}. \tag{6}$$

And, by using $q \leq i$ and $p \geq i + 2$:

$$\frac{p+1}{q+1} \;\geq\; \frac{i+3}{i+1} \;=\; 1 + \frac{2}{i+1}. \tag{7}$$

Hence, by combining Eqs. 5, 6 and 7, we get:

- $1 + \frac{2}{i+1} \leq \frac{3}{2}$. This inequality can be rewritten as $i \geq 3$;
- by using $p \geq i + 2$ and $i \geq 3$ we get $p \geq 5$;
- $\frac{p+1}{q+1} \leq \frac{3}{2}$. This inequality can be rewritten as $q \geq \frac{2p-1}{3}$. By using $p \geq 5$ we obtain the requested lower bound $q \geq 3$.

Now we introduce some notation about chords. For $1 \leq j \leq q$, we denote by v_{l_j} and v_{r_j} the nodes incident to the *leftmost* and *rightmost* chord of v_j, respectively. Formally,

$$l_j = \min\{j' \mid 1 \leq j' \leq p \text{ and } (v_j, u_{j'}) \text{ is a chord of } C_n\},$$

$$r_j = \max\{j' \mid 1 \leq j' \leq p \text{ and } (v_j, u_{j'}) \text{ is a chord of } C_n\}.$$

We assume l_j and r_j undefined when v_j is not incident to a chord of C_n. By definition of chord distance, l_1, r_1, l_q, and r_q are defined.

Now we provide a formula to compute l_j and r_j. Assuming l_j defined for some j such that $1 < j \leq q$, let $C_{n'}$ and $C_{n''}$ be the subgraphs of C_n induced by the nodes $v_0, v_1, v_2, \ldots, v_j, u_{l_j}, u_{l_j-1}, \ldots, u_1$ and $v_j, v_{j+1}, \ldots, v_{q+1}, u_p, u_{p-1}, \ldots, u_{l_j}$, respectively. Informally, $C_{n'}$ and $C_{n''}$ are the cycles obtained by "cutting" C_n by means of chord (v_j, u_{l_j}). Since cycle $C_{n'}$ has $n' = l_j + j + 1$ nodes, then cycle $C_{n''}$ has

$$\begin{aligned} n'' &= n - n' + 2 \\ &\geq 4 + 2i - (l_j + j + 1) + 2 \\ &= 2i + 5 - l_j - j \end{aligned}$$

nodes. Moreover, it can be observed that $cd(C_{n'}) \leq j-1$ and $cd(C_{n''}) \leq q-j+1$.

We now assume $n' < 4+2(j-1)$, otherwise $n' \geq 4+2(j-1)$ and $cd(C_{n'}) \leq j-1$ imply that $C_{n'}$ represent a contradiction to the minimality of C_n. By symmetry, we assume $n'' < 4 + 2(q - j + 1)$. Inequality $n' < 4 + 2(j - 1)$ can be rewritten as $l_j + j + 1 < 4 + 2(j - 1)$, from which $l_j < j + 1$ and hence $l_j \leq j$ follows. Similarly, from $n'' \geq 2i + 5 - l_j - j$ and $n'' < 4 + 2(q - j + 1)$ we get:

$$\begin{gathered} 2i + 5 - l_j - j < 4 + 2(q - j + 1), \\ l_j > 2(i - q) + j - 1, \\ l_j \geq 2(i - q) + j. \end{gathered}$$

Relations $l_j \leq j$ and $l_j \geq 2(i - q) + j$ imply the following equation:

$$l_j = j, \qquad 1 < j \leq q. \tag{8}$$

Symmetrically,

$$r_{q-j} = p - j, \qquad 1 \leq j < q. \tag{9}$$

Now we show that v_2 is incident to a chord of C_n. In fact, if v_2 is not incident to a chord, let $k = \min\{j > 2 \mid l_j \text{ is defined}\}$. Since l_q is defined, then $3 \leq k \leq q$. Now, let $C_{n'''}$ be the subgraphs of C_n induced by the nodes $v_0, v_1, \ldots, v_k, u_{l_k}, u_{l_k-1}, \ldots, u_1$. Since $l_k = k$ and $k \geq 3$, then cycle $C_{n'''}$ has $n''' = 2k + 1 \geq 7$ nodes and chord distance equal to 1 (in $C_{n'''}$, only v_1 may be incident to chords). Since $s(C_{n'''}) > \frac{3}{2}$, this contradicts $C_n \in \mathrm{DH}(\frac{3}{2}, *)$.

As a consequence, l_2 must be defined and, according to Eq. 8, $l_2 = 2$. Chord (v_2, u_2) contributes to form the cycle $C'_5 = (u_2, u_1, v_0, v_1, v_2)$ having chord distance at most 1 (see Figure 7).

Symmetrically, the same arguments can be used to show that r_{q-1} is defined and, according to Eq. 9, $r_{q-1} = p - 1$. Hence, chord $(v_{q-1}, u_{r_{q-1}})$ contributes to form the cycle $C''_5 = (v_{q-1}, v_q, v_{q+1}, u_p, u_{p-1})$ having chord distance at most 1 (possibly, with chords incident to v_q).

Now, cycles C'_5 and C''_5 along with path $(v_2, v_3, \ldots, v_{q-1})$ form the requested clepsydra. Notice that, since $q \geq 3$, path $(v_2, v_3, \ldots, v_{q-1})$ may consists of a single node (namely, v_2). In this case cycles C'_5 and C''_5 share v_2 and they form a clepsydra of order 1.

To conclude the proof we have to show that the constructed clepsydra is indeed an induced subgraph, that is, we have to show that nodes in C'_5, C''_5, and

$v_2, v_3, \ldots, v_{q-1}$ do not induce additional edges except those forming the clepsydra. Remember that, according to definition of chord distance, if such an additional edge exists, then it is incident to a node in $\{v_1, v_2, \ldots, v_q\}$. For sake of convenience, let us denote by X the set containing nodes in C_5', C_5'', and $v_3, v_4, \ldots, v_{q-2}$. Nodes $v_1, v_2, \ldots, v_q$ are analyzed by cases:

- Nodes $v_3, v_4, \ldots, v_{q-2}$.

  Assume that a node v_k, $3 \leq k \leq q-2$ is incident to a chord of C_n. In this case, by applying Eqs. 8 and 9 we get $l_k = k$ and $r_k = r_{q-(q-k)} = p-(q-k) = p-q+k$. Since $3 \leq k \leq q-2$, then $l_k \geq 3$ and $r_k \leq p-2$. By analyzing indexes of nodes in cycles C_5' and C_5'', it follows that chords of v_k cannot be incident to nodes in $X \setminus \{v_k\}$.

- Node v_2 (symmetrically, node v_{q-1}).

  We already know that v_2 is incident to a chord; Eqs. 8 and 9 imply $l_2 = 2$ and $r_2 = p-q+2$.

  Now, if $q = 3$, then cycles C_4' and C_5'' share node v_2. Since $l_2 = 2$ and $r_2 = p-1$, the leftmost and the rightmost chord of v_2 contribute to form the cycles C_5' and C_5'', respectively, while the other chords of v_2 (if any) cannot be incident to nodes in $X \setminus \{v_2\}$.

  If $q \geq 4$, then $l_2 = 2$ and $r_2 = p-q+2 < p-1$; the leftmost chord of v_2 contributes to form the cycle C_5', while the other chords of v_2 (if any) cannot be incident to nodes in $X \setminus \{v_2\}$.

- Node v_1 (symmetrically, node v_q).

  If v_1 is incident to a chord of C_n, then, by Eq. 9, $r_1 = p-q+1$. Since $q \geq 3$, then $r_1 = p-q+1 < p-1$. This implies that chords from v_1 to other nodes of X (if any) may be incident to u_1 or to u_2 only.

This concludes the proof. □

In [8], the authors provided the following characterization for graphs in $\mathrm{DH}(\frac{3}{2}, *)$:

Theorem 6.5 *[8] Let G be a graph. $G \in \mathrm{DH}(\frac{3}{2}, *)$ if and only if the following graphs are not induced subgraphs of G:*

1. *H_n, for each $n \geq 6$;*
2. *cycles C_6 with $cd(C_6) = 1$;*
3. *cycles C_7 with $cd(C_7) = 1$;*
4. *cycles C_8 with $cd(C_8) = 1$ or $cd(C_8) = 2$.*

By Lemma 6.4, characterizations provided by Theorem 6.5 and Lemma 6.2 produce the following corollary. This result states that every graph in $\mathrm{DH}(\frac{3}{2},*)$ either belongs to $\mathrm{DH}(1,+)$ or contains a clepsydra as induced subgraph.

Theorem 6.6 *Let G be a graph. $G \in \mathrm{DH}(1,+)$ if and only if the following graphs are not induced subgraphs of G:*

1. *H_n, for each $n \geq 6$;*
2. *cycles C_6 with $cd(C_6) = 1$;*
3. *cycles C_7 with $cd(C_7) = 1$;*
4. *cycles C_8 with $cd(C_8) = 1$ or $cd(C_8) = 2$;*
5. *clepsydrae.*

Proof. $\Longrightarrow$*:* Holes H_n, $n \geq 6$, have dilation number at least 2. Cycles with 6, 7, or 8 nodes and chord distance 1 have dilation number equal to 2, 3, and 4, respectively. Cycles with 8 nodes and chord distance 2 have dilation number equal to 2. Finally, clepsydrae have dilation number equal to 2.

$\Longleftarrow$*:* By Theorem 6.5, $G \in \mathrm{DH}(\frac{3}{2},*)$. Moreover, since G does not contain clepsydrae, Lemma 6.4 implies that G does not contain a cycle C_n, $n \geq 4+2i$ and $i \geq 1$, having chord distance $cd(C_n) \leq i$. Trivially, G does not contain a cycle C_{2i+4}, $i \geq 2$, having chord distance $cd(C_{2i+4}) = i$. Since G is an induced subgraph of G^*, Lemma 6.2 implies that $G \in \mathrm{DH}(1,+)$. □

7 Recognition problem for $\mathrm{DH}(1,+)$

Theorem 6.6 provides a basis to device a polynomial time algorithm for the recognition of graphs in $\mathrm{DH}(1,+)$. By Lemma 3.6, we know that $\mathrm{DH}(1,+) \subset \mathrm{DH}(\frac{3}{2},*)$; moreover, comparing the characterization of $(\frac{3}{2},*)$–distance-hereditary graphs and $(1,+)$–distance-hereditary graphs provided by Theorems 6.5 and 6.6, respectively, it follows that a graph G belongs to $\mathrm{DH}(1,+)$ if and only if it belongs to $\mathrm{DH}(\frac{3}{2},*)$ and does not contain a clepsydra as induced subgraph. In [9], it is shown that the recognition problem for graphs in $\mathrm{DH}(\frac{3}{2},*)$ can be solved in polynomial time; as a consequence, for our purposes it is sufficient to devise a polynomial time algorithm to check whether G contains a clepsydra $cl(k)$, $k \geq 1$, as induced subgraph.

Checking whether G contains a clepsydra $cl(k)$, $k \leq 2$, can be performed by analyzing every induced subgraph with 9 (case $cl(1)$) or 10 (case $cl(2)$) nodes. Algorithm 7.1 checks whether G contains a clepsydra $cl(k)$, $k > 2$.

Algorithm 7.1 Looking for a clepsydra $cl(k)$, $k > 2$, in a graph G.

Require: A graph G
Ensure: YES, iff a $cl(k)$, $k > 2$, exists as induced subgraph of G
1: **for all** $A \equiv C_5'$, $B \equiv C_5''$ distinct induced subgraphs of G **do**
2:   **if** $cd(A) \leq 1$, $cd(B) \leq 1$, and $\langle A \cup B \rangle$ is not connected **then**
3:     **for all** $x \in A$, $y \in B$ such that $deg_A(x) = deg_B(y) = 2$ **do**
4:       $G_{xy} := G - \{N[(A - x) \cup (B - y)] \setminus \{x, y\}\}$
5:       **if** x and y are connected in G_{xy} **then**
6:         **return** YES
7:       **end if**
8:     **end for**
9:   **end if**
10: **end for**
11: **return** NO

Algorithm 7.1 considers all the possible pairs of distinct cycles A and B with 5 nodes that are induced subgraphs of G. If $A \cup B$ induces a connected subgraph S, then either S is not a clepsydra or S is a clepsydra $cl(k)$ with $k \leq 2$.

If $cd(A) \leq 1$, $cd(B) \leq 1$, and S is not connected (Line 2), then A and B could belong to a clepsydra. To check this, the algorithm properly selects two nodes x and y, each one in a different cycle (Line 3), and it tries to find a path P connecting them. P is looked for in the subgraph G_{xy} obtained by removing from G all the nodes in $A \cup B$ and their neighbors but x and y. Then, if x and y remain connected in G_{xy}, this means that the searched path P exists.

Since it can be easily observed that Algorithm 7.1 works in polynomial time, then we can state the following theorem:

Theorem 7.1 *The recognition problem for the class* DH$(1, +)$ *can be solved in polynomial time.*

Notice that the previous result has only a theoretical value, since the provided algorithm is not efficient. In fact, it is enough to observe that the cycle at Line 1 is executed $O(n^{10})$ times.

8 Conclusions and open problems

In this paper we have introduced, characterized, and provided algorithmic results for $(k, +)$–distance-hereditary graphs, a parametric extension of the class of distance-hereditary graphs. These graphs can model communication networks having desirable connectivity properties. In spite of the results provided in this work, many interesting problems are left open:

1. The recognition problem can be solved in linear time for DH$(0,+)$ [2, 19], in polynomial time for DH$(1,+)$ (Theorem 7.1), and it is Co-NP-complete for the generic case (Corollary 5.3). What is the computational complexity of the recognition problem for $k > 1$, k fixed? If such a problem is hard, what is the largest constant k such that the recognition problem for DH$(k,+)$ can be solved in polynomial time?

2. Can characterization of graphs in DH$(1,+)$ provided by Theorem 6.6 be extended to other classes DH$(k,+)$, $k > 1$?

3. In [7], optimal compact routing schemes are defined for graphs in DH$(0,+)$. Is it possible to define compact routing schemes (or other kinds of routing schemes) for networks based on graphs in DH$(k,+)$, $k > 0$?

4. Several combinatorial problems are solvable in polynomial time for DH$(0,+)$. Can some of these results be extended to DH$(k,+)$, $k > 0$?

During the revision process of this paper we were informed about two related papers. Paper [1] independently studies and characterizes DH$(1,+)$; its main result is a statement which is equivalent to Theorem 6.6. Paper [22], starting from the preliminary version of this work [5], extends results provided by Theorems 6.6 and 7.1 to the case $k = 2$. Hence, paper [22] provides partial answers to questions 1 and 2 above.

A more challenging problem is to study the (s,d)–distance-hereditary graphs, i.e, graphs obtained by composing the notions of $(s,*)$–distance-hereditary and $(d,+)$–distance-hereditary graphs. These graphs form the class DH(s,d), and can be formally defined as follows: Let $s \geq 1$ and $d \geq 0$ be a rational and a natural number, respectively. A graph G is a (s,d)*–distance-hereditary graph* if, for each connected induced subgraph G' of G:

$$d_{G'}(x,y) \leq s \cdot d_G(x,y) + d, \quad \text{for each } x,y \in G'.$$

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