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Modeling, dimension reduction, and nonlinear vibrations of thermomechanically coupled laminated plates

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Abstract

A unified formulation of thermomechanical, geometrically nonlinear, laminated plates that integrates mechanical and thermal aspects is presented. It allows for constructing and comparing a variety of continuous models of different mechanical richness and with full thermoelastic coupling embedded, as well as for deriving minimal reduced order models suitable to provide useful information on fundamental thermomechanical phenomena occurring in the system nonlinear and complex dynamics. Comparative numerical investigations of free and forced vibrations can be carried out through both models of three, fully coupled, ordinary differential equations and simplified, partially coupled, models of two, or even one, ODEs, with the aim to unveil the actual importance of accounting for the various terms to reliably describe the most important thermomechanical effects on the system response.

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Keywords: Thermomechanics; laminated plates; dimension reduction; nonlinear vibrations

1. Introduction

In the framework of the 2D modeling of laminated plates [1,2], different approximations for the deformation of the reference plane and for the shear-warping of the cross section result in distinct geometrically nonlinear and shear deformation models, respectively. In addition to these mechanical features, it is often important to evaluate the effects of thermal phenomena [3-6], by properly selecting some relevant simplifying assumptions (with/without thermomechanical coupling, ratio between current and reference temperatures, order of the temperature distribution,

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etc.). This results in a large number of possible continuous models that can be effectively constructed and compared through a unified formulation. In turn, when aiming at identifying reduced order models to be used for highlighting some main features of nonlinear dynamic response, shear deformations and, possibly, nonlinear curvatures are taken into account [7], whereas thermoelastic coupling is generally overlooked.

This paper presents a unified modeling of thermomechanically coupled laminated plates at both continuous (Sect. 2) and discretized (Sect. 3) levels, providing sample equations and variables of selected reference models, whose reduced response in free linear dynamics (Sect. 4) and forced nonlinear dynamics (Sect. 5) is exemplarily illustrated.

2. Two-dimensional modeling

The structure of many physical theories aimed at building the various fundamental models is independent of the underlying physical content, as it can be highlighted by the classical Tonti diagram [8] which identifies the three basic sets of balance, configuration and phenomenological equations, with the related variables, that give rise to the 3D governing equations of a generic theory.

When dealing with a multiphysics problem in a bidimensional engineering framework, a similar structure can be obtained starting from the basic generalizing assumption 3D→2D

$$\{\text{3D configuration variables}\} = \{\text{shape}\} \times \{\text{generalized configuration variables (2D)}\} \tag{1}$$

that expresses the 3D configuration variables in terms of 2D generalized ones (through shape mathematical functions), and generalizing all the equations and variables of Tonti's decomposition. By applying this procedure to the 2D thermomechanical laminated plate, we obtain the unified modeling scheme in Fig. 1 [6], which integrates mechanical and thermal aspects by addressing them in parallel via the introduction of generalized 2D variables and equations also for the latter.

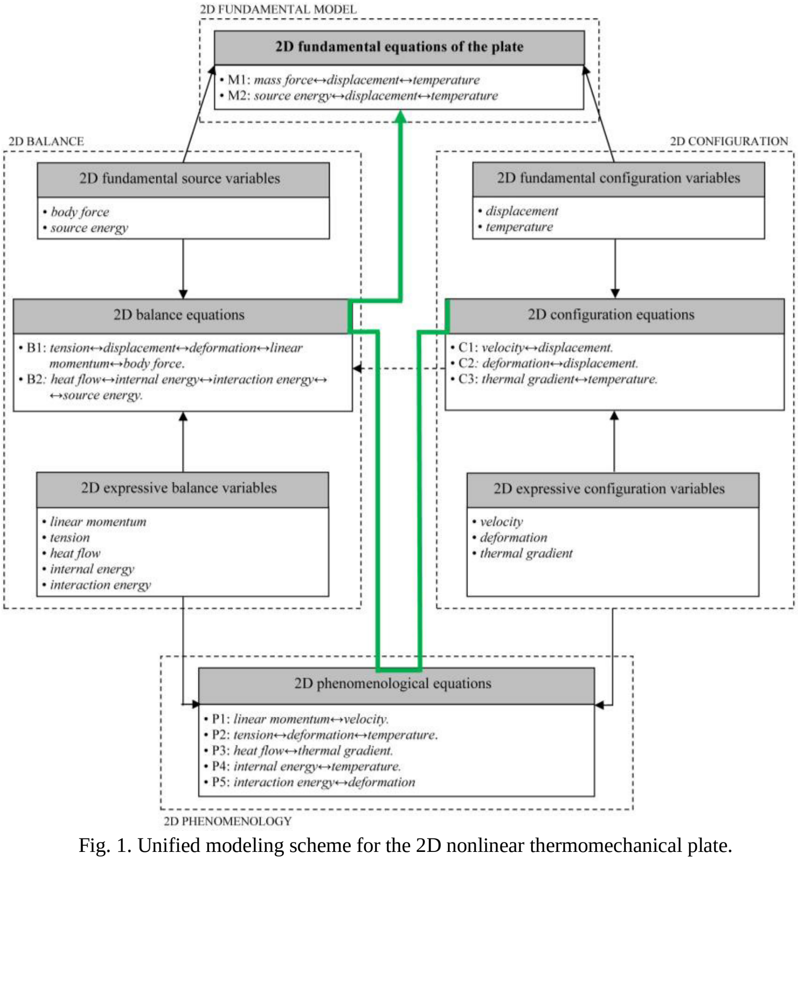

Fig. 1. Unified modeling scheme for the 2D nonlinear thermomechanical plate.

The unified scheme allows to construct and compare different continuous nonlinear models with full thermoelastic coupling, which result from different assumptions about the plate mechanical and thermal configurations. The resulting models are not equally advantageous. As regards geometric nonlinearities, *general* models accounting for all of them [2] involve significant computational difficulties when aiming to obtain minimum order discretized models for the analysis of nonlinear vibrations through the procedure generally used for the *classical* von Karman models. On the other hand, von Karman strains - although involving some nonlinear terms - do not account for the change of structural configuration within the curvature-displacement relationship, because of considering only linear terms in the curvature expressions. Yet, the neglected geometric nonlinearities may entail non-negligible effects in the nonlinear analysis of composite plates. Therefore, in addition to the general and von Karman types of continuous models, *intermediate* models [7] also accounting for nonlinear terms in the curvature expressions can be considered. They retain the great advantage of all von Karman models as regards performing minimal reductions.

Table 1 shows some possible models that, based on a variety of kinematical and thermal features, are classified in the three groups outlined above, i.e. (i) general models (which turn out to be mathematically intractable in a reduced order modeling perspective based on kinematic condensation), (ii) classical von Karman models (mathematically tractable for all types of laminates), and (iii) intermediate models (mathematically tractable only for symmetric laminates).

Table 1**.** Some general, intermediate and classical continuous models.

	General		intermediate		classical (von Karman deformation)		
Features	GTTC	GCTC	MGFTC	MGCTC	TTC	FTC	CTC
in-plane deformation	cubic	cubic	quadratic	quadratic	quadratic	quadratic	quadratic
flexural and twisting curvatures	cubic	cubic	cubic	cubic	linear	linear	linear
spiral curvatures	quadratic	quadratic	absent	absent	absent	absent	absent
shear deformability	cubic	absent	linear	absent	cubic	linear	absent
temperature field	cubic	linear	linear	linear	cubic	linear	linear

By way of example, Figure 2 displays the mathematical relationships C2 and C3 implicitly reported in the configuration block (see Fig. 1) of the TTC (Third-order shear-deformable theory with Thermomechanical Coupling) model (Table 1), that is a geometrically nonlinear model with third order shear deformability and temperature distribution along the thickness which is assumed consistently cubic (Fig. 3), with the associated four thermal variables being reduced to two by imposing the convective boundary conditions on the plate upper and lower surfaces. For the same model, Figure 4 shows the thermal and thermomechanical relationships P3, P4 and P5 of the phenomenological block (remind Fig. 1). In Figs. 2 and 4, the following 2D quantities appear: *u*, *v*, *w*, displacements of the reference plane; ϕ_1 , ϕ_2 , rotations of the cross section; T_0 , T_1 , membrane and bending components of the temperature; $\varepsilon_{ij}^{(0)}$, $\varepsilon_{ij}^{(1)}$, $\varepsilon_{ij}^{(3)}$, von Karman nonlinear membrane strains, Kirchhoff linear bending strains (curvatures), and Reddy higher order bending strains; $\gamma_i^{(0)}$, $\gamma_i^{(1)}$, Mindlin linear transverse shearing strains and Reddy higher order transverse shearing strains; $g_j^{(0)}$, $g_j^{(1)}$, membrane and bending components of the thermal gradient; $\left(q_i^{(0)},\ q_i^{(1)}\right), \left(b^{(0)},\ b^{(1)}\right), \left(a^{(0)},\ a^{(1)}\right)$, membrane and bending components of the heat flow, internal energy and interaction energy; $\left(\lambda_{ij}^A, \lambda_{ij}^B, \lambda_{ij}^D, \lambda_{ij}^E, \lambda_{ij}^F\right)$, $\left(\beta_{ij}^A, \beta_{ij}^B, \beta_{ij}^D, \beta_{ij}^E, \beta_{ij}^F, \beta_{ij}^G, \beta_{ij}^H\right)$, $\left(C^A, C^B, C^D, C^E, C^F\right)$, thermal conductivities, thermal capacities, and thermoelastic stiffnesses of the laminate, which involve membrane (upper-case A), bending-membrane (*D*), bending (*B*), and higher order (*E*, *F*, *G*, *H*) contributions; T_{ref} , reference temperature; c_i , r_i , expressions containing geometrical and phenomenological parameters. All details of the formulation can be found in [9].

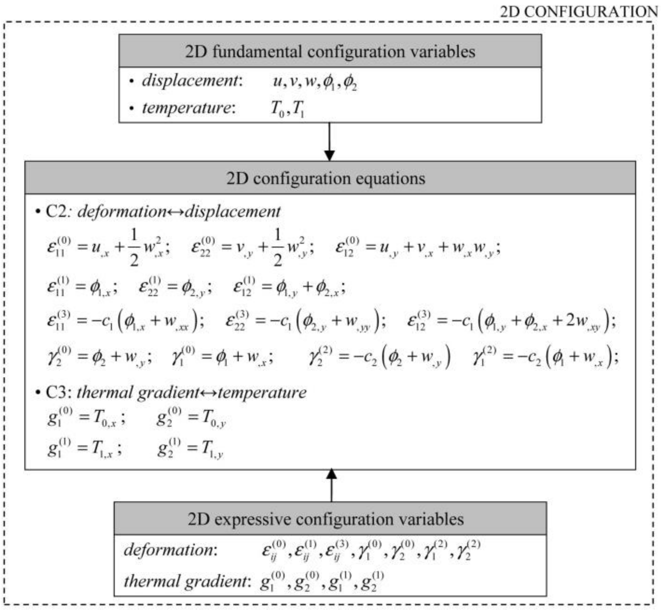

Fig. 2. Some relationships in the configuration block of the TTC model.

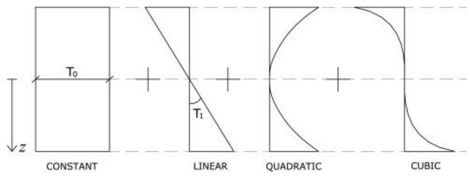

Fig. 3. Single contributions in the overall cubic temperature profile of the TTC model.

3. Zero-dimensional modeling

The dimensional reduction of the previous 2D formulation can be performed starting from the basic assumptions:

$$\{\text{generalized configuration variables (2D)}\} = \{\text{shape}\} \times \{\text{reduced configuration variables (0D)}\} \tag{2}$$

that expresses the 2D generalized variables in terms of 0D reduced ones (through shape mathematical functions).

In order to describe some basic phenomena of the dynamics, an effective minimal dimension reduction of the continuous models can be pursued via a Galerkin procedure [6]. In the case of TTC model, the seven 2D configuration variables $u, v, w, \phi_1, \phi_2, T_0, T_1$ (Fig. 2) can be expressed in terms of seven time-dependent reduced variables $U, V, W, \Gamma_1, \Gamma_2, T_{R0}, T_{R1}$ through only one shape function for each component. Indeed, expressing both in-plane displacement components and out-of-plane shear angles in terms of transverse displacement and of the two (membrane and bending) thermal variables via kinematic condensations performed at the continuum and discrete level, respectively, [9], only three reduced components remain in the problem: the deflection amplitude W and the membrane T_{R0} and bending T_{R1} temperature amplitudes. This allows us to end up with a system of three thermomechanically coupled ordinary differential equations. In Fig. 5, these equations are embedded in the 0D unified thermomechanical scheme which underlies the governing ODEs of the reduced models, similar to the unified scheme (Fig. 1) underlying the PDEs of the continuous models. The coefficients a_{ij} are constant expressions that incorporate the features and physical properties of the model [9]. A mechanical type body diagram of the reduced TTC model, that schematizes in mechanical terms also the thermal aspects of the problem, is reported in Fig. 6.

2D phenomenological equations

• P3: *heat flow↔thermal gradient*

$$\left\{\begin{array}{c}\left\{q_i^{(0)}\right\}\\ \left\{q_i^{(1)}\right\}\end{array}\right\}=\left[\begin{array}{c|c} r_1\left[\lambda_{ij}^A\right]+r_2\left[\lambda_{ij}^B\right]+r_3\left[\lambda_{ij}^D\right]+r_4\left[\lambda_{ij}^E\right] & r_5\left[\lambda_{ij}^A\right]+r_6\left[\lambda_{ij}^B\right]+r_7\left[\lambda_{ij}^D\right]+r_8\left[\lambda_{ij}^E\right]\\ \hline r_1\left[\lambda_{ij}^B\right]+r_2\left[\lambda_{ij}^D\right]+r_3\left[\lambda_{ij}^E\right]+r_4\left[\lambda_{ij}^F\right] & r_5\left[\lambda_{ij}^B\right]+r_6\left[\lambda_{ij}^D\right]+r_7\left[\lambda_{ij}^E\right]+r_8\left[\lambda_{ij}^F\right]\end{array}\right]\left\{\begin{array}{c}\left\{g_j^{(0)}\right\}\\ \left\{g_j^{(1)}\right\}\end{array}\right\}$$

• P4: *internal energy↔temperature*

$$\left\{\begin{array}{c}b^{(0)}\\ b^{(1)}\end{array}\right\}=\left[\begin{array}{c|c} r_1C^A+r_2C^B+r_3C^D+r_4C^E & r_5C^A+r_6C^B+r_7C^D+r_8C^E\\ \hline r_1C^B+r_2C^D+r_3C^E+r_4C^F & r_5C^B+r_6C^D+r_7C^E+r_8C^F\end{array}\right]\left\{\begin{array}{c}T_0\\ T_1\end{array}\right\}+\left\{\begin{array}{c}r_9C^A+r_{10}C^B+r_{11}C^D+r_{12}C^E\\ \hline r_9C^B+r_{10}C^D+r_{11}C^E+r_{12}C^F\end{array}\right\}$$

• P5: *interaction energy↔deformation*

$$\left\{\begin{array}{c}a^{(0)}\\ a^{(1)}\end{array}\right\}=T_{ref}\left[\begin{array}{ccc|ccc|ccc}\beta_{11}^A & \beta_{22}^A & \beta_{12}^A & \beta_{11}^B & \beta_{22}^B & \beta_{12}^B & \beta_{11}^E & \beta_{22}^E & \beta_{12}^E\\ \hline \beta_{11}^B & \beta_{22}^B & \beta_{12}^B & \beta_{11}^D & \beta_{22}^D & \beta_{12}^D & \beta_{11}^F & \beta_{22}^F & \beta_{12}^F\end{array}\right]\left\{\begin{array}{c}\varepsilon_{11}^{(0)}\\ \varepsilon_{22}^{(0)}\\ \varepsilon_{12}^{(0)}\\ \hline \varepsilon_{11}^{(1)}\\ \varepsilon_{22}^{(1)}\\ \varepsilon_{12}^{(1)}\\ \hline \varepsilon_{11}^{(3)}\\ \varepsilon_{22}^{(3)}\\ \varepsilon_{12}^{(3)}\end{array}\right\}$$

Fig. 4. Some relationships in the phenomenological block of the TTC model.
$\left\{q_i\right\}$, $\left\{g_j\right\}$ are vectors while $\left[\lambda_{ij}\right]$ are symmetric matrices (2x2), with i=1,2 and j=1,2.

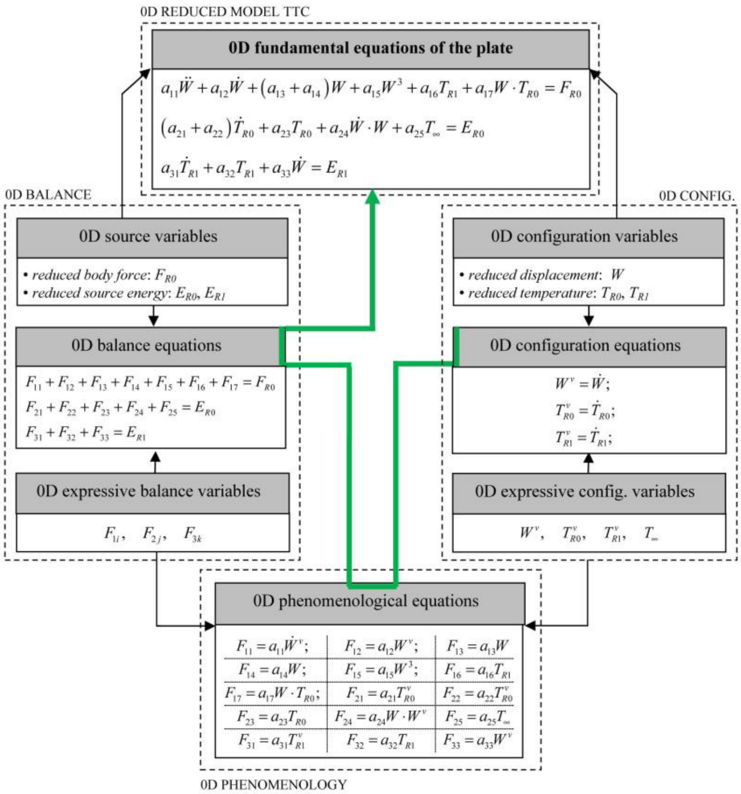

Fig. 5. Unified modeling scheme for the 0D nonlinear thermoelastic plate: reduced TTC model.

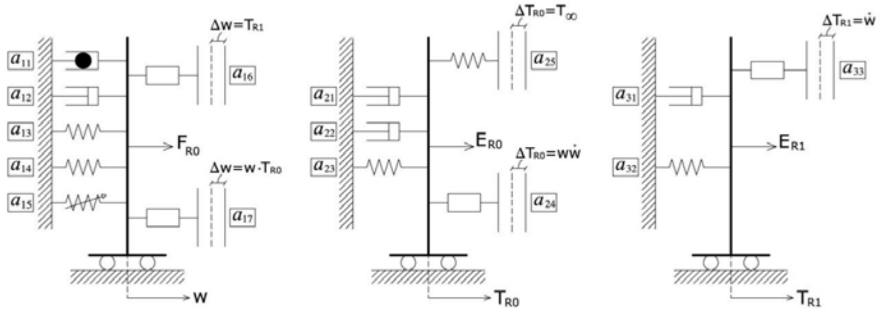

Fig. 6. A mechanical type body diagram of the reduced thermomechanical model in Fig. 5.

The 0D unified framework enables (i) to identify all terms of the reduced model by referring to the underlying continuous one, and (ii) to possibly consider further models variably simplified owing to the presence/absence of mechanical and/or thermal excitations and to consideration of coupling terms. In particular, variably decoupling the elastic aspects from the thermal one and neglecting membrane or/and bending thermal dynamics allows us to end up with two-degree-of-freedom models, or even with a single-degree-of-freedom model, with an actually two-way or a solely single-way thermomechanical coupling. For example, in Table 2, possible reduced models based on the continuous CTC (Classical von Karman theory with Thermomechanical Coupling) model (Table 1) are summarized.

Table 2. Reduced models based on the classical von Karman CTC model.

Features	CTCRa	CTCRb	CTCRc	CTCRd	CTCRe	CTCRf	CTCRg
number of ODEs	3	3	2	2	2	2	1
thermal dynamics	membrane & bending	membrane & bending	membrane	membrane	bending	bending	absent
multiphysicsinteraction	two-way	one-way	two-way	one-way	two-way	one-way	one-way

4. Numerical validation in linear dynamics

Linear models extracted from nonlinear reduced models of the previous section allow to obtain numerical benchmarks as regards linear dynamics. The ensuing frequency values in Table 3 show remarkable agreement with the most refined ones (LD4) obtained for the isotropic plate in [5] through the Carrera Unified Formulation (CUF). Of course, in the case of stubby plates, a good agreement is kept when considering shear deformable models (e.g. TTC). The thermomechanical coupling provides slightly higher frequencies with respect to the pure mechanical case because it acts like a thermal source which leads to a wider global stiffness of the plate [5].

Table 3. Fundamental frequency (*Hz*) for the isotropic plate with several models and thickness ratios *a/h*. (M) means thermal part of the model removed. CLT refers to Kirchhoff plate, LD4 to layer-wise plate with fourth-order expansion of displacements and temperature in thickness direction.

a/h	**5**	**10**	**50**	**100**
CTC(M)	194.16	48.542	1.9416	0.4854
CTC	195.15	48.787	1.9515	0.4878
FTC(M)	175.39	47.228	1.9395	0.4852
TTC(M)	175.16	47.211	1.9394	0.4852
TTC	175.87	47.436	1.9492	0.4877
LD4 [5]	172.40	46.946	1.9390	0.4852
LD4(TM) [5]	173.10	47.158	1.9481	0.4875
CLT [5]	188.08	48.148	1.9411	0.4854
CLT(TM) [5]	189.87	48.607	1.9596	0.4900

5. Nonlinear vibrations

Fundamental aspects of the nonlinear dynamics of continuous models of different richness and accuracy can be highlighted by analyzing the relevant minimal models. By way of example, considering a symmetric cross-play laminate, Figure 7 shows the global bifurcations diagrams of the configuration variables W, T_{R0} and T_{R1} versus the frequency of an applied harmonic transversal mechanical forcing, as obtained by using the fully coupled classical von Karman CTCRa model of Table 2 for a multilayer laminated plate. Typical nonlinear dynamical behaviors including various periodic motions, jump phenomena, sub-harmonics and aperiodic responses are observed. The thermal dynamics T_{R0} and T_{R1} is activated and sustained by the sole mechanical forcing (since there is no external thermal forcing) through the thermomechanical coupling terms of the model; as expected, when the aperiodic plate deflection W appears, the thermal dynamics becomes irregular too.

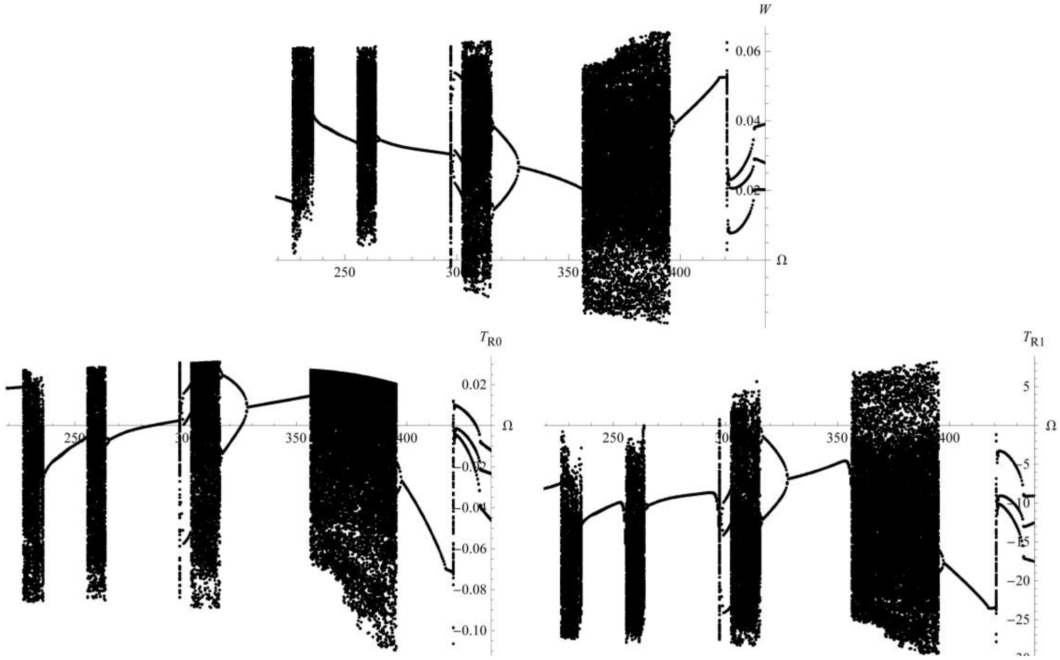

Fig. 7. Global bifurcation diagrams of plate deflection W and temperatures T_{R0}, T_{R1} versus the frequency Ω of an harmonic transversal mechanical forcing (CTC model).

Within the multitude of parameters governing the plate response, some of them are seen to induce a significant thermoelastic damping causing decay of the vibration amplitude, as highlighted by other outcomes (here not reported); in particular, possibly chaotic responses of (partially) uncoupled models are seen to be somehow regularized by the full coupling effects of the complete model. Based on the possible importance of thermomechanical phenomena in free and forced nonlinear vibrations, comparative numerical investigations are presently going on with also a view on the suitability and effectiveness of strongly simplified models in reliably describing thermal effects under variable system and excitation conditions.

The comparative analysis of reduced models is useful also in view of applications, since it may throw useful light on the possibility to exploit thermal effects to favorably affect the system response via an appropriate control of some parameter.

6. Conclusions

A unified formulation of thermomechanical nonlinear laminated plates that integrates mechanical and thermal aspects has been presented. At 2D level, it allows for constructing and comparing different continuous models with full thermoelastic coupling embedded. At 0D level, the unified framework enables to identify all terms of the reduced model by referring to the underlying continuous one, and to possibly consider further variably simplified models.

Fundamental aspects of the nonlinear dynamics of continuous models of different richness and accuracy can be highlighted by analyzing the relevant, variably reduced, minimal models; and the suitability and effectiveness of strongly simplified models in reliably describing thermal effects under variable system and excitation conditions can be analyzed.

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