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Fully inert subgroups of torsion-complete p-groups

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The main result of this paper states that fully inert subgroups of torsion-complete abelian p-groups are commensurable with fully invariant subgroups, which have a satisfactory characterization by a classical result by Kaplansky. As the proof of this fact relies on the analogous result for direct sums of cyclic p-groups, we provide revisited and simplified proofs of the fact that fully inert subgroups of direct sums of cyclic p-groups are commensurable with fully invariant subgroups.

1. Introduction

All groups considered in this paper are assumed to be abelian. Given a group G and an endomorphism $\phi \in \mathrm{End}(G)$, a subgroup H of G is called *ϕ-inert* if it has finite index in $\phi(H) + H$, and it is called *fully inert* if it is ϕ-inert for all endomorphisms ϕ of G.

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Clearly, the notion of fully inert subgroup generalizes that of fully invariant subgroup. Furthermore, two subgroups H and K of G are said to be *commensurable* if $H \cap K$ has finite index both in H and in K. These definitions can be transferred to modules over the ring J_p of the p-adic integers, since the torsion J_p-modules are exactly the p-groups. So a submodule B of a J_p-module A is called *fully inert* if it has finite index in $\psi(B)+B$ for all endomorphisms ψ of A, and two submodules B and C of A are *commensurable* if $B \cap C$ has finite index both in B and in C.

It is worthwhile recalling that the notions of ϕ-inert and fully inert subgroups originated not from a desire to generalize the notions of ϕ-invariant and fully invariant subgroups, but rather they arose naturally in the investigation of dynamical properties of endomorphisms of abelian groups. Actually, ϕ-inert subgroups were a basic tool in the definition of *intrinsic algebraic entropy* introduced in [3], which in turn is a variant of the notion of *algebraic entropy* that was investigated in depth in [5]. Moreover, the intrinsic algebraic entropy of the endomorphisms of some important classes of groups, including groups of finite rank, may be computed just using fully inert subgroups which are independent of the choice of the endomorphisms (see [10]). We refer to our recent paper [9] for a survey of the different notions of entropy in the setting of abelian groups.

There is a parallel between fully inert subgroups of p-groups and fully inert submodules of torsion-free J_p-modules. In fact, the main theorem in [11] states that fully inert subgroups of direct sums of cyclic p-groups are commensurable with fully invariant subgroups, and an example is given of a separable p-group with a fully inert subgroup not commensurable with any fully invariant subgroup. In parallel, in [12] it is proved that a free J_p-module satisfies the property that a fully inert submodule is commensurable with a fully invariant submodule, and an example is furnished of a torsion-free J_p-module not satisfying this property.

This parallel is not complete, since in [12] it is also proved that fully inert submodules of torsion-free J_p-modules which are complete in the p-adic topology are commensurable with fully invariant submodules, while the analogous result for p-groups is not available. The parallel notion of complete torsion-free J_p-modules, in the setting of p-groups, is the notion of torsion-complete p-groups. In fact, these p-groups are complete in the inductive topology or, equivalently, they are isomorphic to the torsion part of the completion in the p-adic topology of any basic subgroup (see [7] and [13]).

The main goal of this paper is to provide this missing parallel result. Thus in Section 3 we will prove the main theorem of this paper, stating that fully inert subgroups of torsion-complete p-groups are commensurable with fully invariant subgroups. The proof of the theorem is split in several parts, depending on the intersection of the fully inert subgroup H with a basic subgroup B. First we prove the theorem when $H \cap B$ is either of finite index in B, or it is finite. When neither of this two cases arise, the two main proofs concern first the case of $H \cap B$ unbounded, and then that of $H \cap B$ bounded. Surprisingly enough, the most challenging case is the latter.

An essential tool in the proof is the analogous result for direct sums of cyclic p-groups. This latter result has a long and elaborate proof divided in two parts, the bounded and the unbounded cases. Unfortunately, there are some gaps in the intermediate results used in the second part of the proof in [11]. Thus in Section 2 we revisit some steps of [11], providing a simplified version of these in the bounded case and a corrected version in the unbounded case.

2. Revisiting fully inert subgroups of direct sums of cyclic p-groups

The standard notation we shall use for a direct sum of cyclic p-groups is:

$$G = \oplus_{n\geq 1} B_n = B_1 \oplus B_2 \oplus \cdots \oplus B_n \oplus B_n^\sharp$$

where $B_n \cong \oplus_{\alpha_n} \mathbb{Z}(p^n)$ for all n (α_n is the n-th Ulm-Kaplansky invariant of G) and $B_n^\sharp = \oplus_{i>n} B_i$, with projections $\pi_i : G \to B_i$ and $\pi_n^\sharp : G \to B_n^\sharp$.

Remark 2.1. In [8, p.168] the subgroup $B_n^\sharp$ is denoted by B_n^*. We have changed this notation, since we will denote by H^* the fully invariant hull of a subgroup H of a group G (see Proposition 2.6).

In [11, Theorem 3.10] it is proved that a fully inert subgroup H of a direct sum of cyclic p-groups G is commensurable with a fully invariant subgroup. The proof is quite elaborate and makes use of sophisticated techniques, both in the bounded and in the unbounded case. Unfortunately, the proof in the unbounded case has some gaps. In this section we furnish a simpler proof for the bounded case, using a result on fully inert subgroups of free groups proved in [6]. Then we simplify the proof for the unbounded case by means of a reduction to box-like subgroups. Recall that a subgroup H of a direct sum of groups $G = \oplus_{i\in I} G_i$ is a *box-like subgroup* if $H = \oplus_{i\in I} H_i$, with $H_i \leq G_i$ for all $i \in I$. Box-like subgroups have been introduced in [4] in the investigation of fully inert subgroups of divisible groups, and used intensively in [2].

The notation used to indicate that two subgroups H and K are commensurable is $H \sim K$. A crucial property of commensurability used throughout paper is that it is transitive, and hence it is an equivalence relation.

2.1. The bounded case

The proof of the next proposition replaces the long proof of [11, Theorems 2.2]; it makes use of a result from [6] on free groups and some general results from [2].

Proposition 2.2. *A fully inert subgroup H of a bounded p-group G is commensurable with a fully invariant subgroup.*

Proof. Step 1. $G = \oplus \mathbb{Z}(p^n)$ is a homogeneous bounded p-group.

Write $G = F/K$, with F a free group and $K = p^n F$, and $H = F'/K$, with F' a free subgroup of F. Every endomorphism ϕ of F induces an endomorphism $\bar{\phi}$ of $F/K = G$, and since H is fully inert in G we have that $(H + \bar{\phi}H)/H \cong (F' + \phi F')/F'$ is finite; therefore F' is fully inert in F. From [6, Theorem 2.8] we derive that $F' \sim mF$ for some integer m, and this implies that $H \sim mG = p^k G$ for some k, as desired.

Step 2. $G = \bigoplus_{1 \le i \le n} \oplus_{\alpha_i} \mathbb{Z}(p^i)$.

Set $\oplus_{\alpha_i} \mathbb{Z}(p^i) = B_i$ and $H_i = H \cap B_i$ for all $i \le n$. By [2, Lemma 2.3] we get that $H \sim H_1 \oplus \cdots \oplus H_n$, and by [4, Proposition 4.2] that each H_i is uniformly fully inert in B_i. By Step 1, $H_i \sim p^{k_i} B_i$ for certain integers $0 \le k_i \le i$. There follows that $H \sim X = p^{k_1} B_1 \oplus \cdots \oplus p^{k_n} B_n$. Now [2, Proposition 2.1] ensures that X is fully inert in G. We claim that this implies that X is commensurable with a fully invariant subgroup.

We can eliminate the B_i's which are finite and prove that the direct sum of the remaining B_j's, which is still fully inert, is fully invariant in G. So, let us assume that $B = B_{c_1} \oplus \cdots \oplus B_{c_r}$, where $c_1 < \cdots < c_r$. Without loss of generality we can assume that all the B_{c_i}'s are countable, since in the general case the argument below is easily adaptable. In view of [11, Lemma 1.6], it is enough to prove that, for all $0 < j < h \le r$:

$$k_{c_j} \le k_{c_h} \le k_{c_j} + c_h - c_j.$$

First assume, by way if contradiction, that $k_{c_h} > k_{c_j} + c_h - c_j$ for some $j < h$, i.e., $c_j - k_{c_j} > c_h - k_{c_h}$. Since $H_{c_h} \cong \oplus \mathbb{Z}(p^{c_h - k_{c_h}})$ and $H_{c_j} \cong \oplus \mathbb{Z}(p^{c_j - k_{c_j}})$, the canonical injection of B_{c_j} into B_{c_h} sends H_{c_j} on an infinite subgroup of B_{c_h} strictly containing H_{c_h}, which is absurd since B is fully inert.

Assume now, by way if contradiction, that $k_{c_h} < k_{c_j}$ for some $j < h$. Then $c_h - k_{c_h} > c_j - k_{c_j}$, hence the canonical surjection of B_{c_h} onto B_{c_j} sends H_{c_h} on an infinite subgroup of B_{c_j} strictly containing H_{c_j}, again absurd since X is fully inert. □

2.2. Reduction to box-like subgroups

In order to provide a correct proof of [11, Theorem 3.10], we restate Lemma 3.3 of [11] in a slight modified version and furnish a new proof of it, since the original statement and its proof made use of an ambiguous notation, namely, $\sum_{i \ge t} \pi_i H$, that may be read either as $\sum_{i \ge t} \pi_i(H)$, or as $(\sum_{i \ge t} \pi_i)(H)$. In our context the correct way is $\sum_{i \ge t} \pi_i(H)$.

Lemma 2.3. *Let H be a fully inert subgroup of a p-group $G = \oplus_{i \in \mathbb{N}} X_i$, $(X_i \ne 0)$, and let $\pi_i : G \to X_i$ $(i \in \mathbb{N})$ be the canonical projections. Then there exists an index $t \in \mathbb{N}$ such that $(\oplus_{i \ge t} \pi_i(H) + H)/H$ is finite.*

Proof. Assume, by way of contradiction, that $(\oplus_{i \ge t} \pi_i(H) + H)/H$ is infinite for all $t \in \mathbb{N}$. We will select inductively a sequence of indices $t_1 < t_2 < t_3 < \cdots$ in $\mathbb{N}$ and a sequence of elements $h_1, h_2, h_3, \cdots$ in H such that the endomorphism $\psi : G \to G$ defined by setting, for each element $(x_i)_{i \in \mathbb{N}} \in \oplus_{i \in \mathbb{N}} X_i$:

$$\psi((x_i)_{i \in \mathbb{N}}) = (x_{t_n})_{n \in \mathbb{N}}$$

satisfies the condition that the elements

$$\psi(h_1) + H,\ \psi(h_2) + H,\ \psi(h_3) + H,\ \cdots$$

are all different. This condition will contradict the hypothesis that $(\psi(H) + H)/H$ is finite.

Fix a $t_1 \in \mathbb{N}$ such that $\pi_{t_1}(H)$ is not contained in H, and an element $\pi_{t_1}(h_1) \in \pi_{t_1}(H) \setminus H$ for some $h_1 \in H$. As the support of h_1 is finite, choose an index $s_1 > t_1$ such that $\pi_t(h_1) = 0$ for all $t \geq s_1$.

As the set $\{\pi_{t_1}(h) + H \mid h \in H\}$ is finite and $(\oplus_{t\geq s_1}\pi_t(H) + H)/H$ is infinite, there are infinitely many elements of the form $\pi_t(h) + H$ $(h \in H)$ with $t \geq s_1$ such that $\pi_{t_1}(h) + \pi_t(h) + H \neq \pi_{t_1}(h_1) + H$. Select one of these elements $\pi_{t_2}(h_2) + H$.

As the set $\{\pi_{t_1}(h) + H,\ \pi_{t_2}(h') + H \mid h, h' \in H\}$ is finite and $(\oplus_{t\geq s_2}\pi_t(H) + H)/H$ is infinite, there are infinitely many elements of the form $\pi_t(h) + H$ $(h \in H)$ with $t \geq s_2$ such that $\pi_{t_1}(h) + \pi_{t_2}(h) + \pi_t(h) + H \neq \pi_{t_1}(h_1) + H, \pi_{t_1}(h_2) + \pi_{t_2}(h_2) + H$. Select one of these elements $\pi_{t_3}(h_3) + H$.

Continuing in this way we obtain the desired sequences, since for each $n \in \mathbb{N}$ we have that $\psi(h_n) = (\pi_{t_1} + \cdots + \pi_{t_n})(h_n)$, since $\pi_t(h_n) = 0$ for all $t \geq t_{n+1}$. $\square$

Thus the statement of [11, Lemma 3.3] is essentially correct, but the next [11, Corollary 3.4] is wrong; it states that, under the hypotheses of Lemma 3.3, there exists an index t such that $H \cap \oplus_{i\geq t} G_i = \oplus_{i\geq t}\pi_i(H)$. The next counter-example shows that this is not true in general.

Example 2.4. Let $G = \oplus_{n\in\mathbb{N}}\langle e_n\rangle$, where $\langle e_n\rangle \cong \mathbb{Z}(p^n)$ for all n, and let $H = \oplus_{n>1}\langle e_n - e_1\rangle$. Then $G/H \cong \mathbb{Z}(p)$, thus H is commensurable with G, hence uniformly fully inert. Clearly $\pi_i(H) = \langle e_i\rangle$ for all $i \in \mathbb{N}$, therefore for every $t \in \mathbb{N}$ we have that $\oplus_{i\geq t}\pi_i(H) = \oplus_{i\geq t}\langle e_i\rangle$ strictly contains its subgroup $H \cap \oplus_{i\geq t}\langle e_i\rangle = \oplus_{i>t}\langle e_i - e_t\rangle$. Note that H is not a box-like subgroup of G with respect to the direct decomposition $G = \oplus_{n\geq 1}\langle e_n\rangle$, but it is box-like if we write G as $G = \langle e_1\rangle \oplus_{n>1} \langle e_n - e_1\rangle$.

The failure of [11, Corollary 3.4] is not serious, since we may reduce to the case of H a box-like subgroup of $\oplus_i B_i$, that is, of the form $\oplus_i H_i$, with $H_i \leq B_i$ for all i, as the next result shows.

Corollary 2.5. *Let H be a fully inert subgroup of an unbounded direct sum of cyclic p-groups $G = \oplus_{i\in\mathbb{N}} B_i$. Then $(\oplus_{i\geq 1}\pi_i(H))/H$ is finite, therefore H is commensurable with a box-like subgroup.*

Proof. By [2, Lemma 2.3],

$$(\pi_1(H) \oplus \cdots \oplus \pi_n(H) \oplus \pi_n^\sharp(H))\ /\ H$$

is finite for every n, where $\pi_n^\sharp : G \to B_n^\sharp = \oplus_{i>n} B_i$ is the canonical surjection; therefore

$$(\pi_1(H) \oplus \cdots \oplus \pi_n(H) + H) \ / \ H$$

is finite for every n.

By Lemma 2.3, there exists a positive integer t such that

$$(\bigoplus_{i \geq t} \pi_i(H) + H)/H$$

is finite. Putting together these two facts, we get that

$$\oplus_{i\geq 1}\pi_i(H)/H$$

is finite. Therefore H is commensurable with the box-like subgroup $\oplus_{i\geq 1}\pi_i(H)$. □

The next result is [11, Proposition 3.5]; we restate it in the present notation and terminology, but we omit the proof that is correctly given in [11]. Recall that the *fully invariant hull* of a subgroup H of a group G, denoted by H^*, is the minimal fully invariant subgroup of G containing H.

Proposition 2.6. *Let $H = \oplus_{i\in I}\pi_i(H)$ be a box-like fully inert subgroup of a p-group $G = \oplus_{i\in I}\langle e_i\rangle$, where $\pi_i : G \to \langle e_i\rangle$ is the canonical projection. Then $H^* = \bigcup_{\phi\in End(G)} \phi(H)$.*

Looking for fully inert subgroups H of unbounded direct sums of cyclic p-groups $G = \oplus_n B_n$, by Corollary 2.5 we can assume, without loss of generality, that

$$H = \oplus_{n\geq 1}\pi_n(H) = \oplus_{n\geq 1}(H \cap B_n)$$

is a box-like subgroup. Now each subgroup $H_n = H \cap B_n$ is fully inert in B_n, but this property is useless if G is semi-standard, i.e., each homogeneous component B_n is finite: in fact, in this situation the subgroups H_n are fully inert in B_n for any box-like subgroup H, so we cannot use the argument applied in Step 2 of the proof of Proposition 2.2. Thus the first case we will consider in the next subsection is that in which G is unbounded but is semi-standard (see Corollary 2.10).

2.3. The general case

First we need the next result, which is essentially [11, Theorem 3.7], but adapted to the box-like setting. As in that theorem, we keep the notation $H^t = \oplus_{i\geq t} H_i$ for $t \geq 1$, and H^{*t} for its fully invariant hull in $B_{t-1}^\sharp = \oplus_{i\geq t} B_i$.; furthermore, if $\psi : \oplus_i B_i \to \oplus_i B_i$ is an endomorphism, we will set $Supp(\psi) = \{i \in \mathbb{N} \mid \psi(B_i) \neq 0\}$.

Lemma 2.7. *Assume that the unbounded direct sum of cyclic p-groups $G = \oplus_{i\in\mathbb{N}} B_i$ contains a box-like fully inert subgroup $H = \oplus_{i\geq 1} H_i$, with $H_i = H \cap B_i$. Then there exists an index $t \in \mathbb{N}$ such that $(H^{*t} + H)/H$ is finite.*

Proof. Assume, by way of contradiction, that $(H^{*t} + H)/H$ is infinite for all $t \in \mathbb{N}$. We will select inductively

- indices $t_1 < t_2 < t_3 < \cdots$ in $\mathbb{N}$
- elements $a_1, a_2, a_3, \cdots$ such that $a_i \in H^{*t_i}$ for all i and $a_i + H \neq a_j + H$ for $i \neq j$
- elements $g_1, g_2, g_3, \cdots$ such that $g_i \in H^{t_i}$ for all i
- endomorphisms $\psi_1, \psi_2, \psi_3, \cdots$ of G such that $\psi_i(g_i) = a_i$ and $Supp(\psi_i)$ contained in $B_{t_i} \oplus B_{t_i+1} \oplus \cdots \oplus B_{t_{i+1}-1}$ for all i.

Pick arbitrary $t_1 \in \mathbb{N}$ and $a_1 \in H^{*t_1} \setminus H$. By Proposition 2.6, $H^{*t_1} = \bigcup_{\phi\in End(B^\sharp_{t_1-1})} \phi(H^{t_1})$, therefore there exist an element $g_1 \in H^{t_1}$ and an endomorphism ϕ_1 of $B^\sharp_{t_1-1}$ such that $\phi_1(g_1) = a_1$. If $g_1 \in B_{t_1} \oplus B_{t_1+1} \oplus \cdots \oplus B_{t_2-1}$ for some $t_2 > t_1$, let ψ_1 be the endomorphism of G which coincides with ϕ_1 on $B_{t_1} \oplus B_{t_1+1} \oplus \cdots \oplus B_{t_2-1}$, and vanishes elsewhere.

As $(H^{*t_2} + H)/H$ is supposed to be infinite, there exists $a_2 \in H^{*t_2}$ such that $a_2 + H \neq a_1 + H$. By Proposition 2.6, there exist an element $g_2 \in H^{t_2}$ and an endomorphism ϕ_2 of $B^\sharp_{t_2-1}$ such that $\phi_2(g_2) = a_2$. If $g_2 \in B_{t_2} \oplus B_{t_2+1} \oplus \cdots \oplus B_{t_3-1}$ for some $t_3 > t_2$, let ψ_2 be the endomorphism of G which coincides with ϕ_2 on $B_{t_2} \oplus B_{t_2+1} \oplus \cdots \oplus B_{t_3-1}$, and vanishes elsewhere. Note that, by construction, $Supp(\psi_1) \cup Supp(\psi_2) = \emptyset$.

Continuing in this way, we obtain the desired families of indices, elements and endomorphisms. Now define the endomorphism $\psi : G = \oplus_{i\in\mathbb{N}} B_i \to G = \oplus_{i\in\mathbb{N}} B_i$ as follows: if $x = (x_i)_{i\in\mathbb{N}} \in G$ ($x_i \in B_i$ for all i), set

$$\psi(x) = \sum_{t_n \leq i < t_{n+1}} \psi_n(x_i).$$

This map is well defined since almost all the coordinates of x are zero and the supports of the maps ψ_n are mutually disjoint. In this way we reach the desired contradiction, as $(\psi(H) + H)/H$ contains the infinite set $\psi(g_i) + H = \psi_i(g_i) + H = a_i + H$, $(i \in \mathbb{N})$. □

The next result is essentially [11, Corollary 3.8], which however uses an incorrect argument when it derives that $(H^{*s} + H)/H = 0$ for some $s \geq t$ from the fact that $\bigcap_{n\geq t} H^{*n} = 0$. Our proof uses a different argument.

Corollary 2.8. *Let $H = \oplus_{i\geq 1} H_i$ be a box-like fully inert subgroup of the unbounded direct sum of cyclic p-groups $G = \oplus_{i\geq 1} B_i$. Then there exists an index $s \in \mathbb{N}$ such that $H^s = \oplus_{i\geq s} H_i$ is fully invariant in $B^\sharp_{s-1} = \oplus_{i\geq s} B_i$.*

Proof. From Lemma 2.7 we know that $(H^{*t} + H)/H$ is finite for a $t \in \mathbb{N}$. This implies that the descending chain: $(H^{*t} + H)/H \geq (H^{*t+1} + H)/H \geq (H^{*t+2} + H)/H \geq \cdots$ is stationary. Now we show that $\bigcap_{i\in\mathbb{N}}(H^{*i} + H) = H$. Let $x = b_1 + \cdots + b_m \in \bigcap_{i\in\mathbb{N}}(H^{*i} + H)$, with $b_j \in B_j$ for all $j \leq m$. Since $H^{*m} + H = H_1 \oplus \cdots \oplus H_m \oplus H^{*m}$, we get that $x = h_1 + \cdots + h_m + h^{*m}$, with $h_j \in h_j$ for all $j \leq m$ and $h^{*m} \in H^{*m}$. Comparing the two expressions of x we deduce that $h^{*m} = 0$, consequently $x \in H$. The preceding arguments show that there exists an $s \in \mathbb{N}$ such that $H^{*s} \leq H \cap B^{\sharp}_{s-1} = H^s$, therefore H^s is fully invariant in $B^{\sharp}_{s-1}$. □

As in [11, Theorem 3.9], one can easily deduce from Corollary 2.8 the main result for semi-standard direct sums of cyclic p-groups.

Theorem 2.9. *A fully inert subgroup H of a semi-standard direct sum of cyclic p-groups G is commensurable with a fully invariant subgroup.*

Proof. By Corollary 2.5 we can assume that H is box-like, thus $H = H_1 \oplus \cdots \oplus H_s \oplus H^s$. From Corollary 2.8 we know that H^s is fully invariant in $B^{\sharp}_{s-1}$ for a certain $s \in \mathbb{N}$. Then [11, Lemma 1.5] ensures that there exists a subgroup C of $B_1 \oplus \cdots \oplus B_s$ such that $C \oplus H^s$ is fully invariant in G. But the semi-standard hypothesis implies that C is finite; since H^s has also finite index in H, we conclude that H is commensurable with the fully invariant subgroup $C \oplus H^s$. □

Now the proof in the general case goes as in [11, Theorem 3.10], using as main ingredient Corollary 2.8. So the main result is the following theorem, whose proof is only sketched. The reader interested in the details is referred to the original proof.

Theorem 2.10. *A fully inert subgroup H of a direct sum of cyclic p-groups G is commensurable with a fully invariant subgroup.*

Sketch of the proof. The difference with respect to the proof of the semi-standard case of Theorem 2.10 is that $H_1 \oplus \cdots \oplus H_s$, which is fully inert in $B_1 \oplus \cdots \oplus B_s$, is no longer finite; however, in view of the solution of the bounded case, $H_1 \oplus \cdots \oplus H_s$ is commensurable with a fully invariant subgroup L of $B_1 \oplus \cdots \oplus B_s$. Clearly H is commensurable with $L \oplus H^s$. The proof of [11, Theorem 3.10] consists in showing that $L \oplus H^s$ is commensurable with a fully invariant subgroup of G using [11, Lemma 1.6], which characterizes fully invariant subgroups of direct sums of cyclic p-groups (recall that [11, Lemma 1.6] rephrases [1, Theorem 2.8]). □

3. Fully inert subgroups of torsion-complete p-groups

In this section we prove a result for fully inert subgroups of torsion-complete p-groups analogous to the results proved in [11] and [12] for direct sums of cyclic p-groups and for complete torsion-free J_p-modules, respectively. The proof in the case of p-groups is more technical, due to the fact that fully invariant subgroups of separable p-groups have a more complex structure than fully invariant submodules of torsion-free J_p-modules.

Let G be a separable p-group with basic subgroup $B = \oplus_{n\geq 1} B_n$, where $B_n \cong \oplus_{\alpha_n} \mathbb{Z}(p^n)$. Recall that the exact sequence $0 \to B \to G \to G/B \to$ is pure, and that $G/B \cong \oplus \mathbb{Z}(p^\infty)$ is divisible. This exact sequence gives rise to the long contravariant exact sequence:

$$0 \to Hom(G/B, G) \to Hom(G, G) \to Hom(B, G) \to Pext(G/B, G) \to \cdots$$

where the first Hom is zero, since G is assumed to be reduced, and $Pext$ denotes the group of the pure-exact sequences. It is a basic result in homological algebra for p-groups that the group G is torsion-complete if and only if $Pext(\mathbb{Z}(p^\infty), G) = 0$ (see [8, Corollary 3.7, p. 314]). Therefore, if G is such a group, we have the isomorphism $Hom(G, G) \cong Hom(B, G)$ induced by the restriction, thus every homomorphism $\phi : B \to G$ extends uniquely to an endomorphism of G; this fact holds, in particular, for every endomorphism of B.

From now on, we will identify a reduced unbounded torsion-complete p-group with basic subgroup B with $\overline{B}$, the torsion subgroup of the p-adic completion of B. Both B and $\overline{B}$ are separable groups, hence, they are fully transitive; therefore, by Kaplansky's theorem [8, Theorem 2.2, p. 307], the fully invariant subgroups of B and $\overline{B}$ are of the form $B(\mathbf{u})$ and $\overline{B}(\mathbf{u})$, respectively, for some increasing sequence of non-negative integers or symbols ∞: $\mathbf{u} = (r_0, r_1, r_2, \cdots)$ (see [8, Chapter 10, Section 2] for the notion of fully transitive group G and for the definition of the subgroups $G(\mathbf{u})$). As B and $\overline{B}$ are unbounded, the non-zero subgroups of the form $B(\mathbf{u})$ and $\overline{B}(\mathbf{u})$ are infinite.

3.1. Preliminary results

We prove now some results on commensurable subgroups of an arbitrary group, on the existence of maps from unbounded direct sums of cyclics, and on fully inert subgroups of torsion-complete groups that will be used in this section.

Lemma 3.1. *Let $H \sim K$ be commensurable subgroups of a group G, and let L a subgroup of G. Then $H \cap L \sim K \cap L$.*

Proof. We have the following relations:

$$((H \cap L) + (K \cap L))/(K \cap L) \cong (H \cap L)/(H \cap L \cap K) \cong ((H \cap L) + K)/K \leq (H + K)/K.$$

As the last quotient is finite, so too is the first. The same relations hold replacing H and K, so both $H \cap L$ and $K \cap L$ have finite index in $(H \cap L) + (K \cap L)$, and we are done. $\square$

Lemma 3.2. *If B is an infinite direct sum of cyclic p-groups and Y is an unbounded p-group, then there exists a homomorphism $\sigma : B \to Y$ such that $\sigma(B[p])$ is infinite.*

Proof. Let $B = \oplus_{i\in I}\langle e_i\rangle$ and let $B' = \oplus_{j\in J}\langle e'_j\rangle$ be a basic subgroup of Y. Select a sequence of cyclic summands $\langle e_{i_n}\rangle$ ($n \in \mathbb{N}$) of B and a sequence of cyclic summands $\langle e'_{j_n}\rangle$ ($n \in \mathbb{N}$) of B' of strictly increasing order, such that the order of e'_{j_n} is greater than or equal to the order of e_{i_n}. Then define σ by embedding each e_{i_n} into e'_{j_n}, and sending to zero all the remaining generators e_i. Clearly $\sigma(B[p]) = \oplus_{n\in\mathbb{N}}\langle e'_{j_n}\rangle[p]$, which is infinite. □

The next result exhibits some simple consequences of full inertia which will be useful in our approach. The results are well known but we include the simple proof for the convenience of the reader.

Lemma 3.3. *(i) If A is a subgroup of the group G having the property that every endomorphism of A extends to an endomorphism of G, then if H is fully inert in G, $H \cap A$ is fully inert in A.*

(ii) If H is a fully inert subgroup of $G = X \oplus Y$, then

$$(H \cap X) \oplus (H \cap Y) \leq H \leq \pi_X(H) \oplus \pi_Y(H),$$

where π_X, π_Y are the canonical projections of G onto X, Y respectively. Furthermore, $(H \cap X) \oplus (H \cap Y) \sim H \sim \pi_X(H) \oplus \pi_Y(H)$.

Proof. Since H is fully inert in G, the quotient $(H+\phi(H))/H$ is finite for all $\phi \in \mathrm{End}(G)$. Now if ψ is an arbitrary endomorphism of A, there is, by assumption, an endomorphism ϕ of G with $\phi \restriction A = \psi$ and so $((H \cap A) + \psi(H \cap A))/H \cap A \leq ((H + \phi(H)) \cap A)$ and since the RHS is isomorphic to a subgroup of the finite group $(H + \phi(H))/H$, the LHS is also finite. Since ψ was arbitrary, $H \cap A$ is fully inert in A.

It follows from part (i) that $H \cap X, H \cap Y$ are fully inert in G and clearly $(H \cap X) \oplus (H \cap Y) \leq H \leq \pi_X(H) \oplus \pi_Y(H)$. To complete the proof it suffices to show that the quotient $\pi_X(H) \oplus \pi_Y(H)/(H \cap X) \oplus (H \cap Y)$ is finite.

Now $H \cap X = H \cap \pi_X(H)$ and $H \cap Y = H \cap \pi_Y(H)$, so that $\pi_X(H)/(H \cap X) \cong (H + \pi_X(H))/H$ is finite since H is fully inert in G; similarly Y. It then follows immediately that $\pi_X(H) \oplus \pi_Y(H)/(H \cap X) \oplus (H \cap Y)$ is also finite. □

Lemma 3.3(i) enables us to derive from [11, Theorem 3.10] or Theorem 2.10 above that $H \cap B$ is commensurable with a fully invariant subgroup of B. Thus, under the hypothesis of Lemma 3.3 we derive that $H \cap B \sim B(\mathbf{u})$ for some sequence $\mathbf{u}$; in particular, the two extreme cases may happen, that is, $H \cap B \sim \{0\}$ (i.e., $H \cap B$ is finite), or $H \cap B \sim B$ (i.e., $H \cap B$ has finite index in B).

The next result presents a sufficient condition and a necessary condition in order that $H \cap B$ is commensurable with a non-zero fully invariant subgroup $B(\mathbf{u})$.

Proposition 3.4. (1) *If H is a subgroup of $\overline{B}$ which is commensurable with the non-zero fully invariant subgroup $\overline{B}(\mathbf{u})$, then $H \cap B$ is commensurable with $B(\mathbf{u})$.*

(2) *If* $H \cap B$ *is commensurable with* $B(\mathbf{u})$*, then* H *is commensurable with the subgroup* $H' = H + B(\mathbf{u})$ *of* $\overline{B}$ *such that* $(H' \cap B)/B(\mathbf{u})$ *is finite.*

Proof. (1) Since $\overline{B}(\mathbf{u}) \cap B = B(\mathbf{u})$, the result follows immediately from Lemma 3.1.

(2) From $H' = H + B(\mathbf{u})$ we have $(H' \cap B)/B(\mathbf{u}) = ((H + B(\mathbf{u})) \cap B)/B(\mathbf{u}) = ((H \cap B) + B(\mathbf{u}))/B(\mathbf{u})$, and the last quotient is finite as $H \cap B \sim B(\mathbf{u})$. To prove that $H \sim H' = H + B(\mathbf{u})$ it is enough to show that $(H + B(\mathbf{u}))/H$ is finite. But $(H + B(\mathbf{u}))/H \cong B(\mathbf{u})/(H \cap B(\mathbf{u})) = B(\mathbf{u})/(H \cap B \cap B(\mathbf{u}))$, where again the last quotient is finite as $H \cap B \sim B(\mathbf{u})$. □

Remark 3.5. A useful consequence of Proposition 3.4 is that we may replace a fully inert subgroup H of $\overline{B}$ such that $H \cap B \sim B(\mathbf{u})$ by a subgroup $H_1 \geq H$ with the quotient H_1/H finite and $B(\mathbf{u})$ a subgroup of finite index in $H_1 \cap B$. Note that H_1 will again be fully inert in $\overline{B}$, as $H_1 \sim H$, by [2, Proposition 2.1]. Consequently, we shall often find it convenient to assume that we are working with a fully inert subgroup H of $\overline{B}$ with $B(\mathbf{u})$ of finite index in $H \cap B$.

3.2. The two extreme cases

We start considering the two extreme cases of $H \cap B \sim B$ (that is, $H \cap B$ has finite index in B), and $H \cap B \sim 0$ (that is, $H \cap B$ is a finite subgroup). These are particular cases of when $H \cap B$ is unbounded or bounded, respectively, that are discussed in the next subsections. But we think that it could be of interest to have easier and direct proofs of these special cases.

Lemma 3.6. *Let* H *be a fully inert subgroup of the reduced unbounded torsion complete* p*-group* $\overline{B}$ *such that* $H \cap B$ *is commensurable with* B*. Then* H *is commensurable with* $\overline{B}$*, that is,* $\overline{B}/H$ *is finite.*

Proof. Since $B/(H \cap B) \cong (H + B)/H$ is finite, there exists a finite subgroup F of B such that $H + B = H + F$. Set $H' = H + F$. As H' is commensurable with H, it is fully inert in $\bar{B}$ (see the Remark 3.5), and since $B \leq H'$, $\overline{B}/H'$ is divisible. We shall prove that $H' = \overline{B}$. Assume by way of contradiction that $\overline{B}/H' \neq 0$. Then a copy of $\mathbb{Z}(p^\infty)$ is contained in $\overline{B}/H'$, so there exists a sequence of distinct cosets $\{x_n + H'\}_{n \in \mathbb{N}}$ $(x_n \in \overline{B})$. Write B as a direct sum of cyclic p-groups $\langle e_i \rangle$, with i ranging over a suitable index set I. The fact that B is unbounded ensures that for each $n \in \mathbb{N}$ there exists a generator e_{i_n} of B of order greater than the order of x_n. Therefore we can define a homomorphism $\phi : \oplus_{n \in \mathbb{N}} \langle e_{i_n} \rangle \to \overline{B}$, by sending each e_{i_n} to x_n. Extend this map to a homomorphism $\psi : B \to \overline{B}$ by sending e_i to 0 for all $i \neq i_n$. Then extend ψ to an endomorphism $\overline{\psi}$ of $\overline{B}$. Then $(H' + \overline{\psi}(H'))/H'$ contains the infinite sequence of distinct cosets $x_n + H'$ $(n \in \mathbb{N})$, contradicting the fact that H' is fully inert. Therefore $\overline{B} = H'$ and H has finite index in $\overline{B}$. □

Consider now the case when $H \cap B$ is finite.

Lemma 3.7. *Let H be a fully inert subgroup of the reduced unbounded torsion complete p-group $\overline{B}$ such that $H \cap B$ is finite. Then H is finite.*

Proof. Assume for a contradiction, that H is infinite. We define a homomorphism $\sigma : \overline{B} \to B$ such that $\sigma(H)$ is infinite. This will give the desired contradiction, since $H \cap \sigma(H)$, being contained in $H \cap B$, is finite, therefore $(H + \sigma(H))/H \cong \sigma(H)/(H \cap \sigma(H))$ would be infinite, contradicting the full inertia of H in $\overline{B}$.

The assumption that H is infinite implies that the socle $H[p]$ is infinite. As $\overline{B}$ is pure-complete, this socle supports a pure subgroup K of $\overline{B}$, by [8, Lemma 5.4, p. 322], and the closure $\overline{K}$ of K in the p-adic topology is a summand of $\overline{B}$, by [8, Corollary 3.9, p. 315].

If K is bounded, then it is an infinite direct sum of cyclic groups and it follows from Lemma 3.2 that there is a homomorphism $\phi : K = \overline{K} \to B$ such that $\phi(K[p]) = \phi(H[p])$ is infinite and this mapping extends to an endomorphism σ of $\overline{B}$. Since $\sigma(H) \geq \sigma(H[p]) = \phi(H[p])$, $\sigma(H)$ is infinite and we are done.

Assume now that K is unbounded, and let B' be a basic subgroup of K such that $\overline{K} = \overline{B}'$. The socle $B'[p]$ is infinite and it is contained in $K[p] = H[p]$. It again follows from Lemma 3.2 that there is a homomorphism $\psi : B' \to B$ such that $\psi(B'[p])$ is infinite and this extends to a homomorphism from $\overline{K}$ to $\overline{B}$, which in turn can be extended to an endomorphism σ of $\overline{B}$, since, as noted above, $\overline{K}$ is a summand of $\overline{B}$. Then as $\sigma(H) \geq \sigma(H[p]) \geq \psi(B'[p])$, we have that $\sigma(H)$ is infinite, as required. $\square$

3.3. The unbounded case

Our goal now is to prove the converse of item (1) in Proposition 3.4 for a fully inert subgroup H, starting with the unbounded case.

A significant simplification in our arguments in the unbounded situation arises from the fact that if $\overline{B}(\mathbf{u})$ is an unbounded fully invariant subgroup of $\overline{B}$, then $\overline{B} = \overline{B}(\mathbf{u}) + B$. This is because $\overline{B}(\mathbf{u})$ is then a large subgroup of $\overline{B}$ in the sense of Pierce - see, for example, the discussion in [7, Chapter 10, Section 2].

Proposition 3.8. *Let H be a fully inert subgroup of a reduced torsion complete p-group $\overline{B}$ such that $H \cap B$ is unbounded for some basic subgroup B. If $H \cap B \sim B(\mathbf{u})$, then $H \sim \overline{B}(\mathbf{u})$.*

Proof. As noted in the Remark 3.5, we can replace H by the commensurable subgroup $H' = H + B(\mathbf{u})$, which contains $B(\mathbf{u})$ and has the property that $(H' \cap B)/B(\mathbf{u})$ is finite: in fact, $(H + B(\mathbf{u}))/H \cong B(\mathbf{u})/(B(\mathbf{u}) \cap H) = B(\mathbf{u})/(B(\mathbf{u}) \cap H \cap B)$ is finite. The subgroup H' is still fully inert in $\overline{B}$ and clearly $H \sim \overline{B}(\mathbf{u})$ if and only if $H' \sim \overline{B}(\mathbf{u})$. Therefore we can identify H with H' and assume, from now on, that $B(\mathbf{u})$ is contained in $H \cap B$ as a subgroup of finite index. We will show that, with this identification, H contains $\overline{B}(\mathbf{u}))$ as a subgroup of finite index, which gives the desired result.

Now $(H + \overline{B}(\mathbf{u}))/H \cong \overline{B}(\mathbf{u})/(H \cap \overline{B}(\mathbf{u}))$, which is an epimorphic image of $\overline{B}(\mathbf{u}))/B(\mathbf{u}))$, since $B(\mathbf{u})) \leq H \cap \overline{B}(\mathbf{u})$. But we have:

$$\overline{B}(\mathbf{u})/B(\mathbf{u}) = \overline{B}(\mathbf{u}))/(\overline{B}(\mathbf{u}) \cap B) \cong (\overline{B}(\mathbf{u}) + B)/B = \overline{B}/B,$$

since, as noted above $\overline{B}(\mathbf{u}) + B = \overline{B}$ in this unbounded situation. Therefore $\overline{B}(\mathbf{u})/B(\mathbf{u})$ is divisible, and consequently so too is its epimorphic image $(H + \overline{B}(\mathbf{u}))/H$. Imitating the proof of Lemma 3.6, we will prove that the factor group $(H + \overline{B}(\mathbf{u}))/H$ is zero. If not, $(H + \overline{B}(\mathbf{u}))/H$ contains a copy of $\mathbb{Z}(p^\infty)$, hence there exists a sequence of distinct cosets $\{x_n + H\}_{n \in \mathbb{N}}$ $(x_n \in \overline{B}(\mathbf{u}))$.

The subgroup $B(\mathbf{u})$ can be expressed as a direct sum $\bigoplus \langle p^{a_i} e_i \rangle$ where the e_i are generators of B; clearly it suffices to consider a countable collection of these. Now assuming the order of e_i is p^{b_i}, note that $B(\mathbf{u})$ unbounded means that $b_i - a_i$ tends to infinity with i. So given any n, there is an i_n such that b_{i_n} is greater than the order of $x_{n+a_{i_n}}$. Now define $\phi : B \to \overline{B}(\mathbf{u})$ by sending e_{i_n} to $x_{n+a_{i_n}}$ and mapping everything else to 0. This means that the image of $p^{a_{i_n}} e_{i_n}$ will be $p^{a_{i_n}} x_{n+a_{i_n}}$ and modulo H one will just pick up the images x_n using the usual relations for $\mathbb{Z}(p^\infty)$.

Extend ϕ to an endomomorphism ψ of $\overline{B}$. Then $(H + \overline{\psi}(H))/H$ contains $(H + \psi(B(\mathbf{u})))/H$, and consequently it contains the infinite sequence $x_n + H$ $(n \in \mathbb{N})$, contradicting the fact that H is fully inert. Therefore $(H + \overline{B}(\mathbf{u}))/H = 0$, or, equivalently, $H \geq \overline{B}(\mathbf{u}))$.

Finally, using the inclusion $H \geq \overline{B}(\mathbf{u}))$, we have:

$$(H \cap B)/B(\mathbf{u}) = (H \cap B)/(\overline{B}(\mathbf{u}) \cap B) \cong ((H \cap B) + \overline{B}(\mathbf{u})))/\overline{B}(\mathbf{u})) =$$
$$(H \cap (B + \overline{B}(\mathbf{u})))/\overline{B}(\mathbf{u})) = (H \cap \overline{B})/\overline{B}(\mathbf{u})) = H/\overline{B}(\mathbf{u})).$$

Since the first term is finite, so too is the last term, therefore $H \sim \overline{B}(\mathbf{u})$. $\square$

Corollary 3.9. *Let H be a fully inert subgroup of a reduced torsion complete p-group $\overline{B}$ such that $H \cap B$ is unbounded (respectively, bounded) for some basic subgroup B. Then $H \cap B'$ is unbounded (respectively, bounded) for any other basic subgroup B' of $\overline{B}$.*

Proof. By Proposition 3.8, if $H \sim B(\mathbf{u})$ is unbounded, H is commensurable with the unbounded subgroup $\overline{B}(\mathbf{u})$, so $H \cap B' \sim \overline{B}(\mathbf{u}) \cap B' = B'(\mathbf{u})$, which is unbounded, hence $H \cap B'$ is unbounded too. The proof for the bounded case follows immediately. $\square$

More information on the fully inert unbounded subgroup H is furnished in the following proposition.

Proposition 3.10. *If H is a fully inert unbounded subgroup of $\overline{B}$, then*

(1) $\overline{B} = X \oplus Z$, *where* $p^k Z = 0$ *for some* $k \geq 0$ *and* $\overline{B}[p] = H[p] \oplus Z[p]$*;*
(2) H *is commensurable with a subgroup* H' *of* X *such that* $p^k \overline{B}[p] \leq H'$*.*

Proof. (1) We have $\overline{B}[p] = H[p] \oplus S$ for some subsocle S. If $S = 0$, then our claim is true for $Z = 0$ and $X = \overline{B}$. If $S \neq 0$, then S supports a pure subgroup Z of the torsion-complete group $\overline{B}$. We shall prove that Z is bounded. Assume, by way of contradiction, that Z is unbounded.

The socle $H[p]$ supports a pure unbounded subgroup K of $\overline{B}$, which has an unbounded basic subgroup C, say. Then, applying Lemma 3.2, we get a homomorphism $\sigma : C \to Z \leq \overline{B}$ such that $\sigma(C[p])$ is infinite. Recall that the closure, $\overline{K}$, of K in $\overline{B}$ in the p-adic topology is a direct summand of $\overline{B}$.

From the pure-exact sequence $0 \to C \to \bar{K} \to D \to 0$, where D is a divisible p-group, we derive the exact sequence $Hom(\overline{K}, \overline{B}) \to Hom(C, \overline{B}) \to Pext^1(D, \overline{B})$. Since the last term is zero, we can extend σ to a homomorphism from $\overline{K}$ to $\overline{B}$, and then to an endomorphism ϕ of $\overline{B}$ by sending a complement of $\overline{K}$ to zero. Then, noting that $H[p] \cap Z = 0$, we get:

$$(H + \phi(H))/H \geq (H + \phi(H[p]))/H \geq (H \oplus \sigma(C[p]))/H \cong \sigma(C[p])/(H \cap \sigma(C[p])$$

that gives the desired contradiction, since $\sigma(C[p])$ is infinite and $H \cap \sigma(C[p]) \leq H[p] \cap Z = 0$. Therefore Z is bounded, i.e., $p^k Z = 0$ for some integer $k \geq 0$. As Z is pure, it is a direct summand, hence $\overline{B} = X \oplus Z$ for some complement X.

(2) Obviously $\pi_X(H) + H = \pi_X(H) \oplus \pi_Z(H)$, where $\pi_Z : \overline{B} \to Z$ is the canonical projection of $\overline{B}$ onto Z. We have the inclusions

$$(H \cap \pi_X(H)) \oplus (H \cap \pi_Z(H)) \leq H \leq \pi_X(H) \oplus \pi_Z(H) = \pi_X(H) + H.$$

As H is fully inert, $(\pi_X(H) + H)/H$ is finite. We prove now that $(\pi_X(H) + H)/\pi_X(H)$ is also finite.

Noting that $\pi_X(H) \cap H = H \cap X$ and that $H/H \cap X \cong \pi_Z(H)$, we have

$$(\pi_X(H) + H)/\pi_X(H) \cong H/(H \cap \pi_X(H)) = H/(H \cap X) \cong \pi_Z(H) \cong (H \oplus \pi_Z(H))/H$$

where the last term is finite since H is fully inert and $H[p] \cap Z[p] = 0$ implies $H \cap Z = 0$. Thus we have proved that $H \sim \pi_X(H)$. Furthermore, noting that $p^k Z = 0$, we have:

$$\begin{aligned} p^k \overline{B}[p] &= (p^k X \oplus p^k Z) \cap (H[p] \oplus Z[p]) \leq p^k X \cap (\pi_X(H[p]) \oplus Z[p]) \\ &\leq p^k X \cap \pi_X(H[p]) \leq \pi_X(H). \end{aligned}$$

Setting $H' = \pi_X(H)$ we get the desired conclusion. □

3.4. The bounded case

The proof of the bounded case may be considerably simplified if we can replace $B(\mathbf{u})$ by $B[p^k]$, that is, if we may assume that the sequence $\mathbf{u}$ is of the form $(0, 1, 2, \cdots, k-1, \infty, \cdots)$. So we start the bounded case by performing this reduction.

Let $B = \oplus_{i\geq 1} B_i$, where $B_i \cong \oplus_{\alpha_i} \mathbb{Z}(p^i)$ for all i (α_i is the i-th Ulm-Kaplansky invariant of B). As recalled in Section 2, for every $n \geq 1$ we have the direct decomposition

$$B = B_1 \oplus B_2 \oplus \cdots \oplus B_n \oplus B_n^\sharp$$

where $B_n^\sharp = \oplus_{i>n} B_i$. Let $\pi_n^\sharp : B \to B_n^\sharp$ and $\tau_n : B \to B_1 \oplus B_2 \oplus \cdots \oplus B_n$ be the canonical projections. Then

$$\overline{B} = B_1 \oplus B_2 \oplus \cdots \oplus B_n \oplus \overline{B}_n^\sharp$$

where $\overline{B}_n^\sharp$ is the torsion-completion of $B_n^\sharp$. Accordingly with the above notation, let $\overline{\tau}_n : \overline{B} \to B_1 \oplus B_2 \oplus \cdots \oplus B_n$ and $\overline{\pi}_n^\sharp : \overline{B} \to \overline{B}_n^\sharp$ be the canonical projections.

Lemma 3.11. *Let $G \cong \oplus_{i\in I} \langle e_i \rangle$, where $\langle e_i \rangle \cong \mathbb{Z}(p^n)$ for all $i \in I$, and let $\mathbf{u}$ be the sequence $(r_0, r_1, \cdots, r_{k-1}, \infty, \cdots)$. If $n \geq r_{k-1} + k$, then $G(\mathbf{u}) = G[p^k]$.*

Proof. As the inclusion $G(\mathbf{u}) \leq G[p^k]$ is trivially true, we prove the converse inclusion. If $x \in G[p^k]$, then $p^k x = 0$ implies that $x = p^r y$ for a suitable $y \in G$, and an integer $r \geq n - k$. But then $r \geq r_{k-1}$, therefore obviously $x \in G(\mathbf{u})$. □

Corollary 3.12. *In the notation above, if $n \geq r_{k-1} + k$, then*

$$B(\mathbf{u}) = (B_1 \oplus B_2 \oplus \cdots \oplus B_n)(\mathbf{u}) \oplus B_n^\sharp[p^k] \quad , \quad \overline{B}(\mathbf{u}) = (B_1 \oplus B_2 \oplus \cdots \oplus B_n)(\mathbf{u}) \oplus \overline{B}_n^\sharp[p^k].$$

Proof. The first equality is an immediate consequence of the equality $B(\mathbf{u}) = (B_1 \oplus B_2 \oplus \cdots \oplus B_n)(\mathbf{u}) \oplus B_n^\sharp(\mathbf{u})$ and of the fact that, by Lemma 3.11, $B_i(\mathbf{u}) = B_i[p^k]$ for each $i > n$. To prove the second equality, it is enough to show that $\overline{B}_n^\sharp(\mathbf{u}) = \overline{B}_n^\sharp[p^k]$. We may use the same proof as in Lemma 3.11. The inclusion $\overline{B}_n^\sharp(\mathbf{u}) \leq \overline{B}_n^\sharp[p^k]$ is trivially true. For the converse inclusion, let $x \in \overline{B}_n^\sharp[p^k]$. Since $\overline{B}_n^\sharp[p^k] = \prod_{i>n} B_i[p^k]$, $p^k x = 0$ implies that $x = p^r y$ for a suitable $y \in \overline{B}_n^\sharp$, and an integer $r \geq n - k$. But then $r \geq r_{k-1}$, therefore obviously $x \in \overline{B}_n^\sharp(\mathbf{u})$. □

Let H be a fully inert subgroup of $\overline{B}$ and assume that $H \cap B \sim B(\mathbf{u})$, where $\mathbf{u}$ is the sequence $(r_0, r_1, \cdots, r_{k-1}, \infty, \cdots)$. Fix a positive integer $n \geq r_{k-1} + k$.

Let $H_1 = H \cap (B_1 \oplus B_2 \oplus \cdots \oplus B_n)$, $H_2 = H \cap B_n^\sharp$ and $\overline{H}_2 = H \cap \overline{B}_n^\sharp$. Then Lemma 3.3(ii) ensures that $\overline{\tau}_n(H)/H_1$ and $\overline{\pi}_n^\sharp(H)/\overline{H}_2$ are finite, and we have the inclusions

$$H_1 \oplus \overline{H}_2 \leq H \leq \overline{\tau}_n(H) \oplus \overline{\pi}_n^\sharp(H).$$

So, for our purposes, since H is commensurable with both $H_1 \oplus \overline{H}_2$ and $\overline{\tau}_n(H) \oplus \overline{\pi}_n^\sharp(H)$, we may replace H by each one of these fully inert subgroups, that is, we may assume that

$$H_1 \oplus \overline{H}_2 = H = \overline{\tau}_n(H) \oplus \overline{\pi}_n^\sharp(H).$$

Then $H \cap B = H_1 \oplus H_2 \sim B(\mathbf{u}) = (B_1 \oplus B_2 \oplus \cdots \oplus B_n)(\mathbf{u}) \oplus B_n^\sharp[p^k]$, hence $H_1 \sim (B_1 \oplus B_2 \oplus \cdots \oplus B_n)(\mathbf{u})$ and $H_2 \sim B_n^\sharp[p^k]$. The consequence of this fact is that, if we want to prove that $H \sim \overline{B}(\mathbf{u})$, it is enough to prove that

$$H_2 \sim B_n^\sharp[p^k] \Rightarrow \overline{H}_2 \sim \overline{B}_n^\sharp[p^k].$$

More generally, we deduce that it is enough to prove that for the fully inert subgroup H of $\overline{B}$ the following implication holds

$$H \cap B \sim B[p^k] \Rightarrow H \sim \overline{B}[p^k].$$

Lemma 3.13. *Let H be a fully inert subgroup of a reduced unbounded torsion complete p-group $\overline{B}$ such that $H \cap B$ is bounded. If $H \cap B \sim B[p^k]$, then $H[p]$ and $\overline{B}[p]$ are commensurable.*

Proof. As noted in Remark 3.5, we will assume that $H \cap B$ contains $B[p^k]$ as a subgroup of finite index. We will prove that $H[p]$ is of finite index in $\overline{B}[p]$. Let $\overline{B}[p] = H[p] \oplus X'$ and suppose for a contradiction that X' is infinite.

Choose a countable set of basis elements $X = \{x_1, x_2, \dots\} \subseteq X'$ and form the set X into a disjoint union of subsets $X^1, X^2, \dots, X^n, \dots$ (there may be infinitely many subsets or just finitely many), where the elements of a given X^n have the property that each $x \in X^n$ has height s_n in $\overline{B}$ for some fixed integer $s_n \geq 0$, and $s_1 < s_2 < \cdots$. Then the subgroups generated by the sets X^i consist of direct sums of the form $\mathcal{X}^i = \bigoplus_{j \in J_i} \langle x_{ij} \rangle$ for some (possibly infinite) index set J_i; note that $\langle X \rangle = \bigoplus_{i \geq 1} \mathcal{X}^i$. Now each x_{ij} gives rise to an element y_{ij} of $\overline{B}$ with $x_{ij} = p^{s_i} y_{ij}$ and each subgroup $\langle y_{ij} \rangle$ is a direct summand of $\overline{B}$. Set $\mathcal{Y}^i = \bigoplus_{j \in J_i} \langle y_{ij} \rangle$, a summand of $\overline{B}$. Note that $|J_i| \leq \alpha_{s_i}(\overline{B})$ (the s_ith Ulm invariant of $\overline{B}$) and since the Ulm invariants of B and $\overline{B}$ are equal, we can find corresponding elements b_{ij} in B and corresponding summands $B^1, B^2, \dots$, where each B^i is homocyclic of exponent $s_i + 1$. Now define a map $\phi : \bigoplus_{i \geq 1} B^i \to \overline{B}$ by sending $b_{ij} \mapsto y_{ij}$; this map extends to an endomorphism ψ of $\overline{B}$ with $\psi(B^i) = \mathcal{Y}^i$. Note that each element $p^{s_i} b_{ij} \in B[p] \leq B \cap H$, as we assumed that $H \cap B$ contains $B[p^k]$, so that each $p^{s_i} b_{ij} \in H[p]$.

Now $H[p] = H \cap \overline{B}[p]$ is fully inert in $\overline{B}$ being the intersection of two fully inert subgroups and so we have that

$$(H[p]) + \psi(H[p])/H[p] \geq (H[p]) + \psi(\oplus p^{s_i} B^i)/H[p].$$

However, the term on the righthand side of the above inequality is isomorphic to $\langle X\rangle/(H[p]\cap\langle X\rangle)$ and the denominator is 0 since $H[p]\cap\langle X\rangle[p]=0$. This gives the required contradiction and we have established that $H[p]$ is of finite index in $\overline{B}[p]$ and consequently that $H[p]\sim\overline{B}[p]$. $\square$

Using induction, we can now solve the bounded case.

Proposition 3.14. *Let H be a fully inert subgroup of a reduced unbounded torsion complete p-group $\overline{B}$ such that $H\cap B$ is bounded. If $H\cap B\sim B[p^k]$, then H and $\overline{B}[p^k]$ are commensurable.*

Proof. The plan of the proof is the following:

- we know from Lemma 3.13 that $H[p]$ has finite index in $\overline{B}[p]$, so we may extend H finitely to obtain a new fully inert subgroup H_1 which is commensurable with H and has the property that $H_1[p]=\overline{B}[p]$;
- we find a fully inert subgroup $H_2\geq H_1$ with $H_2\sim H_1$ such that $H_2[p^2]=\overline{B}[p^2]$;
- we iterate this process to get eventually a fully inert subgroup $H_k\sim H$ such that $H_k[p^k]=\overline{B}[p^k]$;
- finally we prove that $H\sim H[p^k]$, hence $H\sim\overline{B}[p^k]$.

Let $n<k$ and suppose, for an inductive argument, that we have $H_n[p^n]=\overline{B}[p^n]$ for some fully inert subgroup H_n of $\overline{B}$, where $H\sim H_n$. Now consider $\overline{B}[p^{n+1}]$ and assume there are infinitely many elements x_i $(i\in\mathbb{N})$ in $\overline{B}[p^{n+1}]$ which are independent modulo $H_n[p^{n+1}]$; clearly each $x_i\notin\overline{B}[p^n]$. The elements p^nx_i, which are in the socle, are also independent; in fact, assume that $\alpha_1p^nx_1+\cdots+\alpha_rp^nx_r=0$, where $0\leq\alpha_i<p$ for all i. Then $p^n(\alpha_1x_1+\cdots+\alpha_rx_r)=0$ implies that $\alpha_1x_1+\cdots+\alpha_rx_r\in\overline{B}[p^n]=H_n[p^n]$, hence the independence of the x_i modulo $H_n[p^{n+1}]$ ensures that the α_i are all 0.

Now we have $p^nx_i=p^{s_i}p^nz_i$ for suitable elements $z_i\in\overline{B}$ of height 0 and suitable integers $s_i\geq 0$; therefore $x_i=p^{s_i}z_i+w_i$, where $w_i\in\overline{B}[p^n]=H_n[p^n]$. Note that as p^nx_i is an element of order p and height s_i+n, it follows from a result of Prüfer - see Corollary 5.2.2 in [7] - that z_i generates a cyclic direct summand of order p^{s_i+n+1} of $\overline{B}$. Note that the independence of the p^nx_i means that the z_i are also independent and if there are κ many z_i of a given order then the equality of the Ulm invariants of B and $\overline{B}$ will ensure that there are κ many independent elements d_i in B.

Thus $D=\bigoplus\langle d_i\rangle$ is a direct summand of B. Hence we may define a map $\phi:B\to\overline{B}$ by setting $\phi(d_i)=z_i$ and letting ϕ act trivially on a complement of D; clearly ϕ extends to an endomorphism ψ of $\overline{B}$, with $\psi(d_i)=z_i$, noting that $\psi(p^{s_i}d_i)=p^{s_i}z_i=x_i-w_i\in x_i+H_n[p^{n+1}]$.

As H_n is fully inert in $\overline{B}$, the subgroup $H_n[p^{n+1}]$ is also fully inert in $\overline{B}$; we have the inclusion:

$$(H_n[p^{n+1}]+\psi(H_n[p^{n+1}]))/H_n[p^{n+1}]\geq(H_n[p^{n+1}]+\psi(\bigoplus\langle p^{s_i}d_i\rangle))/H_n[p^{n+1}]$$

where the last term contains the infinite set of nonzero elements $\{x_i + H_n[p^{n+1}]\}$ – contradiction. So there are only finitely many elements of $\overline{B}[p^{n+1}]$ which are not in $H_n[p^{n+1}]$; say $X = \{x_1, x_2, \dots x_n\}$ are these elements. If $H_{n+1} = H_n + \langle X \rangle$, then certainly H_{n+1} is fully inert in $\overline{B}$, $H_{n+1} \sim H_n \sim H$, and $\overline{B}[p^{n+1}] = H_{n+1}[p^{n+1}]$.

To conclude, we must prove that $H \sim H[p^k]$. Up to now we have seen that there exists a subgroup H_k of $\overline{B}$ which is commensurable with H, hence fully inert in $\overline{B}$, such that $\overline{B}[p^k] = H_k[p^k]$. So it is enough to show that $H_k/H_k[p^k]$ is finite, or, equivalently, that $(H_k/H_k[p^k])[p] = H_k[p^{k+1}]/H_k[p^k]$ is finite. Assume, by way of contradiction, that there are infinitely many elements $y_i \in H_k[p^{k+1}]$ ($i \in \mathbb{N}$) which are independent modulo $H_k[p^k]$. The elements $p^k y_i$, which are in the socle, are also independent; in fact, assume that $\alpha_1 p^k y_1 + \cdots + \alpha_r p^k y_r = 0$, where $0 \le \alpha_i < p$ for all i. Then $p^k(\alpha_1 y_1 + \cdots + \alpha_r y_r) = 0$ implies that $\alpha_1 y_1 + \cdots + \alpha_r y_r \in \overline{B}[p^k] = H_k[p^k]$, hence the independence of the y_i modulo $H_k[p^k]$ ensures that the α_i are all 0.

Now repeat the proof as before to prove that we reach a contradiction with the property of H_k of being fully inert in $\overline{B}$. □

3.5. The main result for torsion-complete groups

Collecting the results obtained up to now, we get the main announced theorem.

Theorem 3.15. *A fully inert subgroup H of a reduced unbounded torsion complete p-group $\overline{B}$ is commensurable with a fully invariant subgroup of $\overline{B}$.*

Proof. $H \cap B$ is fully inert in B, by Lemma 3.3, so, as proved in [4], it is commensurable with a fully invariant subgroup of B. Such a subgroup of B is of the from $B(\mathbf{u})$. If $B(\mathbf{u}) = B$ or $B(\mathbf{u}) = 0$ the proof follows from Lemma 3.6 and Lemma 3.7, respectively. If $B(\mathbf{u})$ is unbounded, the proof follows by Proposition 3.8. Finally, if $B(\mathbf{u})$ is bounded and non-zero, then the proof follows by Proposition 3.14 □

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