Networks with Small Stretch Number*
(Extended Abstract)

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Abstract. In a previous work, the authors introduced the class of *graphs with bounded induced distance of order* k, (BID(k) for short) to model non-reliable interconnection networks. A network modeled as a graph in BID(k) can be characterized as follows: if some nodes have failed, as long as two nodes remain connected, the distance between these nodes in the faulty graph is at most k times the distance in the non-faulty graph. The smallest k such that $G \in$ BID(k) is called *stretch number* of G. In this paper we give new characterization, algorithmic, and existence results about graphs with small stretch number.

1 Introduction

The main function of a network is to provide connectivity between the sites. In many cases it is crucial that (properties about) connectivity is preserved even in the case of (multiple) faults in sites. Accordingly, a major concern in network design is fault-tolerance and reliability. That means in particular that the network to be constructed shall remain reliable even in the case of site faults.

According to the actual applications and requirements, the term 'reliability' may stand for different features. In this work, it concerns bounded distances, that is our goal is to investigate about networks in which distances between sites remain *small* even in the case of faulty sites. As the underlying model, we use unweighted graphs, and measure the distance in a network in which node faults have occurred by a shortest path in the subnetwork that is induced by the non-faulty components. Using this model, in [7] we have introduced the class BID(k) of *graphs with bounded induced distance* of order k. A network modeled as a graph in BID(k) can be characterized as follows: if some nodes have failed, as long as two nodes remain connected, the distance between these nodes in the faulty graph is at most k times the distance in the non-faulty graph.

Some characterization, complexity, and structural results about BID(k) are given in [7]. In particular, the concept of *stretch number* has been introduced: the stretch number $s(G)$ of a given graph G is the smallest rational number

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k such that G belongs to BID(k). Given the relevance of graphs in BID(k) in the area of communication networks, our purpose is to provide characterization, algorithmic, and existence results about graphs having small stretch number.

Results. We first investigate graphs having stretch number at most 2. In this context we show that: (i) there is no graph G with stretch number $s(G)$ such that $2 - \frac{1}{i} < s(G) < 2 - \frac{1}{i+1}$, for each integer $i \geq 1$ (this fact was conjectured in [7]); (ii) there exists a graph G such that $s(G) = 2 - \frac{1}{i}$, for each integer $i \geq 1$. These results give a partial solution to the following more general problem: Given a rational number k, is k an *admissible stretch number*, i.e., is there a graph G such that $s(G) = k$? We complete the solution to this problem by showing that every rational number $k \geq 2$ is an admissible stretch number. Finally, we give a characterization result in term of forbidden subgraphs for the class BID($2 - \frac{1}{i}$), for each integer $i > 1$. This characterization result allows us to design a polynomial time algorithm to solve the recognition problem for the class BID($2 - \frac{1}{i}$), for each $i \geq 1$ (if k is not fixed, this problem is Co-NP-complete for the class BID(k) [7]). We conclude the paper by showing that such an algorithmic approach cannot be used for class BID(k), for each integer $k \geq 2$.

Related works. In literature there are several papers devoted to fault-tolerant network design, mainly starting from a given desired topology and introducing fault-tolerance to it (e.g., see [4,15,19]). Other works follow our approach.

In [14], a study about our concepts is performed: they give characterizations for graphs in which *no delay* occurs in the case that a *single* node fails. These graphs are called *self-repairing*. In [8], authors introduce and characterize new classes of graphs that guarantee constant stretch factors k even when a multiple number of *edges* have failed. In a first step, they do not limit the number of edge faults at all, allowing for *unlimited* edge faults. Secondly, they examine the case where the number of edge faults is *bounded* by a value ℓ. The corresponding graphs are called k–self-spanners and (k, ℓ)–self-spanners, respectively. In both cases, the names are motivated by strong relationships to the concept of k*–spanners* [21]. Related works are also those concerning distance-hereditary graphs [18]. In fact, distance-hereditary graphs correspond to the graphs in BID(1), and graphs with bounded induced distance can be also viewed as a their parametric extension (in fact, BID(k) graphs are mentioned in the survey [2] as k–distance-hereditary graphs). Distance-hereditary graphs have been investigated to design interconnection network topologies [6,11,13], and several papers have been devoted to them (e.g., see [1,3,5,10,12,16,20,23]).

The remainder of this extended abstract is organized as follows. Notations and basic concepts used in this work are given in Section 2. In Section 3 we recall definitions and results from [7]. Section 4 shows the new characterization results, and in Section 5 we answer the question about admissible stretch numbers. In Section 6 we give the complexity result for the recognition problem for the class BID($2 - \frac{1}{i}$), for every integer $i \geq 1$, and in Section 7 we give some final remarks.

Due to space limitations, some proofs and technical details are omitted and will be provided in the full paper.

2 Notation

In this work we consider finite, simple, loopless, undirected and unweighted graphs $G = (V, E)$ with node set V and edge set E. We use standard terminologies from [2,17], some of which are briefly reviewed here.

A *subgraph* of G is a graph having all its nodes and edges in G. Given a subset S of V, the *induced subgraph* $\langle S \rangle$ of G is the maximal subgraph of G with node set S. $|G|$ denotes the cardinality of V. If x is a node of G, by $N_G(x)$ we denote the *neighbors* of x in G, that is, the set of nodes in G that are adjacent to x, and by $N_G[x]$ we denote the *closed neighbors* of x, that is $N_G(x) \cup \{x\}$. $G - S$ is the subgraph of G induced by $V \setminus S$.

A sequence of pairwise distinct nodes $(x_0, \ldots, x_n)$ is a *path* in G if $(x_i, x_{i+1}) \in E$ for $0 \leq i < n$, and is an *induced path* if $\langle \{x_0, \ldots, x_n\} \rangle$ has n edges. A graph G is *connected* if for each pair of nodes x and y of G there is a path from x to y in G.

A *cycle* C_n in G is a path $(x_0, \ldots, x_{n-1})$ where also $(x_0, x_{n-1}) \in E$. Two nodes x_i and x_j are *consecutive* in C_n if $j = (i + 1) \bmod n$ or $i = (j + 1) \bmod n$. A *chord* of a cycle is an edge joining two non-consecutive nodes in the cycle. H_n denotes an *hole*, i.e., a cycle with n nodes and without chords. The *chord distance* of a cycle C_n is denoted by $cd(C_n)$, and it is defined as the minimum number of consecutive nodes in C_n such that every chord of C_n is incident to some of such nodes. We assume $cd(H_n) = 0$.

The length of a shortest path between two nodes x and y in a graph G is called *distance* and is denoted by $d_G(x, y)$. Moreover, the length of a longest induced path between them is denoted by $D_G(x, y)$. We use the symbols $P_G(x, y)$ and $p_G(x, y)$ to denote a longest and a shortest induced path between x and y, respectively. Sometimes, when no ambiguity occurs, we use $P_G(x, y)$ and $p_G(x, y)$ to denote the sets of nodes belonging to the corresponding paths. $I_G(x, y)$ denotes the set containing all the nodes (except x and y) that belong to a shortest path from x to y.

If x and y are two nodes of G such that $d_G(x, y) \geq 2$, then $\{x, y\}$ is a *cycle-pair* if there exist a path $p_G(x, y)$ and a path $P_G(x, y)$ such that $p_G(x, y) \cap P_G(x, y) = \{x, y\}$. In other words, if $\{x, y\}$ is a cycle-pair, then the set $p_G(x, y) \cup P_G(x, y)$ induces a cycle in G.

Let G_1, G_2 be graphs having node sets $V_1 \cup \{m_1\}$, $V_2 \cup \{m_2\}$ and edge sets E_1, E_2, respectively, where $\{V_1, V_2\}$ is a partition of V and $m_1, m_2 \notin V$. The *split composition* [9] of G_1 and G_2 with respect to m_1 and m_2 is the graph $G = G_1 * G_2$ having node set V and edge set $E = E_1^{'} \cup E_2^{'} \cup \{(x, y) \mid x \in N(m_1), y \in N(m_2)\}$, where $E_i^{'} = \{(x, y) \in E_i \mid x, y \in V_i\}$ for $i = 1, 2$.

3 Basic Definitions and Previous Results

In this section we recall from [7] some definitions and results useful in the remainder of the paper.

Definition 1. [7] *Let k be a real number. A graph $G = (V, E)$ is a* bounded induced distance graph of order k *if for each connected induced subgraph G' of G:*

$$d_{G'}(x, y) \leq k \cdot d_G(x, y), \quad \text{for each } x, y \in G'.$$

The class of all the bounded induced distance graphs of order k is denoted by BID(k)*.*

From the definition it follows that every class BID(k) is hereditary, i.e., if $G \in$ BID(k), then $G' \in$ BID(k) for every induced subgraph G' of G.

Definition 2. [7] *Let G be a graph, and $\{x, y\}$ be a pair of connected nodes in G. Then:*

1. *the* stretch number $s_G(x, y)$ of the pair $\{x, y\}$ *is given by* $s_G(x, y) = \frac{D_G(x,y)}{d_G(x,y)}$*;*
2. *the* stretch number $s(G)$ *of G is the maximum stretch number over all possible pairs of connected nodes, that is, $s(G) = \max_{\{x,y\}} s_G(x, y)$;*
3. *$\mathcal{S}(G)$ is the set of all the pairs of nodes inducing the stretch number of G, that is, $\mathcal{S}(G) = \{\{x, y\} \mid s_G(x, y) = s(G)\}$.*

The stretch number of a graph determines the minimum class which a given graph G belongs to. In fact, $s(G) = \min\{t : G \in \text{BID}(t)\}$. As a consequence, $G \in$ BID(k) if and only if $s(G) \leq k$.

Lemma 1. [7] *Let $G \in$ BID(k), and $s(G) > 1$. Then, there exists a cycle-pair $\{x, y\}$ that belongs to $\mathcal{S}(G)$.*

Theorem 1. [7] *Let G be a graph and $k \geq 1$ a real number. Then, $G \in$ BID(k) if and only if $cd(C_n) > \left\lceil \frac{n}{k+1} \right\rceil - 2$ for each cycle C_n, $n > 2k + 2$, of G.*

4 New Characterization Results

Graphs in BID(1) have been extensively studied and different characterizations have been provided. One of these characterization is based on forbidden induced subgraphs [1], and in [7] this result has been extended to the class BID($\frac{3}{2}$). In this section we further extend this characterization to the class BID($2 - \frac{1}{i}$), for every integer $i \geq 2$.

Lemma 2. *Let G be a graph with $1 < s(G) < 2$, and let $\{x, y\} \in \mathcal{S}(G)$ be a cycle-pair. If C is the cycle induced by $p_G(x, y) \cup P_G(x, y)$, then every internal node of $p_G(x, y)$ is incident to a chord of C.*

Proof. Omitted. □

Theorem 2. *Given a graph G and an integer $i \geq 2$, then $G \in \mathrm{BID}(2 - \frac{1}{i})$ if and only if the following graphs are not induced subgraphs of G:*

1. *H_n, for each $n \geq 6$;*
2. *cycles C_6 with $cd(C_6) = 1$;*
3. *cycles C_7 with $cd(C_7) = 1$;*
4. *cycles C_8 with $cd(C_8) = 1$;*
5. *cycles C_{3i+2} with $cd(C_{3i+2}) = i$.*

Proof. Only if part. Holes H_n, $n \geq 6$, have stretch number at least 2. Cycles with 6, 7, or 8 nodes and chord distance 1 have stretch number equal to 2, 5/2, and 3, respectively. Cycles C_{3i+2} with chord distance equal to i have stretch number at least $\frac{2i+1}{i+1} = 2 - \frac{1}{i+1}$. Since the considered cycles have stretch number greater than $2 - \frac{1}{i}$, then they are forbidden induced subgraphs for every graph belonging to $\mathrm{BID}(2 - \frac{1}{i})$.

If part. Given an arbitrary integer $i \geq 2$, we prove that every graph $G \notin \mathrm{BID}(2 - \frac{1}{i})$ contains one of the forbidden subgraphs or a proper induced subgraph $G' \notin \mathrm{BID}(2 - \frac{1}{i})$. In the latter case, we can recursively apply to G' the following proof.

If $G \notin \mathrm{BID}(2 - \frac{1}{i})$ then, by Theorem 1, G contains a cycle C_n, $n \geq 6$, as induced subgraph such that $0 \leq cd(C_n) \leq \left\lceil \frac{i \cdot n}{3i-1} \right\rceil - 2$. This means that a cycle-pair $\{x, y\} \in \mathcal{S}(G)$ generates C_n. In particular, we can assume that C_n is induced by the nodes of the two internal node-disjoint paths $P_G(x, y) = (x, u_1, u_2, \ldots, u_p, y)$ and $p_G(x, y) = (x, v_1, v_2, \ldots, v_q, y)$, such that $p + q + 2 = n$ and $cd(C_n) = q$.

If $q = 0$ then we obtain the holes H_n, $n \geq 6$. If $q = \left\lceil \frac{i \cdot n}{3i-1} \right\rceil - 2$ and $n = 6, 7, 8, 3i + 2$, then we obtain the other forbidden subgraphs.

Now, we show that if $n \geq 9$, $n \neq 3i + 2$, and q fulfills $1 \leq q \leq \left\lceil \frac{i \cdot n}{3i-1} \right\rceil - 2$, then C_n contains one of the given forbidden subgraphs or an induced subgraph G' such that $G' \notin \mathrm{BID}(2 - \frac{1}{i})$.

By Lemma 2, every node v_k, $1 \leq k \leq q$, must be incident to a chord of C_n, otherwise C_n has a stretch number greater or equal to 2 and hence it is itself a forbidden subgraph of G. As a consequence, we can denote by r_j the largest index j' such that v_j and $u_{j'}$ are connected by a chord of C_n, i.e. $r_j = \max\{j' \mid (v_j, u_{j'}) \text{ is a chord of } C_n\}$. Informally, r_j gives the *rightmost* chord connecting v_j to some vertex of $P_G(x, y)$.

Notice that, if $r_1 > 3$ then the subgraph of C_n induced by the nodes $v_1, x, u_1, \ldots, u_{r_1}$ is forbidden, since it is a cycle with at least 6 nodes and chord distance at most 1. Hence, in the remainder of this proof we assume that $r_1 \leq 3$.

Let us now analyze two distinguished cases for C_n, according whether the chord distance q of C_n either (i) fulfills $1 \leq q < \left\lceil \frac{i \cdot n}{3i-1} \right\rceil - 2$, or ($ii$) is equal to $\left\lceil \frac{i \cdot n}{3i-1} \right\rceil - 2$.

(*i*) We consider C_n with $n \geq 9$ and chord distance q such that $1 \leq q < \left\lceil \frac{i \cdot n}{3i-1} \right\rceil - 2$. If $C_{n'}$ denotes the subgraph induced by the nodes of C_n except the nodes $x, u_1, \ldots, u_{r_1-1}$, then $C_{n'}$ is a cycle with $n' \geq n-3$ nodes and chord distance at most $q-1$. To prove that $C_{n'}$ is forbidden, we have to show that $\left\lceil \frac{i \cdot n'}{3i-1} \right\rceil - 2 \geq q-1$:

$$\left\lceil \frac{i \cdot n'}{3i-1} \right\rceil - 2 \geq \left\lceil \frac{i \cdot n - 3i}{3i-1} \right\rceil - 2 \geq q - 1,$$

$$\left\lceil \frac{i \cdot n - 3i}{3i-1} \right\rceil - 2 > q - 2,$$

$$\left\lceil \frac{i \cdot n - 3i}{3i-1} + 2 \right\rceil - 2 > q,$$

$$\left\lceil \frac{i \cdot n + 4i - 2}{3i-1} \right\rceil - 2 > q.$$

The last inequality holds because $4i - 2 \geq 0$ for each integer $i \geq 1$, and $\left\lceil \frac{i \cdot n}{3i-1} \right\rceil - 2 > q$.

(*ii*) We consider C_n with $n \geq 9$ and chord distance q such that $q = \left\lceil \frac{i \cdot n}{3i-1} \right\rceil - 2$. In this case q is given whenever a fixed value for n is chosen. In general, since $n \geq 9$, it follows that $q \geq 2$.

Let us analyze again the cycle $C_{n'}$. Recalling that $n' \geq n - 3$ and $cd(C_{n'}) \leq q - 1$, then

$$\left\lceil \frac{i \cdot n'}{3i-1} \right\rceil - 2 \geq \left\lceil \frac{i \cdot n - 3i}{3i-1} \right\rceil - 2 \geq q - 1$$

is equivalent to

$$\left\lceil \frac{i \cdot n - 1}{3i-1} \right\rceil - 2 \geq q.$$

In the following we show that, for every n such that $9 \leq n \leq 6i$, either this relation holds or n is equal to $3i + 2$. This means that the cycle $C_{n'}$ is forbidden for each cycle C_n, $9 \leq n \leq 6i$.

Since $\left\lceil \frac{i \cdot n}{3i-1} \right\rceil - 2 = q$ holds by hypothesis, we have to study when $\left\lceil \frac{i \cdot n - 1}{3i-1} \right\rceil \geq \left\lceil \frac{i \cdot n}{3i-1} \right\rceil$. This relation does not hold if and only if there exists an integer m such that $\frac{i \cdot n - 1}{3i-1} \leq m < \frac{i \cdot n}{3i-1}$, that is $\frac{i \cdot n - 1}{3i-1} = m$. Then, since this equality is equivalent to $n = 3m - \frac{m-1}{i}$, m can be equal to $\ell \cdot i + 1$ only, for each integer $\ell \geq 0$. As a consequence, $n = 3m - \frac{m-1}{i} = 3(\ell \cdot i + 1) - \ell$, $\ell \geq 0$. For $\ell = 0$ we obtain $n = 3$ (but we are considering $n \geq 9$), for $\ell = 1$ and $\ell = 2$ the value of n is $3i + 2$ and $n = 6i + 1$, respectively. The cycle with $3i + 2$ nodes is one of the forbidden cycles in the statement of the theorem. As a conclusion, the induced cycle $C_{n'}$ shows that C_n contains a forbidden induced subgraph when $9 \leq n \leq 6i$.

It remains to be considered the case when $n \geq 6i+1$. In this case $q = \left\lceil \frac{i \cdot n}{3i-1} \right\rceil - 2$ implies $q \geq 2i$, and hence we can compute the value r_i. Since $v_i, v_{i-1}, \ldots, v_1, x, u_1, \ldots, u_{r_i}$ induce a cycle with chord distance i, then it has at most $3i+1$ nodes otherwise it is forbidden. In other words, $r_i \leq 2i$. The subgraph $C_{n''}$ induced by the nodes of C_n except the nodes $v_{i-1}, \ldots, v_1, x, u_1, \ldots, u_{r_i-1}$ is a cycle with $n'' \geq n - 3i + 1$ nodes and chord distance at most $q - i$. To prove that $C_{n''}$ is forbidden, let us show that $\left\lceil \frac{i \cdot n''}{3i-1} \right\rceil - 2 \geq q - i$. The inequality

$$\left\lceil \frac{i \cdot n''}{3i-1} \right\rceil - 2 \geq \left\lceil \frac{i \cdot (n-3i+1)}{3i-1} \right\rceil - 2 \geq q - i$$

is equivalent to

$$\left\lceil \frac{i \cdot n}{3i-1} \right\rceil - 2 \geq q.$$

The last relation holds by hypothesis, and this concludes the proof □

5 Admissible Stretch Numbers

In [7], it was conjectured that there exists no graph G such that $2 - \frac{1}{i} < s(G) < 2 - \frac{1}{i+1}$, for each integer $i \geq 1$. In this section we show that the conjecture is true. Moreover, we extend the result by showing that it is possible to answer to the following general question: Given a rational number $t \geq 1$, is there a graph G such that $s(G) = t$?. In other words, we can state when a given positive rational number is an *admissible* stretch number.

Definition 3. *A positive rational number t is called* admissible stretch number *if there exists a graph G such that $s(G) = t$.*

In the remainder of this section we first show that the conjecture recalled above is true, and then we show that each positive rational number greater or equal than 2 is an admissible stretch number.

Lemma 3. *If p and q are two positive integers such that $2 - \frac{1}{i} < \frac{p}{q} < 2 - \frac{1}{i+1}$, for some integer $i \geq 1$, then $q > i$.*

Proof. Omitted. □

Theorem 3. *If t is a rational number such that $2 - \frac{1}{i} < t < 2 - \frac{1}{i+1}$, for some integer $i \geq 1$, then t is not an admissible stretch number.*

Proof. We have to show that there exists no graph G such that $2 - \frac{1}{i} < s(G) < 2 - \frac{1}{i+1}$, for each integer $i \geq 1$.

By contradiction, let us assume that there exist an integer $i \geq 1$ and a graph G such that $2 - \frac{1}{i} < s(G) < 2 - \frac{1}{i+1}$. By Lemma 1 there exists a cycle-pair $\{x, y\} \in \mathcal{S}(G)$. If we assume that $P_G(x, y) = (x, u_1, u_2, \ldots, u_{p-1}, y)$ and $p_G(x, y) = (x, v_1, v_2, \ldots, v_{q-1}, y)$, then $p_G(x, y) \cup P_G(x, y)$ induces a cycle C, and $s(G) = \frac{p}{q}$. By Lemma 3, the relation $q > i$ holds; then, the node v_i exists in the path $p_G(x, y)$. By Lemma 2, the node v_i is incident to a chord of C, and hence we can define the integer r, $1 \leq r \leq q-1$, such that

$$r = \max\{j \mid (v_i, u_j) \text{ is a chord of } C\}.$$

Let us now denote by C_L the cycle induced by the nodes $v_i, v_{i-1}, \ldots, v_1, x, u_1, u_2, \ldots, u_r$, and by C_R the cycle induced by the nodes $v_i, v_{i+1}, \ldots, v_{q-1}, y, u_{p-1}, u_{p-2}, \ldots, u_r$. In other words, the chord (v_i, u_r) divides C into the *left* cycle C_L, and the *right* cycle C_R.

First of all, let us compute the stretch number of the cycle C_R. Since $p_G(x, y) = (x, v_1, v_2, \ldots, v_{q-1}, y)$ then $p_{C_R}(v_i, y) = (v_i, v_{i+1}, \ldots, v_{q-1}, y)$. Moreover, since the path $(v_i, u_r, u_{r+1}, \ldots, u_{p-1}, y)$ is induced in C, then its length implies $D_{C_R}(v_i, y) \geq p - r + 1$. Then

$$s(C_R) \geq s_{C_R}(v_i, y) \geq \frac{p-r+1}{q-i}.$$

Since C_R is an induced subgraph of G then

$$\frac{p-r+1}{q-i} \leq \frac{p}{q}.$$

This inequality is equivalent to

$$\frac{p}{q} \leq \frac{r-1}{i}.$$

From the relations

$$2 - \frac{1}{i} < \frac{p}{q} \leq \frac{r-1}{i}$$

we obtain that $r > 2i$, that is $r \geq 2i + 1$.

Let us now compute the stretch number of the cycle C_L when $r \geq 2i+1$. In this case, $p_{C_L}(x, u_r) = (x, v_1, v_2, \ldots, v_i, u_r)$ and $P_{C_L}(x, u_r) = (x, u_1, u_2, \ldots, u_r)$. Then

$$s(C_L) \geq s_{C_L}(x, u_r) = \frac{r}{i+1} \geq \frac{2i+1}{i+1} \geq 2 - \frac{1}{i+1}.$$

The obtained relation implies that $s(C_L) > s(G)$. This is a contradiction since C_L is an induced subgraph of G. □

In order to show that each rational number equal or greater than 2 is admissible as stretch number, let us consider the graph $G(n_1, n_2, \ldots, n_t)$ obtained by composing t holes $H_{n_1}, H_{n_2}, \ldots, H_{n_t}$ by split composition, where $n_i \geq 5$ for $1 \leq i \leq t$. The holes correspond to the following chordless cycles (as an example, see Figure 1, where $t = 5$) :

- $H_{n_1} = (l_1, x_0, x_1, m_1', r_1, \ldots)$;
- $H_{n_i} = (l_i, m_i, x_i, m_i', r_i, \ldots)$, for each i such that $1 < i < t$;
- $H_{n_t} = (l_t, m_t, x_t, x_{t+1}, r_t, \ldots)$.

The holes are composed by means of the split composition as follows:

$$G(n_1, n_2, \dots, n_t) = H_{n_1} * H_{n_2} * \dots * H_{n_t},$$

and the marked nodes between H_{n_i} and $H_{n_{i+1}}$ are m'_i and m_{i+1}, $1 \le i < t$, respectively.

In the following, we denote by V_1 (V_t, resp.) the set containing all the nodes of the hole H_{n_1} (H_{n_t}, resp.) but x_0, x_1, and m'_1 (m_t, x_t, and x_{t+1}, resp.); we denote by V_i the set containing all the nodes of the hole H_{n_i} but m_i, x_i, and m'_i, $1 \le i \le t$. Finally, we denote by X the set $\{x_0, x_1, \dots, x_{t+1}\}$.

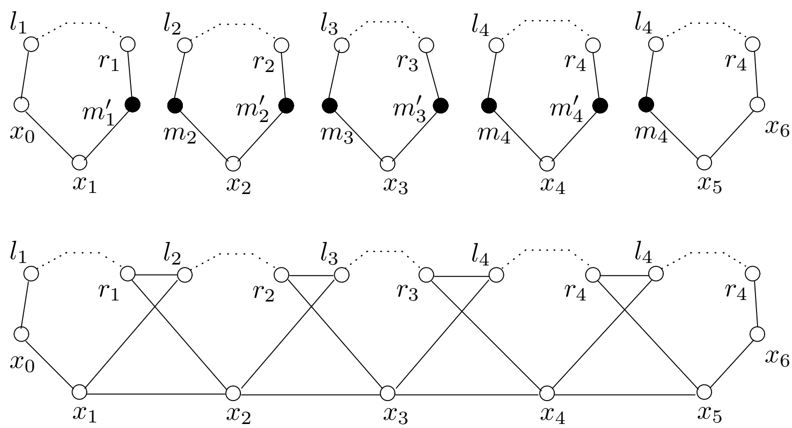

Fig. 1. *The graph $G(n_1, n_2, n_3, n_4, n_5)$ obtained by the split composition of 5 holes. The i-th hole has $n_i \ge 5$ nodes.*

Lemma 4. *Given the graph $G = G(n_1, n_2, \dots, n_t)$, the following facts hold:*

1. $s_G(x_0, x_{t+1}) = \frac{\sum_{i=1}^{t} n_i - 3t + 1}{t+1}$;
2. *if* $i < j$ *then* $p_G(x_i, x_j) \cup P_G(x_i, x_j)$ *induces a subgraph isomorphic to* $G(n_i, n_{i+1}, \dots, n_j)$;
3. *if* $n_t \ge \frac{\sum_{i=1}^{t-1} n_i}{t-1}$ *then* $s_G(x_0, x_{t+1}) > s_G(x_0, x_t)$;
4. *there exists a pair* $\{u, v\} \in \mathcal{S}(G)$ *such that* $u \in X$ *and* $v \in X$;
5. *if* $n_i = n$ *for some fixed integer* n, $1 \le i \le t$, *then* $s(G) = s_G(x_0, x_{t+1}) = \frac{nt - 3t + 1}{t+1}$.

Proof. Omitted. □

Notice that the stretch number of nodes x_0 and x_{t+1} in $G(n_1, n_2, \dots, n_t)$ does not depend on how many nodes are in each hole; it depends only on the total number of nodes in $G(n_1, n_2, \dots, n_t)$ and on the number t of used holes.

Theorem 4. *If t is a rationale number such that $t \ge 2$, then t is an admissible stretch number.*

Sketch of the Proof. Let us suppose that $t = p/q$ for two positive integers p and q. If $q = 1$ then $G = H_{2p+2}$, if $q = 2$ then $G = H_{p+2}$. When $q \geq 3$ we show that the graph G is equal to $G(n_1, n_2, \ldots, n_{q-1})$ for suitable integers $n_1, n_2, \ldots, n_{q-1}$.

Let $b = 3 + \left\lfloor \frac{p-1}{q-1} \right\rfloor$ and $r = (p-1) \bmod (q-1)$. Let us choose the sizes of the holes $H_{n_1}, H_{n_2}, \ldots, H_{n_{q-1}}$ according to the following strategy: r holes contain exactly $b+1$ nodes, while the remaining $q-1-r$ contain exactly b nodes. By Fact 1 of Lemma 4 it follows that $s_G(x_0, x_q) = \frac{\sum_{i=1}^{q-1} n_i - 3(q-1)+1}{q} = \frac{p}{q}$. To prove that $S(G) = p/q$, by Fact 4 of Lemma 4, we have to prove that $s_G(x_i, x_j) \leq p/q$, $1 \leq i, j \leq q-2$. This property holds only if we are able to determine a deterministic method to decide whether the hole H_{n_i}, $1 \leq i \leq q-1$, contains either b or $b+1$ nodes. In the full paper we show that such a deterministic method exists. □

Corollary 1. *For each integer $i \geq 1$, $2 - \frac{1}{i}$ is an admissible stretch number.*

Proof. From Fact 5 of Lemma 4, it follows that $G = G(n_1, n_2, \ldots, n_{i-1})$ such that $n_j = 5$ for each $1 \leq j \leq i-1$, has stretch equal to $2 - \frac{1}{i}$. □

The results provided by Corollary 1, Theorem 3, and Theorem 4 can be summarized in the following two corollaries.

Corollary 2. *Let t be an admissible stretch number. Then, either $t \geq 2$ or $t = 2 - \frac{1}{i}$ for some integer $i \geq 1$.*

Corollary 3. *For every admissible stretch number t, split composition can be used to generate a graph G with $s(G) = t$.*

Notice that, by Theorem 4, we can also use every irrational number greater than 2 to define graph classes containing graphs with bounded induced distance. For instance, $\mathrm{BID}(\pi) \neq \mathrm{BID}(k)$ for every rational number k.

6 Recognition Problem

The recognition problem for BID(1) can be solved in linear time [1,16]. In [7], this problem has been shown Co-NP-complete for the generic case (i.e., when k is not fixed), and the following question has been posed: What is the largest constant k such that the recognition problem for $\mathrm{BID}(k)$ can be solved in polynomial time?

In this section we show that Theorem 2 can be used to devise a polynomial algorithm to solve the recognition problem for the class $\mathrm{BID}(k)$, for every $k < 2$.

Lemma 5. *There exists a polynomial time algorithm to test whether a given graph G contains, as induced subgraph, a cycle C_n with $n \geq 6$ and $cd(C_n) \leq 1$.*

Proof. Omitted. □

Theorem 5. *For any fixed integer $i \geq 1$, the recognition problem for the class $\mathrm{BID}(2 - \frac{1}{i})$ can be solved in polynomial time.*

Proof. For $i = 1$ the problem can be solved in linear time [1,16]. By Theorem 2, a brute-force, rather naive algorithm for solving the recognition problem for the class BID$(2 - \frac{1}{i})$, $i > 1$, is: test if G contains, as induced subgraph, (1) a cycle C_n with $n \geq 6$ and $cd(C_n) \leq 1$, or (2) a cycle C_{3i+2} with chord distance equal to i. To perform Test 1 above, we can use the algorithm of Lemma 5, and to perform Test 2 we can check whether any subset of $3i + 2$ nodes of G forms a cycle with chord distance equal to i. The latter test can be implemented in polynomial time since the number of subsets of nodes with $3i + 2$ elements is bounded by n^{3i+2}. □

7 Conclusions

In this paper we provide new results about graph classes that represent a parametric extension of the class of distance-hereditary graphs. In any graph G belonging to the generic new class BID(k), the distance between every two connected nodes in every induced subgraph of G is at most k times their distance in G.

The recognition problem for BID$(2 - \frac{1}{i})$ can be solved in polynomial time (Theorem 5), and the corresponding algorithm is based on Theorem 2. Can the same approach be used in order to solve the same problem for class BID(k), $k \geq 2$? In other words, if $k \geq 2$ is an integer, is it possible to characterize BID(k) by listing all its forbidden induced subgraphs? For instance, holes H_n with $n \geq 2k + 3$, and a *finite* number of cycles having different chord distance. Unfortunately, the following theorem states that it is not possible.

Theorem 6. *For each integers $k \geq 2$ and $i \geq 2$, there exists a minimal forbidden induced cycle for the class* BID(k) *with chord distance equal to i.*

Proof. Omitted. □

Many problems are left open. For instance, what is the largest constant k such that the recognition problem for BID(k) can be solved in polynomial time? Moreover, several algorithmic problems are solvable in polynomial time for BID(1). Can some of these results be extended to BID(k), $k > 1$?

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