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Bipartite finite Toeplitz graphs[☆]

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ABSTRACT

Let $n, a_1, \ldots, a_k$ be distinct positive integers. A finite Toeplitz graph $T_n(a_1, \ldots, a_k) = (V, E)$ is a graph where $V = \{v_0, \ldots, v_{n-1}\}$ and $E = \{(v_i, v_j) : |i-j| \in \{a_1, \ldots, a_k\}\}$. In this paper, we characterize bipartite finite Toeplitz graphs with $k \le 3$. In most cases, the characterization takes $O(\log a_3)$ arithmetic steps; in the remaining cases, it takes $O(a_1)$. A consequence of the proposed results is the complete characterization of the chromatic number of finite Toeplitz graphs with $k \le 3$. In addition, we characterize some classes of infinite bipartite Toeplitz graphs with $k \ge 4$.

1. Introduction

Let $n, a_1, a_2, \ldots, a_k$ be distinct positive integers such that $1 \le a_1 < a_2 < \cdots < a_k < n$. By $T_n(a_1, a_2, \ldots, a_k) = (V, E)$ we denote the (simple undirected) *finite Toeplitz graph* where $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E = \{(v_i, v_j)$, for $|i-j| \in \{a_1, a_2, \ldots, a_k\}\}$ (see Fig. 1). The numbers $a_1, a_2, \ldots, a_k$ are called *entries*. The name of this class of graphs is due to the fact that the adjacency matrix is a Toeplitz matrix, i.e., each of its descending diagonals from left to right is constant. In the literature, Toeplitz graphs with an infinite number of vertices are also defined (*infinite Toeplitz graphs*).

In this paper, we shall mainly focus on Toeplitz graphs with two or three entries: we shall denote them by $T_n(a, b)$ and $T_n(a, b, c)$, respectively, where $1 \le a < b < c < n$. Consider an arbitrary edge (v_i, v_j) of $T_n(a, b)$ or $T_n(a, b, c)$: if $|i-j| = a, b, c$, respectively, we shall say that the vertices v_i and v_j are *a-, b-, c-adjacent*, respectively, and that $(v_i, v_j) \in E$ is an *a-, b-, c-edge*, respectively. By *a-path* A_p, for $p = 0, 1, \ldots, a-1$, we denote the path containing vertex v_p and made of *a*-edges only: the vertices of A_p are $v_p, v_{p+a}, \ldots, v_{p+ta}$, where t is such that $n - a \le p + ta < n$ (that is to say, all the vertices v_x verifying $x \bmod a = p$ belong to A_p).

The problem we study is the bipartiteness of Toeplitz graphs. We remark that if $k = 1$ the problem is trivial: in fact, the Toeplitz graph $T_n(a)$ is bipartite, because it consists of a collection of a vertex-disjoint paths.

In [5], the author proposes a procedure to test a finite $T_n(a, b)$ for bipartiteness (the procedure has an arithmetic complexity of $O(\log(b+1))$), and states some results for three particular subclasses of finite non-bipartite $T_{2a+1}(a, b, c)$ with $1 \le a < b < c < n = 2a + 1$. From these results, one can derive the non-bipartiteness of the Toeplitz graphs $T_n(a, b, c)$ with $1 \le a < b < c < 2a + 1 \le n$. Infinite bipartite Toeplitz graphs are characterized in [7].

In Section 2, we recall some results on the connectivity of Toeplitz graphs with two entries. In Section 3, we state an $O(\log b)$ closed-form condition to test a finite $T_n(a, b)$ for bipartiteness. In Section 4, we characterize the whole family of

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1 Dedicated to my mum.

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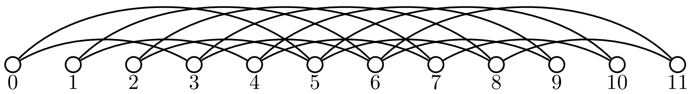

Fig. 1. The Toeplitz graph $T_{12}(3, 5)$.

finite bipartite Toeplitz graphs $T_n(a, b, c)$, thereby providing an answer to the problem posed in [5]. The Toeplitz graphs $T_n(a, b, c)$ have been partitioned into three subsets. For two of them (dealt with in Sections 4.1 and 4.2), the problem reduces to verifying a simple closed-form condition, which takes $O(\log c)$ arithmetic operations. Section 4.3 is devoted to studying the graphs in the third subset: we prove that they can be checked for bipartiteness by determining whether a suitable diophantine equation admits a solution verifying prescribed constraints or not (we recall that in [5] the author uses diophantine equations to characterize three particular subclasses of finite non-bipartite Toeplitz graphs $T_{2a+1}(a, b, c)$, but he states that the framework of diophantine equations, though necessary, is not sufficient to completely characterize the non-bipartiteness of Toeplitz graphs). We also show that the problem of finding a constrained solution to the diophantine equation can be reduced to testing for feasibility an integer program in three variables and eight linear constraints. This last problem requires $O(\log c)$ arithmetic operations for its solution, but determining some of its data requires $O(a)$ operations. Nevertheless, in the same section, several graphs in the third subclass are characterized, whose bipartiteness can be determined in $O(\log c)$ arithmetic operations. In Section 5, the proved results are extended to characterize some subclasses of bipartite Toeplitz graphs $T_n(a_1, \ldots, a_k)$ with $k \geq 4$, to infinite bipartite Toeplitz graphs, and to some integer distance graphs with two or three entries (an *integer distance graph* $G_{\mathbb{Z}}(a_1, a_2, \ldots, a_k)$ is a graph with an infinite number of vertices $\{\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots\}$, where two vertices v_x and v_y are adjacent if and only if $|x - y| \in \{a_1, a_2, \ldots, a_k\}$; see [1,2,4]).

A consequence of our results and those in [10] is an efficient answer to the open problem of providing a complete characterization of the chromatic number of Toeplitz graphs with three entries.

2. Connectivity of the $T_n(a, b)$

The present section recalls some results on the connectivity of the Toeplitz graphs $T_n(a, b)$.

Theorem 1 ([8,10]). *Let $T_n(a, b)$ be a Toeplitz graph with $1 \leq a < b < n$.*

- *$T_n(a, b)$ is connected if and only if $\gcd(a, b) = 1$ and $n \geq a + b - 1$.*

In addition we have the following.

- *If $\gcd(a, b) = 1$ and $n = a + b - 1$, then $T_n(a, b)$ is a Hamiltonian path.*
- *If $\gcd(a, b) = 1$ and $n = a + b$, then $T_n(a, b)$ is a Hamiltonian cycle.*
- *A $T_n(a, b)$ with $\gcd(a, b) = 1$ has $\max\{1, a + b - n\}$ connected components.*

We derive the following two corollaries.

Corollary 1. *Vertices $v_0, \ldots, v_{a+b-1}$ of a Toeplitz graph $T_n(a, b)$ with $\gcd(a, b) = 1$ and $n \geq a + b$ induce a cycle.*

In Proposition 7 of [11], it is proved that $T_{a+b+1}(a, b)$ with $\gcd(a, b) = 1$ has a Hamiltonian path with endpoints v_0 and v_{a+b}. This result follows immediately from the result above: delete edge (v_0, v_a) from the cycle, and add edge (v_a, v_{a+b}) to get the path.

Corollary 2. *Consider a Toeplitz graph $T_n(a, b)$ with $1 \leq a < b < n$.*

- *If $a + b - \gcd(a, b) + 1 \leq n \leq a + b$, then $T_n(a, b)$ is a collection of $n + \gcd(a, b) - a - b$ vertex-disjoint cycles and $a + b - n$ vertex-disjoint paths.*
- *$T_n(a, b)$ has $\max\{\gcd(a, b), a + b - n\}$ connected components, which are all paths or isolated vertices if $n \leq a + b - \gcd(a, b)$.*

In the remainder of the paper we shall often make use of particular subgraphs of $T_n(a, b)$, namely the graphs T^p induced by the set V^p of vertices v_x with $x \bmod \gamma = p$, for $p = 0, \ldots, \gamma - 1$, where $\gamma = \gcd(a, b)$. By construction, the graph T^p induced by V^p is isomorphic to $T_{n^p}(\frac{a}{\gamma}, \frac{b}{\gamma})$, where $n^p = \left\lceil \frac{n}{\gamma} \right\rceil$ for $p = 0, \ldots, n \bmod \gamma - 1$ and $n^p = \left\lfloor \frac{n}{\gamma} \right\rfloor$ for $p = n \bmod \gamma, \ldots, \gamma - 1$. Notice that $T_n(a, b)$ is the union of all the subgraphs T^p.

3. Bipartite Toeplitz graphs $T_n(a, b)$

In the present section, by means of a short and combinatorial proof, we state some easy necessary and sufficient conditions for a connected $T_n(a, b)$ to be bipartite.

From Corollary 2, the following theorem is immediately derived.

Theorem 2. *A Toeplitz graph $T_n(a, b)$ with $n \leq a + b - \gcd(a, b)$ is bipartite.*

For all the remaining cases, we have the following theorem.

Theorem 3. *A Toeplitz graph $T_n(a, b)$ with $n \geq a + b - \gcd(a, b) + 1$ is bipartite if and only if $\frac{a}{\gcd(a,b)}$ and $\frac{b}{\gcd(a,b)}$ are odd.*

Proof (*If Part*)**.** If all the entries are odd, then every edge connects an even-indexed vertex to an odd-indexed one.
Only if part. Let $\gamma = \gcd(a, b)$, and consider the subgraph T^0. By definition, T^0 is isomorphic to $T_{\lceil \frac{n}{\gamma} \rceil}(\frac{a}{\gamma}, \frac{b}{\gamma})$. Since $\lceil \frac{n}{\gamma} \rceil \geq \frac{a}{\gamma} + \frac{b}{\gamma}$, by Corollary 1, T^0 contains a cycle on the first $\frac{a}{\gamma} + \frac{b}{\gamma}$ vertices. The assumption that $T_n(a, b)$ is bipartite implies that T^0 is bipartite, and thus that $\frac{a}{\gamma} + \frac{b}{\gamma}$ is even; that is to say, $\frac{a}{\gamma}$ and $\frac{b}{\gamma}$ have the same parity. By definition of $\gamma = \gcd(a, b)$, they are both odd. □

The most expensive operation for checking this condition is to compute $\gcd(a, b)$. Since it can be done in $O(\log b)$ arithmetic computations, this is the computational complexity of checking a given $T_n(a, b)$ for bipartiteness.

4. Bipartite Toeplitz graphs $T_n(a, b, c)$

The present section is devoted to characterizing the bipartite Toeplitz graphs $T_n(a, b, c)$. The following preliminary results hold.

Theorem 4. *A Toeplitz graph $T_n(a, b, c)$ with a, b, c odd is bipartite.*

Proof. If all the entries are odd, then every edge connects an even-indexed vertex to an odd-indexed one. □

Theorem 5. *If b or c is an even multiple of a, then a Toeplitz graph $T_n(a, b, c)$ is non-bipartite.*

Proof. Let h be an even integer. If $b = ha$, then $\{(v_0, v_a), (v_a, v_{2a}), \ldots, (v_{(h-1)a}, v_{ha}), (v_{ha}, v_0)\}$ is an odd cycle. Similarly if $c = ha$. □

Theorem 6. *A Toeplitz graph $T_n(a, b, c)$ is bipartite if and only if $T_{\lceil \frac{n}{\gcd(a,b,c)} \rceil}\left(\frac{a}{\gcd(a,b,c)}, \frac{b}{\gcd(a,b,c)}, \frac{c}{\gcd(a,b,c)}\right)$ is bipartite.*

Proof. Let $\delta = \gcd(a, b, c)$. Consider the set $\overline{V}^q$ of all the vertices v_t with $t \bmod \delta = q$, for $q = 0, \ldots, \delta - 1$. By construction, the graph $\overline{T}^q$ induced by $\overline{V}^q$ is isomorphic to $T_{n^q}(\frac{a}{\delta}, \frac{b}{\delta}, \frac{c}{\delta})$, where $n^q = \lceil \frac{n}{\delta} \rceil$ for $q = 0, \ldots, n \bmod \delta - 1$ and $n^q = \lfloor \frac{n}{\delta} \rfloor$ for $q = n \bmod \delta, \ldots, \delta - 1$. By definition of $\gcd(a, b, c)$, and since $\overline{T}^1, \overline{T}^2, \ldots, \overline{T}^{\delta-1}$ are all isomorphic to subgraphs of $\overline{T}^0$, the claim follows. □

Thanks to this theorem, in what follows we shall limit ourselves to considering the Toeplitz graphs $T_n(a, b, c)$ with $\gcd(a, b, c) = 1$: each of the following subsections proves the bipartiteness of the Toeplitz graphs $T_n(a, b, c)$ according to three different value ranges for n. In addition, as a consequence of Theorems 4 and 5, it is convenient to apply the results proved in the next three subsections to Toeplitz graphs with at least one even entry, and such that neither b nor c is an even multiple of a.

4.1. Toeplitz graphs $T_n(a, b, c)$ with $\gcd(a, b, c) = 1$ and $n \leq c + \gcd(a, b) - 1$

We start by proving that the bipartiteness of the Toeplitz graphs $T_n(a, b, c)$ with $\gcd(a, b, c) = 1$ and $n \leq c + \gcd(a, b) - 1$ depends on the bipartiteness of $T_n(a, b)$.

Lemma 1. *A Toeplitz graph $T_n(a, b, c)$ with $n \leq c + \gcd(a, b) - 1$ and $\gcd(a, b, c) = 1$ is bipartite if and only if $T_n(a, b)$ is bipartite.*

Proof. Let $\gamma = \gcd(a, b)$. $T_n(a, b, c)$ has $n - c$ c-edges. The assumption that $n \leq c + \gamma - 1$ implies that the graph has at most $\gamma - 1$ c-edges. Since $\gcd(c, \gamma) = 1$ (as implied by $\gcd(a, b, c) = 1$), no cycle containing a c-edge exists when $T_n(a, b, c)$ contains fewer than γ c-edges (notice that when $\gamma = 1$ the graph has no c-edges at all). Hence $T_n(a, b, c)$ is bipartite if and only if $T_n(a, b)$ is bipartite. □

An example is shown in Fig. 2.
Combining the lemma above with Theorems 2 and 3 of Section 3 we get the following results.

Theorem 7. *A Toeplitz graph $T_n(a, b, c)$ with $\gcd(a, b, c) = 1$ and $n \leq \min\{a + b - \gcd(a, b); c + \gcd(a, b) - 1\}$ is bipartite.*

Theorem 8. *A Toeplitz graph $T_n(a, b, c)$ with $\gcd(a, b, c) = 1$ and $a + b - \gcd(a, b) + 1 \leq n \leq c + \gcd(a, b) - 1$ is bipartite if and only if $\frac{a}{\gcd(a,b)}$ and $\frac{b}{\gcd(a,b)}$ are odd.*

In order to check whether a given $T_n(a, b, c)$ undergoes the stated conditions, the most expensive operation is computing $\gcd(a, b, c)$ and $\gcd(a, b)$. Since $\gcd(a, b, c) = \gcd(\gcd(a, b), c)$, we get an overall arithmetic complexity of $O(\log c)$.

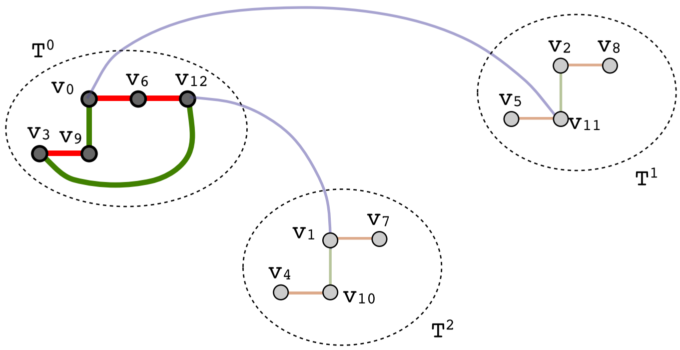

Fig. 2. The Toeplitz graph $T_{13}(6, 9, 11)$: in red, green, and blue, respectively, the a-, b-, and c-edges. (For interpretation of the references to colour in this figure legend, please observe that a-edges are represented horizontally, b-edges vertically, and c-edges neither horizontally nor vertically; otherwise, refer to the web version of this article.)

4.2. Toeplitz graphs $T_n(a, b, c)$ with $\gcd(a, b, c) = 1$ and $n \geq \max\{a + b - \gcd(a, b) + 1; c + \gcd(a, b)\}$

In this section we prove the following.

Theorem 9. *A Toeplitz graph $T_n(a, b, c)$ with $\gcd(a, b, c) = 1$ and $n \geq \max\{a + b - \gcd(a, b) + 1; c + \gcd(a, b)\}$ is bipartite if and only if a, b, c are odd.*

Proof. The *if part* follows from Theorem 4, so let us turn to prove the *only if part*. The assumption that $T_n(a, b, c)$ is bipartite implies that $T_n(a, b)$ is bipartite, too. Let $\gamma = \gcd(a, b)$. By Theorem 3, it follows that $\frac{a}{\gamma}, \frac{b}{\gamma}$ are odd. We distinguish two cases: $\gamma = 1$ and $\gamma > 1$.

If $\gamma = 1$, by Theorem 1, $T_n(a, b)$ is connected, and the (unique) bipartition of the set of vertices of $T_n(a, b)$ is the one which separates the odd-indexed vertices from the even-indexed ones. Since we are assuming that $T_n(a, b, c)$ is bipartite, it must be the case that c is odd, too.

Now consider the case $\gamma > 1$, and consider the subgraphs T^p of $T_n(a, b)$, for $p = 0, \ldots, \gamma - 1$. The assumption that $n \geq a + b - \gamma + 1$ implies that each T^p is connected, by Theorem 1. In addition, each c-edge connects distinct subgraphs T^p, because $\gcd(c, \gamma) = 1$ (as implied by $\gcd(a, b, c) = 1$). Since $n \geq c + \gamma$, the graph $T_n(a, b, c)$ has $n - c \geq \gamma$ c-edges. The first γ of them are found in a cycle $\mathcal{C}$ which alternates a (suitable) c-edge with a path (made of a- and b-edges, only) within a (suitable) T^p. Precisely: $\mathcal{C}$ contains the c-edge (v_0, v_c), then a path within $T^{c \bmod \gamma}$ from v_c to $v_{c \bmod \gamma}$, then the c-edge $(v_{c \bmod \gamma}, v_{c \bmod \gamma + c})$, then a path within $T^{((c \bmod \gamma + c) \bmod \gamma)}$, and so on, until a (suitable) c-edge brings $\mathcal{C}$ back to a vertex $v_y \in T^0$, and a path from v_y to v_0 within T^0 closes it.

Let us determine the number of edges of $\mathcal{C}$. $\mathcal{C}$ contains γ c-edges and γ paths within the subgraphs T^p. The bipartiteness of $T_n(a, b)$ implies that all the paths connecting two given vertices v_z and v_p in T^p do have the same parity. For example, consider a path of T^p, say W_z, with i_z a-edges and j_z b-edges, where i_z and j_z are such that $p + i_z a + j_z b = z$. We get $i_z \frac{a}{\gamma} + j_z \frac{b}{\gamma} = \frac{z-p}{\gamma}$ (notice that the definition of T^p implies that $\frac{z-p}{\gamma}$ is integer). Recalling that $\gamma = \gcd(a, b)$, we know that $\frac{a}{\gamma}$ and $\frac{b}{\gamma}$ are integer, coprime, and odd. Thus the length $i_z + j_z$ of path W_z and the quantity $i_z \frac{a}{\gamma} + j_z \frac{b}{\gamma} = \frac{z-p}{\gamma}$ have the same parity. In particular: for the $(\gamma - c \mod \gamma)$ vertices $v_c \in T^{c \bmod \gamma}, \ldots, v_{c+\gamma-1-c \bmod \gamma} \in T^{\gamma-1}$, the parity of the corresponding paths $W_c, \ldots, W_{c+\gamma-1-c \bmod \gamma}$ is the same as the parity of $\frac{c-c \bmod \gamma}{\gamma}$; and, for the $(c \mod \gamma)$ vertices $v_{c+\gamma-c \bmod \gamma} \in T^0, \ldots, v_{c+\gamma-1} \in T^{c \bmod \gamma-1}$, the parity of the corresponding paths $W_{c+\gamma-1-c \bmod \gamma}, \ldots, W_{c+\gamma-1}$ is the same as the parity of $\frac{c+\gamma-c \bmod \gamma}{\gamma}$. Thus, the parity of the number of edges in $\mathcal{C}$ is the same as the parity of $\gamma + (\gamma - c \mod \gamma)\frac{c-c \bmod \gamma}{\gamma} + (c \mod \gamma)\frac{c+\gamma-c \bmod \gamma}{\gamma} = \gamma + c$. Since $T_n(a, b, c)$ is bipartite, the quantity $\gamma + c$ is even; that is to say, γ and c have the same parity. Recalling that $\gamma = \gcd(a, b)$ and $\gcd(a, b, c) = 1$, γ and c are both odd. The conditions that γ is odd and that $\frac{a}{\gamma}, \frac{b}{\gamma}$ are odd imply that a, b are odd, and prove the theorem. $\square$

An example is drawn in Fig. 3.

The computational complexity of checking the proposed condition for an arbitrary $T_n(a, b, c)$ is $O(\log c)$ in the arithmetic model.

4.3. Toeplitz graphs $T_n(a, b, c)$ with $\gcd(a, b, c) = 1$ and $c + \gcd(a, b) \leq n \leq a + b - \gcd(a, b)$

All the remaining cases are the Toeplitz graphs $T_n(a, b, c)$ with $\gcd(a, b, c) = 1$ verifying $c + \gcd(a, b) \leq n \leq a + b - \gcd(a, b)$, which we now focus on.

If b (c, respectively) is an even multiple of a, Theorem 5 applies, and the problem is solved. In all other cases we define $E_n(a, b, c) = [e_{i,j}]$ (see Fig. 4) as the matrix whose rows and columns are indexed from 0 to $a - 1$, and from b to $n - 1$,

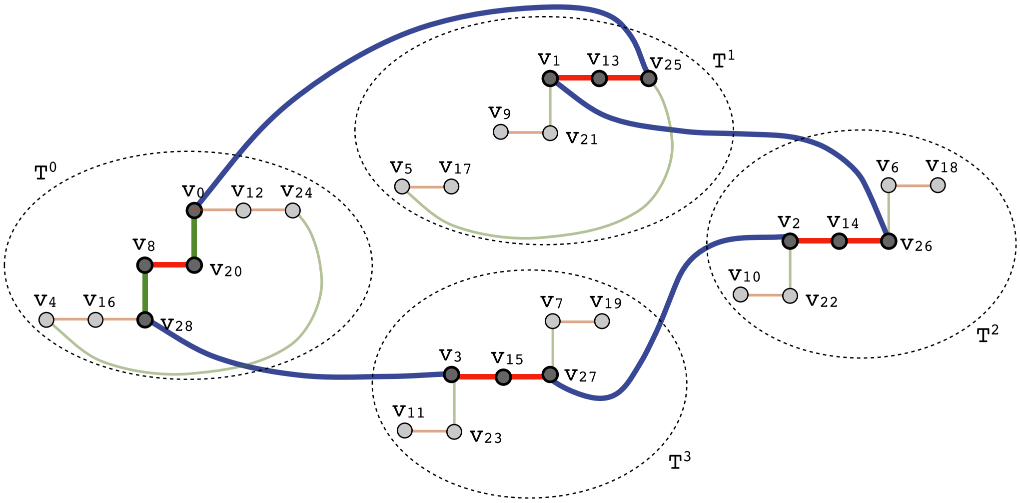

Fig. 3. The Toeplitz graph $T_{29}(12, 20, 25)$; in bold, the edges belonging to the odd cycle C_{13}.

a

	11	12	13	14	15	16
0	B	C		2A		
1		B	C		2A	
2			B	C		2A
3				B	C	
4	1A				B	C
5		1A				B
6			1A			

b

	19	20	21	22	23	24
0	B		3A			C
1		B		3A		
2			B		3A	
3				B		3A
4					B	
5	2A					B
6		2A				

Fig. 4. (a) The type 1 matrix $E_{17}(7, 11, 12)$ and (b) the type 2 matrix $E_{25}(7, 19, 24)$.

respectively, where $e_{i,j}$ = "qA" if $|i - j| = qa$, for some integer q, and $e_{i,j}$ = "B" ("C", respectively) if $|i - j| = b$ ($|i - j| = c$, respectively). If b (c, respectively) is an (odd) multiple of a, we set $e_{i,j}$ = "pA, B" ($e_{i,j}$ = "pA, C", respectively). The *multiplicity* of $e_{i,j}$ is the number of edges it represents (when $e_{i,j}$ = "pA, B" or $e_{i,j}$ = "pA, C", one can fix the multiplicity either to p or 1, which have the same parity, by hypothesis). In what follows, by *non-empty* we mean containing a "pA", a "B", a "C", or a "$*$".

$E_n(a, b, c)$ is a diagonal matrix (see Fig. 4). Precisely, in E we find (not necessarily distinct): the leftmost A-diagonal of elements "qA", (the A$^{\text{left}}$-*diagonal*); a diagonal of "B"-elements (the B-*diagonal*); a diagonal of "C"-elements (the C-*diagonal*); and the rightmost diagonal of elements "$(q + 1)$ A" (the A$^{\text{right}}$-*diagonal*), if and only if $n \geq a(1 + \lfloor \frac{b}{a} \rfloor) + 1$; all the remaining diagonals are empty. By definition, one has $q = \lfloor \frac{b}{a} \rfloor$. If q is even (odd, respectively), then the A$^{\text{left}}$-diagonal is an *even-diagonal* (*odd-diagonal*, respectively); if both A-diagonals are present, either one is an even-diagonal and the other one is an odd-diagonal.

We distinguish two types of matrix: if $\lfloor \frac{b}{a} \rfloor = \lfloor \frac{c}{a} \rfloor$, then $E_n(a, b, c)$ is of *type* 1; if $\lfloor \frac{b}{a} \rfloor < \lfloor \frac{c}{a} \rfloor$, then $E_n(a, b, c)$ is of *type* 2. In a type 1 matrix, from left to right, we find the A$^{\text{left}}$-diagonal, the B-diagonal, the C-diagonal, and, if $n \geq a(1 + \lfloor \frac{b}{a} \rfloor) + 1$, the A$^{\text{right}}$-diagonal (see $E_{17}(7, 11, 12)$ in Fig. 4(a)); in a type 2 matrix, from left to right, we find the A$^{\text{left}}$-diagonal, the B-diagonal, the A$^{\text{right}}$-diagonal, and the C-diagonal (see $E_{25}(7, 19, 24)$ in Fig. 4(b)).

Since the subgraph induced by all the a-edges is a collection of disjoint a-paths, a cycle in the graph must necessarily contain some b- or c-edge (this fact explains why matrix $E_n(a, b, c)$ is defined on columns b to $n - 1$). The following lemma holds.

Lemma 2. *Consider a Toeplitz graph $T_n(a, b, c)$. If $n \leq a + b - \gcd(a, b)$, then every b-edge and every c-edge connects the smallest-indexed vertex of an a-path with the largest-indexed vertex of a different a-path, and every vertex of the graph has at most one b-edge and at most one c-edge incident to it.*

Proof. As for the first claim, focus on the b-edges. Let $\gamma = \gcd(a, b)$. We distinguish two cases: $\gamma = 1$ and $\gamma > 1$. If $\gamma = 1$, consider the graph $T_n(a, b)$ and an arbitrary b-edge (v_x, v_{x+b}). One has $0 \leq x \leq n - 1 - b = a - 2$; otherwise v_x, v_{x+b} are not in $T_n(a, b)$. The inequality $x \leq a - 2$ shows that $x \bmod a = x$; thus vertex v_x is the smallest-indexed vertex of a-path A_x. On the other hand, no a-edge (v_{x+b}, v_{x+b+a}) exists in $T_n(a, b)$, as $x + b + a > n - 1$, for all $x \geq 0$, showing that v_{x+b} is the largest-indexed vertex of $A_{(x+b) \bmod a}$. Notice that A_x and $A_{(x+b) \bmod a}$ are not the same path, since $\gcd(a, b) = 1$. If $\gamma > 1$,

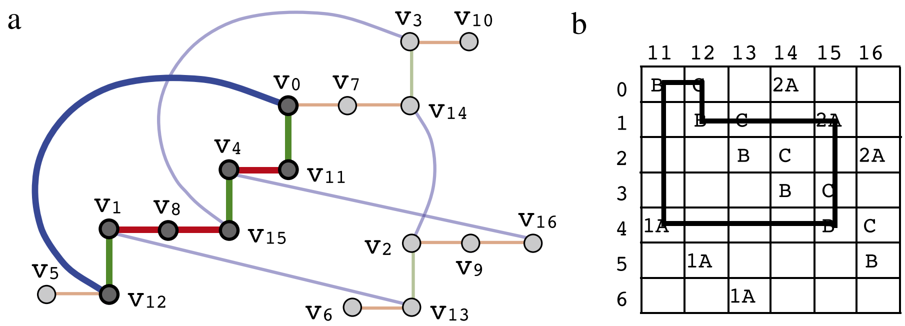

Fig. 5. (a) The Toeplitz graph $T_{17}(7, 11, 12)$ (in bold, the edges belonging to C_7) and (b) the matrix $E_{17}(7, 11, 12)$ with the CMC corresponding to the odd cycle C_7.

consider graphs T^p, for $p = 0, \dots, \gamma - 1$. The assumption that $n \le a+b-\gamma$ implies that $n^0 = \lceil \frac{n}{\gamma} \rceil \le \frac{a}{\gamma} + \frac{b}{\gamma} - 1$. This proves that each T^p is isomorphic to a subgraph of $T_{\frac{a}{\gamma}+\frac{b}{\gamma}-1}(\frac{a}{\gamma}, \frac{b}{\gamma})$, which the proof above applies to, and the first claim follows for the b-edges. The same result holds for c-edges, as $n \le a+b-\gamma \le a+b-1 \le a+c-1$.

As for the second claim, consider the b-edges (*a fortiori* the same result holds for c-edges). An arbitrary vertex v_x has two b-edges incident to it if and only if vertices v_{x-b} and v_{x+b} are both in the graph. That is to say, if both $x \ge b$ and $x+b \le n-1$ hold. By definition, $b \ge a+1$, and we get $n \ge a+b+2$, which contradicts the assumption. □

There are two consequences of the lemma above. One is that every cycle $\mathcal{C}$ consists of maximal sequences of consecutive a-edges, separated by non-empty paths alternating b- and c-edges; the other is that every maximal sequence of consecutive a-edges is an a-path. Hence $\mathcal{C}$ can be written as $A_{i_1}, F_{i_2}, A_{i_3}, F_{i_4}, \dots, A_{i_p}$, where each A_{i_q} is an a-path, and each F_{i_q} is a non-empty path alternating a b- and a c-edge. These two facts show that on matrix $E_n(a, b, c)$ all the cycles of $T_n(a, b, c)$ are easily represented, as we now explain.

A cycle $\mathcal{C}$ of $T_n(a, b, c)$ can be mapped onto a Closed Manhattan Curve (CMC, for short) with corners in the non-empty elements of $E_n(a, b, c)$, and vice versa. A CMC has a corner in an element $e_{i,h}$ if and only if all the edges the element represents, belong to $\mathcal{C}$ (see Fig. 5). The number of edges in $\mathcal{C}$ can be easily computed by summing up the multiplicity of the elements which are corners of the corresponding CMC. In the example of Fig. 5, $\mathcal{C}$ is defined on $2+1+1+1+1+1 = 7$ edges.

In order to prove the bipartiteness of $T_n(a, b, c)$, we are interested in finding the CMCs corresponding to odd cycles. As observed above, the number of edges in a CMC can be obtained by computing the sum of the multiplicity of the elements in its corners. This can be done, for example, by computing the sum of the multiplicity of the elements which are endpoints of each vertical segment of a CMC. In both matrices, the sum of the multiplicity of the endpoints of a vertical segment is an odd quantity if and only if exactly one endpoint is in an even diagonal. Since the only even diagonal is the even A-diagonal, and we are interested in the parity of the number of edges in a cycle, and not in the number itself, we can observe the following.

Observation 1. *An odd cycle in a Toeplitz graph $T_n(a, b, c)$ is represented by a CMC of $E_n(a, b, c)$ with an odd number of vertical segments incident to elements of the even* A*-diagonal.*

An important issue coming from the geometry of a CMC concerns the length of a vertical segment. In fact, since the corners of a CMC are placed in non-empty elements of $E_n(a, b, c)$, the difference among the row indices of the endpoints (the *length*) of any vertical segment can assume one of few values, which are the distances among two consecutive non-empty diagonals and the sum of two of them having a non-empty diagonal in common. These quantities depend on the matrix type. Precisely, for a type 1 matrix, we distinguish the following segments:

- s_1, of length $k_1 = b \bmod a$, which connects an element of the $\mathrm{A}^{\mathrm{left}}$-diagonal with an element of the B-diagonal;
- s_2, of length $k_2 = c - b$, which connects an element of the B- with an element of the C-diagonal;
- if $n \ge a(1 + \lfloor \frac{b}{a} \rfloor) + 1$, we also define s_3, of length $k_3 = a - c \bmod a$, which connects an element of the C-diagonal with an element of the $\mathrm{A}^{\mathrm{right}}$-diagonal;

and for a type 2 matrix:

- s_1, of length $k_1 = b \bmod a$, which connects an element of the $\mathrm{A}^{\mathrm{left}}$-diagonal with an element of the B-diagonal;
- s_2, of length $k_2 = a - b \bmod a = a \lceil \frac{b}{a} \rceil - b$, which connects an element of the B- with an element of the $\mathrm{A}^{\mathrm{right}}$-diagonal;
- s_3, of length $k_3 = c \bmod a$, which connects an element of the C-diagonal with an element of the $\mathrm{A}^{\mathrm{right}}$-diagonal.

Observe that for a type 1 matrix one has $k_1 + k_2 + k_3 = a$ if s_3 is defined, and $k_1 + k_2 \le a$ if s_3 is not defined (in $T_{17}(7, 11, 12)$ one has $(k_1, k_2, k_3) = (4, 1, 2)$, as shown in Fig. 5(b), where the alignment of the segments s_2 and s_3 gives a vertical segment of length $3 = k_2 + k_3 = 1 + 2$), and for a type 2 matrix one has $k_1 + k_2 = a$ (in $T_{25}(7, 19, 24)$ one has $(k_1, k_2, k_3) = (5, 2, 3)$).

From the geometry of a CMC we also derive another important issue. Without loss of generality, define a direction on $\mathcal{C}$, and let D (U, respectively) be the sum of the lengths of all the vertical segments taken downwards (upwards, respectively).

a

	11	12	13	14	15	16
0	B	C		2A		
1		B	C		2A	
2			B	C		2A
3				B	C	
4	1A				B	C
5		1A				B
6						

b

	19	20	21	22	23	24
0	B					C
1						
2						
3						*
4						
5	2A					B
6						

Fig. 6. (a) The matrix $\widehat{E}_{17}(7, 11, 12)$ and (b) the matrix $\widehat{E}_{25}(7, 19, 24)$.

Since a CMC is a closed curve, we must have $D - U = 0$ (the same holds for the (rightwards and leftwards) horizontal segments).

Let y_i^d and y_i^u denote the number of vertical segments of type s_i taken downwards and upwards, respectively, in a CMC, and let $y_i = y_i^d - y_i^u$ be the net number of them, for $i = 1, 2, 3$. As a consequence of the two issues above, the following diophantine equation holds:

$$k_1 y_1 + k_2 y_2 + k_3 y_3 = 0, \tag{1}$$

where $y_3 = 0$ when s_3 is not defined.

A solution (y_1, y_2, y_3) to Eq. (1) corresponds to a CMC in $E_n(a, b, c)$ if it verifies some constraints (without loss of generality, set $y_3 = 0$ if k_3 is not defined). First of all, we have to ensure that the number of vertical segments available in $E_n(a, b, c)$ is not exceeded. In addition, since we are looking for odd cycles of $T_n(a, b, c)$, we require that a solution contains an *odd* number of vertical segments incident to the *even* A-diagonal (see Observation 1).

In order to compute the maximum number of available vertical segments, we have to identify those elements of $E_n(a, b, c)$, if any, which will never be corners of a CMC. Consider a row or a column with one non-empty element only, say $e_{i,j}$. No *closed* curve can exist with a corner in $e_{i,j}$ (unless we allow the curve to go back where it was coming from, obtaining an even subcycle, which is clearly useless both from the point of view of a CMC and of its parity) (see $e_{4,23}$ in $E_{25}(7, 19, 24)$). Each of these elements, in fact, represents a *dangling* path, that is, an a-path, a b-edge, or a c-edge with an endpoint of degree 1. For this reason, we can delete it from the matrix. Such deletion may cause other elements to be deleted for the same reason (in $E_{25}(7, 19, 24)$, after $e_{4,23}$, we delete $e_{2,23}$, $e_{2,21}$, and $e_{0,21}$). The process can be repeated until no row or column of $E_n(a, b, c)$ contains a unique element (as a consequence, in $E_{25}(7, 19, 24)$ we also delete $e_{6,20}$, $e_{1,20}$, $e_{1,22}$, $e_{3,22}$, and $e_{3,24}$). The resulting matrix will be denoted by $\widehat{E}_n(a, b, c)$ (see Fig. 6).

The number of available vertical segments can now be computed. The number of available vertical segments of type s_1 is the number U_1 of columns of $\widehat{E}_n(a, b, c)$ containing both the endpoints of a segment s_1, namely a "B" and an element of the A^{left}-diagonal; similarly for segments of type s_2 and s_3 in either type of matrix (these upper bounds will be called U_2 and U_3, respectively). For the graph $T_{17}(7, 11, 12)$ we get $(U_1, U_2, U_3) = (2, 5, 3)$ (see Fig. 6(a)).

Actually, in a type 2 matrix things are slightly more complicated. Consider a column, say j, of a type 2 matrix $\widehat{E}_n(a, b, c)$ having exactly a "B" and a "C" (hence $j \geq c$). This column represents vertex v_j which is the endpoint of a b-edge and of a c-edge, and is the largest-indexed vertex of a dangling a-path: this a-path cannot belong to any cycle, but the b- and c-edges incident to v_j may. Precisely, column j tells us that the three vertices v_{j-c}, v_j, v_{j-b} might belong to a cycle $\mathcal{C}$ if and only if they are consecutive in $\mathcal{C}$, as well as the corresponding two edges (v_{j-c}, v_j) and (v_j, v_{j-b}). By definition of s_2 and s_3, any such column j allows for increasing U_2, U_3 by one unit, and the fact that the two edges (v_{j-c}, v_j) and (v_j, v_{j-b}) are consecutive corresponds to constraining y_2, y_3 to have the same sign, that is to say, $y_2 y_3 \geq 0$ (in fact, the "B" and "C" elements of a column containing only these two elements can be corners of a CMC if and only if the segments s_2 and s_3 are taken either both upwards or both downwards). We shall denote by w the number of columns of $\widehat{E}_n(a, b, c)$ having exactly a "B" and a "C" element. It is convenient to keep this information in the matrix: we do that by placing a "$*$" in correspondence of the deleted A^{right} element in every column of $\widehat{E}_n(a, b, c)$ having exactly a "B" and a "C" element (see the"$*$" in $e_{3,24}$ of $\widehat{E}_{25}(7, 19, 24)$ in Fig. 6(b)). Clearly, w is exactly the number of "$*$" in $\widehat{E}_n(a, b, c)$. For the graph $T_{25}(7, 19, 24)$, we get $(U_1, U_2, U_3) = (1, 0, 0)$ and $w = 1$ (see Fig. 6(b)).

As a result, for a type 1 matrix we get the set of constraints in Fig. 7(a); for a type 2 matrix, if $w = 0$, we have to use the set of constraints in Fig. 7(b) and, if $w > 0$, we can also make use of the set of constraints in Fig. 7(c) (where the bounds on $|y_2|$ and $|y_3|$ are looser, but y_2 and y_3 are requested to have the same sign).

Let us define *constrained* a solution (y_1, y_2, y_3) of the diophantine equation (1) which verifies the constraints above. We can prove the following.

Lemma 3. *Consider a Toeplitz graph $T_n(a, b, c)$ with $\gcd(a, b, c) = 1$ and $c + \gcd(a, b) \leq n \leq a + b - \gcd(a, b)$. Every constrained solution (y_1, y_2, y_3) of the diophantine equation (1) corresponds to an odd cycle of $T_n(a, b, c)$, and vice versa.*

Proof. First, we show how to derive a constrained solution from an odd cycle $\mathcal{C}$ of a Toeplitz graph $T_n(a, b, c)$ undergoing the hypothesis. Map $\mathcal{C}$ onto the corresponding CMC, as described above. Arbitrarily define a direction on it, and count the

a	b	c
$\lvert y_1\rvert \le U_1$	$\lvert y_1\rvert \le U_1$	$\lvert y_1\rvert \le U_1$
$\lvert y_2\rvert \le U_2$	$\lvert y_2\rvert \le U_2$	$\lvert y_2\rvert \le U_2 + w$
$\lvert y_3\rvert \le U_3$	$\lvert y_3\rvert \le U_3$	$\lvert y_3\rvert \le U_3 + w$
y_1 odd if $\left\lfloor \frac{b}{a} \right\rfloor$ even	y_1 odd if $\left\lfloor \frac{b}{a} \right\rfloor$ even	y_1 odd if $\left\lfloor \frac{b}{a} \right\rfloor$ even
y_3 odd if $\left\lfloor \frac{b}{a} \right\rfloor$ odd	$y_2 + y_3$ odd if $\left\lfloor \frac{b}{a} \right\rfloor$ odd	$y_2 + y_3$ odd if $\left\lfloor \frac{b}{a} \right\rfloor$ odd
		$y_2 y_3 \ge 0$

Fig. 7. The sets of constraints for (a) a type 1 matrix, (b) a type 2 matrix with $w = 0$, and (c) a type 2 matrix with $w > 0$.

number of vertical segments of type s_1, s_2, and s_3 which CMC is made of. Precisely, set the corresponding counters c_1, c_2, c_3 to 0, and start following CMC: if the current segment is vertical and is of type j, increase (decrease, respectively) c_j by one unit if the segment is taken downwards (upwards, respectively). By construction, (c_1, c_2, c_3) is a solution to (1), and verifies all the required constraints.

We now show how to derive an odd cycle from an arbitrary constrained solution (y_1, y_2, y_3) of the diophantine equation (1).

Consider a type 1 matrix (similar reasonings hold for a type 2 matrix). Let $\underline{s}_j$, $\bar{s}_j$, for $j = 1, 2, 3$, represent a vertical segment of type j directed downwards and upwards, respectively. Call *head* (*tail*, respectively) the element of the matrix where the segment is directed to (originates from). Define Σ_M as a multiset of $|y_1| + |y_2| + |y_3|$ segments. Precisely, Σ_M contains $|y_i|$ occurrences of $\underline{s}_i$ if $y_i \ge 0$ or $|y_i|$ occurrences of $\bar{s}_i$ if $y_i \le 0$, for $i = 1, 2, 3$. We also define the set Σ_F of the segments one might need to construct a CMC in addition to the mandatory segments in Σ_M. Precisely, in Σ_F we find $\left\lfloor \frac{U_i - |y_i|}{2} \right\rfloor$ copies of the pair $\underline{s}_i$, $\bar{s}_i$, for $i = 1, 2, 3$.

For each non-empty row r of $\widehat{E}_n(a, b, c)$, define the *segment set* S_r as follows. For $p = 1, 2, 3$, segment $\underline{s}_p$ ($\bar{s}_p$, respectively) belongs to S_r iff $e_{r,j}$ and $e_{r+k_p,j}$ ($e_{r-k_p,j}$, respectively) are both non-empty (in $E_{25}(7, 19, 24)$, for example, the only segment sets are $S_1 = \{\underline{s}_1, \underline{s}_3\}$, $S_3 = \{\underline{s}_2, \bar{s}_3\}$, and $S_5 = \{\bar{s}_1, \bar{s}_2\}$). Notice that, by definition, given the row index r and a segment $s \in S_r$, the row and column indices of the tail and head of s are fixed: for example, for a type 1 matrix, if $s = \underline{s}_1$, then its tail and head are found in $e_{r,b+r}$ and $e_{r+k_1,b+r}$, respectively, and, if $s = \bar{s}_3$, then its tail and head are found in $e_{r,c+r}$ and $e_{r-k_3,c+r}$, respectively.

In order to define a (directed) CMC corresponding to the given solution (y_1, y_2, y_3), apply the following algorithm. Let S_ρ be a segment set having non-empty intersection with Σ_M, if any, and choose a segment $\sigma \in \Sigma_M \cap S_\rho$; otherwise, consider the solution $(-y_1, -y_2, -y_3)$ and start again. Fix the tail of CMC in row ρ, its current head in the head of σ, and (vertically) connect the tail of CMC to its head; update ρ to the row index of the head of CMC; remove σ from Σ_M; while $\Sigma_M \ne \emptyset$ do: choose a segment $\sigma \in \Sigma_M \cap S_\rho$, if any; if such a segment does not exist, let σ be a segment in $\Sigma_F \cap S_\rho$, and let σ' be the same segment as σ but in the opposite direction, insert σ' into Σ_M, and remove both σ and σ' from Σ_F; (horizontally) connect the current head of CMC to the tail of σ, and then (vertically) to the head of σ, update ρ and the current head of CMC to the row index of the head of σ. Finally, connect the current head of CMC to its tail, and derive from CMC the corresponding odd cycle. Since 0 is an integer linear combination of the constrained solution (y_1, y_2, y_3), the assertion follows. $\square$

While deriving a CMC from a given constrained solution, as in the proof above, it may happen that the row index of the current head of the CMC equals the row index of its tail. In this case, we have found a subcycle $\mathcal{C}'$ of the odd cycle corresponding to the computed CMC. With a backwards reasoning, the solution (y'_1, y'_2, y'_3) corresponding to $\mathcal{C}'$ can be constructed by considering the number of segments, and their direction. If (y'_1, y'_2, y'_3) happens to be constrained, then $\mathcal{C}'$ is an odd cycle; otherwise it is even, and all the remaining segments form an odd cycle.

A consequence of Lemma 3 is the following.

Theorem 10. *A Toeplitz graph $T_n(a, b, c)$ with $\gcd(a, b, c) = 1$ and $c + \gcd(a, b) \le n \le a + b - \gcd(a, b)$ is bipartite if and only if the diophantine equation (1) has no constrained solution.*

We remark that the results above apply to *all* the graphs studied in the paper, that is, to all the Toeplitz graphs $T_n(a, b)$, and to all the Toeplitz graphs $T_n(a, b, c)$. In the next section, we discuss a method to check whether the diophantine equation (1) admits a constrained solution, and its computational complexity. It sometimes happens that the odd cycles can be recognized by applying Observation 1 instead of Theorem 10: in Section 4.3.2, we discuss these easy cases.

4.3.1. Algorithm and complexity

The diophantine equation (1) always admits a solution, as $\gcd(k_1, k_2, k_3)$ divides 0; thus it admits an infinite number of them [3].

Consider the diophantine equation $k_1 y_1 + k_2 y_2 = k_3 h$ for $h \in \mathbb{Z}$. It has a solution if and only if $k_3 h$ is a multiple of $\gcd(k_1, k_2)$, that is to say, if $\frac{k_3 h}{\gcd(k_1,k_2)} \in \mathbb{Z}$. Under this condition, all the infinite solutions (y_1, y_2, y_3) to (1) are represented by the following triples:

$$\left(\frac{k_3 p_0}{\gcd(k_1, k_2)} h + \frac{k_2}{\gcd(k_1, k_2)} g, \frac{k_3 q_0}{\gcd(k_1, k_2)} h - \frac{k_1}{\gcd(k_1, k_2)} g, -h \right) \quad \text{for all } h \in \mathbb{Z} \text{ and } g \in \mathbb{Z},$$

where (p_0, q_0) is a solution to the diophantine equation $k_1 p + k_2 q = \gcd(k_1, k_2)$.

Consider a type 1 matrix, first, and let $\left\lfloor \frac{b}{a} \right\rfloor$ be odd. In this case, a constrained solution has to verify the first three constraints and the fifth one of the set in Fig. 7(a). Representing the solution as above, our problem turns out to be that of verifying the existence of two integer numbers g and h such that

$$\begin{aligned}
&-U_1 \le \frac{k_3 p_0}{\gcd(k_1,k_2)} h + \frac{k_2}{\gcd(k_1,k_2)} g \le U_1 \\
&-U_2 \le \frac{k_3 q_0}{\gcd(k_1,k_2)} h - \frac{k_1}{\gcd(k_1,k_2)} g \le U_2 \\
&-U_3 \le h \le U_3 \\
&h \text{ odd.}
\end{aligned}$$

The addition of another (integer) variable $t \in \mathbb{Z}$ allows us to rewrite the constraint *h odd* as $h = 2t + 1$. Thus, the problem of finding a constrained solution to (1) consists of checking whether the following integer program $\mathcal{I}_n(a, b, c)$ has a feasible solution:

$$\begin{aligned}
&\frac{k_3 p_0}{\gcd(k_1,k_2)} h + \frac{k_2}{\gcd(k_1,k_2)} g \le U_1 \quad h \le U_3 \\
&\frac{k_3 p_0}{\gcd(k_1,k_2)} h + \frac{k_2}{\gcd(k_1,k_2)} g \ge -U_1 \quad h \ge -U_3 \\
&\frac{k_3 q_0}{\gcd(k_1,k_2)} h - \frac{k_1}{\gcd(k_1,k_2)} g \le U_2 \quad h = 2t+1 \\
&\frac{k_3 q_0}{\gcd(k_1,k_2)} h - \frac{k_1}{\gcd(k_1,k_2)} g \ge -U_2 \quad h, k, t \in \mathbb{Z}.
\end{aligned}$$

When the matrix is of type 1 or 2, and $\left\lfloor \frac{b}{a} \right\rfloor$ is even, in $\mathcal{I}_n(a, b, c)$ we have to replace the constraint $h = 2t + 1$ with the constraint

$$\frac{k_3 p_0}{\gcd(k_1,k_2)} h + \frac{k_2}{\gcd(k_1,k_2)} g = 2t + 1.$$

Finally, when the matrix is of type 2 and $\left\lfloor \frac{b}{a} \right\rfloor$ is odd, in $\mathcal{I}_n(a, b, c)$ we have to replace the constraint $h = 2t + 1$ with the constraint

$$\left(\frac{k_3 q_0}{\gcd(k_1,k_2)} - 1\right) h - \frac{k_1}{\gcd(k_1,k_2)} g = 2t + 1.$$

We recall that, if the matrix is of type 2 and the corresponding integer program $\mathcal{I}_n(a, b, c)$ has no feasible solution, according to Fig. 7(c), we can write a new integer program $\mathcal{I}'_n(a, b, c)$, derived from $\mathcal{I}_n(a, b, c)$ replacing U_2, $-U_2$, U_3, and $-U_3$ with $U_2 + w$, 0, $U_3 + w$, and 0, respectively, and check whether $\mathcal{I}'_n(a, b, c)$ has a feasible solution.

We can thus conclude the following.

Theorem 11. *A Toeplitz graph $T_n(a, b, c)$ with $\gcd(a, b, c) = 1$ and $c + \gcd(a, b) \le n \le a + b - \gcd(a, b)$ is bipartite if and only if the corresponding integer program $\mathcal{I}_n(a, b, c)$ and possibly $\mathcal{I}'_n(a, b, c)$ have no feasible solution.*

For example, consider $T_{17}(7, 11, 12)$: the corresponding diophantine equation is $4y_1 + y_2 + 2y_3 = 0$, and an odd cycle is represented by (y_1, y_2, y_3) such that $(|y_1|, |y_2|, |y_3|) \le (U_1, U_2, U_3) = (2, 5, 3)$ and y_3 is odd; the corresponding integer program is $\mathcal{I}_{17}(7, 11, 12) = \{(h, g, t) : g \le 2; g \ge -2; 2h - 4g \le 5; 2h - 4g \ge -5; h \le 3; h \ge -3; h = 2t + 1; h, g, t \text{ integer}\}$, as $(p_0, q_0) = (0, 1)$ is a solution to the diophantine equation $4p + q = 1$. The integer program $\mathcal{I}_{17}(7, 11, 12)$ admits eight integer feasible solutions, for example $(h, g, t) = (1, 1, 0)$, which corresponds to the solution $(y_1, y_2, y_3) = (1, -2. -1)$ representing the odd cycle in Fig. 5(b).

In [9], it is proved that an integer feasibility problem with m constraints, each with binary encoding length $O(s)$, can be solved with $O(m + s)$ arithmetic operations on rational numbers of size $O(s)$.

Our integer program is defined on three integer variables and eight inequalities. In addition, $U_1, U_2, U_3, w, k_1, k_2, k_3, p_0, q_0$ are all not larger than a; thus $\frac{k_1}{\gcd(k_1,k_2)}, \frac{k_2}{\gcd(k_1,k_2)}, \frac{k_3}{\gcd(k_1,k_2)}$ are all not larger than a, and $\frac{k_3 p_0}{\gcd(k_1,k_2)}, \frac{k_3 q_0}{\gcd(k_1,k_2)}$ are all not larger than a^2, resulting in a binary encoding length of each constraint of $O(\log a)$. As a consequence of the result in [9], given U_1, U_2, U_3, and w, our integer feasibility problem can be solved with at most $O(\log a)$ arithmetic operations on rational numbers of size $O(\log a)$.

The overall computational complexity of checking for bipartiteness a given Toeplitz graph $T_n(a, b, c)$ is thus given by the largest complexity among three, namely: $O(\log c)$, to check whether $\gcd(a, b, c) = 1$ and $c + \gcd(a, b) \le n \le a + b - \gcd(a, b)$; $O(\log a)$, to solve the corresponding integer feasibility problem; and the complexity of determining U_1, U_2, U_3, and w.

Therefore we are left with analyzing the complexity of computing U_1, U_2, U_3, and w. These quantities can be computed by deriving $\widehat{E}_n(a, b, c)$ from $E_n(a, b, c)$ in $n - b \le a - \gcd(a, b) < a$ steps, as described in the previous section, and we believe that it is not possible to compute them in logarithmic time, because their values depend on which "B" elements are present in $\widehat{E}_n(a, b, c)$, and not on their number.

By all the results above, we can conclude that a Toeplitz graph $T_n(a, b, c)$ in this class can be checked for bipartiteness in $O(a)$ operations. In the next section, we shall characterize several graphs in this class, whose bipartiteness can be checked without solving the corresponding diophantine equation. In all these cases, the complexity drops to $O(\log c)$ operations again.

To conclude, we focus on the convenience of computing U_1, U_2, U_3, and w from $\widehat{E}_n(a, b, c)$: we shall see that the larger $\lfloor \frac{b}{a} \rfloor$ is, the larger the convenience of computing U_1, U_2, U_3, and w from $\widehat{E}_n(a, b, c)$. Let $q = \lfloor \frac{b}{a} \rfloor$; then $qa \le b < (q+1)a$, and thus $qa \le b < c + \gcd(n, a) \le n \le\le a + b - gcd(a, b) < (q+2)a$; that is to say, $\frac{1}{q+2} < \frac{a}{n} < \frac{1}{q}$, proving the claim. In fact, notice that a $T_n(a, b, c)$ verifying $c + \gcd(a, b) \le n \le a + b - \gcd(a, b)$ gives rise to the same integer feasibility problem originated by $T_{n'}(a', b', c')$, where $a' = a$, $b' = a + b \bmod a$ if $\lfloor \frac{b}{a} \rfloor$ is odd and $b' = 2a + b \bmod a$ if $\lfloor \frac{b}{a} \rfloor$ is even, $c' = b' + (c - b)$, and $n' = b' + (n - b)$.

4.3.2. Some easy cases

In the last section, we showed that computing some of the data of the constraints which a solution of the diophantine equation has to satisfy takes $n - b$ steps. In this section, we characterize some cases whose CMCs are easily identified in the matrix $E_n(a, b, c)$. Observation 1, in particular, allows for immediately recognizing the CMCs corresponding to odd cycles. In other words, we now show that there are some cases where we do not need to solve Eq. (1), because we know *a priori* whether it has a constrained solution or not, and Theorem 10 applies. For all these cases, the dominating computational complexity is that of computing $\gcd(a, b, c)$, namely $O(\log c)$.

For the Toeplitz graphs $T_n(a, b, c)$ giving rise to matrices of either type, we can prove the following.

Lemma 4. *Consider a Toeplitz graph $T_n(a, b, c)$ verifying $\gcd(a, b, c) = 1$ and $c + \gcd(a, b) \le n \le a + b - \gcd(a, b)$. If $k_1 > n - b$ and $\lfloor \frac{b}{a} \rfloor$ is even, then $T_n(a, b, c)$ is bipartite.*

Proof. The assumption that $\lfloor \frac{b}{a} \rfloor$ is even implies that the $\mathrm{A}^{\mathrm{left}}$-diagonal is the even A-diagonal. The assumption that $k_1 > n-b$ implies that in the matrix $E_n(a, b, c)$ the elements of the even A-diagonal are alone in a row (see Fig. 8(a)); thus no CMC exists having a corner in one of these elements. The thesis follows from Observation 1. □

The following lemma applies to the Toeplitz graphs $T_n(a, b, c)$ giving rise to matrices of type 1 with no $\mathrm{A}^{\mathrm{right}}$-diagonal.

Lemma 5. *Consider a Toeplitz graph $T_n(a, b, c)$ verifying $\gcd(a, b, c) = 1$, $\lfloor \frac{b}{a} \rfloor = \lfloor \frac{c}{a} \rfloor$, and $c + \gcd(a, b) \le n \le \min\{a + b - \gcd(a, b); a(1 + \lfloor \frac{b}{a} \rfloor)\}$.*

- *If $\lfloor \frac{b}{a} \rfloor$ is odd, then $T_n(a, b, c)$ is bipartite.*
- *If k_2 is an odd multiple of k_1 and $\lfloor \frac{b}{a} \rfloor$ is even, then $T_n(a, b, c)$ is non-bipartite.*
- *If $k_1 \le n - b - 1$, k_1 is a multiple of k_2, and $\lfloor \frac{b}{a} \rfloor$ is even, then $T_n(a, b, c)$ is non-bipartite.*

Proof. We shall give a sketch of the proof for the third case only (all the proofs are very easy and similar). A possible schema for this case is depicted in Fig. 8(b), where $k_1 = 3k_2$. Consider the CMC drawn in the figure and represented by elements $e_{0,b}, e_{0,c}, e_{k_2,c}, e_{k_2,c+k_2}, e_{2k_2,c+k_2}, e_{2k_2,c+2k_2}, e_{3k_2,c+2k_2}, e_{k_1,b}$. We claim that it corresponds to an odd cycle. Since $e_{k_1,b}$ is the only element of the CMC belonging to the (unique) even diagonal of $E_n(a, b, c)$, the thesis follows. □

An example fitting into the third case of the lemma above is shown in Fig. 8(b). The corresponding diophantine equation is $(3k_2)y_1 + k_2y_2 + k_3y_3 = 0$. A solution (y_1, y_2, y_3) to it corresponds to an odd cycle iff y_1 is odd. It is trivial to see that $(y_1, y_2, y_3) = (1, -3, 0)$ is a feasible solution, and that it corresponds to the CMC in the figure. $T_{58}(11, 50, 52)$ and $T_{19}(6, 15, 16)$ are examples of Toeplitz graphs fitting into this structure.

Corollary 3. *Consider a Toeplitz graph $T_n(a, b, c)$ verifying $\gcd(a, b, c) = 1$, $\lfloor \frac{b}{a} \rfloor = \lfloor \frac{c}{a} \rfloor$, and $c + \gcd(a, b) \le n \le \min\{a + b - \gcd(a, b); a(1 + \lfloor \frac{b}{a} \rfloor)\}$. If $k_1 = k_2$ and $\lfloor \frac{b}{a} \rfloor$ is even, then $T_n(a, b, c)$ is non-bipartite.*

Finally, for the Toeplitz graph $T_n(a, b, c)$ giving rise to matrices of type 2, we have the following result.

Lemma 6. *Consider a Toeplitz graph $T_n(a, b, c)$ verifying $\gcd(a, b, c) = 1$, $\lfloor \frac{b}{a} \rfloor < \lfloor \frac{c}{a} \rfloor$ and $c + \gcd(a, b) \le n \le a + b - \gcd(a, b)$.*

- *If $k_1 = k_2 + k_3$ and $\lfloor \frac{b}{a} \rfloor$ is even, then $T_n(a, b, c)$ is non-bipartite.*
- *If k_2 is a multiple of k_1 and $\lfloor \frac{b}{a} \rfloor$ is odd, then $T_n(a, b, c)$ is non-bipartite.*
- *If k_2 is an odd multiple of k_1 and $\lfloor \frac{b}{a} \rfloor$ is even, then $T_n(a, b, c)$ is non-bipartite.*
- *If $k_1 \le n - b - 1$, k_1 is a multiple of k_2, and $\lfloor \frac{b}{a} \rfloor$ is even, then $T_n(a, b, c)$ is non-bipartite.*
- *If $k_1 \le n - b - 1$, k_1 is an odd multiple of k_2, and $\lfloor \frac{b}{a} \rfloor$ is odd, then $T_n(a, b, c)$ is non-bipartite.*
- *If $k_2 \le \frac{n-b-1}{2}$, k_2 is an even multiple of k_3, and $\lfloor \frac{c}{a} \rfloor$ is even, then $T_n(a, b, c)$ is non-bipartite.*
- *If k_3 is an even multiple of k_2 and $\lfloor \frac{c}{a} \rfloor$ is even, then $T_n(a, b, c)$ is non-bipartite.*

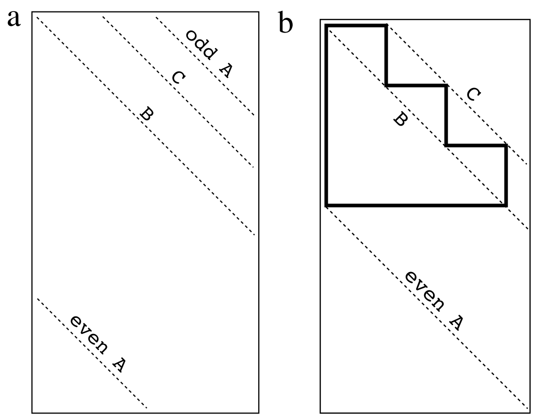

Fig. 8. Schema of the matrix $E_n(a, b, c)$ used in the proofs (a) of Lemma 4, and (b) of the third case of Lemma 5.

The Toeplitz graphs $T_{41}(11, 31, 40)$ and $T_{71}(16, 61, 70)$ are examples fitting into the lemma above (in the first case, and in the last one, respectively).

Corollary 4. *Consider a Toeplitz graph $T_n(a, b, c)$ verifying $\gcd(a, b, c) = 1$, $\lfloor \frac{b}{a} \rfloor < \lfloor \frac{c}{a} \rfloor$, and $c + \gcd(a, b) \le n \le a + b - \gcd(a, b)$. If $k_1 = k_2$, then $T_n(a, b, c)$ is non-bipartite.*

5. Generalizations

In this section, we show that some of the previous results can be extended to Toeplitz graphs with $k \ge 4$ entries, to infinite Toeplitz graphs, or to integer distance graphs. We observe that in all cases their complexity is $O(\log a_k)$.

The following is an extension of Lemma 1 to the Toeplitz graphs $T_n(a_1, \dots, a_k)$ with $k \ge 4$.

Lemma 7. *A Toeplitz graph $T_n(a_1, \dots, a_k)$ with $k \ge 3$, $\gcd(a_1, a_2, a_i) = 1$ for each $i \in \{3, \dots, k\}$ and $n \le a_3 + \gcd(a_1, a_2) - 1$ is bipartite if and only if $T_n(a_1, a_2)$ is bipartite.*

Combining the lemma above with Theorems 2 and 3, we get the following two theorems.

Theorem 12. *A Toeplitz graph $T_n(a_1, \dots, a_k)$ with $k \ge 3$, $\gcd(a_1, a_2, a_i) = 1$ for each $i \in \{3, \dots, k\}$, and $n \le \min\{a + b - \gcd(a_1, a_2); a_3 + \gcd(a_1, a_2) - 1\}$ is bipartite.*

Theorem 13. *A Toeplitz graph $T_n(a_1, \dots, a_k)$ with $k \ge 3$, $\gcd(a_1, a_2, a_i) = 1$ for each $i \in \{3, \dots, k\}$, and $a_1 + a_2 - \gcd(a_1, a_2) + 1 \le n \le a_3 + \gcd(a_1, a_2) - 1$ is bipartite if and only if $\frac{a_1}{\gcd(a_1, a_2)}$ and $\frac{a_2}{\gcd(a_1, a_2)}$ are odd.*

Notice that in Lemma 7 and Theorems 12 and 13 the upper bound on n limits the number and the value of the entries larger than a_3.

Finally, Theorem 9 can be generalized to Toeplitz graphs with $k \ge 4$ entries, as follows.

Theorem 14. *Consider a Toeplitz graph $T_n(a_1, \dots, a_k)$ with $k \ge 3$, and let $\alpha_1, \alpha_2, \alpha_3 \in \{a_1, \dots, a_k\}$ be three distinct entries verifying $\gcd(\alpha_1, \alpha_2, \alpha_3) = 1$. Let $n \ge \max\{\alpha_i + \alpha_j - \gcd(\alpha_i, \alpha_j) + 1; \alpha_h + \gcd(\alpha_i, \alpha_j)\}$ for $\{i, j, h\} = \{1, 2, 3\}$. Then $T_n(a_1, \dots, a_k)$ is bipartite if and only if $a_1, \dots, a_k$ are odd.*

Clearly, if more than one triple $\alpha_1, \alpha_2, \alpha_3$ exists verifying the conditions in the theorem above, the strongest lower bound for n is the one given by a triple which minimizes $\max\{\alpha_i + \alpha_j - \gcd(\alpha_i, \alpha_j) + 1; \alpha_h + \gcd(\alpha_i, \alpha_j)\}$.

Some of the results of the previous sections can be extended to other graph classes: Theorem 13 applies to integer distance graphs $G_{\mathbb{Z}}(a_1, a_2, \dots, a_k)$ (the result was already known: see [1,2,4]) and Theorem 9 applies to connected integer distance graphs $G_{\mathbb{Z}}(a, b, c)$.

In addition, since $\infty > \max\{a + b - \gcd(a, b) + 1; c + \gcd(a, b)\}$, from Theorems 6 and 9, we immediately get the next result, which follows directly also from the results in [7].

Corollary 5. *An infinite Toeplitz graph with three entries a, b, c is bipartite if and only if $\frac{a}{\gcd(a,b,c)}$, $\frac{b}{\gcd(a,b,c)}$, $\frac{c}{\gcd(a,b,c)}$ are odd.*

6. Conclusions

In this paper, we first provide an $O(\log b)$ closed-form characterization for bipartite Toeplitz graphs $T_n(a, b)$, based on easy topological properties of such graphs.

Building on it, we also completely characterize bipartite Toeplitz graphs with three entries. The proposed result allows for checking the bipartiteness of any given Toeplitz graph $T_n(a, b, c)$. In particular, the Toeplitz graphs $T_n(a, b, c)$ are subdivided into three different subclasses. In the first two of them, the characterization consists in verifying a simple condition on n, a, b, c, which takes $O(\log c)$ arithmetic operations. The bipartiteness of the graphs in the third subclass is proved to depend on the existence of a suitable solution to a linear diophantine equation in three variables. Finding such a solution takes $O(\log c)$ operations, but computing the maximum values the variables can assume takes $O(a)$. Some particular cases in the third subclass are also identified: detecting their bipartiteness takes $O(\log c)$ operations, because it does not require the solution of the diophantine equation. We remark that all the Toeplitz graphs with two or three entries can be analyzed in the context of constrained diophantine equations. Nevertheless, the specific conditions that we propose for some particular cases in Sections 3, 4.1, 4.2, and 4.3.2 are much easier to verify.

The results proved in the paper and those in [10] completely solve the open problem of determining the chromatic number of Toeplitz graphs with three entries.

Some of the results proved in the paper are extended to Toeplitz graphs with $k \geq 4$ entries, to infinite Toeplitz graphs, and to integer distance graphs. It seems to us that the framework of constrained diophantine equations, too, can be generalized to Toeplitz graphs with $k \geq 4$ entries, but the exact analysis of this problem is left for a forthcoming paper.

Finally, we remark that in Example 1 at the end of [5] the Toeplitz graphs $T_{2a+1}(a, b, c)$ verifying $a < b < c < n = 2a + 1$, $a = (c - b) + a \bmod (c - b)$, and $a < 3(b - a)$ (thus, $c - b \leq a \leq \min\{2(c - b) - 1, 3(b - a) - 1\}$) are said to be bipartite. However, there are *non-bipartite* Toeplitz graphs fitting into these assumptions (for example, $T_{17}(8, 11, 16)$, where vertices v_0, v_8, v_{16} induce an odd cycle). This fact has been confirmed in [6].

As for Example 2 at the end of [5], it recognizes bipartite Toeplitz graphs $T_{2a+1}(a, b, c)$ fitting into given assumptions, but from this result we cannot derive that $T_n(a, b, c)$ for $n \geq 2a+1$ is bipartite: as an example, $T_{37}(18, 23, 35)$ is a bipartite graph fitting into the assumptions of Example 2 of [5], but from Theorem 9 of the present paper we know that there must exist a threshold value μ such that $T_n(18, 23, 35)$ with $n \geq \mu$ is non-bipartite (in this case, by Theorem 9 we get that $T_n(18, 23, 35)$ is non-bipartite for $n \geq 41$; actually, in application of Theorem 10, if we search for a constrained solution to the diophantine equation (1) we find that $T_n(18, 23, 35)$ is bipartite for $n \leq 38$ and non-bipartite for $n \geq 39$).

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References

[1] G. Chappell, Coloring distance graphs on the integers, 1998. Manuscript. ArXiv:math/9805084v1.
[2] J. Chen, G. Chang, K. Huang, Integral distance graphs, J. Graph Theory 25 (1997) 287–294.
[3] H. Cohen, Number Theory: Vol. I: Tools and Diophantine Equations, in: Springer Graduate Texts in Mathematics, vol. 239, 2007.
[4] R. Eggleton, P. Erdös, D. Skilton, Colouring the real line, J. Combin. Theory Ser. B 39 (1985) 86–100.
[5] R. Euler, Characterizing bipartite Toeplitz graphs, Theoret. Comput. Sci. 263 (2001) 47–58.
[6] R. Euler, Private communication, July 2011.
[7] R. Euler, H. Le Verge, T. Zamfirescu, A characterization of infinite, bipartite Toeplitz graphs, in: Ku Tung-Hsin (Ed.), Combinatorics and Graph Theory'95, Vol. 1, 1995, pp. 119–130.
[8] C. Heuberger, On Hamiltonian Toeplitz graphs, Discrete Math. 245 (2002) 107–125.
[9] H. Lenstra, Integer programming with a fixed number of variables, Math. Oper. Res. 8 (1983) 538–548.
[10] S. Nicoloso, U. Pietropaoli, On the chromatic number of Toeplitz graphs, Discrete Appl. Math. (2011) http://dx.doi.org/10.1016/j.dam.2011.07.012, in press.
[11] L.D. Penso, D. Rautenbach, J. Szwarcfiter, Long cycles and paths in distance graphs, Discrete Math. 310 (2010) 3417–3420.