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A new analysis for Klein-Gordon model with local fractional derivative

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Abstract This work adopts Yang's local fractional derivative to define the fractional Klein-Gordon equation in a fractal space or microgravity space. The variational principle of local fractional Klein-Gordon equation is successfully established via fractional semi-inverse transform method and the classical He's variational iteration method (HVIM) is used to obtain its approximate analytical solution.

1. Introduction

Fractional calculus has a history of over 300 years. Fractional calculus is an excellent mathematical tool to elaborate mathematical modeling. At present, fractional calculus has found applications in the fields of physics and materials science, such as nanotechnology, artificial intelligence, and diffusion processes and so on. There are many different definitions of fractional calculus. By far the most common are Riemann-Liouville definition, Caputo definition, He's fractional definition, etc [1–3]. The approximate analytical solution of a fractional differential equation can be gained by the Homotopy perturbation method (HPM), Homotopy analysis method (HPM), Taylor series method (TSM) and so on [4–9].

The classical Klein-Gordon equation is given as follows [10,11]

$$\frac{D^2\omega(x,t)}{Dt^2}-\frac{D^2\omega(x,t)}{Dx^2}-\lambda\omega(x,t)-\beta\omega^2(x,t)-\gamma\omega^3(x,t)=0, \tag{1.1}$$

with the initial condition

$$\omega(x,0)=\omega_0, \tag{1.2}$$

where λ,β,γ are the known to be constants. The traditional Klein-Gordon equation is very important in physics. It is well known that many problems in quantum mechanics and classical mechanics, soliton and material physics can be modeled by the linear and nonlinear Klein-Gordon equations. For example, the classical nonlinear Klein-Gordon equation can be used to elaborate the dislocations in crystals, Josephson junction, and a rigid swing attached to a drawn wire. When a rigid pendulum attached to a stretched wire vibrates in a fractal or microgravity space [12–14]. These traditional derivatives will not elaborate on the new physical model. For example, in Fig. 1, A spring oscillates in a fractal space that can not be described by conventional derivative. The Yang's local fractional derivative [15–20] will have to be adopted to establish the new mathematical oscillatory model as follows

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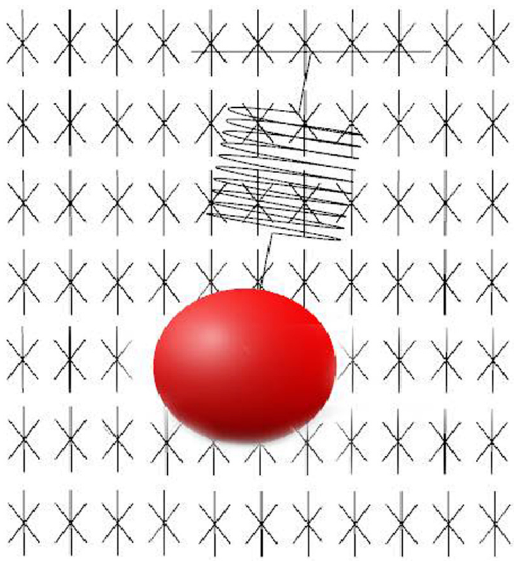

Fig. 1 A spring oscillates in a fractal space.

$$\frac{{}^{Y\rho}D^{2\rho}\Im}{{}_{Loc}Dt^{2\rho}}+\lambda\Im=0, \tag{1.3}$$

where the ${}^{Y\rho}D^{\rho}\Im/{}_{Loc}Dt^{\rho}$ is called Yang's local fractional derivative.

Therefore, when a rigid swing attached to a drawn wire vibrates in microgravity space, the Eq. (1.1) does not perfectly describe this physical phenomenon. Yang's local fractional derivative can be adopted to elaborate on the modeling. The local fractional Klein-Gordon equation is established as follows

$$\frac{{}^{Y\varsigma}D^{2\varsigma}\omega(x,t)}{{}_{Loc}Dt^{2\varsigma}}-\frac{D^{2}\omega(x,t)}{Dx^{2}}-F(\omega)=0, \tag{1.4}$$

with the initial condition

$$\omega(x,0)=\omega_0, \tag{1.5}$$

where ${}^{Y\varsigma}D^{\varsigma}\omega/{}_{Loc}Dt^{\varsigma}$ is the Yang's local fractional derivative and $F(\omega)$ is a function.

In this work, we will adopt the semi-inverse method [21–27] to obtain the variational principle for the local fractional Klein-Gordon equation, and He's variational iteration method (HVIM) [28–37] is used to gain its approximate analytical solution in a fractal or microgravity space [38].

2. Yang's local fractional derivative

Definition 1. The definition of Yang's local fractional derivative is given as follows [15–20]

$$LocY\varepsilon D_{\xi}^{(\varepsilon)}\wp(\xi_0)=\wp^{(\varepsilon)}(\xi_0)=\left.\frac{d^{\varepsilon}\wp(\xi)}{d\xi^{\varepsilon}}\right|_{\xi=\xi_0}=\lim_{\xi\to\xi_0}\frac{\Delta^{\varepsilon}(\wp(\xi)-\wp(\xi_0))}{(\xi-\xi_0)^{\varepsilon}}, \tag{2.1}$$

where $\Delta^{\varepsilon}(\wp(\xi)-\wp(\xi_0))\cong\Gamma(1+\varepsilon)\Delta(\wp(\xi)-\wp(\xi_0))$.

Definition 2. Some special functions of Yang's local fractional derivative is defined as follows

$$LocY\beta E_{\beta}(\tau^{\beta})=\sum_{\lambda=0}^{\infty}\frac{\tau^{\lambda\beta}}{\Gamma(1+\lambda\beta)}, \tag{2.2}$$

$$LocY\beta\sin_{\beta}(\tau^{\beta})=\sum_{\eta=0}^{\infty}(-1)^{\eta}\frac{\tau^{(2\eta+1)\beta}}{\Gamma[1+(2\eta+1)\beta]}, \tag{2.3}$$

$$LocY\beta Cos_{\beta}(\tau^{\beta})=\sum_{\eta=0}^{\infty}(-1)^{\eta}\frac{\tau^{2\eta\beta}}{\Gamma(1+2\eta\beta)}, \tag{2.4}$$

$$LocY\beta Sinh_{\beta}(\tau^{\beta})=\frac{LocY\beta E_{\beta}(\tau^{\beta})+LocY\beta E_{\beta}(-\tau^{\beta})}{2},$$

$$LocY\beta Cosh_{\beta}(\tau^{\beta})=\frac{LocY\beta E_{\beta}(\tau^{\beta})-LocY\beta E_{\beta}(-\tau^{\beta})}{2}, \tag{2.5}$$

where $\tau\in R$, $0<\beta<1$.

Definition 3. The Yang's local fractional calculus of some non-differentiable functions is as follows

$$\frac{{}^{Y\varsigma}D^{\varsigma}}{{}_{Loc}D\tau^{\varsigma}}\left(\frac{\tau^{n\varsigma}}{\Gamma(1+n\varsigma)}\right)=\frac{\tau^{(n-1)\varsigma}}{\Gamma(1+(n-1)\varsigma)}, \tag{2.6}$$

$$\frac{{}^{Y\varsigma}D^{\varsigma}}{{}_{Loc}D\tau^{\varsigma}}\left(LocY\varsigma E_{\varsigma}(\tau^{\varsigma})\right)=LocY\varsigma E_{\varsigma}(\tau^{\varsigma}), \tag{2.7}$$

$$\frac{{}^{Y\varsigma}D^{\varsigma}}{{}_{Loc}D\tau^{\varsigma}}\left(LocY\varsigma Sin_{\varsigma}(\tau^{\varsigma})\right)=LocY\varsigma Cos_{\varsigma}(\tau^{\varsigma}), \tag{2.8}$$

$$\frac{{}^{Y\varsigma}D^{\varsigma}}{{}_{Loc}D\tau^{\varsigma}}\left(LocY\varsigma Cos_{\varsigma}(\gamma^{\varsigma})\right)=LocY\varsigma Sin_{\varsigma}(\tau^{\varsigma}), \tag{2.9}$$

$$LocY\varsigma J_{\gamma}^{\varsigma}\left(\frac{\tau^{n\varsigma}}{\Gamma(1+n\varsigma)}\right)=\frac{\tau^{(n+1)\varsigma}}{\Gamma(1+(n+1)\varsigma)}. \tag{2.10}$$

3. The He's variational iteration method (HVIM)

The He's Variational iteration method (HVIM) is an excellent mathematical tool to solve the linear or nonlinear fractional differential equation. In this work, the HVIN is used to solve the local fractional differential equation.

Consider the following differential equation:

$$\ell\varpi+\Xi\varpi=\Omega(x), \tag{3.1}$$

where ℓ and Ξ are linear and nonlinear operators, respectively.

According to the He's variational theory, we can easily gain a correct equation for Eq. (3.1), which reads such that

$$\varpi_{n+1}(x)=\varpi_n(x)+\int_0^x\chi\left\{\ell\Re_n(\xi)+\Xi\hat{\Re}_n(\xi)-\Omega(\xi)\right\}d\xi. \tag{3.2}$$

In Eq. (3.2), the χ can be gained by the variational theory. The right of Eq.(3.2) is the correction term. The $\hat{\varpi}_n$ is the restricted variation. The $\hat{\varpi}_n$ satisfies the following formula

$$\delta\hat{\varpi}_n=0. \tag{3.3}$$

Therefore, the approximate analytical solution of Eq. (3.1) is gained as follows

$$\varpi=\lim_{n\to\infty}\varpi_n. \tag{3.4}$$

4. Variational principle for local fractional Klein-Gordon equation

The variational principle of local fractional Klein-Gordon can be organized by a semi-inverse transform method.

Consider the local fractional Klein-Gordon equation as follows

$$\frac{{}^{Y_\varsigma}D^{2\varsigma}\omega(x,t)}{{}_{Loc}Dt^{2\varsigma}} - \frac{D^2\omega(x,t)}{Dx^2} - F(\omega) = 0. \quad (4.1)$$

Let $F(\omega) = a\omega(x,t) + b\omega^2(x,t) + c\omega^3(x,t)$, and the Eq. (4.1) can be rewritten into the following form

$$\frac{{}^{Y_\varsigma}D^{2\varsigma}\omega(x,t)}{{}_{Loc}Dt^{2\varsigma}} - \frac{D^2\omega(x,t)}{Dx^2} - a\omega(x,t) - b\omega^2(x,t) - c\omega^3(x,t) = 0. \quad (4.2)$$

The variational principle of Eq. (4.2) can be given according to the semi-inverse transform method as follows

$$J(\omega) = \int_{t_0}^{t_1}\int\left\{\left\{-\frac{1}{2}\left(\frac{{}^{Y_\varsigma}D^\varsigma\omega}{{}_{Loc}Dt^\varsigma}\right)^2 + \frac{1}{2}\left(\frac{D\omega}{Dx}\right)^2 - \frac{a}{2}\omega^2 - \frac{b}{3}\omega^3 - \frac{c}{4}\omega^4\right\}dt^\varsigma\right\}dx. \quad (4.3)$$

The Euler-Lagrange equation of Eq. (4.3) can be written in the following form

$$\frac{{}^{Y_\varsigma}D^\varsigma}{{}_{Loc}Dt^\varsigma}\left(\frac{{}^{Y_\varsigma}D^\varsigma\omega}{{}_{Loc}Dt^\varsigma}\right) - \frac{D}{Dx}\left(\frac{D\omega}{Dx}\right) - a\omega - b\omega^2 - c\omega^3 = 0. \quad (4.4)$$

According to chain ruler, Eq. (4.4) is similar to Eq. (4.1).

5. Variational iteration method for local fractional Klein-Gordon equation

Consider the local fractional Klein-Gordon equation as follows

$$\frac{{}^{Y_\varsigma}D^{2\varsigma}\omega}{{}_{Loc}Dt^{2\varsigma}} - \frac{D^2\omega}{Dx^2} + \frac{3\omega}{4} - \frac{3\omega^3}{2} = 0, \quad (5.1)$$

with the initial conditions

$$\omega(x,0) = -\mathrm{sech}(x), \quad \omega_t(x,0) = \frac{1}{2}\mathrm{sech}(x)\tanh(x). \quad (5.2)$$

We adopt HVIM to obtain the approximate analytical solution for Eq. (5.1). Now, we establish a new correction equation as follows

$$\omega_{n+1}(x,t) = \omega_n(x,t) + \int_0^t \chi\left\{\frac{{}^{Y_\varsigma}D^{2\varsigma}\omega(x,\xi)}{{}_{Loc}Dt^{2\varsigma}} - \frac{D^2\tilde{\omega}(x,\xi)}{Dx^2} + \frac{3}{4}\tilde{\omega}(x,\xi) - \frac{3}{2}\tilde{\omega}^3(x,\xi)\right\}d\xi. \quad (5.3)$$

By the variational method, we have

$$\chi''(\xi) = 0, 1 - \chi'|_{\xi=t} = 0, \chi_{\xi=x} = 0. \quad (5.4)$$

So, we have

$$\chi(\xi) = \xi - t. \quad (5.5)$$

We obtain the variational iteration formula as follows

$$\omega_{n+1}(x,t) = \omega_n(x,t) + \int_0^t (\xi - t)\left\{\frac{D^{2\varsigma}\omega(x,\xi)}{Dt^{2\varsigma}} - \frac{D^2\tilde{\omega}(x,\xi)}{Dx^2} + \frac{3}{4}\tilde{\omega}(x,\xi) - \frac{3}{2}\tilde{\omega}^3(x,\xi)\right\}d\xi, \quad (5.6)$$

$$\omega_0(x,t) = -\mathrm{sech}(x) + \frac{t^\varsigma}{2\Gamma(1+\varsigma)}\mathrm{sech}(x)\tanh(x). \quad (5.7)$$

According to Eq. (5.6) and Eq. (5.7), we obtain the following results by MAPLE.17 software

$$\omega_1(x,t) = -\mathrm{sech}(x) + \frac{t^\varsigma}{2\Gamma(1+\varsigma)}\mathrm{sech}(x)\tanh(x) - \frac{t^{2\varsigma}}{8\Gamma(1+2\varsigma)}(\mathrm{sech}(x) - 2\mathrm{sech}^3(x)) + \frac{t^{3\varsigma}}{48\Gamma(1+3\varsigma)}\mathrm{sech}(x)\tanh(x)(1 - 6\mathrm{sech}^2(x)) + \cdots,$$

$$\omega_2(x,t) = -\mathrm{sech}(x) + \frac{t^\varsigma}{2\Gamma(1+\varsigma)}\mathrm{sech}(x)\tanh(x) - \frac{t^{2\varsigma}}{8\Gamma(1+2\varsigma)}(\mathrm{sech}(x) - 2\mathrm{sech}^3(x)) + \frac{t^{3\varsigma}}{48\Gamma(1+3\varsigma)}\mathrm{sech}(x)\tanh(x)(1 - 6\mathrm{sech}^2(x)) - \frac{t^{4\varsigma}\mathrm{sech}(x)}{384\Gamma(1+4\varsigma)}(1 - 20\mathrm{sech}^2(x) + 24\mathrm{sech}^4(x)) + \frac{t^{5\varsigma}}{3840\Gamma(1+5\varsigma)}\mathrm{sech}(x)\tanh(x)(1 - 60\mathrm{sech}^2(x) + 120\mathrm{sech}^4(x)) + \cdots,$$

$$\begin{aligned}\omega_3(x,t) = &-\mathrm{sec}\ h(x) + \tfrac{t^\varsigma}{2\Gamma(1+\varsigma)}\mathrm{sec}\ h(x)\tanh(x) \\ &- \tfrac{t^{2\varsigma}}{8\Gamma(1+2\varsigma)}(\mathrm{sec}\ h(x) - 2\mathrm{sec}\ h^3(x)) \\ &+ \tfrac{t^{3\varsigma}}{48\Gamma(1+3\varsigma)}\mathrm{sec}\ h(x)\tanh(x)(1 - 6\mathrm{sec}\ h^2(x)) \\ &- \tfrac{t^{4\varsigma}\mathrm{sec}\ h(x)}{384\Gamma(1+4\varsigma)}(1 - 20\mathrm{sec}\ h^2(x) + 24\mathrm{sec}\ h^4(x)) \\ &+ \tfrac{t^{5\varsigma}}{3840\Gamma(1+5\varsigma)}\mathrm{sec}\ h(x)\tanh(x)(1 - 60\mathrm{sec}\ h^2(x) \\ &+ 120\mathrm{sec}\ h^4(x)) \\ &- \tfrac{t^{6\varsigma}}{46080\Gamma(1+6\varsigma)}\mathrm{sec}\ h(x)(1 - 182\mathrm{sec}\ h^2(x) \\ &+ 840\mathrm{sec}\ h^4(x) - 720\mathrm{sec}\ h^6(x)) \\ &- \tfrac{t^{6\varsigma}\mathrm{sec}\ h(x)\tanh(x)}{46080\Gamma(1+6\varsigma)}(1 - 546\mathrm{sec}\ h^2(x) + 4200\mathrm{sec}\ h^4(x) \\ &- 5040\mathrm{sec}\ h^6(x)) + \cdots\end{aligned} \quad (5.8)$$

When $\varsigma = 1$ the Eq. (5.1) is the classical Klein-Gordon equation, and it has the exact solution as follows

$$\omega(x,t) = -\mathrm{sech}(x + 0.5t). \quad (5.9)$$

Fig. 2 shows the error between the exact solution and the approximate analytical solution when $\varsigma = 1$. In this work, we have only calculated three times, if we want to improve the accuracy of our analytical solution, we only need to calculate more times. Figs. 3–6 shows that the variation of approximate analytical solution for different ς. These fully demonstrate that our method is accurate and efficient.

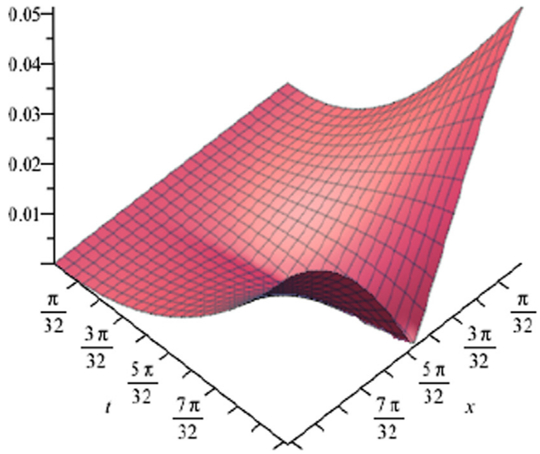

Fig. 2 The error between the exact solution and the approximate solution.

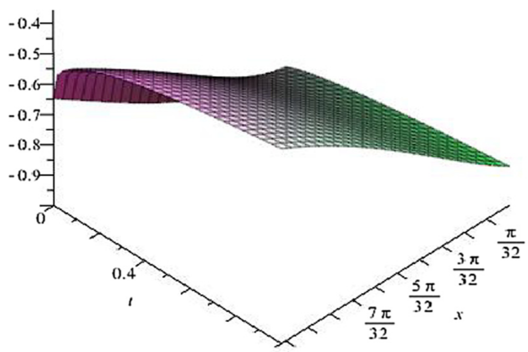

Fig. 3 $\varsigma = 0.3$.

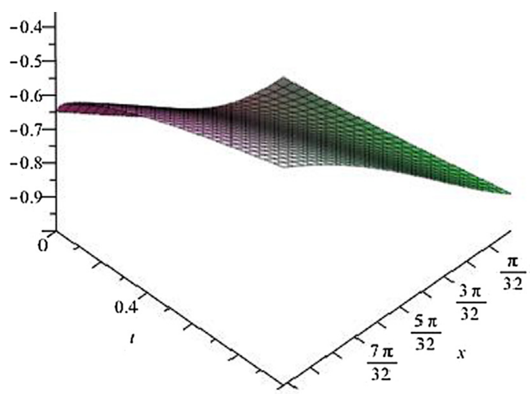

Fig. 4 $\varsigma = 0.6$.

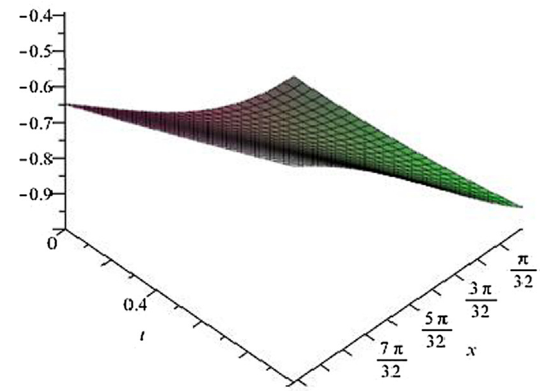

Fig. 5 $\varsigma = 0.8$.

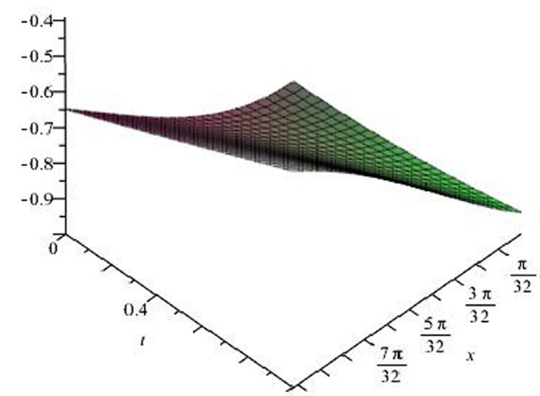

Fig. 6 $\varsigma = 1$.

6. Conclusion

In this work, a new modeling of a rigid swing attached to a drawn wire vibrates in microgravity space is successfully organized on the basis of Yang's local fractional derivative. The variational principle of the local fractional modeling via fractional semi-inverse transform method, and He's variational iteration method (HVIM) is adopted to find its approximate analytical solution. The variational principle can be adopted to construct conservation laws and suggest solution structures of the solitary waves. We will discuss this property in a forthcoming paper.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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