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The interplay of different symmetries in quantum mechanical potentials

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Abstract. We construct an so(2,2) potential algebra and discuss how it is influenced when $\mathcal{PT}$ symmetry is imposed on the potential. We illustrate the procedure with the $\mathcal{PT}$ symmetric Scarf II potential.

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1. $\mathcal{PT}$ symmetry of potentials

With the introduction of $\mathcal{PT}$ symmetric quantum mechanics [1] the investigation of non-Hermitian quantum mechanical problems has received much attention in the past couple of years. In $\mathcal{PT}$ symmetric quantum mechanics the potentials defined in one dimension are invariant under the simultaneous action of the space and time reflection operations $\mathcal{P}$ and $\mathcal{T}$, and have the property $[V(-x)]^* = V(x)$. The first notable finding was that despite being complex, these potentials often have real bound-state energy spectrum, and this was interpreted as the consequence of $\mathcal{PT}$ symmetry. However, it was soon noticed that $\mathcal{PT}$ symmetry is neither a necessary, nor a sufficient condition for having real energy spectrum in a complex potential. It is not a sufficient condition, because the energy eigenvalues may also appear in complex conjugated pairs, in which case the eigenfunctions cease to be eigenfunctions of the $\mathcal{PT}$ operator, and this scenario has been interpreted as the spontaneous breakdown of PT symmetry [1]. Neither is $\mathcal{PT}$ symmetry a necessary condition, because there are complex non-$\mathcal{PT}$ symmetric potentials with real energy eigenvalues [2].

More recently $\mathcal{PT}$ symmetric quantum mechanics was put into a more general context as the special case of pseudo-Hermiticity [3], and this also accounted for the modified inner product and the pseudo-norm [4, 5] used in $\mathcal{PT}$ symmetric quantum mechanics.

After the first numerical examples [1], a number of exactly solvable $\mathcal{PT}$ symmetric potentials have been derived, mainly as the $\mathcal{PT}$ symmetric versions of conventional solvable potentials (see e.g. [6] and references therein). The analysis of these problems showed that due to the generally less strict boundary conditions, $\mathcal{PT}$ symmetric potentials have two sets of normalizable solutions, which can be distinguished with the introduction of the quasi-parity quantum number $q = \pm 1$ [7].

2. Potential algebras and $\mathcal{PT}$ symmetry

A more conventional symmetry concept related to quantum mechanical potentials is that of the potential algebra [8]. Potential algebras are somewhat similar to degeneracy algebras in the sense that the elements of the algebra connect degenerate levels which, however, belong to different Hamiltonians, i.e. potentials of the same type with different depth. Furthermore, in the case of non-compact potential algebras scattering and resonance states can also be discussed in a group theoretical framework, in addition to bound states.

It is an interesting task to investigate how the potential algebras are influenced by $\mathcal{PT}$ symmetry. In this respect the doubling of the normalizable states is an especially interesting question. In the first investigations combining these two symmetry concepts the sl(2,C) [9] and the su(1,1)$\simeq$so(2,1) [10] algebras were used, which contain only a single pair of ladder operators. However, one expects that another set of them should be defined in relation with the second set of normalizable solutions, so the potential algebra should be enlarged. The most obvious choice is considering the so(2,2)$\simeq$so(2,1)$\oplus$so(2,1) algebra. This algebra has been studied in the case of Hermitian (real) potentials [8], and here we discuss it for $\mathcal{PT}$ symmetric potentials.

Let us consider the commutation relations defining the so(2,2) algebra [11]

$$[J_z, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = -2aJ_z \ , \qquad [J_i, K_j] = 0 \tag{1}$$

$$[K_z, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = -2bK_z \ , \qquad i, j = +,\ -,\ z \ , \tag{2}$$

which also includes the so(4) and so(3,1) algebras for $a = b = -1$ and $a = -b = \pm 1$, respectively. A straightforward coordinate realization of this algebra can be made by

$$J_\pm = \mathrm{e}^{\pm \mathrm{i}\phi}\left(\pm h_1(x)\frac{\partial}{\partial x} \pm g_1(x) + f_1(x)J_z + c_1(x) + k_1(x)K_z\right) \ , \qquad J_z = -\mathrm{i}\frac{\partial}{\partial \phi} \ , \tag{3}$$

$$K_\pm = \mathrm{e}^{\pm \mathrm{i}\chi}\left(\pm h_2(x)\frac{\partial}{\partial x} \pm g_2(x) + f_2(x)J_z + c_2(x) + k_2(x)K_z\right) \ , \qquad K_z = -\mathrm{i}\frac{\partial}{\partial \chi} \ . \tag{4}$$

We find that the algebra defined in (1) and (2) is obtained if the following relations hold:

$$k_2^2 - h_2 k_2' = b \qquad h_2 f_2' - f_2 k_2 = 0 \qquad k_2^2 - f_2^2 = b \qquad c_1 = c_2 = 0 \ , \tag{5}$$

$$h_1 = Ah_2 \qquad f_1 = Ak_2 \qquad k_1 = Af_2 \qquad g_1 = Ag_2 \qquad A^2 = \frac{a}{b} = \pm 1 \ . \tag{6}$$

The Casimir invariant

$$C_2^{(JK)} = 2C_2^{(J)} + 2C_2^{(K)} \equiv 2\left(-aJ_+J_- + J_z^2 - J_z - bK_+K_- + K_z^2 - K_z\right) \tag{7}$$

is then a second-order differential operator and its eigenfunctions $\Psi \equiv \Psi(x,\phi,\chi) = \mathrm{e}^{\mathrm{i}(m\phi+m'\chi)}\psi(x)$ are also those of J_z and K_z. Here $\psi(x)$ is the physical wavefunction depending on the coordinate x, while ϕ and χ are auxiliary variables, which are multiplied with m and m', the eigenvalues of generators J_z and K_z, respectively.

With the additional condition $g_2 = \frac{1}{2}(k_2 - h_2')$ the linear derivative can be eliminated and the eigenvalue equation of $C_2^{(JK)}$ then takes the form of the Schrödinger-like equation

$$\begin{aligned} C_2^{(JK)}\Psi &= 4bh_2^2\Psi'' + [b((h_2')^2 + k_2^2 - 2h_2''h_2) - 2 \\ &\quad +4(1-bk_2^2)(J_z^2+K_z^2) - 8bf_2k_2J_zK_z]\Psi \\ &= \omega(\omega+2)\Psi \ . \end{aligned} \tag{8}$$

When h_2 is a constant, the Hamiltonian is simply proportional with the Casimir operator, and thus there will be a set of degenerate energy levels of *different* Hamiltonians connected by the generators, and we obtain a potential algebra. (When these Hamiltonians are identical, we get a symmetry or degeneracy algebra, but this cannot occur in the case of one-dimensional potentials, because these are forbidden under rather general conditions.) Before closing this section we mention that the generators transform under the $\mathcal{PT}$ operation as

$$\mathcal{PT}(J/K)_\pm(\mathcal{PT})^{-1} = (J/K)_\mp \ , \qquad \mathcal{PT}(J/K)_z(\mathcal{PT})^{-1} = -(J/K)_z \ . \tag{9}$$

3. An illustration: the Scarf II potential

The Scarf II potential is obtained by substituting $h_2 = 1$, $g_2 = -\frac{1}{2}\tanh x$, $f_2 = \mathrm{i}/\cosh x$, $k_2 = -\tanh x$ and $a = b = 1$ [11]:

$$V(x) = -\left(m^2 + m'^2 - \frac{1}{4}\right)\frac{1}{\cosh^2 x} - 2\mathrm{i}mm'\frac{\sinh x}{\cosh^2 x} \tag{10}$$

and the so(2,2) generators are

$$J_\pm = \mathrm{e}^{\pm\mathrm{i}\phi}\left(\pm\frac{\partial}{\partial x} - \tanh x(J_z \pm \frac{1}{2}) + \frac{\mathrm{i}}{\cosh x}K_z\right) \ , \tag{11}$$

$$K_\pm = \mathrm{e}^{\pm\mathrm{i}\chi}\left(\pm\frac{\partial}{\partial x} - \tanh x(K_z \pm \frac{1}{2}) + \frac{\mathrm{i}}{\cosh x}J_z\right) \ . \tag{12}$$

If m and m' are real, potential (10) is $\mathcal{PT}$-symmetric, while its Hermitian version is obtained if m or m' is imaginary.

The first independent solution is

$$F_1(x) = (1+\mathrm{i}y)^{\frac{m'-m}{2}+\frac{1}{4}}(1-\mathrm{i}y)^{-\frac{m+m'}{2}+\frac{1}{4}}F(-m+\frac{1}{2}-\mathrm{i}k, -m+\frac{1}{2}+\mathrm{i}k, m'-m+1; \frac{1+\mathrm{i}y}{2}) \tag{13}$$

with $y = \mathrm{i}\sinh x$, while the second one is obtained by the $m \leftrightarrow m'$ transformation. These solutions lead to discrete eigenvalues when $k = \mathrm{i}(m - n - \frac{1}{2})$ and $k = \mathrm{i}(m' - n - \frac{1}{2})$ holds for the two solutions, respectively [10].

Multiplying (13) with the usual phase factors, we find that the effect of the ladder operators on the $\Psi_1(m, m'; x) = \mathrm{e}^{\mathrm{i}(m\phi + m'\chi)} F_1(x)$ is $J_{\pm}\Psi_1(m, m'; x) \rightarrow \Psi_1(m \pm 1, m'; x)$, $K_{\pm}\Psi_1(m, m'; x) \rightarrow \Psi_1(m, m' \pm 1; x)$, while the action of the generators on the second independent solution is obtained by the replacements: $J \leftrightarrow K$, $m \leftrightarrow m'$. It has also been shown that the effect of the ladder operators (11) and (12) is the same as that of the supersymmetric shift operators, defined also in two varieties, correspondig to quasi-parity $q = \pm 1$ [12].

Finally, we note that the transmission and reflection coefficients of the Scarf II potential have been derived [10], and the spontaneous breakdown of $\mathcal{PT}$ symmetry has also been studied for it [13].

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