The Geometry of classical change of signature

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Abstract

The proposal of the possibility of change of signature in quantum cosmology has led to the study of this phenomenon in classical general relativity theory, where there has been some controversy about what is and is not possible. We here present a new analysis of such a change of signature, based on previous studies of the initial value problem in general relativity. We emphasize that there are various continuity suppositions one can make at a classical change of signature, and consider more general assumptions than made up to now. We confirm that in general such a change can take place even when the second fundamental form of the surface of change does not vanish.

1 Introduction

Following on recent developments in quantum cosmology [1-3], a subject of some interest is the possibility of a change of signature in a classical space-time [4-12]. We discuss here in depth the geometry associated with such a classical change of signature. The results obtained differ depending on what smoothness assumptions one makes. We look at the most general case, resulting from concentrating on the 3-dimensional surface where the change of signature occurs, rather than on either the Lorentzian (hyperbolic) or Riemannian (positive definite) enveloping space (the latter is often referred to as Euclidean; however we prefer Riemannian, as 'Euclidean' suggests that the space is flat) .

In our approach we emphasize the initial value problem associated with signature change and the dynamical content of the theory, rather than regarding the problem as just a generalisation of the well-known Israel junction conditions [13]. There are more than junction conditions involved. In the case of the surface of a star, junction conditions are rather separated from the role of the initial value problem (because the surface is timelike). In the case of a change of signature, this must take place on a spacelike surface and so is essentially tied in to the nature of the initial value problem. Junction conditions play here a kinematical role, while the real dynamics of the change of signature are captured by the constraints associated with the field equations. This understanding underlies the approach we adopt.

The first fundamental form must be continuous. The continuity of the the second fundamental form as seen from both sides, is only assumed up to the action of an infinitesimal diffeomorphism corresponding to a Lie derivative. This allows a kink in the geometry - not allowed in the more restrictive assumptions considered up to now. We insist that the constraints are valid for both enveloping metrics. Further junction conditions only arise if the matter is assumed to be smoothly behaved - which may not be required.

These conditions thus generalise those considered by Ellis et al [5,6,11], which in turn are more general than those considered by Hayward et al [7,8,10] on the basis of their more restricted approach (placing more stringent restrictions on what is allowed). Our stand- point is that one can adopt any of these views - they are based on different philosophies of how one should approach junction conditions - or indeed one can question whether there should be any conditions other than a gluing condition, such as is adopted here.

We avoid use of specific coordinate systems, as well as use of abstract notation such as is employed by Hayward [7]. Rather we follow the notation of Hawking and Ellis [14] and of Fisher and Marsden [15].

2 Approach Taken

We let $\mathcal{S}$ denote a compact oriented three-manifold, and let

$$\Theta : \mathcal{S} \to (M^{(4)}, g) \equiv M \tag{1}$$

be an embedding of $\mathcal{S}$ in a Lorentzian manifold $(M^{(4)}, g)$ such that the imbedded manifold $\Theta(\mathcal{S})$ is space-like, that is the pull-back

$$\Theta^*(g) \equiv h \tag{2}$$

is a Riemannian metric on $\mathcal{S}$.

Similarly we define

$$\hat{\Theta} : \mathcal{S} \to (M^{(4)}, \hat{g}) \equiv \hat{M} \tag{3}$$

as an embedding of $\mathcal{S}$ in the *same* 4-dimensional manifold, $M^{(4)}$, but now endowed with a Riemannian metric $\hat{g}$, *viz.*, $(M^{(4)}, \hat{g})$.

Our strategy is to think of the metrics g and $\hat{g}$ as living on the same portion of manifold, and in order to avoid misunderstandings, we wish to stress that M and $\hat{M}$ are just a shorthand notation for the same underlying four-manifold $M^{(4)}$ with different metrics, with g Lorentzian, whereas $\hat{g}$ is Riemannian. As we are not concerned with global problems we may restrict ourselves to a a tubular neighborhood of $\Theta(\mathcal{S})$ (containing also $\hat{\Theta}(\mathcal{S})$). For the moment g and $\hat{g}$ are arbitrary. This coexistence of both Riemannian and Lorentzian metrics on the same region of the manifold will in our opinion avoid a lot of problems when thinking of the geometry involved.

We are going to identify - modulo the action of the diffeomorphisms- the Lorentzian and Riemannian geometry along a common imbedded spacelike hypersurface, determined by the constraints associated with the Einstein equations.

3 Geometry

In order to define the variables of interest, we need to characterise the foliations employed and the related lapse and shift in both the Riemannian and Lorentzian cases.

Let $E^{\infty}(\mathcal{S}, \hat{M})$ and $E^{\infty}(\mathcal{S}, M)$ denote the sets of all spacelike imbeddings of $\mathcal{S}$ in $\hat{M}$ and M respectively.

Suppose we have a curve in each of these imbedding spaces: namely a one-parameter (λ) family of spacelike imbeddings of $\mathcal{S}$ into M, and a similar one-parameter (λ) family of imbeddings of $\mathcal{S}$ into $\hat{M}$. Explicitly, $\Theta_\lambda: \mathcal{S} \times I \rightarrow M$ and $\hat{\Theta}_\lambda: \mathcal{S} \times I \rightarrow \hat{M}$, where $I \equiv (-\epsilon, \epsilon)$ for a suitably small $\epsilon > 0$. This family of imbeddings defines a corresponding one-parameter family of vector fields $X^{(4)}_\lambda: \mathcal{S} \rightarrow TM^{(4)}$ and $\hat{X}^{(4)}_\lambda: \mathcal{S} \rightarrow T\hat{M}^{(4)}$ by

$$\frac{d\Theta_\lambda}{d\lambda}(p) = X^{(4)}_\lambda(\Theta_\lambda(p)) \tag{4}$$

and

$$\frac{d\hat{\Theta}_\lambda}{d\lambda}(p) = \hat{X}^{(4)}_\lambda(\hat{\Theta}_\lambda(p)) \tag{5}$$

as p varies over $\mathcal{S}$.

In order to simplify the notation a bit, we shall denote them simply by X_λ and $\hat{X}_\lambda$. Roughly speaking, either in M or in $\hat{M}$ these vectors connect the point $\Theta_\lambda(p)$ with $\Theta_{\lambda+d\lambda}(p)$ (and similarly for $\hat{\Theta}$); namely the images of a given point p in $\mathcal{S}$ under two infinitesimally near imbeddings.

If n and $\hat{n}$ respectively denote the forward-pointing unit normals to $\Theta(\mathcal{S})$ and $\hat{\Theta}(\mathcal{S})$ (so $n^a n^b g_{ab} = -1$; $\hat{n^a}\hat{n}^b\hat{g}_{ab} = +1$), we can as usual decompose the vector fields X and $\hat{X}$ into their normal and tangential components:

$$X_\lambda = N_\lambda \hat{n} + \beta_\lambda \tag{6}$$

$$\hat{X}_\lambda = \hat{N}_\lambda \hat{n} + \hat{\beta}_\lambda \tag{7}$$

which define the corresponding family of lapse functions on $\mathcal{S}$, *i.e.*, $N_\lambda \colon \mathcal{S} \to R$ and a corresponding family of shift vector fields again on $\mathcal{S}$, namely $\beta_\lambda \colon \mathcal{S} \to T\mathcal{S}$. We wish to stress the fact (slightly obscured by our simplified notation) that the family of lapse functions N_λ are defined on the abstract manifold $\mathcal{S}$, and similarly the family of shift vector fields β_λ are defined over $\mathcal{S}$; similarly for the lapse $\hat{N}_\lambda$ and shift $\hat{\beta}_\lambda$. Here "the lapse and the shift are seen in their proper geometric roles - describ- ing the hypersurface deformations in the enveloping geometries - rather than as pieces of the metric" (Isenberg and Nester [16]).

The metric interpretation comes about for instance if we use the maps

$$F : I \times \mathcal{S} \to M \tag{8}$$

defined by

$$(\lambda, p) \mapsto \Theta_\lambda(p) \tag{9}$$

as a diffeomorphism of $I \times \mathcal{S}$ onto a tubular neighbourhood of $\Theta_0(\mathcal{S})$. We can then pull back the metric g onto $I \times \mathcal{S}$ and get the usual expression

$$(F^*g)_{\alpha\beta} dx^\alpha dx^\beta = -(N_\lambda^2 - \beta_i \beta^i) d\lambda^2 + 2\beta_i dx^i d\lambda + h_{ij} dx^i dx^j \tag{10}$$

where indices α and β run from 1 to 4, i and j run from 1 to 3, $\{x^i\}$ are local coordinates on $\mathcal{S}$, and h_{ij} is the $\lambda-$dependent one-parameter family of metrics on $\mathcal{S}$. A similar analysis holds for $\hat{F}$, leading to

$$(\hat{F}^*\hat{g})_{\alpha\beta}dy^\alpha dy^\beta = +(\hat{N}_\lambda^2 + \hat{\beta}_i\hat{\beta}^i)d\lambda^2 + 2\hat{\beta}_i dy^i d\lambda + \hat{h}_{ij}dy^i dy^j \qquad (11)$$

with the obvious meaning of the symbols.

There are a number of general comments that should be made at this stage. In particular, we wish to caution the reader to not confuse the abstract manifold $\mathcal{S}\times I$ with its images $\Theta_\lambda(\mathcal{S})$ and $\hat{\Theta}_\lambda(\mathcal{S})$ in $M = (M^{(4)}, g)$ and $\hat{M} = (M^{(4)}, \hat{g})$, respectively. Typically, when dealing with the initial value problem, one is accustomed to do so for obvious reasons, and this identification is usually harmless. However making clear the distinction is more than a technical convenience here. By identifying $\mathcal{S}\times I$ with $\Theta_\lambda(\mathcal{S})$ and $\hat{\Theta}_\lambda(\mathcal{S})$ one is lead to an incorrect interpretation of the vector field $\partial/\partial\lambda$, which is defined on $\mathcal{S}\times I$, in terms of which the initial value formalism is phrased. Observe that the parameter λ is the natural label for all the fields h_λ, $\hat{h}_\lambda$, N_λ, $\hat{N}_\lambda$, β_λ, $\hat{\beta}_\lambda$, and the extrinsic curvatures (defined below), if they are referred either to the Lorent- zian or to the Riemannian case. This is a rather obvious statement when things are correctly seen, as they should be, on $\mathcal{S}\times I$. It is not an obvious statement at all if we identify $\mathcal{S}\times I$ with its images under Θ_λ and $\hat{\Theta}_\lambda$. In this case, since the foliations $\Theta_\lambda(\mathcal{S})$ and $\hat{\Theta}_\lambda\mathcal{S})$ are different, and with different deformation vectors X_λ and $\hat{X}_\lambda$, one is incorrectly led to believe that these deformation vectors must be tangent to different deformation coordinates, namely $X_\lambda = \partial/\partial\lambda$ and $\hat{X}_\lambda = \partial/\partial\omega$, for some other defor- mation parameter ω. As stressed before, this is usually harm- less in standard situations where one has just one enveloping space- time, but it is fatal here where the enveloping geometries are two and quite distinct.

The source of the error, in proceeding as above, lies in the fact that one is identifying vectors living on different spaces, since the family of vector fields $\partial/\partial\lambda$ is defined on $\mathcal{S}$, while the deformation vectors X_λ and $\hat{X}_\lambda$ are defined on M and $\hat{M}$, respectively. If these two latter are different, their intuitive identification with vectors tangent to a deformation coordinate, (*i.e.*, with $\partial/\partial\lambda$), is problematic and very confusing.

One must clearly separate the role of the vector tangent to the deformation coordinate, which is $\partial/\partial\lambda$, and which is defined on $\mathcal{S}$, from the vectors X_λ and $\hat{X}_\lambda$ which are respectively associated to the imbeddings Θ_λ and $\hat{\Theta}_\lambda$, (these vector fields can be thought of as the vector fields covering the two distinct family of imbeddings Θ_λ and $\hat{\Theta}_\lambda$ of $\mathcal{S}$).

It is our strategy to address the geometry of signature change exclu- sively in terms of quantities defined on $\mathcal{S}$ and this should be clearly kept in mind when deciding which quantities should be continuous through a surface of signature change. For instance it would be very unnatural from our viewpoint to assume the continuity of the unit normals, for these quantities live in the embedding spacetimes M and $\hat{M}$, and this is something that an observer living in $\mathcal{S}$ does not know *a priori.* It is much more natural for him to assume the continuity of the vector $\partial/\partial\lambda$ and of the lapse function and of the shift vector fields, since all such quantities are well defined on $\mathcal{S}$ and they provide him the complete kinematical framework for describing – from his standpoint – the deformations of $\mathcal{S}$ which may be compatible with a Riemannian geometry on one side and with a Lorentzian geometry on the other.

With these general remarks out of the way, we recall that in order to describe the imbeddings Θ and $\hat{\Theta}$, besides introducing the 3-metrics h and $\hat{h}$ we must also introduce, on $\mathcal{S}$, two symmetric tensor fields K and $\hat{K}$ to be interpreted as the second fundamental forms of $\Theta_\lambda(\mathcal{S})$ and $\hat{\Theta}_\lambda(\mathcal{S})$ respectively. In our notation, they are defined, at the generic point $x \in \mathcal{S}$, and for any pair of vectors u and v in $T_x\mathcal{S}$ by

$$K_x(u,v) =< T_x\Theta \circ u|\nabla^{(4)}(T_x\Theta \circ v)n >_g (\Theta(x)) \qquad (12)$$

where $\nabla^{(4)}$ denotes the covariant derivative operator in M, the brackets $<\cdot|\cdot>_g (\Theta(x))$ stand for the inner product in the Lorentzian metric g evaluated at the point $\Theta(x) \in M$, and $T_x\Theta$ stands for the tangential mapping, at $x \in \mathcal{S}$, associated to the embedding Θ.

Similarly, and with an obvious meaning of the symbols,

$$\hat{K}_x(u,v) =< T_x\hat{\Theta} \circ u|\hat{\nabla}^{(4)}(T_x\hat{\Theta} \circ v)n >_{\hat{g}} (\hat{\Theta}(x)) \qquad (13)$$

For each given value of λ the fields (h, K) and $(\hat{h}, \hat{K})$ cannot be arbitrarily prescribed. From the Gauss-Codazzi relation, one gets that such fields must satisfy four compatibility conditions, namely in the Riemannian case

$$R(\hat{h}) - (\hat{K}^{dc}\hat{h}_{dc})^2 + \hat{K}^{ab}\hat{K}^{cd}\hat{h}_{ac}\hat{h}_{bd} = -2\Theta^*(G_{\mu\nu}\hat{n}^\mu\hat{n}^\nu) \tag{14}$$

where $\hat{G}_{\mu\nu}$ is the Einstein tensor of $\hat{g}$ and

$$\hat{D}_a\hat{K}^{ac}\hat{h}_{cb} - \hat{D}_b\hat{K}^{cd}\hat{h}_{cd} = \hat{\Theta}^*[R_{\mu\nu}(\hat{g})\hat{n}^\nu\perp^\mu_b] \tag{15}$$

where $\hat{D}$ is the covariant derivative in $(\mathcal{S}, \hat{h})$ and $R_{\mu\nu}$ is the Ricci tensor of the metric $\hat{g}$.

In the Lorentzian case, we obtain

$$R(h) + (K^{dc}h_{dc})^2 - K^{ab}K^{cd}h_{ac}h_{bd} = 2\Theta^*(G_{\mu\nu}n^\mu n^\nu) \tag{16}$$

where $G_{\mu\nu}$ is the Einstein tensor of g and

$$D_aK^{ac}h_{cb} - D_bK^{cd}h_{cd} = \Theta^*[R_{\mu\nu}(g)n^\nu\perp^\mu_b]\,. \tag{17}$$

4 Change of Signature

Now we are ready to discuss the possibility of change of signature through a regular hypersurface. Till now the embedded hypersurfaces $\Theta_\lambda(\mathcal{S})$ and $\hat{\Theta}_\lambda(\mathcal{S})$ were kept distinct. The basic condition we need in order to be able to speak of a signature change is to choose one of the $\Theta_\lambda(\mathcal{S})$ to 'coincide' with one of the manifolds of the family $\hat{\Theta}_\lambda(\mathcal{S})$.

Asking directly, as often is implicitly done, that for a given range of λ, say $-\epsilon < \lambda < \epsilon$, $\Theta_\lambda(\mathcal{S}) \equiv \hat{\Theta}_\lambda(\mathcal{S})$, is too restrictive. And this is partially the reason for having unnecessary stringent constraints on the second fundamental form on the hypersurface of signature change. It is more natural to assume, at leas t a priori, that the identification between $\Theta_\lambda(\mathcal{S})$ and $\hat{\Theta}_\lambda(\mathcal{S})$, $-\epsilon < \lambda < \epsilon$, occurs modulo the action of diffeomorphisms of the manifold

$\mathcal{S}$. More particularly, we consider a λ dependent family of diffeomorphisms $\phi_\lambda: \mathcal{S} \to \mathcal{S}$, smoothly varying as $-\epsilon < \lambda < \epsilon$, and such that for a given value of λ, say $\lambda = 0$,

$$\hat{\Theta}_0(p) = \Theta_0(p), \forall p \in \mathcal{S} \tag{18}$$

namely, it is only required that $\phi_\lambda = id_{\mathcal{S}}$ for $\lambda = 0$. The strategy will be to use these diffeomorphisms to glue the bottom (Riemannian) region with the top (Lorentzian) region. This will mean - remembering that there are two metrics on $\mathcal{S} \times I$ - we designate the metric $\hat{g}$ as the physical metric in the lower region $\mathcal{S} \times (0, -\beta)$ and the metric g as the physical metric in the upper region $\mathcal{S} \times (\beta, 0)$. On the zero section, $\mathcal{S} \times \{0\}$, of $\mathcal{S} \times I$, the constraints associated to the Lorentzian and to the Riemannian imbedding must be simultaneously satisfied.

It is clear that as far as the three-metrics h and $\hat{h}$ are concerned, the action of the one-parameter group of diffeomorphisms ϕ_λ is simply that of having

$$\hat{h} = \phi^*_\lambda h \tag{19}$$

for $-\epsilon < \lambda < \epsilon$, and in particular, $\hat{h} = h$ for $\lambda = 0$.

The situation is less dull as far as concerns the tensor fields K and $\hat{K}$ yielding the second fundamental forms. In order to see how the action of ϕ_λ relates K and $\hat{K}$ on $\mathcal{S}$ let us write the explicit expressions of K and $\hat{K}$ in terms of the three-metrics h, $\hat{h}$, and of the vector field (defined over $\mathcal{S} \times I$), $\frac{\partial}{\partial \lambda}$. We get

$$K_{ij} = N^{-1}_\lambda [\frac{\partial}{\partial \lambda} h_{ij} - L_{\beta_\lambda} h_{ij}] \tag{20}$$

and similarly

$$\hat{K}_{ij} = \hat{N}^{-1}_\lambda [\frac{\partial}{\partial \lambda} \hat{h}_{ij} - L_{\hat{\beta}_\lambda} \hat{h}_{ij}] \tag{21}$$

where $L.$ denotes Lie differentiation along the vector field indicated.

For $-\epsilon < \lambda < \epsilon$, we have $\hat{h}_{ij} = (\phi^*_\lambda h)_{ij}$ thus

$$\hat{K}_{ij} = \hat{N}^{-1}_\lambda [\frac{\partial}{\partial \lambda}(\phi^*_\lambda h)_{ij} - L_{\hat{\beta}_\lambda}(\phi^*_\lambda h)_{ij}] \tag{22}$$

A direct computation shows (see *e.g.*, DeTurck [17]),

$$\frac{\partial}{\partial \lambda}[(\phi^*_\lambda h)_{ij}(p)] = \phi^*_\lambda [\frac{\partial}{\partial \lambda} h_{ij}(\phi_\lambda(p))] + \phi^*_\lambda [L_{v_\lambda} h_{ij}(\phi_\lambda(p))] \tag{23}$$

where the vector field v_λ is the generator of the one-parameter group of diffeomorphisms ϕ_λ according to

$$\frac{\partial}{\partial \lambda}\phi_\lambda(p) = v_\lambda(\lambda, \phi_\lambda(p)) \tag{24}$$

with the initial condition $\phi_\lambda|_{\lambda=0} = id_{\mathcal{S}}$.

Thus

$$\hat{K}_{ij} = \hat{N}^{-1}_\lambda \phi^*_\lambda [\frac{\partial}{\partial \lambda} h_{ij}(\phi_\lambda(p)) + L_{v_\lambda} h_{ij}(\phi_\lambda(p)) - L_{\hat{\beta}_\lambda} h_{ij}(\phi_\lambda(p))] \tag{25}$$

In particular, for $\lambda = 0$, we get

$$\hat{K}_{ij} = \hat{N}^{-1}_\lambda [\frac{\partial}{\partial \lambda} h_{ij} + L_{v_\lambda} h_{ij} - L_{\hat{\beta}_\lambda} h_{ij}] \tag{26}$$

which shows that if, as argued in the previous paragraph, we assume continuity of the lapse and the shift for $\lambda = 0$:

$$\hat{N}_\lambda = N_\lambda, \quad \hat{\beta}_\lambda = \beta_\lambda, \tag{27}$$

and assuming also continuity of $\frac{\partial}{\partial \lambda} h_{ij}$, then

$$\hat{K}_{ij} = K_{ij} + N^{-1}_\lambda L_{v_\lambda} h_{ij} \tag{28}$$

(a similar relation holds for any $-\epsilon < \lambda < \epsilon$ provided that we act by ϕ^*_λ). Thus, on the hypersurface $\Theta_0(\mathcal{S}) = \hat{\Theta}_0(\mathcal{S})$ where we seek a change of signature, we may assume that the corresponding second fundamental forms coincide only up the Lie derivative term $N^{-1}_\lambda L_{v_\lambda} h^{ij}$.

We wish to stress that by forcing ϕ_λ to be the identity for all λ, one may obviously achieve equality between the second fundamental forms on the transition hypersurface. But fixing *a priori* the three degrees of freedom (per space point) associated with ϕ_λ will be a very bad investment when dealing with the constraints.

One may also argue that equation (26) is equally compatible with having continuity of the second fundamental form, provided that one allows for a discontinuous shift vector field, namely $\beta_\lambda = \hat{\beta}_\lambda - v_\lambda$. Further impositio n of the continuity of the shift would then yield $v_\lambda = 0$, and the former case of freezing the diffeomorphism group is then recovered. All this is actually related to what one considers standard junction conditions in the setting of signature changes. In ordinary situations, such conditions require the continuity of the four-metric and of the second fundamental form. But whether or not a such conditions can be extended at face value to the case of surfaces of signature change is a very delicate issue. Continuity of the four-metric leads to vanishing of the lapse function, which is quite disturbing. Moreover, the tensor fields K and $\hat{K}$, when interpreted as second fundamental forms, are to be thought of as defined in terms of a unit normal (to the surface of signature change) whose norm changes sign at the junction. Thus it is not obvious at all that the continuity of the second fundamental form is a *natural* requirement in the case of surfaces of signature change.

In this respect, it is often argued that the correct answer must come from the field equations. More precisely, one should impose the validity of the field equations everywhere, in particular on the surface of signature change. This point of view is apparently reasonable and interesting, but implies very severe constraints on the resulting solutions. We offer here an alternative point of view, namely we do not force the validity of the full four dimensional field equations on the surface of signature change, but rather we concentrate on the validity of that part of the field equations which is really *intrinsic* to the surface of signature change, namely we impose the consistency among the four constraints associated with the field equations. In our view, this is a minimal necessary requirement, the basic one. Further restrictions can come only if one has some input from the matter fields present, in particular on how they behave on the surface of signature change; and that is a matter for debate.

We wish also to stress the following point. From the point of view of analysis and physics, partial differential equations of mixed type, where the type (elliptic, hyperbolic, or parabolic) of the equation is a function of position, are rather familiar. The added difficulty here, in considering surfaces of signature change, lies exactly in the diffeomorphism invariance of the theory. By considering the full field equations at once everywhere, one is behaving as if there exists a general theory of boundary value problems independent of the type of the equation, which is very bold, to say the least. Even in the simplest cases in hydrodynamics, such a theory is very delicate, and general results exist only for equations of special types. The situation becomes hopeless in a general relativity setting. Indeed, Einstein's equations in the Riemannian regime are a strongly overdetermined elliptic system (owing to diffeomorphism invariance), and the problem of finding a metric with a preassigned Einstein or Ricci tensor is often obstructed even at an infinitesimal level, (*i.e.*, there are even obstructions to finding a metric, around a given point, with prescribed Ricci tensor, see [17]). The situation changes drastically in the Lorentzian regime. Thus it is fair to say that the study of mixed type Einstein equations is a completely open problem. It follows that forcing the validity of the field equations everywhere, in the case of a surface of signature change, is a formal procedure not really justified from an existing theory, and to which one should give the same *interlocutory* status as other proposals. In our approach, restricting attention to the constraints forced on the surface of signature change, one is considering what kind of initial data is compatible with a signature change in terms of partial differential equations which do *not* change type on the surface of signature change. Furthermore, these contain the essential dynamical equations of the theory (for example in the Robertson-Walker case, they include the Friedmann equation), which lead to the Wheeler-de Witt equation which underlies quan- tum cosmology.

As a final remark, notice that at first reading one may think there is a surface layer present in our formalism because of the allowed discontinuity of the second fundamental form. However, there is no variance with the essence of the junction conditions of Israel [13], since we are assuming the continuity of the proper dynamical variables, which are $\frac{\partial}{\partial\lambda}h_{ij}$. These conditions are usually written down in terms of adapted coordinates such that the second fundamental form is the time derivative, and so do not allow for the action of a diffeomorphism which is responsible for the Lie derivative terms. Actually, in the geometrical setting discussed here, as stressed above, they should not be taken at face value, since the general remarks discussed in the previous paragraph apply also here. In our setting, the proper variables to match are the lapse N_λ, the shift β_λ, the three-metric h_λ and its derivative $\frac{\partial}{\partial\lambda}h$ – as we have done, and no surface layer is present as is clearly shown by imposing the constraints.

4.1 Constraints

The constraints, both in their Lorentzian and Riemannian version, must hold for $\lambda = 0$.

Let us start from the momentum (or divergence) constraint. We assume that on M both $\hat{g}$ and g satisfy the corresponding form of Einstein field equations, the Riemannian form for the former, the standard Lorentzian form for the latter. Thus in the Riemannian case

$$\hat{R}_{\alpha\beta} = \hat{T}_{\alpha\beta} - \frac{1}{2}\hat{g}_{\alpha\beta}\hat{g}^{\gamma\delta}\hat{T}_{\gamma\delta} \tag{29}$$

where $\hat{T}_{\alpha\beta}$ are the components of the Riemannian energy-momentum tensor. Relative to the slicing $\hat{\Theta}_\lambda(\mathcal{S})$ we shall write

$$\hat{T}_{\alpha\beta} = \hat{\mu}\hat{n}_\alpha\hat{n}_\beta + \hat{j}_\alpha\hat{n}_\beta + \hat{j}_\beta\hat{n}_\alpha + \hat{s}_{\alpha\beta} \tag{30}$$

where $\hat{\mu}$, $\hat{j}_\alpha$, and $\hat{s}_{\alpha\beta}$ respectively are the normal-normal, normal-tangential, and tangential-tangential projections of $\hat{T}_{\alpha\beta}$ with respect to $\hat{\Theta}_\lambda(\mathcal{S})$. In the

Lorentzian case, we shall similarly write

$$R_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta} T_{\gamma\delta} \tag{31}$$

where $T_{\alpha\beta}$ are the components of the energy-momentum tensor which, relative to the slicing $\Theta_\lambda(\mathcal{S})$, can be decomposed according to

$$T_{\alpha\beta} = \mu n_\alpha n_\beta + j_\alpha n_\beta + j_\beta n_\alpha + s_{\alpha\beta} \tag{32}$$

where μ, j_α, and $s_{\alpha\beta}$ respectively are the relative density of mass-energy, the relative density of momentum, and the relative spatial stress tensor with respect to $\Theta_\lambda(\mathcal{S})$.

In general, there is no a priori need to assume that for $\lambda = 0$ the *matter* variables are continuous. From a phenomenological point of view, there is no obvious evidence that one should assume continuity of the stress tensor components at the change of signature, although one might make that assumption if given no further information. On the other hand, if one has a more fundamental description of the stress tensor, for example as arising from a scalar field, one can work out the continuity properties of the stress tensor components from that description. This was done in [5,6] for the case of a classical scalar field. Then the obvious continuity conditions are that the fundamental variables associated with the more fundamental description are continuous, and satisfy whatever requirements there may be to give a good set of initial data for the matter field equations on either side of the signature change surface.

In general this will result in discontinuous stress tensor components. This is not unreasonable in view of the fact that the usual conservation laws for energy and momentum break down at a change of signature surface [12]. The fundamental underlying point is that it is difficult to understand physics in the positive definite region, indeed classical physics in the usual sense will not exist there (although quantum physics will be fine!). Thus one must be open-minded as to what conditions should be imposed on 'matter' in the positive definite regime, in a classical discussion of signature change.

Without making specific assumptions, the momentum constraint forced on $\mathcal{S} \times \{0\}$ by the Riemannian side is

$$\hat{D}^a \hat{K}_{ab} - \hat{D}_b \hat{k} = \hat{j}^b \qquad (33)$$

where $\hat{k} \equiv \hat{h}^{cd} \hat{K}_{cd}$ is the rate of volume expansion (the trace of the second fundamental form). Since, for $\lambda = 0$, $\hat{D}^a = D^a$ and $\hat{K}_{ab} = K_{ab} + N^{-1} L_v h_{ab}$ we get

$\mathrm{D}^a K_{ab} + D^a(N^{-1} L_v h_{ab}) - D_b k - D_b(h^{cd} N^{-1} L_v h_{cd}) = \hat{j}_b$(34)(where as above $k \equiv h^{cd} K_{cd}$). But the momentum constraint forced on $\mathcal{S} \times \{0\}$ by the Lorentzian side implies that

$$D^a K_{ab} - D_b k = -j^b \qquad (35)$$

which, when introduced in the previous expression, yields

$$D^a(N^{-1} L_v h_{ab}) - D_b(h^{cd} N^{-1} L_v h_{cd}) = j_b + \hat{j}_b \qquad (36)$$

Given h_{ab}, the lapse function N, and the momentum densities j_b, $\hat{j}_b$ the above is a system of partial differential equations determining the vector field v which generates the gluing one-parameter group of diffeomorphisms ϕ_λ in the neighbourhood of $\lambda = 0$. Notice however that this system is elliptic (*i.e.*, (36) can be actually inverted) only if the vector field v is divergence-free, $D^a v_a = 0$. This further requirement implies that k, the trace of the second fundamental form, is continuous through the surface of signature change S, namely

$$k = \hat{k} \qquad (37)$$

This result is quite satisfactory since in an initial value approach, the rate of volume expansion is to be considered as a kinematical variable selecting the family of hypersurfaces along which we are following the dynamics of the gravitational field.

Next we can impose that both the Riemannian and the Lorentzian version of the Hamiltonian constraint hold for $\mathcal{S} \times \{0\}$. This yields

$$2\mu + 2\hat{\mu} + \frac{2}{3}k^2 +$$

$\mathrm{h}^{ac}h^{bd}(\tilde{K}_{ab} + N^{-1}L_v h_{ab})(\tilde{K}_{cd} + N^{-1}L_v h_{cd}) + \tilde{K}^{ab}\tilde{K}_{ab} = 0$(38)where $\tilde{K}_{ab}$ denotes the trace-free part of K_{ab}.

If we assume that $\hat{\mu} \geq 0$, then the above condition, being the sum of algebraically independent non-negative terms, is only compatible with the vanishing of each summand. Thus, in this cas e from $\hat{\mu} \geq 0$ we actually get $\hat{\mu} = 0$, $\mu = 0$, $k = 0$, $\tilde{K}_{ab} = 0$, and $N^{-1}L_v h_{ab} = 0$. Thus, if we require continuity of the matter variables through a surface of signature change, we found, as expected, that the second fundamental form must vanish correspondingly. Notice that this result follows without requiring the a priori continuity of the 4-metric or the continuity of the second fundamental form. Actually, it is precisely the continuity of the matter variables which forces such a result. It is not in the geometry, and as argued in the previous paragraph, there is no a priori need to assume that for $\lambda = 0$ the matter variables are continuous, or satisfy energy conditions reminiscent of the Lorentzian regime.

In general, without imposing any continuity or sign restriction on $\hat{\mu}$, equation (37) must be considered as a constraint on the Lorentzian rate of volume expansion k. In other words, the above compatibility condition between the Hamiltonian constraints sets an origin for the *extrinsic time* k which parametrizes the time evolution in the Lorentzian region. Geometrically speaking, this condi- tion is simply selecting the hypersurface where the signature change can occur [5,6].

One could use an approach even more closely tuned to the spirit of the initial value problem by using as dynamical variables the conformal part of the 3-metric, and the scaled divergence-free trace-free part of the second fundamental form. However this would complicate the equations without throwing much light on the basic issues we are addressing. We have therefore avoided these complications here, although this more detailed analysis shows signs of raising interesting questions.

5 Relation to other approaches

It is essential to our approach that the 3-metric is continuous through the change of signature. Others have emphasized [7,8,10] their belief in the importance of using coordinate systems where *all* the covariant components of the metric are continuous at a change of signature surface. We have not adopted this view, *inter alia* because then some of the contravariant metric components will diverge at the surface of change, leading *inter alia* to the divergence of various Christoffel terms; so the appearance of continuity is somewhat misleading.

What we do believe is important is that the kinematics should be well-behaved there; this means we demand a well behaved shift and lapse, which determine the 4-dimensional metric structure. In particular the lapse should not go to zero because if it does then one halts the evolution in the coordinate system thereby defined. This means in turn that while the 3-metric components and their first 'time' derivative can always be chosen continuous up to a diffeomorphism, if the lapse is regular then the 4-dimensional metric tensor components associated will have a discontinuous component (the time-time component, which is not dynamical).

In our geometric approach, there is no need to assume *a priori* that the 4-dimensional metric is continuous, because we have shown that one can match the Lorentzian and Riemannian spacetimes without making such an assumption, by having a perfectly well behaved kinematical description (the lapse and shift are well-behaved in our approach). The kinematics through a signature change surface should as far as possible be free from particular coordinate choices, and one should be free to choose the kinematical data (the lapse and shift) as desired, not forced to make them go to zero.

The approach of [7] is based on a different view: emphasizing more the role of the full space-time metric than the view used here. It also assumes additional differentiability for the solutions, and is therefore more restrictive than the view adopted here; it is not surprising that the results obtained are more restrictive than if one does not impose these extra conditions. However that view also implies the lapse function goes to zero as one approaches the change surface. This 'collapse of the lapse' may be expected to cause problems for the dynamics [18].

It will be clear from the above that the generic situation does not require a vanishing of the second fundamental form at the surface of change, which is required for example in both the distributional [7] and the Hartle-Hawking approach [19] (which uses a complex time variable). Our hope is that the present geometrical analysis of the classical case will be of help in understanding the full generality of what may be possible in the quantum case, through first clarifying the full generality of the analogous classical situation.

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