VALIDITY CONDITIONS OF THE HYDROSTATIC APPROACH FOR SELF-GRAVITATING SYSTEMS: A MICROCANONICAL ANALYSIS

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I. Introduction and outline

♠ Hydrostatic approach widely used in Astrophysics [BINNEY-TREMAINE, 1987]

- Isothermal sphere for point particles [ANTONOV,1964],[LYNDEN-BELL,1968]
- Introduction of short-range repulsion [ARONSON-HANSEN,1972]

Key ingredients: **Mean-field treatment of gravitational** + **Local equilibrium**

♠ Statistical mechanics descriptions [PADMANABHAN, 1990] $\Rightarrow$
Mathematical proofs in specific limits [MESSER-SPOHN, 1982], [KIESSLING-PERCUS, 1995]

RELIABILITY for a FINITE system ?

- II. Model and auxilary systems
- III. Scaling properties of the auxilary systems
- IV. Emergence of thermalization in the infinite system
- V. Hydrostatic description of the infinite system
- VI. Validity conditions for the finite system
- VII. Concluding comments

II.1. Hard spheres with gravitational interactions

- We consider a **gravitational model** $\mathcal{S}$ made with
  - N identical hard spheres (m,σ)
  - enclosed in a spherical box $(\Lambda = 4\pi R^3/3)$
  - particle density $n = N/\Lambda$ and mass density $\rho = mn$

- The corresponding **Hamiltonian** reads

$$H_N = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + \frac{1}{2} \sum_{i \neq j} v(|\mathbf{r}_i - \mathbf{r}_j|)$$

$$v(r) = \infty \text{ for } r < \sigma \ \ , \ \ v(r) = -Gm^2/r \text{ for } r > \sigma$$

- No **dispersion** in shapes, sizes and masses

- No **sticking** leading to agregation

II.2. Microcanonical description

$\mathcal{S}$ **isolated**, with **fixed energy** E and no other **conserved quantity**.

♠ Microcanonical ensemble :

- Distribution in phase space

$$f_{\mathrm{micro}}(\mathbf{r}_1, ..., \mathbf{r}_N, \mathbf{p}_1, ..., \mathbf{p}_N) = \delta(E - H_N)$$

- Number of microstates

$$\Omega(E, N, \Lambda) = C_N \int_{\Lambda^N \times R^{3N}} \prod_i \mathrm{d}^3\mathbf{r}_i \mathrm{d}^3\mathbf{p}_i \delta(E - H_N)$$

Equilibrium state of $\mathcal{S}$ depends on $N, \varepsilon = E/(GM^2/R), \eta = \pi n \sigma^3/6$

◇ $\mathrm{f}_{\mathrm{micro}}$ is a **stationary** solution of evolution equations

◇ $\Omega(E, N, \Lambda)$ is **finite** for $\sigma > 0$; it **diverges** for $\sigma = 0$ and $N \geq 3$ [Pomeau, 2007]

II.3. Auxiliary systems and the scaling continuous limit

♠ Sequence of similar **auxiliary systems** $\mathcal{S}_{\mathrm{a}}$ with $\mathrm{N}_{\mathrm{a}} \to \infty$:

- $R_{\mathrm{a}} = (N_{\mathrm{a}}/N)^{1/5} R$
- $m_{\mathrm{a}} = (N_{\mathrm{a}}/N)^{-2/5} m$
- $\sigma_{\mathrm{a}} = (N_{\mathrm{a}}/N)^{-2/15} \sigma$
- $E_{\mathrm{a}} = (N_{\mathrm{a}}/N) E$

That **scaling limit** (SL) defines an **infinite continuous medium** $\mathcal{S}_\infty$ in a stationary state controlled by the two independent dimensionless parameters :

- Dimensionless energy per particle $\varepsilon = E_{\mathrm{a}}/(GM_{\mathrm{a}}^2/R_{\mathrm{a}}) = E/(GM^2/R)$
- Packing fraction $\eta = \pi n_{\mathrm{a}} \sigma_{\mathrm{a}}^3/6 = \pi n \sigma^3/6$.

◇ Inspired from the usual TL with now **mass density fixed** $\rho_{\mathrm{a}} = m_{\mathrm{a}} n_{\mathrm{a}} = \rho$

◇ Other limits in the **canonical ensemble** [MESSER-SPOHN, 1982]
and **grand-canonical ensemble** [KIESSLING-PERCUS,1995] with G **rescaled**

III.1. Bounds for the potential energy in the SL

♠ For any allowed configuration, the potential energy of $\mathcal{S}_{\rm a}$

$$V_{N_{\rm a}} = -\frac{1}{2}\sum_{i\neq j}\frac{Gm_{\rm a}^2}{|\mathbf{r}_i - \mathbf{r}_j|}$$

is **larger** than that of the collapsed configuration where the $N_{\rm a}$ hard spheres make a single cluster with size $L_{\rm coll} \sim N_{\rm a}^{1/3}\sigma_{\rm a}$, which is of order $-Gm_{\rm a}^2 N_{\rm a}^2/L_{\rm coll}$. In the scaling limit, this provides the **classical version of H-stability**

$$V_{N_{\rm a}} \geq -C_{\rm HS}\frac{GN^{2/3}m^2}{\sigma}N_{\rm a}$$

♠ For any allowed configuration, the potential energy should be **smaller** than that of a homogeneous surface mass distribution $N_{\rm a}m_{\rm a}/(4\pi R_{\rm a}^2)$,

$$V_{N_{\rm a}} \leq -\frac{GNm^2}{2R}N_{\rm a}$$

III.2. Extensivity of potential energy in the SL

- Thanks to the extensivity of its upper and lower bounds, the average potential energy of $\mathcal{S}_{\rm a}$

$$\langle V_{N_{\rm a}} \rangle = -\frac{1}{2} \int_{\Lambda_{\rm a}^2} \mathrm{d}^3\mathbf{r}\mathrm{d}^3\mathbf{r}' \rho_{\rm a}^{(2)}(\mathbf{r},\mathbf{r}') \frac{G}{|\mathbf{r}-\mathbf{r}'|}$$

should also be **extensive** in the scaling limit (like the potential energy of an homogeneous sphere with mass density ρ).

- Extensivity consistent with the expected scaling behaviours for $\mathbf{q}$,$\mathbf{q}'$,... fixed

$$\lim_{\rm SL} \rho_{\rm a}(qR_{\rm a}) = \rho g(q;\varepsilon,\eta)$$

$$\lim_{\rm SL} \rho_{\rm a}^{(2)}(R_{\rm a}\mathbf{q},R_{\rm a}\mathbf{q}') = \rho^2 g^{(2)}(\mathbf{q},\mathbf{q}';\varepsilon,\eta)$$

III.3. Fluctuations of the potential energy in the SL

♠ The fluctuations $\langle V_{N_{\rm a}}^2 \rangle - [\langle V_{N_{\rm a}} \rangle]^2$ can be expressed as spatial integrals of $1/|\mathbf{r}-\mathbf{r}'|^2$, $1/|\mathbf{r}-\mathbf{r}'||\mathbf{r}-\mathbf{r}''|$, and $1/|\mathbf{r}-\mathbf{r}'||\mathbf{r}''-\mathbf{r}'''|$ weighted respectively by **two-, three- and four-body** distribution functions. A simple estimation within the considered scaling limit provides

$$\langle V_{N_{\rm a}}^2 \rangle - [\langle V_{N_{\rm a}} \rangle]^2 = o(N_{\rm a}^2)$$

♠ Accordingly, we will use in further estimations of averages involving $V_{N_{\rm a}}$ the ansatz :

$$V_{N_{\rm a}} \to \langle V_{N_{\rm a}} \rangle + W_{N_{\rm a}}$$

for most contributing configurations with $W_{N_{\rm a}} = o(N_{\rm a})$ when $N_{\rm a} \to \infty$

◇ **Non-rigorous** although quite plausible (possible subtle correlations with other variables)

◇ **Confirmed** by *a posteriori* estimations

IV.1. The inhomogeneous mass density

♠ The mass density of $\mathcal{S}_{\rm a}$ is

$$\rho_{\rm a}(\mathbf{r}) = m_{\rm a}\langle\sum_{i=1}^{N_{\rm a}}\delta(\mathbf{r}_i - \mathbf{r})\rangle$$

Using $f_{\rm micro}$, the standard integration over the momenta $\mathbf{p}_i$ leads to

$$\rho_{\rm a}(\mathbf{r}) = B(E_{\rm a}, N_{\rm a}, \Lambda_{\rm a}) \int_{\Lambda_{\rm a}^{N_{\rm a}-1}, |\mathbf{r}_i - \mathbf{r}_j| > \sigma_{\rm a}} \prod_{i=2}^{N_{\rm a}} \mathrm{d}^3\mathbf{r}_i [E_{\rm a} - V_{N_{\rm a}}(\mathbf{r}, \mathbf{r}_2, ..., \mathbf{r}_{N_{\rm a}})]^{3N_{\rm a}/2-1}$$

♠ Introduce the gravitational potential $\Phi(\mathbf{r}|\mathbf{r}_2, ..., \mathbf{r}_{N_{\rm a}}) = \Phi_{N_{\rm a}-1}(\mathbf{r})$ at $\mathbf{r}$ created by the $(N_{\rm a} - 1)$ particles located at $\mathbf{r}_2, ..., \mathbf{r}_{N_{\rm a}}$. Since

$$V_{N_{\rm a}}(\mathbf{r}, \mathbf{r}_2, ..., \mathbf{r}_{N_{\rm a}}) = V_{N_{\rm a}-1}(\mathbf{r}_2, ..., \mathbf{r}_{N_{\rm a}}) + m_{\rm a}\Phi(\mathbf{r}|\mathbf{r}_2, ..., \mathbf{r}_{N_{\rm a}})$$

we obtain in the SL

$$\rho_{\rm a}(\mathbf{r}) \sim \mathrm{cst} \int_{\Lambda^{N_{\rm a}-1}} \mathrm{d}\mu_{N_{\rm a}-1} \prod_{i=2}^{N_{\rm a}} \theta(|\mathbf{r}_i - \mathbf{r}|/\sigma_{\rm a} - 1)\, [E_{\rm a} - V_{N_{\rm a}-1}]^{3/2} \left[1 - \frac{m_{\rm a}\Phi_{N_{\rm a}-1}(\mathbf{r})}{E_{\rm a} - V_{N_{\rm a}-1}}\right]^{3N_{\rm a}/2-}$$

IV.2. Emergence of thermalization in the infinite system

- Rewrite

$$\left[1 - \frac{m_\mathrm{a}\Phi_{N_\mathrm{a}-1}(\mathbf{r})}{E_\mathrm{a} - V_{N_\mathrm{a}-1}}\right]^{3N_\mathrm{a}/2-1} = \exp\left\{(3N_\mathrm{a}/2 - 1)\ln\left[1 - \frac{m_\mathrm{a}\Phi_{N_\mathrm{a}-1}(\mathbf{r})}{E_\mathrm{a} - V_{N_\mathrm{a}-1}}\right]\right\}$$

Since $m_\mathrm{a}\Phi_{N_\mathrm{a}-1}(\mathbf{r}) = O(1)$ and $E_\mathrm{a} - V_{N_\mathrm{a}-1} = O(N_\mathrm{a})$, the expansion of the logarithm leads to

$$\left[1 - \frac{m_\mathrm{a}\Phi_{N_\mathrm{a}-1}(\mathbf{r})}{E_\mathrm{a} - V_{N_\mathrm{a}-1}}\right]^{3N_\mathrm{a}/2-1} \sim \exp\left\{-\frac{3N_\mathrm{a}m_\mathrm{a}\Phi_{N_\mathrm{a}-1}(\mathbf{r})}{2(E_\mathrm{a} - V_{N_\mathrm{a}-1})}\right\}$$

- Applying the fluctuation ansatz, we find in the SL

$$\rho_\mathrm{a}(\mathbf{r}) \sim \mathrm{cst}\left[\int_{\Lambda_\mathrm{a}^{N-1}} \mathrm{d}\mu_{N_\mathrm{a}-1}\prod_{i=2}^{N_\mathrm{a}}\theta(|\mathbf{r}_i - \mathbf{r}|/\sigma_\mathrm{a} - 1)\right]\exp\left\{-\frac{3N_\mathrm{a}m_\mathrm{a}\phi_\mathrm{a}(\mathbf{r})}{2(E_\mathrm{a} - \langle V_{N_\mathrm{a}}\rangle)}\right\}$$

with $\phi_\mathrm{a}(\mathbf{r}) = \langle m_\mathrm{a}\Phi_{N_\mathrm{a}-1}(\mathbf{r})\rangle$.

$\Rightarrow$ **THERMALIZATION** with $\mathrm{T}_\infty = \lim_{SL} 2(E - \langle V_{N_\mathrm{a}}\rangle)/(3N_\mathrm{a})$

V.1. Hydrostatic picture for the infinite system

♠ The hydrostatic approach for $\mathcal{S}_\infty$ is justified thanks to

- **Local thermodynamical equilibrium** is ensured by hard-core repulsion entirely.
- At the local scale, particles feel the **mean-field gravitational potential**

$$\phi_{\rm a}(\mathbf{r}) = -\int_{\Lambda_{\rm a}} {\rm d}^3\mathbf{r}' \rho_{\rm a}(\mathbf{r}') \frac{G}{|\mathbf{r}' - \mathbf{r}|}$$

- The local correlation length $\lambda_{\rm HS}$ is much **smaller** than the characteristic variation length $R_{\rm a}$ of $\rho_{\rm a}(\mathbf{r})$.

♠ Accordingly, the hydrostatic equilibrium reads for the rescaled quantities $g(\mathbf{q};\varepsilon,\eta) = \lim_{\rm SL} \rho_{\rm a}(R_{\rm a}\mathbf{q})/\rho$ and $\psi(\mathbf{q};\varepsilon,\eta) = \lim_{\rm SL} \phi_{\rm a}(R_{\rm a}\mathbf{q})/(GM_{\rm a}/R_{\rm a})$ of $\mathcal{S}_\infty$, as

$$\nabla_\mathbf{q}\left[g(\mathbf{q};\varepsilon,\eta)p_{\rm HS}(\eta g(\mathbf{q};\varepsilon,\eta))\right] = -g(\mathbf{q};\varepsilon,\eta)\nabla_\mathbf{q}\psi(\mathbf{q};\varepsilon,\eta)/T^*(\varepsilon,\eta)$$

where $p_{\rm HS}$ is the dimensionless hard-sphere pressure (**no gravitation**) and $T^*(\varepsilon,\eta) = T_\infty/(GM^2/NR)$.

V.2. A few remarks about the hydrostatic equations

- Once the SL has been taken, we can take the limit $\eta \to 0$ where $p_{\rm HS}(\eta\rho(\mathbf{r})/\rho) \to 1$.

$\Rightarrow$ **ISOTHERMAL SPHERE**
[EMDEN,1907],[ANTONOV,1964],[LYNDEN-BELL,1968]

- **Multiplicity** of solutions for the hydrostatic equations

$\Rightarrow$ **PHASE TRANSITIONS** [CHAVANIS, 2006]

- Breakdowns of the hydrostatic approach
  - For $\eta = 0$, when $\varepsilon < -0.335...$ $\longrightarrow$ **No solutions**
  - For ε sufficiently negative and/or η sufficiently large $\longrightarrow$ **local cristalisation**

V.3. Correlations and fluctuations in the SL

- Within the hydrostatic approach :
  - Mass distribution $\rho_{\rm a}(\mathbf{r})$ varies on the scale $R_{\rm a}$
  - Correlations, like $[\rho_{\rm a}^{(2)}(\mathbf{r},\mathbf{r}') - \rho_{\rm a}(\mathbf{r})\rho_{\rm a}(\mathbf{r}')]$, decay over the hard-sphere local correlation length $\lambda_{\rm HS}$ of order $\sigma_{\rm a}$

- This implies :
  - The average potential energy is indeed **extensive**, $\langle V_{N_{\rm a}}\rangle = V_{\rm self} + V_{\rm corr}$ with $V_{\rm self} = O(N_{\rm a})$ and $V_{\rm corr} = O(N_{\rm a}^{1/3})$.
  - Fluctuations behave as $\langle V_{N_{\rm a}}^2\rangle - [\langle V_{N_{\rm a}}\rangle]^2 = O(N_{\rm a})$

◇ Fluctuations similar to that of an **ordinary system** with short-range interactions at **thermodynamical equilibrium**.

VI.1. Extensivity condition for the finite system

♠ According to the analysis for $\mathcal{S}_{\rm a}$, Boltzmann-like factors also emerge for the finite system $\mathcal{S}$ if

$$N\left[\frac{m\Phi_{N-1}(\mathbf{r})}{E-V_{N-1}}\right]^2 \ll 1$$

♠ That condition has to be fullfilled by the most probable configurations which determine the eq equilibrium state of $\mathcal{S}$. This provides

$$\frac{[\psi(0;\varepsilon,\eta)]^2}{N[T^*(\varepsilon,\eta)]^2} \ll 1$$

◇ Fixed ε and $\eta \longrightarrow N$ **large** enough

◇ Fixed $N \longrightarrow T^*(\varepsilon,\eta)$ **large** enough

VI.2. Fluctuation condition for the finite system

♠ Another crucial step relies in the estimation of the *a priori* **fluctuating** exponential

$$\exp\left[-\frac{3Nm\Phi_{N-1}(\mathbf{r})}{2(E-V_{N-1})}\right]$$

♠ Since a typical fluctuation of $m\Phi_{N-1}(\mathbf{r})$ is of order Gm^2/σ, we obtain the **weak-coupling** condition

$$\frac{Gm^2}{\sigma T} \ll 1$$

which can be recast as

$$\frac{1}{N^{2/3}\eta^{1/3}T^*(\varepsilon,\eta)} \ll 1$$

◇ Fixed N and $\eta \longrightarrow T^*(\varepsilon,\eta)$ **large** enough
◇ Fixed N and $T^*(\varepsilon,\eta) \longrightarrow \eta$ **cannot be too small**
◇ Analysis of validity of Antonov's mean-field theory requires to introduce a **finite** η

VI.3. Astrophysical examples

♠ **Globular clusters** : $R \simeq 50 \text{ pc}$, $m \simeq 1 \text{ M}_{\odot}$, $N \simeq 6\ 10^5$, $\delta v \simeq 7 \text{ kms}^{-1}$
[Binney-Tremaine, 1987]

- Extensivity → **YES**
- Fluctuations → **NO**

The hydrostatic approach **FAILS** in relation with the formation of **binaries**

♠ **Gas of dust** : $\sigma \simeq 15\ \mu\text{m}$, $m \simeq 10^{-9} \text{ g}$, $N \simeq 10^{34}$
[Kalas-Graham-Clampin, 2005]

- Extensivity → **YES**
- Fluctuations → **YES**

The hydrostatic approach **WORKS** because objects are sufficiently **light**

VII. Concluding comments

♠ Validity conditions of the hydrostatic approach for the **hard-spheres model**
→ **Useful insights** for astrophysical applications

A more complete analysis requires :

♠ Reliability of the model

- **Choice** of σ ?
- Box *versus* **self-confinement**?
- Conservation of the total angular momentum → **global rotation**
  [VOTYAKOV-HIDMI-DE MARTINO-GROSS,2002]

♠ Reliability of a microcanonical description and dynamical limitations

- **Relaxation times** *versus* the age of the system
- Existence of **quasi-stationary states**
  [ANTONI-RUFFO-TORCINI,2004],[CHAVANIS,2005]
- Possible **ergodicity breaking**
  [CHABANOL-CORSON-POMEAU,2000],[POSCH-THIRRING,2000]