Counting non planar diagrams: an exact formula

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Abstract

We present an explicit solution of a simply stated, yet unsolved, combinatorial problem, of interest both in quantum field theory (Feynman diagrams enumeration, beyond the planar approximation) and in statistical mechanics (high temperature loop expansion of some frustrated lattice spin model).

PACS: 02.10.Eb, 02.10.Nj, 05.50.+q

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The problem of counting the number of rooted loops (closed path starting from a given point) according to their length and area, on an hypercubic lattice, in the limit of infinite dimensionality, was addressed by Parisi *et al.* [1, 2]. They were investigating spin models with frustration but without any quenched disorder, in order to test the conjecture that such deterministic models could behave at low temperature as some suitably chosen spin-glass model with quenched disorder. They considered the frustrated Spherical and XY models in the limit of large dimensionality D of the lattice, where the saddle point approximation becomes exact. In their analysis of these models, they showed that the high temperature expansion (*i.e.* loop expansion) can be nicely rewritten by using the q-oscillator algebra [3], where q measures the frustration per plaquette and varies continuously on the real interval $[-1, +1]$, between the fully-frustrated case ($q = -1$, fermionic algebra) and the ferromagnetic case ($q = 1$, bosonic algebra). The frustration was induced on the infinite-dimensional hypercubic lattice through an applied Abelian lattice gauge field producing a static and constant external magnetic field suitably oriented to give the same magnetic flux $\pm B$ for any plaquette of the lattice. Such a magnetic field having the same projection over all the axis, up to the sign, in Refs. [1, 2] these signs were chosen randomly in order to avoid the selection of a preferred direction. The usual unfrustrated ferromagnetic spin interaction is obtained for $B = 0$. Non-vanishing values of B induce a frustration around each plaquette, which is maximal for $B = \pi$, the fully-frustrated case.

In the framework of the high temperature expansion, the free energy of such models is expressed as a sum over the contribution of loops of increasing length $2k$:

$$\beta F = \sum_{k=0}^{\infty} \frac{\beta^{2k}}{2k} G_k \ . \qquad (1)$$

Each loops encloses a number of plaquettes; in the case of the models considered in [1, 2], for each loop the magnetic field yields a weight, proportional to $\exp(iBA)$, where A is the sum of plaquettes with signs depending on the orientations. The total contribution of all loops of length $2k$ is given by G_k. Due to the average over orientations and loops, the quantity G_k is a polynomial in the variable $q = \cos B$, the coefficient G_{kl} of q^l being given by the number of loops of length $2k$ and area l [†]; $G_k(q)$ is of order $k(k-1)/2$, given by the maximal area encloseable by a loop of length $2k$. The coefficients G_{kl} can also be interpreted as the number of Feynman diagrams with $2k$ external points, which are joined pairwise by lines (propagators) intersecting l times. These diagrams also occur in the topological (large N) expansion of Matrix Models [4], where the planar limit corresponds to no intersections, i.e. to the $q = 0$ case. Equivalently, in simple graphical terms, they just can be seen as the number of way of connecting pairwise $2k$ points on a circle with k chords intersecting exactly l times. In the following we shall refer to this last picture. In Ref. [1], the enumeration of such diagrams was investigated. In particular, a recursion relation was found for the coefficients of the polynomial $G_k(q)$ – a sort of Wick theorem – which can be nicely expressed by the algebra of the q-oscillators $a_q, a_q^\dagger$:

$$a_q a_q^\dagger - q a_q^\dagger a_q = 1 \ . \tag{2}$$

These operators [3] act on the Hilbert space spanned by the vectors: $|m\rangle$, $m = 0, \ 1, \ ...,$ as follows,

$$\begin{aligned} a_q|m\rangle &= \sqrt{[m]_q} \quad |m-1\rangle \ , & a_q|0\rangle &= 0 \ , \\ a_q^\dagger|m\rangle &= \sqrt{[m+1]_q}\ |m+1\rangle \ , & [m]_q &= \frac{1-q^m}{1-q} \ . \end{aligned} \tag{3}$$

[†]The area of a loop is defined as the minimal area of a surface of lattice plaquettes which have that loop as boundary.

Using the recursion relation, the weighted multiplicities of the diagrams of eq. (1) were neatly written as an expectation value over the ground state of the q-oscillators [1, 2]:

$$G_k(q) \;=\; \langle 0|(a_q^\dagger + a_q)^{2k}|0\rangle \;. \tag{4}$$

The authors of Refs. [1, 2] did not exploit the consequences of this result, preferring to turn themselves to numerical investigation of the models they were interested in. As a consequence, till now only the two limiting cases $G_k(0)$ and $G_k(1)$ were explicitely known; when $q = 1$ it is only a matter of counting the way of connecting $2k$ points on a circle, with no restrictions, and this is simply the number of pairings of $2k$ objects: $(2k-1)!!$. When $q = 0$ we are in fact evaluating the planar limit of the zero-dimensional $2k$-point Green function of a matrix model in the limit of vanishing interaction [4]. In simple graphical terms this correspond to the number of way of joining pairwise $2k$ points on a circle with non intersecting chords. In other word, this is just one of the many possible definition of Catalan numbers (see, *e.g.* [5]), given as:

$$G_k(0) = \frac{(2k)!}{k!(k+1)!} \;. \tag{5}$$

But the number G_{kl} of way of connecting pairwise $2k$ points on a circle, with exactly l intersections (*i.e.* the coefficient of q^l in $G_k(q)$) is not so easily accessible. In Refs. [1, 2], they were found by direct enumeration of the graphs on a computer. In [6], an explicit form of a generating function for coefficients G_{kl} was presented, but then the explicit evaluation of such coefficients relied upon heavy symbolic manipulations. We shall now present instead a simple, easy to evaluate, formula for a generic coefficient G_{kl}.

To this purpose let us first introduce the x_q coordinate representation, $x_q = a_q^\dagger + a_q$, $x_q|x\rangle = x|x\rangle$, which is given by the so-called continuous q-Hermite polynomials [8, 9].

These are defined by:

$$H_n(x) = \langle x|n\rangle \; \mathcal{C}_n \; , \qquad \mathcal{C}_n = \left([n]_q!\right)^{1/2} \; \mathcal{C}_0 \; , \tag{6}$$

where the normalization constant $\mathcal{C}_0$ is fixed by $H_0(x) = 1$ and the q-factorial is

$$[n]_q! = [n]_q \; [n-1]_q \ldots [1]_q \; , \qquad [1]_q = [0]_q = 1 \; . \tag{7}$$

These polynomials satisfy, of course, a three-term recursion relation in the index n:

$$xH_n(x) = H_{n+1}(x) + [n]_q \; H_{n-1}(x) \; , \qquad n \geq 1 \; . \tag{8}$$

x ranges over the interval $x \in \left[-2/\sqrt{1-q}, 2/\sqrt{1-q}\right]$, and a convenient convenient parametrisation is

$$x = \frac{2}{\sqrt{1-q}} \cos\theta \; , \qquad \theta \in [0, \pi]. \tag{9}$$

More properties of these q-Hermite polynomials can be found in Ref. [8], where they are defined as $\mathcal{H}_n(\cos\theta) = (1-q)^{n/2} H_n(x)$. The most important property for us is the orthogonalizing measure $\nu_q(x)$ [7, 8, 9]:

$$\int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}} \nu_q(x) \; dx \; H_n(x) \; H_m(x) \;\; = \;\; \delta_{n,m} \; [n]_q! \; , \tag{10}$$

$$\begin{aligned} \nu_q(x) \;\; &= \;\; \frac{\sqrt{1-q}}{2\pi} \; q^{-1/8} \; \Theta_1\left(\frac{\theta}{\pi}, q\right) \\ &= \;\; \frac{\sqrt{1-q}}{\pi} \; \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \; \sin[(2n+1)\theta] \end{aligned} \tag{11}$$

where $\Theta_1(z, q)$ is the first Jacobi theta function.

Polynomial $G_k(q)$ may now be rewritten as

$$G_k(q) = \int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}} \nu_q(x) \; x^{2k} \; dx \; . \tag{12}$$

Indeed, using the explicit form (11) of the integration measure, performing the integration and playing a little bit with indices, we may write

$$G_k(q) = \left(\frac{1}{1-q}\right)^k \sum_{l=0}^{k} (-1)^l \binom{2k+1}{k-l} \frac{2l+1}{2k+1} q^{l(l+1)/2} . \tag{13}$$

We now only need to perform explicitely the division of the polynomial of degree $k(k+1)/2$ defined by the sum in the previous formula. To this purpose, let us state the following

Theorem[‡]: *Let $P(q) = \sum_{l=0}^{N} p_l q^l$ be an integer coefficient polynomial of degree N in q, exactly divisible by $(1-q)^k$. Then $A(q) = P(q)/(1-q)^k$ is an integer coefficient polynomial of degree $N-k$ whose coefficients are simply expressed in terms of the p_l's as follows:*

$$A(q) = \sum_{l=0}^{N-k} a_l q^l , \qquad a_l = \sum_{i=0}^{l} \binom{k+l-1-i}{k-1} p_i . \tag{14}$$

We therefore readily get a simple closed expression for G_{kl}:

$$G_{kl} = \sum_{i=0}^{i_{max}} (-1)^i \binom{k+l-1-i(i+1)/2}{k-1} \binom{2k+1}{k-i} \frac{2i+1}{2k+1} , \qquad l \leq \frac{k(k-1)}{2} , \tag{15}$$

where i_{max} is the largest integer i satisfying $i(i+1)/2 \leq l$. Because of their definition as counting numbers, all G_{kl} should be positive integers. This fact is indeed not apparent from (15), and we are not able to present a rigorous proof of this statement, nevertheless be are convinced of its validity. Sensibleness arguments rely upon the intrinsic positivity of (4) and its derivatives with respect to q. In our opinion, the main flaw of (15) is however that the G_{kl} are expressed in terms of differences of large numbers, and this makes their evaluation for asymptotically large values of k an hard task.

[‡] We are unfortunately unable to quote any reference. We are of course convinced this theorem has been enounced long time ago. The only proof we have been able to give, absolutely inelegant, is by direct verification. We do not believe it is worth being presented here.

We have computed explicitely with Mathematica [10], through formula (15), which is very efficient, all values G_{kl} for k ranging from 1 to 9, and verified that they indeed match the results of Ref. [2] which were found by direct enumeration of the graphs on a computer.

In conclusion, we have presented an explicit solution to a combinatorial problem which could hardly be addressed directly; our solution, which makes use of the mapping to the physical problem of the q-oscillator proposed in [1], is somewhat indirect; in this very fact resides, in our opinion, its main interest: a new strategy is proposed to attack non trivial combinatorial problems, whenever recurrence relations are known, but not explicitely solved: the mapping upon the known solution of some physical problem opens the way to an explicit analytic solution.

Acknowledgements

We would like to thank Andrea Cappelli and Francesco Bonechi for useful discussions.

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