Cent. Eur. J. Math. • 11(10) • 2013 • 1774-1784
DOI: 10.2478/s11533-013-0288-7

VERSITA

Central European Journal of **Mathematics**

Abstract Korovkin-type theorems in modular spaces and applications

Research Article

Carlo Bardaro[1*], Antonio Boccuto[1†], Xenofon Dimitriou[2‡], Ilaria Mantellini[1§]

1 *Department of Mathematics and Computer Sciences, University of Perugia, via Vanvitelli 1, 06123 Perugia, Italy*

2 *Department of Mathematics, University of Athens, Panepistimiopolis, Athens 15784, Greece*

Received 25 October 2012; accepted 18 December 2012

Abstract: We prove some versions of abstract Korovkin-type theorems in modular function spaces, with respect to filter convergence for linear positive operators, by considering several kinds of test functions. We give some results with respect to an axiomatic convergence, including almost convergence. An extension to non positive operators is also studied. Finally, we give some examples and applications to moment and bivariate Kantorovich-type operators, showing that our results are proper extensions of the corresponding classical ones.

MSC: 40A35, 41A35, 46E30

Keywords: Modular space • Linear operator • Korovkin theorem • Filter convergence • Almost convergence

1. Introduction

The Korovkin theorem is the object of study of many mathematicians. In the classical Korovkin theorem [21] the uniform convergence in $\mathcal{C}([a,b])$, the space of all continuous real-valued functions defined on the compact interval $[a,b]$, is proved for a sequence of positive linear operators, assuming the convergence only on the test functions $1, x, x^2$. There are also trigonometric versions of this theorem, with the test functions $1, \sin x, \cos x$. One more set of test functions in abstract contexts was suggested in [3, 4]. Recently some versions of Korovkin theorems were proved in the setting of modular spaces, which include as particular cases L^p, Orlicz and Musielak–Orlicz spaces [9, 28]. Another direction is to consider more general kinds of convergences for the operator sequence involved: for example, convergence generated by a regular

* *E-mail: bardaro@unipg.it*
† *E-mail: boccuto@yahoo.it, boccuto@dmi.unipg.it*
‡ *E-mail: xenofon11@gmail.com, dxenof@math.uoa.gr*
§ *E-mail: mantell@dmi.unipg.it*

summability matrix method, statistical and filter convergence [1, 5–8, 10, 12, 13, 18]. Some investigation was performed in fractional and fuzzy Korovkin theory, e.g. Baskakov-type extensions of Korovkin theorems and related applications were obtained, see for instance [5] and its bibliography. In [2] and [5] some versions of Korovkin-type theorems were obtained for not necessarily positive operators.

In this paper we prove some Korovkin-type theorems with respect to filter convergence, introduced in [19], in the context of modular spaces for positive linear operators whose domain is a subspace of the set of all measurable functions, defined in topological spaces, and we consider several classes of test functions, satisfying suitable properties. Also the case of not necessarily positive operators is considered, following an approach given in [5]. Our results extend Korovkin-type theorems given in [8, 10, 17, 18] in the context of modular spaces and in [16] in the setting of ideal convergence. Note that at least the results concerning positive operators can be extended to more general kinds of convergence, not necessarily generated by free filters or regular matrix methods: among them we recall almost convergence [25]. Finally we give some examples and applications.

2. Preliminaries

We begin with recalling some properties of the filters of $\mathbb{N}$. A nonempty family $\mathcal{F}$ of subsets of $\mathbb{N}$ is called a *filter* of $\mathbb{N}$ iff $\emptyset \notin \mathcal{F}$, $A \cap B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and $B \supset A$ we get $B \in \mathcal{F}$. A sequence $(x_n)_n$ in $\mathbb{R}$ is said to be $\mathcal{F}$-*convergent* to $x \in \mathbb{R}$ (and we write $x = (\mathcal{F}) \lim_n x_n$) iff for every $\varepsilon > 0$ we get $\{n \in \mathbb{N} : |x_n - x| \le \varepsilon\} \in \mathcal{F}$. Let $\underline{\mathrm{x}} = (x_n)_n$ be a sequence in $\mathbb{R}$, and set

$$A_{\underline{\mathrm{x}}} = \big\{a \in \mathbb{R} : \{n \in \mathbb{N} : x_n \ge a\} \notin \mathcal{F}\big\}, \qquad B_{\underline{\mathrm{x}}} = \big\{b \in \mathbb{R} : \{n \in \mathbb{N} : x_n \le b\} \notin \mathcal{F}\big\}.$$

The $\mathcal{F}$-*limit superior* of $(x_n)_n$ is defined by

$$(\mathcal{F}) \limsup_n x_n = \begin{cases} \sup B_{\underline{\mathrm{x}}} & \text{if} \quad B_{\underline{\mathrm{x}}} \neq \emptyset, \\ -\infty & \text{if} \quad B_{\underline{\mathrm{x}}} = \emptyset. \end{cases} \tag{1}$$

The $\mathcal{F}$-*limit inferior* of $(x_n)_n$ is given by

$$(\mathcal{F}) \liminf_n x_n = \begin{cases} \inf A_{\underline{\mathrm{x}}} & \text{if} \quad A_{\underline{\mathrm{x}}} \neq \emptyset, \\ +\infty & \text{if} \quad A_{\underline{\mathrm{x}}} = \emptyset. \end{cases} \tag{2}$$

Examples 2.1.

The filter $\mathcal{F}_{\mathrm{cofin}}$ of all subsets of $\mathbb{N}$ whose complement is finite is called the *Fréchet filter*. Note that the limit, limit superior and limit inferior with respect to $\mathcal{F}_{\mathrm{cofin}}$ coincide with the usual ones [15].

We denote by $\mathcal{F}_d$ the filter associated with the statistical convergence, that is the set of all subsets of $\mathbb{N}$ whose asymptotic density is 1 [22].

A filter $\mathcal{F}$ on $\mathbb{N}$ is said to be *free* if it contains the Fréchet filter. In what follows we always deal with free filters.

3. The structural assumptions and modulars

We assume that G is a locally compact Hausdorff topological space, endowed with a uniform structure $\mathcal{U} \subset 2^{G\times G}$ which generates the topology of G, see [23]. Let $\mathcal{B}$ be the σ-algebra of all Borel subsets of G, and $\mu \colon \mathcal{B} \to \mathbb{R}$ be a positive σ-finite regular measure. We denote by $L^0(G)$ the space of all real-valued μ-measurable functions on G with identification up to sets of measure μ zero, by $\mathcal{C}_b(G)$ the space of all real-valued continuous and bounded functions on G, and by $\mathcal{C}_c(G)$ the subspace of $\mathcal{C}_b(G)$ of all functions with compact support on G.

Let us recall the notion of modular space [9, 28]. A functional $\rho \colon L^0(G) \to \widetilde{\mathbb{R}}_0^+$ is called a *modular* on $L^0(G)$ if it satisfies the following properties:

i) $\rho[f] = 0 \Leftrightarrow f = 0$, μ-almost everywhere on G;

ii) $\rho[-f] = \rho[f]$ for every $f \in L^0(G)$;

iii) $\rho[af + bg] \leq \rho[f] + \rho[g]$ for every $f, g \in L^0(G)$ and for each $a \geq 0$, $b \geq 0$ with $a + b = 1$.

A modular ρ is said to be *convex* if it satisfies conditions i), ii) and

iii') $\rho[af + bg] \leq a\rho[f] + b\rho[g]$ for all $f, g \in L^0(G)$ and for every $a, b \geq 0$ with $a + b = 1$.

Let $Q \geq 1$ be a constant. We say that a modular ρ is *Q-quasi semiconvex* if $\rho[af] \leq Qa\rho[Qf]$ for all $f \in L^0(G)$, $f \geq 0$ and $0 < a \leq 1$ [8].

We associate to the modular ρ the modular space $L^\rho(G)$ generated by ρ, defined by

$$L^\rho(G) = \left\{ f \in L^0(G) : \lim_{\lambda \to 0^+} \rho[\lambda f] = 0 \right\},$$

and the *space of the finite elements of* $L^\rho(G)$, defined by $E^\rho(G) = \left\{ f \in L^\rho(G) : \rho[\lambda f] < +\infty \text{ for all } \lambda > 0 \right\}$.

We will use the following notions. A modular ρ is said to be *monotone* if $\rho[f] \leq \rho[g]$ for all $f, g \in L^0(G)$ with $|f| \leq |g|$. A modular ρ is *finite* if χ_A (the characteristic function associated with A) belongs to $L^\rho(G)$ whenever $A \in \mathcal{B}$ with $\mu(A) < +\infty$. A modular ρ is *strongly finite* if χ_A belongs to $E^\rho(G)$ for all $A \in \mathcal{B}$ with $\mu(A) < +\infty$. A modular ρ is said to be *absolutely continuous* if there is a positive constant a with the property: for all $f \in L^0(G)$ with $\rho[f] < +\infty$,

- for each $\varepsilon > 0$ there exists a set $A \in \mathcal{B}$ with $\mu(A) < +\infty$ and $\rho[af\chi_{G \setminus A}] \leq \varepsilon$,
- for every $\varepsilon > 0$ there is a $\delta > 0$ with $\rho[af\chi_B] \leq \varepsilon$ for every $B \in \mathcal{B}$ with $\mu(B) < \delta$.

Example 3.1 ([9, 28]).

Let Φ be the set of all continuous non-decreasing functions $\varphi\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\varphi(0) = 0$, $\varphi(u) > 0$ for all $u > 0$ and $\lim_{u \to +\infty} \varphi(u) = +\infty$ in the usual sense, and let $\widetilde{\Phi}$ be the set of all elements of Φ which are convex functions. For all $\varphi \in \Phi$ (resp. $\widetilde{\Phi}$), the functional ρ^φ defined by

$$\rho^\varphi[f] = \int_G \varphi(|f(s)|)\, d\mu(s), \qquad f \in L^0(G), \tag{3}$$

is a (resp. convex) modular on $L^0(G)$ and $L^\varphi(G) = \{f \in L^0(G) : \rho^\varphi[\lambda f] < +\infty \text{ for some } \lambda > 0\}$ is the Orlicz space generated by φ.

We now define the modular and strong convergence in the context of the filter convergence (for the classical cases see [9, 28]). A sequence $(f_n)_n$ of functions in $L^\rho(G)$ is $\mathcal{F}$*-modularly convergent* to $f \in L^\rho(G)$ if there is $\lambda > 0$ with

$$(\mathcal{F}) \lim_n \rho[\lambda(f_n - f)] = 0. \tag{4}$$

Note that the $\mathcal{F}_{\text{cofin}}$-modular convergence coincides with the usual modular convergence. A sequence $(f_n)_n$ in $L^\rho(G)$ is $\mathcal{F}$*-strongly convergent* to $f \in L^\rho(G)$ if (4) holds for every $\lambda > 0$. Observe that $\mathcal{F}_{\text{cofin}}$-strong convergence is equivalent to usual strong convergence.

Given a subset $\mathcal{A} \subset L^\rho(G)$ and $f \in L^\rho(G)$, we say that $f \in \overline{\mathcal{A}}$ (that is, f is *in the modular closure of* $\mathcal{A}$) if there is a sequence $(f_n)_n$ in $\mathcal{A}$ such that $(f_n)_n$ is modularly convergent to f in the usual sense. We recall the following result.

Proposition 3.2 ([27, Theorem 1]).

Let ρ be a monotone, strongly finite and absolutely continuous modular on $L^0(G)$. Then $\overline{\mathcal{C}_c(G)} = L^\rho(G)$ with respect to the modular convergence in the ordinary sense.

4. The main results

In this section we prove some Korovkin-type theorems with respect to an abstract finite set of test functions $e_0, \dots, e_m$ in the context of the filter convergence.

In [10, 17, 18] some versions of the Korovkin theorem were given, with respect to methods of convergence, generated by a suitable non-negative regular summability matrix A. Note that for every such method there is a filter $\mathcal{F}$ with the property that the convergence generated by the matrix A is equivalent to the $\mathcal{F}$-convergence, but the converse is in general not true [20, Lemma 4, Corollary 1].

Let $\mathbf{T}$ be a sequence of linear operators $T_n : \mathcal{D} \to L^0(G)$, $n \in \mathbb{N}$, with $\mathcal{C}_b(G) \subset \mathcal{D} \subset L^0(G)$. Here the set $\mathcal{D}$ is the domain of operators T_n. We say that the sequence $\mathbf{T}$, together with the modular ρ, satisfies the *property* (ρ)-$(*)$ if there exist a subset $X_{\mathbf{T}} \subset \mathcal{D} \cap L^\rho(G)$ with $\mathcal{C}_b(G) \subset X_{\mathbf{T}}$ and a positive real constant N with $T_n f \in L^\rho(G)$ for all $f \in X_{\mathbf{T}}$ and $n \in \mathbb{N}$, and $(\mathcal{F}) \limsup_n \rho[\tau(T_n f)] \le N\rho[\tau f]$ for every $f \in X_{\mathbf{T}}$ and $\tau > 0$. Some examples in which property (ρ)-$(*)$ is fulfilled can be found, for instance, in [8].

Set $e_0(t) \equiv 1$ for all $t \in G$, let e_i, $i = 1, \dots, m$, and a_i, $i = 0, \dots, m$, be functions in $\mathcal{C}_b(G)$. Put

$$P_s(t) = \sum_{i=0}^{m} a_i(s) e_i(t), \qquad s, t \in G, \tag{5}$$

and suppose that $P_s(t)$, $s, t \in G$, satisfies the following properties:

(P1) $P_s(s) = 0$ for all $s \in G$;

(P2) for every neighborhood $U \in \mathcal{U}$ there is a positive real number η with $P_s(t) \ge \eta$ whenever $s, t \in G$, $(s, t) \notin U$.

We now give some examples of P_s for which properties (P1) and (P2) are fulfilled.

Examples 4.1.

(a) Let $G = I^m$ be endowed with the usual norm $\|\cdot\|_2$, where $I \subset \mathbb{R}$ is a connected set, $\phi : I \to \mathbb{R}$ be monotone and such that ϕ^{-1} is uniformly continuous on I. Examples of such functions are $\phi(t) = t$ or $\phi(t) = e^t$ when I is a bounded interval. For every $t = (t_1, \dots, t_m) \in G$ set $e_i(t) = \phi(t_i)$, $i = 1, \dots, m$, and $e_{m+1}(t) = \sum_{i=1}^m [\phi(t_i)]^2$. For all $s = (s_1, \dots, s_m) \in G$ put $a_0(s) = \sum_{i=1}^m [\phi(s_i)]^2$, $a_i(s) = -2\phi(s_i)$, $i = 1, \dots, m$, and $a_{m+1}(s) \equiv 1$. We get

$$P_s(t) = \sum_{i=0}^{m+1} a_i(s) e_i(t) = \sum_{i=1}^{m} [\phi(s_i) - \phi(t_i)]^2.$$

It is readily seen that $P_s(s) = 0$ for all $s \in G$, that is (P1). Moreover, by our hypotheses on ϕ and since the norm $\|\cdot\|_2$ is uniformly continuous, it follows that to every $\delta > 0$ there corresponds $\eta > 0$ with $P_s(t) \ge \eta$ whenever $\|s - t\|_2 \ge \delta$, and so (P2) holds.

(b) (see [5, 26]) Let $G = [0, a]$ with $0 < a < \pi/2$, $e_1(t) = \cos t$, $e_2(t) = \sin t$, $t \in G$. Set $a_0(s) \equiv 1$, $a_1(s) = -\cos s$, $a_2(s) = -\sin s$, $s \in G$. For all $s, t \in G$ we get

$$P_s(t) = 1 - \cos s \cos t - \sin s \sin t = 1 - \cos(s - t).$$

Clearly, property (P1) holds. Moreover, it is not difficult to see that for every $\delta > 0$ there is $\eta > 0$ with $P_s(t) \ge \eta$ whenever $s, t \in G$, $|s - t| \ge \delta$, that is (P2) is satisfied.

(c) Let $r \in \mathbb{N}$ be fixed, $G = [0, a/r]^r$, with $0 < a < \pi/2$, $t = (t_1, \dots, t_r)$, $s = (s_1, \dots, s_r) \in G$, $e_{2j-1}(t) = \cos jt_j$, $e_{2j}(t) = \sin jt_j$, $j = 1, \dots, r$. Put $a_0(s) \equiv r$, $a_{2j-1}(s) = -\cos js_j$, $a_{2j}(s) = -\sin js_j$, $j = 1, \dots, r$. For all $s, t \in G$ we have

$$P_s(t) = \sum_{i=0}^{2r} a_i(s) e_i(t) = r - \sum_{j=1}^{r} \cos js_j \cos jt_j - \sum_{j=1}^{r} \sin js_j \sin jt_j = r - \sum_{j=1}^{r} \cos(j(s_j - t_j)).$$

Arguing analogously as in (b), it is possible to check that $P_s(t)$ satisfies (P1) and (P2).

In order to obtain the main theorem, we begin with the following preliminary result.

Theorem 4.2.
Let ρ be a strongly finite, monotone and Q-quasi semiconvex modular. Assume that e_i and a_i, $i = 0, \dots, m$, satisfy properties (P1) and (P2). Let T_n, $n \in \mathbb{N}$ be a sequence of positive linear operators satisfying property (ρ)–$(*)$. If $T_n e_i$ is $\mathcal{F}$-modularly convergent to e_i, $i = 0, \dots, m$, in $L^\rho(G)$, then $T_n f$ is $\mathcal{F}$-modularly convergent to f in $L^\rho(G)$ for every $f \in \mathcal{C}_c(G)$. If $T_n e_i$ is $\mathcal{F}$-strongly convergent to e_i, $i = 0, \dots, m$, in $L^\rho(G)$, then $T_n f$ is $\mathcal{F}$-strongly convergent to f in $L^\rho(G)$ for every $f \in \mathcal{C}_c(G)$.

Proof. Let $f \in \mathcal{C}_c(G)$. Since G is endowed with the uniformity $\mathcal{U}$, f is uniformly continuous and bounded on G. Fix arbitrarily $\varepsilon > 0$. Without loss of generality we can suppose $0 < \varepsilon \le 1$. By uniform continuity of f there exists an element $U \in \mathcal{U}$ with the property that $|f(s) - f(t)| \le \varepsilon$ whenever $s, t \in G$, $(s, t) \in U$.

For all $s, t \in G$ let $P_s(t)$ be as in (5), and in correspondence with U let $\eta > 0$ satisfy condition (P2). If $M = \sup_{t\in G} |f(t)|$, then we get

$$|f(s) - f(t)| \le 2M \le \frac{2M}{\eta} P_s(t) \qquad \text{whenever} \quad s, t \in G, \quad (s, t) \notin U.$$

In any case we have $|f(s) - f(t)| \le \varepsilon + 2MP_s(t)/\eta$ for all $s, t \in G$, that is

$$-\varepsilon - \frac{2M}{\eta} P_s(t) \le f(s) - f(t) \le \varepsilon + \frac{2M}{\eta} P_s(t) \qquad \text{for all} \quad s, t \in G. \tag{6}$$

Since T_n is a positive linear operator, by applying T_n to (6), for all $n \in \mathbb{N}$ and $s \in G$ we get

$$-\varepsilon(T_n e_0)(s) - \frac{2M}{\eta}(T_n P_s)(s) \le f(s)(T_n e_0)(s) - (T_n f)(s) \le \varepsilon(T_n e_0)(s) + \frac{2M}{\eta}(T_n P_s)(s),$$

and hence

$$\begin{aligned} |(T_n f)(s) - f(s)| &\le \left|(T_n f)(s) - f(s)(T_n e_0)(s)\right| + \left|f(s)(T_n e_0)(s) - f(s)\right| \\ &\le \varepsilon(T_n e_0)(s) + \frac{2M}{\eta}(T_n P_s)(s) + M|(T_n e_0)(s) - e_0(s)|. \end{aligned} \tag{7}$$

Let now $\gamma > 0$. By applying the modular ρ, from (7) for all $n \in \mathbb{N}$ we get

$$\rho[\gamma(T_n f - f)] \le \rho[3\gamma\varepsilon(T_n e_0)] + \rho[3\gamma M(T_n e_0 - e_0)] + \rho\left[6\gamma \frac{M}{\eta}(T_n P_{(\cdot)})(\cdot)\right] = J_1 + J_2 + J_3. \tag{8}$$

So, in order to prove the theorem, it is enough to demonstrate the existence of a positive real number γ with $(\mathcal{F})\lim_n \rho[\gamma(T_n f - f)] = 0$. Indeed, let $\lambda > 0$ be such that $(\mathcal{F})\lim_n \rho[\lambda(T_n e_i - e_i)] = 0$ for all $i = 0, \dots, m$: such λ, by hypothesis, does exist. Pick $N > 0$ with $|a_i(s)| \le N$ for each $i = 0, \dots, m$ and $s \in G$, and let $\gamma > 0$ be with $\max\{3\gamma M, 6\gamma(M/\eta)(m+1)N\} \le \lambda$. Taking into account property (P1), for all $n \in \mathbb{N}$ we get

$$J_3 = \rho\left[6\gamma\frac{M}{\eta}(T_n P_{(\cdot)})(\cdot)\right] = \rho\left[6\gamma\frac{M}{\eta}(T_n P_{(\cdot)})(\cdot) - P_{(\cdot)}(\cdot)\right] \le \sum_{i=0}^{m} \rho\left[6\gamma\frac{M}{\eta}(m+1)N(T_n e_i - e_i)\right] \le \sum_{i=0}^{m} \rho[\lambda(T_n e_i - e_i)].$$

So, $(\mathcal{F})\lim\limits_n J_3 = 0$. Moreover, by the choice of λ and γ, it is easy to deduce that $(\mathcal{F})\lim\limits_n J_2 = 0$.

Since ρ is Q-quasi semiconvex and $0 < \varepsilon \le 1$, we have

$$\rho[3\gamma\varepsilon e_0] \le Q\varepsilon\rho[3\gamma Q e_0]. \tag{9}$$

By applying the limit superior and taking into account property (ρ)-$(*)$, from (8) and (9) we obtain

$$0 \le (\mathcal{F}) \limsup_n \rho[\gamma(T_n f - f)] \le (\mathcal{F}) \limsup_n \rho[3\gamma\varepsilon(T_n e_0)] \le N\rho[3\gamma\varepsilon e_0] \le NQ\varepsilon\rho[3\gamma Q e_0]. \tag{10}$$

From (10), by arbitrariness of ε and strong finiteness of ρ, we get $(\mathcal{F}) \limsup_n \rho[\gamma(T_n f - f)] = 0$, and hence $(\mathcal{F}) \lim_n \rho[\gamma(T_n f - f)] = 0$, by virtue of the properties of the filter limit and limit superior. This means that $T_n f$, $n \in \mathbb{N}$, is $\mathcal{F}$-modularly convergent to f in $L^\rho(G)$. The proof of the last part of the theorem is analogous. □

We now give the main theorem of this section, which is an extension of [8, Theorem 1] and [10, Theorem 2.6].

Theorem 4.3.

Let ρ be a monotone, strongly finite, absolutely continuous and Q-quasi semiconvex modular on $L^0(G)$, and T_n, $n \in \mathbb{N}$, be a sequence of positive linear operators satisfying property (ρ)-$(*)$. If $T_n e_i$ is $\mathcal{F}$-strongly convergent to e_i, $i = 0, \dots, m$, in $L^\rho(G)$, then $T_n f$ is $\mathcal{F}$-modularly convergent to f in $L^\rho(G)$ for all $f \in L^\rho(G) \cap \mathcal{D}$ with $f - \mathcal{C}_b(G) \subset X_\mathrm{T}$, where $\mathcal{D}$ and X_T are as before.

Proof. Let $f \in L^\rho(G) \cap \mathcal{D}$ with $f - \mathcal{C}_b(G) \subset X_\mathrm{T}$. By Proposition 3.2, there are $\lambda > 0$ and a sequence f_k, $k \in \mathbb{N}$, in $\mathcal{C}_c(G)$ with $\rho[3\lambda f] < +\infty$ and $\lim_k \rho[3\lambda(f_k - f)] = 0$ in the usual sense. Fix arbitrarily $\varepsilon > 0$ and pick a positive integer $\overline{k}$ with

$$\rho\big[3\lambda(f_{\overline{k}} - f)\big] \le \varepsilon. \tag{11}$$

For all $n \in \mathbb{N}$ we get

$$\rho[\lambda(T_n f - f)] \le \rho\big[3\lambda(T_n f - T_n f_{\overline{k}})\big] + \rho\big[3\lambda(T_n f_{\overline{k}} - f_{\overline{k}})\big] + \rho\big[3\lambda(f_{\overline{k}} - f)\big]. \tag{12}$$

By virtue of Theorem 4.2, we have

$$0 = (\mathcal{F}) \lim_n \rho\big[3\lambda(T_n f_{\overline{k}} - f_{\overline{k}})\big] = (\mathcal{F}) \limsup_n \rho\big[3\lambda(T_n f_{\overline{k}} - f_{\overline{k}})\big]. \tag{13}$$

By property (ρ)-$(*)$, there exists an $N > 0$ with

$$(\mathcal{F}) \lim_n \rho\big[3\lambda(T_n f - T_n f_{\overline{k}})\big] \le N\rho\big[3\lambda(f - f_{\overline{k}})\big] \le N\varepsilon. \tag{14}$$

From (11)–(14) and subadditivity of the $(\mathcal{F})\limsup$, we obtain

$$0 \le (\mathcal{F}) \limsup_n \rho[\lambda(T_n f - f)] \le \varepsilon(N+1). \tag{15}$$

From (15) and arbitrariness of $\varepsilon > 0$ it follows that $(\mathcal{F}) \limsup_n \rho[\lambda(T_n f - f)] = 0$, and hence $(\mathcal{F}) \lim_n \rho[\lambda(T_n f - f)] = 0$, that is the assertion. □

Remarks 4.4.

Note that, in Theorem 4.3, in general it is not possible to obtain $\mathcal{F}$-strong convergence unless the modular ρ satisfies the Δ_2-regularity condition, i.e. there exists a positive real number c_0 with $\rho[2f] \le c_0\rho[f]$ for every $f \in L^0(G)$ (for the classical frame see e.g. [28]).

Using a similar technique, we can prove an analogous result in the space $\mathcal{C}_{2\pi}(\mathbb{R})$ of all continuous real-valued functions on $\mathbb{R}$ of period 2π, by the homeomorphic identification of $\mathcal{C}_{2\pi}([0, 2\pi])$ with $\mathcal{C}(S^1)$.

Examples 4.5.

We now give some examples and applications of our results, showing that in general they are proper extensions of the corresponding classical ones.

(a) Let $G = [0,1]$ be endowed with the Lebesgue measure. Let Φ be as in Example 3.1, and for all $\varphi \in \Phi$, let $\rho = \rho^\varphi$ be as in (3). For every $t \in G$, set $e_i(t) = t^i$, $i = 0,1,2$. Note that (P1) and (P2) are fulfilled (see also Example 4.1 (a)).

Let $\mathcal{F} \neq \mathcal{F}_{\text{cofin}}$ be any free filter, and H be an infinite set, such that $\mathbb{N} \setminus H \in \mathcal{F}$. Since $\mathcal{F} \neq \mathcal{F}_{\text{cofin}}$, then H does exist. We consider the following linear positive operator:

$$M_n(f)(x) = \int_G K_n(t) f(tx)\, dt, \qquad n \in \mathbb{N}, \quad x \in G,$$

for every f belonging to the domain of M_n, where $K_n(t) = (n+1)t^n$ if $n \in \mathbb{N} \setminus H$, $K_n(t) = (n+1)^2 t^n$ if $n \in H$.

Proceeding analogously as in [7], it is not difficult to check that M_n satisfy all the hypotheses of Theorem 4.3. However, for every $\lambda > 0$, we get

$$\rho^\varphi[\lambda(M_n(e_0) - e_0)] = \begin{cases} 0 & \text{if} \quad n \in \mathbb{N} \setminus H, \\ \rho^\varphi[\lambda n] & \text{if} \quad n \in H. \end{cases}$$

So, the sequence $M_n(e_0)$, $n \in \mathbb{N}$, is not modularly convergent in the usual sense. Thus M_n do not fulfil the classical modular Korovkin theorem and so Theorems 4.2 and 4.3 are strict extensions of the corresponding classical ones.

(b) Let us consider bivariate Kantorovich-type operators. Let $\mathcal{F} \neq \mathcal{F}_{\text{cofin}}$ and H be an infinite set with the property that $\mathbb{N} \setminus H \in \mathcal{F}$. Let $G = [0,1]^2$, Φ be as in Example 3.1 and $\rho = \rho^\varphi$ be as in (3). Proceeding as in [10], for every locally integrable function $f \in L^0(G)$, $n \in \mathbb{N}$ and $x, y \in [0,1]$ set

$$P_n(f)(x,y) \;=\; (n+1)^2 \sum_{k,j=0,\dots,n,\,k+j\le n} p_{n,k,j}(x,y) \int_{k/(n+1)}^{(k+1)/(n+1)} \int_{j/(n+1)}^{(j+1)/(n+1)} f(u,v)\, du\, dv,$$

where

$$p_{n,k,j} = \frac{n!}{k!\,j!\,(n-k-j)!} x^k y^j (1-x-y)^{n-k-j}, \qquad k,j \ge 0, \quad x,y \ge 0, \quad x+y \le 1.$$

Let $(s_n)_n$ be the sequence defined by $s_n = 1$ if $n \in \mathbb{N} \setminus H$, $s_n = 0$ if $n \in H$. For all $n \in \mathbb{N}$ and $x, y \ge 0$ with $x + y \le 1$, set $P_n^*(f)(x,y) = s_n P_n(f)(x,y)$. For $u, v \in [0,1]$, set $e_0(u,v) = 1$, $e_1(u,v) = u$, $e_2(u,v) = v$, $e_3(u,v) = u^2 + v^2$. Proceeding analogously as in [10], it is possible to check that the sequence P_n^*, $n \in \mathbb{N}$, satisfies all hypotheses of Theorem 4.3, but not the classical Korovkin theorem.

5. An extension to non-positive operators

One can ask, whether it is possible, in the Korovkin theorems, to relax the positivity condition on the linear operators involved. In [5] there are given some positive answers with respect to the statistical convergence. Following this approach, we now give a Korovkin-type theorem for not necessarily positive linear operators, which is an extension of [5, Theorem 9.1] to the setting of filter convergence.

Let $\mathcal{F}$ be any fixed free filter of $\mathbb{N}$, I be a bounded interval of $\mathbb{R}$, $\mathcal{C}^2(I)$ (resp. $\mathcal{C}_b^2(I)$) be the space of all functions defined on I, (resp. bounded and) continuous together with their first and second derivatives, $\mathcal{C}_+ = \{f \in \mathcal{C}_b^2(I) : f \ge 0\}$, $\mathcal{C}_+^2 = \{f \in \mathcal{C}_b^2(I) : f'' \ge 0\}$.

Let e_i, $i = 1,\dots,m$, and a_i, $i = 0,\dots,m$, be functions in $\mathcal{C}_b^2(I)$, $P_s(t)$, $s,t \in I$, be as in (5), and suppose that $P_s(t)$ satisfies the properties (P1), (P2) and

(P3) there is a positive real constant C_0 with $P_s''(t) \ge C_0$ for all $s,t \in I$. (Here the second derivative is intended with respect to t).

We now give some examples in which property (P3) is fulfilled together with (P1) and (P2).

Examples 5.1.

(a) Note that (P3) is clearly satisfied when $P_s(t) = (s-t)^2$. (See also Example 4.1 (a)).

(b) Let $I = [0, \log 3/2]$, and $P_s(t) = (e^s - e^t)^2$, $s, t \in I$. It is easy to check that $P''_s(t) = 4e^{2t} - 2e^s e^t = 2e^t(2e^t - e^s)$, and so there exists $C_0 > 0$ satisfying (P3) for all $s, t \in I$, since $e^t \geq 1$ and $2e^t - e^s \geq 1/2$ for all $s, t \in I$.

(c) Let $I = [0, a]$ with $0 < a < 2\pi$, and e_i, a_i as in Example 4.1 (b). Then we get $P_s(t) = 1 - \cos(s-t)$, $P''_s(t) = \cos(s-t)$, $t, s \in I$. Analogously as in Example 4.1 (b) it is not difficult to check that $P_s(t)$ satisfies (P3).

We now prove the following Korovkin-type theorem for not necessarily positive linear operators.

Theorem 5.2.

Let $\mathcal{F}$ be any free filter of $\mathbb{N}$, ρ be as in Theorem 4.2, and assume that e_i, a_i, $i = 0, \dots, m$, and $P_s(t)$, $s, t \in I$, satisfy properties (P1), (P2) and (P3). Let T_n, $n \in \mathbb{N}$, be a sequence of linear operators, satisfying property (ρ)-$(*)$ with respect to $\mathcal{F}$-convergence. Suppose that $\{n \in \mathbb{N} : T_n(\mathcal{C}_+ \cap \mathcal{C}^2_+) \subset \mathcal{C}_+\} \in \mathcal{F}$. If $T_n e_i$ is $\mathcal{F}$-modularly convergent to e_i, $i = 0, \dots, m$, in $L^\rho(I)$, then $T_n f$ is $\mathcal{F}$-modularly convergent to f in $L^\rho(I)$, for every $f \in \mathcal{C}^2_b(I)$. If $T_n e_i$ is $\mathcal{F}$-strongly convergent to e_i, $i = 0, \dots, m$, in $L^\rho(I)$, then $T_n f$ is $\mathcal{F}$-strongly convergent to f in $L^\rho(I)$, for every $f \in \mathcal{C}^2_b(I)$. Furthermore, if ρ is absolutely continuous and $T_n e_i$ is $\mathcal{F}$-strongly convergent to e_i, $i = 0, \dots, m$, in $L^\rho(I)$, then $T_n f$ is $\mathcal{F}$-modularly convergent to f in $L^\rho(I)$ for every $f \in L^\rho(I) \cap \mathcal{D}$ with $f - \mathcal{C}_b(I) \subset X_T$.

Proof. Let $f \in \mathcal{C}^2_b(I)$. Note that f is uniformly continuous and bounded on I. Fix arbitrarily $\varepsilon > 0$. Without loss of generality we can suppose $0 < \varepsilon \leq 1$. By uniform continuity of f there exists a $\delta > 0$ with $|f(s) - f(t)| \leq \varepsilon$ for all $s, t \in I$, $|s - t| \leq \delta$.

Let $P_s(t)$, $s, t \in I$, be as in (5), and let $\eta > 0$ be associated with δ, satisfying (P2). By arguing analogously as in the proof of Theorem 4.2, for every $\beta \geq 1$ and $s, t \in I$ we get

$$-\varepsilon - \frac{2M\beta}{\eta} P_s(t) \leq f(s) - f(t) \leq \varepsilon + \frac{2M\beta}{\eta} P_s(t), \tag{16}$$

where $M = \sup_{t \in I} |f(t)|$. From (16) it follows that

$$h_{1,\beta}(t) = \frac{2M\beta}{\eta} P_s(t) + \varepsilon + f(t) - f(s) \geq 0, \tag{17}$$

$$h_{2,\beta}(t) = \frac{2M\beta}{\eta} P_s(t) + \varepsilon - f(t) + f(s) \geq 0 \tag{18}$$

for all $\beta \geq 1$ and s, $t \in I$. Let C_0 satisfy (P3). For each $t \in I$ we have

$$h''_{1,\beta}(t) \geq \frac{2M\beta C_0}{\eta} + f''(t), \qquad h''_{2,\beta}(t) \geq \frac{2M\beta C_0}{\eta} - f''(t).$$

Since f'' is bounded on I, we can choose $\beta \geq 1$ in such a way that $h''_{1,\beta}(t) \geq 0$, $h''_{2,\beta}(t) \geq 0$ for all $t \in I$. From now on we always consider such a choice of β. Thus $h_{1,\beta}, h_{2,\beta} \in \mathcal{C}_+ \cap \mathcal{C}^2_+$. Let $K_0 = \{n \in \mathbb{N} : T_n(\mathcal{C}_+ \cap \mathcal{C}^2_+) \subset \mathcal{C}_+\}$: by hypothesis we get $K_0 \in \mathcal{F}$ and

$$T_n(h_{j,\beta})(s) \geq 0 \qquad \text{for all} \quad n \in K_0, \quad s \in I, \quad j = 1, 2. \tag{19}$$

From (17)–(19) and the linearity of T_n, for all $n \in K_0$ and $s \in I$ we have

$$\frac{2M\beta}{\eta}(T_n P_s)(s) + \varepsilon(T_n e_0)(s) + (T_n f)(s) - f(s)(T_n e_0)(s) \geq 0,$$

$$\frac{2M\beta}{\eta}(T_n P_s)(s) + \varepsilon(T_n e_0)(s) - (T_n f)(s) + f(s)(T_n e_0)(s) \geq 0,$$

and hence

$$-\varepsilon(T_n e_0)(s) - \frac{2M\beta}{\eta}(T_n P_s)(s) \le f(s)(T_n e_0)(s) - (T_n f)(s) \le \varepsilon(T_n e_0)(s) + \frac{2M\beta}{\eta}(T_n P_s)(s).$$

By proceeding similarly as in the proof of Theorem 4.2, applying the modular ρ and taking into account that $K_0 \in \mathcal{F}$, we obtain the assertion of the first part. The other parts follow by arguing analogously as in the proofs of Theorems 4.2 and 4.3. □

We give an example in which the operators involved are not necessarily positive, the hypotheses of Theorem 5.2 are fulfilled and the classical Korovkin theorem is not satisfied.

Example 5.3.
Fix arbitrarily a free filter $\mathcal{F} \neq \mathcal{F}_{\text{cofin}}$, and pick an infinite set H, with the property that $\mathbb{N} \setminus H \in \mathcal{F}$. For all $f \in \mathcal{C}^2([0,1])$ and $x \in [0,1]$ put

$$T_n(f)(x) = \begin{cases} L_n^*(f)(x) & \text{if} \quad n \in H, \\ B_n(f)(x) & \text{if} \quad n \in \mathbb{N} \setminus H, \end{cases}$$

where B_n, $n \in \mathbb{N}$, denote the Bernstein polynomials

$$B_n(f)(x) = \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} f\left(\frac{i}{n}\right), \qquad x \in [0,1],$$

and L_n^* is defined as follows:

$$L_n^*(f)(x) = \int_0^{+\infty} K_n(t) f(tx)\, dt,$$

where $K_n(t) = (1-n)t^n \chi_{]0,1[}(t)$, $t \in [0, +\infty[$. As in [5, p. 129], it is not difficult to check that the hypotheses of Theorem 5.2 are satisfied with respect to $\mathcal{F}$-convergence. Moreover, for all $n \in H$ and $x \in [0,1]$ we have

$$L_n^*(e_0)(x) = \int_0^1 K_n(t)\, dt = \frac{1-n}{n+1},$$

and so the operators T_n are not positive. Thus we get

$$T_n(e_0)(x) = \begin{cases} \dfrac{1-n}{n+1} & \text{if} \quad n \in H, \\ 1 & \text{if} \quad n \in \mathbb{N} \setminus H. \end{cases}$$

Hence for all $x \in [0,1]$, the sequence $T_n(e_0)(x)$, $n \in \mathbb{N}$, is not convergent in the usual sense. Thus, the classical Korovkin theorem is not satisfied.

6. Further remarks and extensions

With suitable modifications and analogous techniques, Theorems 4.2 and 4.3 remain true even if we consider an axiomatic abstract convergence [11, 23].

Let $\mathcal{T}$ be the set of all real-valued sequences $(x_n)_n$. A *convergence* is a pair $(\mathcal{S}, \ell)$, where $\mathcal{S}$ is a linear subspace of $\mathcal{T}$ and $\ell\colon \mathcal{S} \to \mathbb{R}$ is a function, satisfying the following axioms:

(a) $\ell\big((a_1 x_n + a_2 y_n)_n\big) = a_1 \ell((x_n)_n) + a_2 \ell((y_n)_n)$ for every pair of sequences $(x_n)_n, (y_n)_n \in \mathcal{S}$ and for each $a_1, a_2 \in \mathbb{R}$ (*linearity*).

(b) If $(x_n)_n, (y_n)_n \in \mathcal{S}$ and $x_n \le y_n$ definitely (i.e., $x_n \le y_n$, $n \ge n_0$, for some $n_0 \in \mathbb{N}$), then $\ell((x_n)_n) \le \ell((y_n)_n)$ (*monotonicity*).

(c) If $(x_n)_n$ satisfies $x_n = l$ definitely, then $(x_n)_n \in \mathcal{S}$ and $\ell((x_n)_n) = l$.

(d) If $(x_n)_n \in \mathcal{S}$, then $(|x_n|)_n \in \mathcal{S}$ and $\ell((|x_n|)_n) = |\ell((x_n)_n)|$.

(e) Given three sequences $(x_n)_n, (y_n)_n, (z_n)_n$, satisfying $(x_n)_n, (z_n)_n \in \mathcal{S}$, $\ell((x_n)_n) = \ell((z_n)_n)$ and $x_n \leq y_n \leq z_n$ definitely, then $(y_n)_n \in \mathcal{S}$.

Note that $\mathcal{S}$ is the space of all convergent sequences, and ℓ will be the "limit" according to this approach. Observe that the filter convergence satisfies the above axioms (a)–(e) [11].

We now give the axiomatic definition of the operators "limit superior" and "limit inferior" related with a convergence $(\mathcal{S}, \ell)$. Let $\mathcal{T}, \mathcal{S}$ be as above. We define two functions $\overline{\ell}$, $\underline{\ell}\colon \mathcal{T} \to \widetilde{\mathbb{R}}$, satisfying the following axioms:

(f) If $(x_n)_n, (y_n)_n \in \mathcal{T}$, then $\underline{\ell}((x_n)_n) \leq \overline{\ell}((x_n)_n)$ and $\overline{\ell}((x_n)_n) = -\underline{\ell}((-x_n)_n)$.

(g) If $(x_n)_n \in \mathcal{T}$, then

(g_1) $\overline{\ell}\big((x_n + y_n)_n\big) \leq \overline{\ell}((x_n)_n) + \overline{\ell}((y_n)_n)$ (*subadditivity*);

(g_2) $\underline{\ell}\big((x_n + y_n)_n\big) \geq \underline{\ell}((x_n)_n) + \underline{\ell}((y_n)_n)$ (*superadditivity*).

(h) If $(x_n)_n, (y_n)_n \in \mathcal{T}$ and $x_n \leq y_n$ definitely, then $\overline{\ell}((x_n)_n) \leq \overline{\ell}((y_n)_n)$ and $\underline{\ell}((x_n)_n) \leq \underline{\ell}((y_n)_n)$ (*monotonicity*).

(i) A sequence $(x_n)_n \in \mathcal{T}$ belongs to $\mathcal{S}$ if and only if $\overline{\ell}((x_n)_n) = \underline{\ell}((x_n)_n)$.

It is easy to see that the $\mathcal{F}$-limit superior and the $\mathcal{F}$-limit inferior defined in (1) and (2) satisfy the above axioms (f)–(i), see [15, Theorems 3 and 4], [24, Theorem 5]. We now show that these axioms are satisfied also by other kinds of convergences. We say that a sequence $(x_n)_n$ in $\mathbb{R}$ *almost converges* to $x_0 \in \mathbb{R}$ ($(A)\lim_n x_n = x_0$) if

$$\lim_{n\to 0} \frac{x_{m+1} + x_{m+2} + \cdots + x_{m+n}}{n} = x_0$$

uniformly with respect to m, see [25]. It is readily seen that almost convergence satisfies axioms (a)–(i).

A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *$\mathcal{F}$-continuous at* $x_0 \in \mathbb{R}$ if $(\mathcal{F})\lim_n f(x_n) = f(x_0)$ whenever $(\mathcal{F})\lim_n x_n = x_0$, and is called *$A$-continuous at* $x_0 \in \mathbb{R}$ if $(A)\lim_n f(x_n) = f(x_0)$ whenever $(A)\lim_n x_n = x_0$. In [22, Proposition 3.3] it is shown that for every free filter $\mathcal{F}$-continuity is equivalent to usual continuity, while in [14, Theorem 1] it is proved that every A-continuous function at any fixed point x_0 is linear. Thus the concepts of A- and $\mathcal{F}$-continuity do not coincide, and hence almost convergence is not generated by any free filter.

Acknowledgements

Research supported by Universities of Perugia and Athens.

References

[1] Agratini O., On statistical approximation in spaces of continuous functions, Positivity, 2009, 13(4), 735–743

[2] Agratini O., Statistical convergence of a non-positive approximation process, Chaos Solitons Fractals, 2011, 44(11), 977–981

[3] Altomare F., Korovkin-type theorems and approximation by positive linear operators, Surv. Approx. Theory, 2010, 5, 92–164

[4] Altomare F., Campiti M., Korovkin-type Approximation Theory and its Applications, de Gruyter Stud. Math., 17, Walter de Gruyter, Berlin, 1994

[5] Anastassiou G.A., Duman O., Towards Intelligent Modeling: Statistical Approximation Theory, Intell. Syst. Ref. Libr., 14, Springer, Berlin, 2011
[6] Bardaro C., Boccuto A., Dimitriou X., Mantellini I., Modular filter convergence theorems for abstract sampling-type operators, Appl. Anal. (in press), DOI: 10.1080/00036811.2012.738480
[7] Bardaro C., Mantellini I., Multivariate moment type operators: approximation properties in Orlicz spaces, J. Math. Inequal., 2008, 2(2), 247–259
[8] Bardaro C., Mantellini I., A Korovkin theorem in multivariate modular function spaces, J. Funct. Spaces Appl., 2009, 7(2), 105–120
[9] Bardaro C., Musielak J., Vinti G., Nonlinear Integral Operators and Applications, De Gruyter Ser. Nonlinear Anal. Appl., 9, Walter de Gruyter, Berlin, 2003
[10] Belen C., Yildirim M., Statistical approximation in multivariate modular function spaces, Comment. Math., 2011, 51(1), 39–53
[11] Boccuto A., Candeloro D., Integral and ideals in Riesz spaces, Inform. Sci., 2009, 179(17), 2891–2902
[12] Boccuto A., Dimitriou X., Modular filter convergence theorems for Urysohn integral operators and applications, Acta Math. Sinica, 2013, 29(6), 1055–1066
[13] Boccuto A., Dimitriou X., Modular convergence theorems for integral operators in the context of filter exhaustiveness and applications, Mediterr. J. Math., 2013, 10(2), 823–842
[14] Borsík J., Šalát T., On F-continuity of real functions, Tatra Mt. Math. Publ., 1993, 2, 37–42
[15] Demirci K., $\mathcal{I}$-limit superior and limit inferior, Math. Commun., 2001, 6(2), 165–172
[16] Duman O., Özarslan M.A., Erkuş-Duman E., Rates of ideal convergence for approximation operators, Mediterr. J. Math., 2010, 7(1), 111–121
[17] Gadjiev A.D., Orhan C., Some approximation theorems via statistical convergence, Rocky Mountain J. Math., 2002, 32(1), 129–138
[18] Karakuş S., Demirci K., Duman O., Statistical approximation by positive linear operators on modular spaces, Positivity, 2010, 14(2), 321–334
[19] Katětov M., Product of filters, Comment. Math. Univ. Carolinae, 1968, 9(1), 173–189
[20] Komisarski A., Pointwise $\mathcal{I}$-convergence and $\mathcal{I}$-convergence in measure of sequences of functions, J. Math. Anal. Appl., 2008, 340(2), 770–779
[21] Korovkin P.P., On convergence of linear positive operators in the spaces of continuous functions, Doklady Akad. Nauk SSSR (N.S.), 1953, 90, 961–964 (in Russian)
[22] Kostyrko P., Šalát T., Wilczyński W., $\mathcal{I}$-convergence, Real Anal. Exchange, 2000/01, 26(2), 669–685
[23] Kuratowski K., Topology I–II, Academic Press/PWN, New York–London/Warsaw, 1966/1968
[24] Lahiri B.K., Das P., I and I^*-convergence in topological spaces, Math. Bohem., 2005, 130(2), 153–160
[25] Lorentz G.G., A contribution to the theory of divergent sequences, Acta Math., 1948, 80, 167–190
[26] Maligranda L., Korovkin theorem in symmetric spaces, Comment. Math. Prace Mat., 1987, 27(1), 135–140
[27] Mantellini I., Generalized sampling operators in modular spaces, Comment. Math. Prace Mat., 1998, 38, 77–92
[28] Musielak J., Orlicz Spaces and Modular Spaces, Lecture Notes in Math., 1034, Springer, Berlin, 1983