Research Article

Irina Astashova, Miroslav Bartušek, Zuzana Došlá*, and Mauro Marini

Asymptotic proximity to higher order nonlinear differential equations

https://doi.org/10.1515/anona-2022-0254
received November 16, 2021; accepted May 2, 2022

Abstract: The existence of unbounded solutions and their asymptotic behavior is studied for higher order differential equations considered as perturbations of certain linear differential equations. In particular, the existence of solutions with polynomial-like or noninteger power-law asymptotic behavior is proved. These results give a relation between solutions to nonlinear and corresponding linear equations, which can be interpreted, roughly speaking, as an asymptotic proximity between the linear case and the nonlinear one. Our approach is based on the induction method, an iterative process and suitable estimates for solutions to the linear equation.

Keywords: higher order differential equation, unbounded solutions, nonoscillatory solution, asymptotic behavior, topological methods

MSC 2020: Primary 34C10, Secondary 34C15

1 Introduction

Consider the higher order nonlinear equation

$$u^{(n)} + q(t)u^{(n-2)} = r(t)|u|^{\lambda}\operatorname{sgn} u, \quad n \geq 2, \quad \lambda > 0, \tag{1}$$

where the functions r and q are continuous and $q(t) > 0$ for $t \geq 1$.

Our aim is to study some asymptotic properties of solutions to (1), by considering (1) as a perturbation of the two-term linear equation

$$y^{(n)} + q(t)y^{(n-2)} = 0. \tag{2}$$

Equation (1) is a particular case of the more general equation

$$u^{(n)} + \sum_{j=0}^{n-1} a_j(t)u^{(j)} = r(t)|u|^{\lambda}\operatorname{sgn} u, \quad n \geq 2, \ \lambda > 0, \tag{3}$$

where the functions a_j, $j = 0, \ldots, n-1$, are continuous for $t \geq 1$.

*** Corresponding author: Zuzana Došlá,** Department of Mathematics and Statistics, Masaryk University, Kotlárrská 2, CZ-61137 Brno, Czech Republic, e-mail: dosla@math.muni.cz
Irina Astashova: Lomonosov Moscow State University, Faculty of Mechanics and Mathematics, Leninskie Gory, 1, Main Building, Moscow, 119991, Russia; Plekhanov Russian University of Economics, Stremyanny Lane, 36, Moscow, 117997, Russia, e-mail: ast.diffiety@gmail.com
Miroslav Bartušek: Department of Mathematics and Statistics, Masaryk University, Kotlárrská 2, CZ-61137 Brno, Czech Republic, e-mail: bartusek@math.muni.cz
Mauro Marini: Department of Mathematics and Informatics "Ulisse Dini", University of Florence, I-50139 Florence, Italy, e-mail: mauro.marini@unifi.it

By a solution to (1) [(3)] we mean a function u differentiable up to order n, which satisfies (1) [(3)] on $[T_x, \infty)$, $T_x \geq 1$ and such that

$$\sup\{|u(t)| : t \geq T\} > 0 \quad \text{for any } T \geq T_x.$$

As usual, a solution u to (1) [(3)] is said to be *oscillatory* if u changes its sign for arbitrary large t and *nonoscillatory* if u is different from zero for any sufficiently large t.

These equations have been investigated from different points of view as a generalization of Emden-Fowler-type differential equation

$$u^{(n)} = r(t)|u|^{\lambda}\operatorname{sgn} u, \quad \lambda > 0, \quad \lambda \neq 1, \tag{4}$$

see, for example, [13, Chapter IV], [4, Ch.I], and references therein for more details.

The problem of the proximity of solutions of two differential equations has a long history and has been studied in various directions for a large variety of equations. Here, we recall the monograph [13], previous papers [1–3,5–7,9,14,16,17,20], and references therein. More precisely, in [1,2] sufficient conditions are obtained for the existence of solutions of (3) which are close in a neighborhood of infinity to any nonzero constant or, more generally to a polynomial of j degree, $j = 0, \dots, n-1$. In [3], the problem of asymptotic equivalence of (4) and its perturbations is studied. In [5], an asymptotic classification of solutions of (3) with $n = 3, 4$ is given by topological methods. In [6,7], motivated by [12], some asymptotic relationships between (1) and (2) are obtained by using a topological approach. In [9], the existence, uniqueness, and the asymptotic equivalence of a linear differential system and one of its nonlinear perturbation with advanced and retarded argument is considered. In [14], some general relationship between solutions of a linear differential system and its perturbed form with maxima are given. In [16,17], a certain type of asymptotic equivalence between a dynamic equation and its perturbations has been considered. In [20], a type of an asymptotic equivalence between two different differential systems has been examined by using the concept of reflecting functions and these results are applied to the search of periodic solutions.

In particular, in [6,7] an important role is played by the second-order linear equation

$$h'' + q(t)h = 0. \tag{5}$$

If the function q is bounded away from zero, then (5) is oscillatory, that is, any of its solution is oscillatory. Under this additional assumption, in [6, Theorem 1] it is shown that solutions of (2) are, roughly speaking, in asymptotic proximity with solutions of (1) and in [7, Theorem 2, Theorem 3] their boundedness is examined. If the function q tends to some positive constant, then (2) has $(n-2)$ parametric set of polynomial solutions, and (1) with $\lambda = 1$ has $(n-2)$ parametric set of polynomial-like solutions, see [7, Theorem 5].

Here we consider the case in which the linear equation (5) is nonoscillatory, that is, any of its solution is nonoscillatory. Under this assumption, a direct computation shows that (1) can be written for large t as the two-term equation

$$\left(h^2(t)\left(\frac{x^{(n-2)}(t)}{h(t)}\right)'\right)' = h(t)r(t)|u|^{\lambda}\operatorname{sgn} u,$$

where h is a nonoscillatory solution of (5). Using this fact and an iterative method, which is similar to the one given in [6], we study the existence of a n-parameter set of nonoscillatory solutions to equation (1) with polynomial and noninteger power-law growth, which are, roughly speaking, in asymptotic proximity with solutions of the linear unperturbed equation (2).

As usual, we use the following notation.

Let g_i, $i = 1, 2, 3$ be continuous functions and $g_3 \neq 0$ near infinity. Then:

– The symbol $g_1 = O(g_2)$ means that ultimately $|g_1(t)| \leq M|g_2(t)|$ for some constant M, that is, there exist $T \geq 0$ and $M > 0$ such that

$$|g_1(t)| \leq M|g_2(t)| \quad \text{on } [T, \infty);$$

- The symbol $g_1 = o(g_3)$ means that $\lim_{t\to\infty} g_1(t)/g_3(t) = 0$;
- The symbol $g_1 \sim g_3$ means that $\lim_{t\to\infty} g_1(t)/g_3(t) = 1$;
- For $\lambda > 0$ and $u \in \mathbb{R}$ the symbol $[u]_\pm^\lambda$ means $|u|^\lambda \operatorname{sgn} u$, for short.

Equation (3) has been studied in the literature as a perturbation of the linear differential equation

$$y^{(n)} + \sum_{j=0}^{n-1} a_j(t) y^{(j)} = 0. \tag{6}$$

We start by referring Sobol's result [18] obtained for equation (6).

Theorem A. [18, Theorem 1] *Under the condition*

$$\int_1^\infty t^{n-j-1}|a_j(t)|\mathrm{d}t < \infty \quad \textit{for } j = 0, 1, \dots, n-1, \tag{7}$$

equation (6) *has a fundamental family of solutions* y_j *satisfying for large* t *the conditions*

$$y_j(t) = t^j(1 + o(1)) \quad \textit{for } j = 0, 1, \dots, n-1. \tag{8}$$

In this paper, it is also proved that the derivatives of such solutions also have a power-law growth.

This result has been extended by I. Kiguradze to equation (4) and to a more general higher order equation.

Theorem B. [11, Theorem 1] *Let* $\lambda \geq 1$ *and* $1 \leq k \leq n$. *In order equation* (4) *has solutions* $u_i(t)$ $(i = 1, 2, \dots, k)$ *such that for* $t \to \infty$

$$u_i(t) \sim c_i t^{i-1}, \quad u_i'(t) \sim (i-1)c_i t^{i-2}, \dots u_i^{(i-1)}(t) \sim (i-1)! \quad c_i \neq 0,$$

it is sufficient, and if r *does not change its sign, it is also necessary that*

$$\int_1^\infty |r(t)| t^{n-1+(\lambda-1)(k-1)} \mathrm{d}t < \infty.$$

The next result follows from [13, Corollary 8.2] applying to equation

$$u^{(n)} = q(t, u, u', \dots, u^{(n-1)}), \tag{9}$$

where $q(t, x_1, \dots, x_n)$ is a continuous function on $[1, \infty) \times \mathbb{R}^n$.

Theorem C. *Let there exist* $l \in \{1, \dots, n\}$, $\mu \in (0, \infty)$, $\mu \neq 0$, *and a continuous function* $q^* : [T, \infty) \to \mathbf{R}_+$, $T \geq 1$, *such that*

$$|q(t, x_1, \dots, x_n)| \leq q^*(t) \quad \textit{for } t \geq T,$$
$$\left| x_k - \alpha \frac{(l-1)!}{(l-k)!} t^{l-k} \right| \leq \mu t^{l-k} \quad (k = 1, \dots, l),$$
$$|x_k| \leq \mu t^{l-k} \quad (k = l+1, \dots, n),$$

and

$$\int_T^\infty t^{n-l} q^*(t) \mathrm{d}t < \infty.$$

Then equation (9) *possesses a solution u having the asymptotic representation*

$$u^{(k-1)}(t) = \alpha\frac{(l-1)!}{(l-k)!}t^{l-k} + o(t^{l-k}) \quad (k = 1, \ldots, l),$$
$$u^{(k-1)}(t) = o(t^{l-k}) \quad (k = l+1, \ldots, n).$$

Using this result it is easy to give conditions if (3) has solutions with a polynomial power-law growth at infinity. These results are presented in Section 3, along with a discussion on the proximity between equations (1) and (2). Finally, in Section 4 some comments, examples, and suggestions for future research studies are presented.

2 Asymptotic representation of solutions to equation (1)

Our main result in this section concerns the existence of solutions with polynomial-like or noninteger power-law asymptotic behavior and reads as follows.

Theorem 1. *Let the second-order differential equation* (5) *be nonoscillatory. Assume that for some real number* $m \in [0, n-1]$

$$\int_1^\infty t^{n-1+m\lambda+i_q}|r(t)|\mathrm{d}t < \infty, \tag{10}$$

where $i_q = 0$ *in the case*

$$\int_1^\infty tq(t)\mathrm{d}t < \infty \tag{11}$$

and $i_q = 1$ *in the case*

$$\int_1^\infty tq(t)\mathrm{d}t = \infty. \tag{12}$$

Then for any solution y to (2) *such that* $y(t) = O(t^{\mathrm{m}})$ *there exists a solution u to* (1) *such that for large t*

$$u^{(i)}(t) = y^{(i)}(t) + \varepsilon_i(t), \quad i = 0, 1, \ldots, n-1, \tag{13}$$

where all ε_i *are functions of bounded variation and* $\lim_{t\to\infty}\varepsilon_i(t) = 0$.

To prove Theorem 1, we use a similar approach to that given in [6], which is based on the induction method, an iterative process, and suitable estimates for solutions to (2).

Equations (2) and (5) are strictly related. When $q(t) \equiv 0$, the space of solutions to (2) has a fundamental system of solutions consisting of

$$t^j, \quad j = 0, 1, \ldots, n-1.$$

In the general case, it is easy to see that the space of solutions to (2) has a fundamental system of solutions consisting of

$$\Gamma_1, \Gamma_2, t^j, \quad j = 0, 1, \ldots, n-3, \tag{14}$$

where t^j are missing in case $n = 2$, and

$$\Gamma_1(t) = \int_1^t (t-s)^{n-3}h_1(s)\mathrm{d}s, \quad \Gamma_2(t) = \int_1^t (t-s)^{n-3}h_2(s)\mathrm{d}s, \tag{15}$$

in case $n \geq 3$,

$$\Gamma_1(t) = h_1(t), \quad \Gamma_2(t) = h_2(t) \quad \text{in case } n = 2;$$

h_1, h_2 are two independent solutions to (5), see [6].

Let h_1 and h_2 be two linearly independent solutions to (5) with Wronskian $d > 0$. Put

$$w(s,t) = h_1(s)h_2(t) - h_1(t)h_2(s), \quad z(s,t) = \frac{\partial}{\partial t} w(s,t). \tag{16}$$

Lemma 1. *Let* (5) *be nonoscillatory.*

(*i*) *If* (11) *holds, then there exists a constant* $K > 0$ *such that*

$$|w(s,t)| \leq Ks, \quad |z(s,t)| \leq Ks \quad \textit{for } s \geq t \geq 1,$$

and

$$\Gamma_1(t) = O(t^{n-2}), \quad \Gamma_2(t) = O(t^{n-1}).$$

(*ii*) *If* (12) *holds, then there exists a constant* $K > 0$ *such that*

$$|w(s,t)| \leq Ks^2, \quad |z(s,t)| \leq Ks \quad \textit{for } s \geq t \geq 1,$$

and

$$\Gamma_1(t) = O(t^{n-1}), \quad \Gamma_2(t) = O(t^{n-1}).$$

Proof. Let h_1, h_2 be two linearly independent solutions to (5) with Wronskian $d > 0$. According to [8], Theorem 1(i_2), (i_3), and Theorem 2(i_2), (i_3), we can choose

$$\lim_{t\to\infty} |h_1(t)| = c_1 > 0, \quad \lim_{t\to\infty} h_1'(t) = 0, \quad \lim_{t\to\infty} |h_2(t)| = \infty, \quad \lim_{t\to\infty} |h_2'(t)| = c_2 > 0$$

in case (11). Hence, $|h_2(t)| \leq c_2 t$ and from (16) we obtain $|w(s,t)| \leq Ks$ for $t \geq 1$ and $K = c_1 c_2$. Similarly,

$$\lim_{t\to\infty} |h_i(t)| = \infty, \quad \lim_{t\to\infty} h_i'(t) = 0, \quad i = 1, 2,$$

in case (12). From these, (15), and (16) the conclusions follow. □

The following lemma is a modification of [6, Lemma 1], where it has been stated for (1) in the case that (5) is oscillatory.

Lemma 2. *Let equation* (5) *be nonoscillatory and let* $\{u_k\}_{k\in\mathbb{N}}$ *be a sequence of continuous functions on* $[t_0, \infty)$, $t_0 \geq 1$. *Consider for* $k \geq 2$ *and* $t \geq t_0$ *the sequence* $\{\alpha_k\}$ *is given by*

$$\alpha_k(t) = \frac{(-1)^{n+1}}{d} \int_t^{T_k} \frac{(\sigma - t)^{n-3}}{(n-3)!} \int_\sigma^{T_k} r(s)\, [u_{k-1}(s)]_{\pm}^{\lambda} w(s,\sigma) \mathrm{d}s \mathrm{d}\sigma,$$

in case $n \geq 3$ *and*

$$\alpha_k(t) = -\frac{1}{d} \int_t^{T_k} r(s)[u_{k-1}(s)]_{\pm}^{\lambda} w(s,t) \mathrm{d}s$$

in case $n = 2$, *where* $T_k = t_0 + k$.

(*i*) *If* (11) *holds, then, for* $t \geq t_0$,

$$|\alpha_k^{(i)}(t)| \leq M \left| \int_t^{T_k} s^{n-1-i} |r(s)|\, |u_{k-1}(s)|^{\lambda} \mathrm{d}s \right|, \quad i = 0, 1, \ldots, n-3, \tag{17}$$

$$|\alpha_k^{(i)}(t)| \le M \left| \int_t^{T_k} s|r(s)|\, |u_{k-1}(s)|^\lambda \mathrm{d}s \right|, \quad i = n-2, n-1; \tag{18}$$

(ii) If (12) *holds, then, for* $t \ge t_0$,

$$\begin{aligned} |\alpha_k^{(i)}(t)| &\le M \left| \int_t^{T_k} s^{n-i}|r(s)|\, |u_{k-1}(s)|^\lambda \mathrm{d}s \right|, \quad i = 0, 1, \ldots, n-3, \\ |\alpha_k^{(n-2)}(t)| &\le M \left| \int_t^{T_k} s^2|r(s)|\, |u_{k-1}(s)|^\lambda \mathrm{d}s \right|, \\ |\alpha_k^{(n-1)}(t)| &\le M \left| \int_t^{T_k} s|r(s)|\, |u_{k-1}(s)|^\lambda \mathrm{d}s \right|, \end{aligned} \tag{19}$$

where $M = K/d$ *and* K *is given in* Lemma 1. *Note that* (17) *and* (19) *are missing if* $n = 2$.

Proof. Suppose (11) holds, case (12) can be treated similarly. For $n \ge 3$, $t \ge t_0$, and $i = 1, 2, \ldots, n-3$ we have

$$\alpha_k^{(i)}(t) = (-1)^{n+i+1}\frac{1}{d}\int_t^{T_k} \frac{(\sigma - t)^{n-3-i}}{(n-3-i)!}\int_\sigma^{T_k} r(s)[u_{k-1}(s)]_\pm^\lambda w(s, \sigma)\mathrm{d}s\mathrm{d}\sigma. \tag{20}$$

Hence, for $n \ge 2$

$$\alpha_k^{(n-2)}(t) = -\frac{1}{d}\int_t^{T_k} r(s)[u_{k-1}(s)]_\pm^\lambda w(s, t)\mathrm{d}s, \tag{21}$$

$$\alpha_k^{(n-1)}(t) = -\frac{1}{d}\int_t^{T_k} r(s)[u_{k-1}(s)]_\pm^\lambda z(s, t)\mathrm{d}s, \tag{22}$$

$$\alpha_k^{(n)}(t) = r(t)[u_{k-1}(t)]_\pm^\lambda - q(t)\alpha_k^{(n-2)}(t). \tag{23}$$

If $n = 2$ the statement follows from (21), (22), and Lemma 1.

Let $n \ge 3$. From (20) and Lemma 1(i) we have

$$|\alpha_k^{(n-3)}(t)| \le \frac{K}{d}\left| \int_t^{T_k} s^2\ |r(s)||u_{k-1}\ |^\lambda \mathrm{d}s \right|$$

and the conclusion holds for $i = n-3$. Furthermore, for $i = 0, 1, \ldots, n-4$ we obtain

$$|\alpha_k^{(i)}(t)| \le \left| \int_t^{T_k} |\alpha_k^{(i+1)}(s)|\mathrm{d}s \right| \le \frac{K}{d}\left| \int_t^{T_k} (s-t)^{n-1-i}|r(s)||u_{k-1}\ |^\lambda \mathrm{d}s \right|. \tag{24}$$

Thus, the conclusion holds for these i. Finally, the estimates for $i = n-2, n-1$, follow from (21), (22), and Lemma 1(i). □

Proof of Theorem 1. The idea of the proof is similar to that of Theorem 1 in [6].

Suppose (11) holds, i.e., $i_q = 0$. Since $y(t) = O(t^m)$, there exists $L > 0$ such that for $t \ge 1$

$$|y(t)| < Lt^m. \tag{25}$$

Put $\overline{L} = L + 1/2$ and choose $t_0 > 1$ such that

$$\frac{\overline{L}K}{d}\int_{t_0}^{\infty} s^{n-1+\lambda m}|r(s)|\mathrm{d}s \le \frac{1}{2}, \tag{26}$$

where K is given by Lemma 1. Put $T_k = t_0 + k$. On $[t_0, \infty)$, consider the sequence $\{u_k\}$ defined by

$$u_1(t) = y(t),$$
$$u_k(t) = y(t) + \frac{(-1)^{n+1}}{d}\int_t^{T_k} \frac{(\sigma - t)^{n-3}}{(n-3)!}\int_\sigma^{T_k} r(s)[u_{k-1}(s)]_\pm^\lambda (s, \sigma)\mathrm{d}s\mathrm{d}\sigma$$

in case $n \ge 3$, and

$$u_k(t) = y(t) - \frac{1}{d}\int_t^{T_k} r(s)[u_{k-1}(s)]_\pm^\lambda w(s, t)\mathrm{d}s$$

in case $n = 2$.

For the functions α_k defined on $[t_0, \infty)$ in Lemma 2, we have

$$\alpha_k^{(i)}(t) = u_k^{(i)}(t) - y^{(i)}(t), \quad i = 0, \dots, n-1,$$

and, in view of (23),

$$\alpha_k^{(n)}(t) = r(t)[u_{k-1}(t)]_\pm^\lambda - q(t)u_k^{(n-2)}(t) + q(t)y_k^{(n-2)}(t) = u_k^{(n)}(t) - y^{(n)}(t). \tag{27}$$

Thus, for $t \ge t_0$ we have

$$u_k^{(n)}(t) + q(t)u_k^{(n-2)}(t) = r(t)[u_{k-1}(t)]_\pm^\lambda. \tag{28}$$

We show that for all $i = 0, \dots, n$ the sequences $\{u_k^{(i)}\}$ are uniformly bounded and equicontinuous on each finite subinterval of $[t_0, \infty)$.

First, let us show that

$$|u_k(t) - y(t)| \le \frac{1}{2}, \quad t \in [t_0, \infty).$$

Clearly, this holds for $k = 1$. We proceed by induction and assume that

$$|u_{k-1}(t) - y(t)| \le \frac{1}{2}, \quad t \in [t_0, \infty).$$

From this and (25), we have $|u_{k-1}(t)| < \overline{L}t^m$. In view of Lemma 2(i) and (25), we obtain

$$|\alpha_k(t)| \le \frac{K\overline{L}^\lambda}{d}\int_t^{T_k} s^{n-1+\lambda m}|r(s)|\mathrm{d}s.$$

Thus, in view of (26), we obtain

$$|u_k(t) - y(t)| \le \frac{K\overline{L}^\lambda}{d}\int_t^{\infty} s^{n-1+\lambda m}|r(s)|\mathrm{d}s \le \frac{1}{2}. \tag{29}$$

Similarly, using again Lemma 2(i),

$$|u_k^{(i)}(t) - y^{(i)}(t)| = |\alpha_k^{(i)}(t)| \le M_1\int_t^{\infty} s^{n-1-i+\lambda m}|r(s)|\mathrm{d}s \tag{30}$$

for $i = 1, \dots, n-2$, and $n \ge 3$, and

$$|u_k^{(n-1)}(t) - y^{(n-1)}(t)| = |\alpha_k^{(n-1)}(t)| \leq M_1 \int_t^{\infty} s^{1+\lambda m}|r(s)|\mathrm{d}s \tag{31}$$

for $n \geq 2$, where $M_1 = K\overline{L}\,/\,d$ and K is given by Lemma 1.

Hence, $\{u_k^{(i)}\}$, $i = 0, 1, \ldots, n-1$, are uniformly bounded on each finite subinterval of $[t_0, \infty)$. Moreover, in view of (28), the same holds for $\{u_k^{(n)}\}$. Then $\{u_k^{(i)}\}$, $i = 0, \ldots, n-1$, are equicontinuous on each finite subinterval in $[t_0, \infty)$ and, from (28), the same holds for $\{u_k^{(n)}\}$. Hence, $\{u_k\}$ admits a converging subsequence $\{u_{k_j}\}$ such that $\{u_{k_j}^{(i)}\}$, $i = 0, \ldots, n$, uniformly converge to a function $u^{(i)}$ on each finite subinterval of $[t_0, \infty)$.

Again from (28) we obtain that u is a solution to (1). From (10), (30), and (31), using the Lebesgue dominated convergence theorem, we obtain

$$\int_{t_0}^{\infty} |u^{(i)}(s) - y^{(i)}(s)|\mathrm{d}s < \infty, \quad i = 1, 2, \ldots, n-1,$$

whence all $u^{(i)} - y^{(i)}$, $i = 0, \ldots, n-2$, are of bounded variation in a neighborhood of infinity. Taking into account (27), we obtain

$$|u^{(n)}(t) - y^{(n)}(t)| \leq |r(t)||u^{\lambda}(t)| + q(t)|u^{(n-2)}(t) - y^{(n-2)}(t)|.$$

Since $|u(t)| < \overline{L}t^m$, we obtain

$$|u^{(n)}(t) - y^{(n)}(t)| \leq \overline{L}^{\lambda}|r(t)|t^{\lambda m} + q(t)|u^{(n-2)}(t) - y^{(n-2)}(t)|.$$

Thus, $u^{(n-1)} - y^{(n-1)}$ is also of bounded variation in a neighborhood of infinity. Finally, from (29), (30), and (31) we obtain (13).

If (12) holds, then the proof is similar, we use Lemma 2(ii) instead of Lemma 2(i). □

From Theorem 1 we obtain the following.

Corollary 1. *Let* (5) *be nonoscillatory and* $n \geq 3$. *Assume that* (10) *holds with* i_q *as in* Theorem 1 *and some* $m \in \{0, 1, \ldots, n-3\}$. *Then for any polynomial* Q *with* $\deg Q \leq m$ *there exists a solution* u *of* (1) *such that for large* t

$$u^{(i)}(t) = Q^{(i)}(t) + \varepsilon_i(t), \quad i = 0, 1, \ldots, n-1,$$

where all ε_i *are functions of bounded variation and* $\lim_{t\to\infty}\varepsilon_i(t) = 0$.

Proof. Since Q is a solution of (2), the conclusion follows from Theorem 1. □

Corollary 2. *Let* (5) *be nonoscillatory. Assume that*

$$\int_1^{\infty} t^{(n-1)(\lambda+1)+i_q}|r(t)|\mathrm{d}t < \infty, \tag{32}$$

where i_q *is as in* Theorem 1. *Then for any polynomial* Q *with* $\deg Q \leq n-3$ *in case* $n \geq 3$, $Q \equiv 0$ *in case* $n = 2$, *there exist solutions* u to (1) *such that for large* t

$$u^{(i)}(t) = (c_1\Gamma_1(t) + c_2\Gamma_2(t) + Q(t))^{(i)} + \varepsilon_i(t), \quad i = 0, \ldots, n-1,$$

where Γ_1 *and* Γ_2 *are given by* (15), c_1 *and* c_2 *are arbitrary constants, and* ε_i *are functions of bounded variation and* $\lim_{t\to\infty}\varepsilon_i(t) = 0$.

Proof. Take the fundamental system of solutions to (2) given in (14). Applying Theorem 1 for $m = n-1$, we obtain the conclusion. □

We complete this section by considering the special case of (1)

$$u^{(n)}(t) + \frac{\varepsilon}{t^2} u^{(n-2)}(t) = r(t)|u|_{\pm}^{\lambda}, \tag{33}$$

and the related linear equation

$$y^{(n)}(t) + \frac{\varepsilon}{t^2} y^{(n-2)}(t) = 0, \tag{34}$$

where $\varepsilon \in (0, 1/4)$. Then the estimations given in Lemma 1(ii) can be improved and we obtain the following results.

Theorem 2. *Let $\varepsilon \in (0, 1/4)$, and, for some real number $m \in [0, n-1]$,*

$$\int_1^\infty t^{n-1+m\lambda} \, |r(t)| \mathrm{d}t < \infty. \tag{35}$$

Then for any solution y to (34) such that $y(t) = O(t^m)$ there exists a solution u to (33) such that for large t

$$u^{(i)}(t) = y^{(i)}(t) + \varepsilon_i(t), \quad i = 0, 1, \ldots, n-1,$$

where all ε_i are functions of bounded variation and $\lim_{t\to\infty} \varepsilon_i(t) = 0$.

Proof. The corresponding second-order linear equation

$$h'' + \frac{\varepsilon}{t^2} h = 0$$

is nonoscillatory and has a fundamental system of solutions $h_1(t) = t^{\mu_1}$, $h_2(t) = t^{\mu_2}$, where

$$\mu_1 = (1 - \sqrt{1-4\varepsilon})/2, \quad \mu_2 = (1 + \sqrt{1-4\varepsilon})/2,$$

see, e.g., [19, Chapter 2.1]. Take the fundamental system of solutions to (34) given in (14). If $n \geq 3$, then

$$\Gamma_1(t) = \int_1^t (t-s)^{n-3} s^{\mu_1} \mathrm{d}s = O(t^\beta), \tag{36}$$

$$\Gamma_2(t) = \int_1^t (t-s)^{n-3} s^{\mu_2} \mathrm{d}s = O(t^\gamma), \tag{37}$$

where

$$\beta = n - \frac{3}{2} - \frac{\sqrt{1-4\varepsilon}}{2}, \quad \gamma = n - \frac{3}{2} + \frac{\sqrt{1-4\varepsilon}}{2}.$$

If $n = 2$,

$$\Gamma_1(t) = h_1(t) = O(t^\beta), \quad \Gamma_2(t) = h_2(t) = O(t^\gamma). \tag{38}$$

Furthermore,

$$|w(s,t)| = h_1(s)h_2(t) - h_1(t)h_2(s) \leq 2s \quad \text{for } s \geq t \geq 1.$$

Now we proceed by the similar way as in the proof of Theorem 1 replacing the estimation of w from Lemma 1(ii) by $|w(s,t)| \leq 2s$. ☐

From Theorem 2 we obtain immediately the following.

Corollary 3. *Assume that*

$$\int_1^\infty t^{n-1+\gamma\lambda}|r(t)|\mathrm{d}t < \infty,$$

where $\gamma = n - \frac{3}{2} + \frac{\sqrt{1-4\varepsilon}}{2}$.

Then for any polynomial Q *with* $\deg\, Q \le n-3$ *in case* $n \ge 3$, $Q \equiv 0$ *in case* $n = 2$, *there exist solutions* u *of* (33), *such that for large* t

$$u^{(i)}(t) = (c_1\Gamma_1(t) + c_2\Gamma_2(t) + Q(t))^{(i)} + \varepsilon_i(t), \quad i = 0,\ldots,n-1,$$

where Γ_1 *and* Γ_2 *are given by* (36), (37), *and* (38), c_1 *and* c_2 *are arbitrary constants, and* ε_i *are functions of bounded variation and* $\lim_{t\to\infty}\varepsilon_i(t) = 0$.

Proof. We apply Theorem 2 with $m = \gamma$. □

3 Asymptotic representation of solutions to equation (3)

In this section, we study some asymptotic properties of equation (3) and discuss the proximity of this equation to the unperturbed linear equation (6). We start by presenting conditions under which equation (3) has solutions with asymptotic growth of polynomial type. This result can be obtained as a corollary of Theorem C.

Corollary 4. *Suppose that the continuous functions* $a_0,\ldots,a_{n-1}$ *satisfy* (7) *and for some integer number* $m \in \{0, 1, \ldots, n-1\}$

$$\int_1^\infty t^{n-1+(\lambda-1)m}|r(t)|\mathrm{d}t < \infty. \tag{39}$$

Then for any $C \ne 0$ *there exists a solution* u *to equation* (3) *satisfying for large* t

$$\begin{aligned} u^{(j)}(t) &= \frac{Cm!}{(m-j)!}t^{m-j}\,(1+o(1)) \qquad \text{for } j = 0, 1, \ldots, m, \\ u^{(j)}(t) &= o(t^{m-j}) \quad \text{for } j = m+1, \ldots, n-1. \end{aligned} \tag{40}$$

In particular, if

$$\int_1^\infty t^{w}|r(t)|\mathrm{d}t < \infty, \tag{41}$$

where $w = \lambda(n-1)$ if $\lambda \ge 1$ *and* $w = n-1$ *if* $\lambda < 1$, *then for any solution* y *of* (6) *there exists a solution* u *of* (3) *such that for large* t

$$u(t) = y(t)(1+o(1)), \tag{42}$$

$$u^{(i)}(t) = y^{(i)}(t)(1+o(1)) + o(t^{-1}), \quad i = 1,\ldots,n-1. \tag{43}$$

Proof. In Theorem C, choose $l = m+1$, $k = j+1$, $\alpha = C$, $\mu = 1$. Put

$$q(t, x_1, \ldots, x_n) = r(t)|x_1|_{\pm}^{\lambda} - \sum_{k=1}^{n} a_{k-1}(t)x_k,$$

$$q^*(t) = c_1 t^{\lambda(l-1)}|r(t)| + c_2\sum_{k=1}^{n}|a_{k-1}(t)|t^{l-k},$$

and $c_1 = (1 + |\alpha|)^\lambda$, $c_2 = 1 + |\alpha|(l-1)!$.

We claim that

$$|q(t, x_1, \dots, x_n)| \le q^*(t)$$

for $t \ge 1$ and

$$|x_k - \frac{\alpha\,(l-1)!}{(l-k)!} t^{l-k}| \le t^{l-k}, \quad k = 1, \dots, l,$$
$$|x_k| \le t^{l-k}, \quad k = l+1, \dots, n.$$

Indeed, we have

$$|q(t, x_1, \dots, x_n)| \le |r(t)|(t^{l-1} + |\alpha| t^{l-1})^\lambda + \sum_{k=1}^{l} |a_{k-1}(t)| \left(t^{l-k} + |\alpha| \frac{(l-1)!}{(l-k)!} t^{l-k} \right) + \sum_{k=l+1}^{n} |a_{k-1}(t)| t^{l-k}$$
$$\le c_1 t^{\lambda(l-1)} |r(t)| + c_2 \sum_{k=1}^{n} |a_{k-1}(t)| t^{l-k} = q^*(t).$$

Thus, using (7) and (39)

$$\int_1^\infty t^{n-l} q^*(t) \mathrm{d}t = c_1 \int_1^\infty t^{n-l+\lambda(l-1)} |r(t)| \mathrm{d}t + c_2 \int_1^\infty \sum_{k=1}^{n} |a_{k-1}(t)| t^{n-k} \mathrm{d}t < \infty.$$

Hence, both relations in (40) hold.

In order to complete the proof, let us show that (42) and (43) are satisfied. Assume that (41) holds. Then (39) is valid for $m = 0, 1, \dots, n-1$. Indeed, denoting $w_m = n - 1 + (\lambda - 1)m$, we have

$$\max_{m=0,1,\dots,n-1} w_m = w_{n-1} = \lambda(n-1) \quad \text{if } \lambda \ge 1,$$
$$\max_{m=0,1,\dots,n-1} w_m = w_0 = n-1 \quad \text{if } \lambda < 1.$$

Let y be an arbitrary solution of (6). Then according to the proved part of Corollary 4 with $r(t) \equiv 0$, there exist a fundamental system of solutions to (6), say $y_0(t), \dots, y_{n-1}(t)$ such that

$$\begin{aligned} y_l^{(j)}(t) &= \frac{m!}{(m-j)!} t^{m-j}\,(1 + o(1)) \quad \text{for } j = 0, 1, \dots, l \\ y_l^{(j)}(t) &= o(t^{m-j}) \quad \text{for } j = l+1, \dots, n-1, \end{aligned} \tag{44}$$

where $l \in 0, 1, \dots, n-1$, and

$$y(t) = \sum_{l=0}^{n-1} c_l\, y_l(t), \quad t \ge 1,$$

where c_l, $l = 0, 1, \dots, n-1$ are suitable constants. Let $m \in \{0, 1, \dots, n-1\}$ be such that $c_m \ne 0$, $c_l = 0$ for $l > m$. Using (44) we have

$$y^{(j)}(t) = \sum_{l=0}^{m} c_l y_l^{(j)}(t) = c_m \frac{m!}{(m-j)!} t^{m-j}(1 + o(1)) \quad \text{for } j = 1, \dots, m,$$
$$y^{(j)}(t) = \sum_{l=0}^{m} c_l\, y_l^{(j)}(t) = o(t^{m-j}) \quad \text{for } j = m+1, \dots, n-1.$$

From here and (40), equation (3) has a solution u with the same asymptotics, letting $C = c_m$. Therefore, for large t

$$u^{(k-1)}(t) = y^{(k-1)}(t)(1 + o(1)) \quad \text{for } k = 1, \dots, m,$$
$$u^{(k-1)}(t) = y^{(k-1)}(t) + o(t^{m-k}) = y^{(k-1)}(t) + o(t^{-1}) \quad \text{for } k = m+1, \dots, n.$$

If $k = 1$, then (42) holds, in other cases we obtain (43). □

Observe that if $y^{(i)}$ for some $i \in \{1, \ldots, n-1\}$ is bounded away from zero, then (43) is equivalent to

$$u^{(i)}(t) = y^{(i)}(t)(1 + o(1)).$$

If $y^{(i)} = o(t^{-1})$, then (43) is equivalent to

$$u^{(i)}(t) = y^{(i)}(t) + o(t^{-1}).$$

Remark 1. If (7) and (41) hold, then for any $C \neq 0$ and every $m = 0, 1, \ldots, n-1$ there exist solutions $y_1, \ldots, y_n$ of equation (6) with different asymptotic representation (40), and by Corollary 4 there exists n-parametric set of solutions of equation (3).

Remark 2. In [12, Section 8.3, p. 163], there is an example of Emden-Fowler-type equation (4) having solution u with a noninteger power-law asymptotic behavior. Thus, the corresponding linear equation $y^{(n)} = 0$ has no solution y satisfying $\lim_{t\to\infty}|u(t) - y(t)| = 0$.

Remark 3. Applying Corollary 4 to equation (1), we obtain the following comparison between Corollary 4 and Theorem 1: Condition (7) of Corollary 4 reads as condition (11) of Theorem 1. If (39) holds but not (10), Corollary 4 is applicable while Theorem 1 is not applicable. The same holds if q changes its sign. If m is not integer or (12) holds, Theorem 1 is applicable, while the application of Corollary 4 is not possible.

Remark 4. In [2, Theorem 2.4] it is proved that if $\lambda > 1$, (7) and (32) with $i_q = 0$ hold, then there exist solutions u to equation (3) such that for large t

$$u(t) = \sum_{j=0}^{n-1} C_j y_j(t) + o(1), \tag{45}$$

where C_j are arbitrary constants and the functions y_j are a fundamental system of solutions to equation (6) such that for large t

$$y_j(t) = \frac{t^j}{j!}(1 + o(1)).$$

Corollary 2 extends this result for equation (1).

4 Examples and suggestions

The following examples illustrate Theorems 1, 2, and Corollary 4.

Example 1. Let $\lambda > 0$ and consider the nonlinear equation for $t \geq 1$

$$u^{(4)} + \frac{1}{t^2 \log et} u^{(2)} = \frac{e^{-t}(t^2 \log et + 1)}{(1 + e^{-t})^\lambda t^2 \log et} |u|_\pm^\lambda. \tag{46}$$

A standard calculation shows that

$$u(t) = t + e^{-t} \tag{47}$$

is a solution of (46). Setting

$$q(t) = \frac{1}{t^2 \log et}, \quad r(t) = \frac{e^{-t}(t^2 \log et + 1)}{(1 + e^{-t})^\lambda t^2 \log et},$$

we obtain that (5) is nonoscillatory and (12) is valid. Moreover, we have for any $\sigma > 0$

$$\int_1^\infty t^\sigma r(t)\mathrm{d}t < \infty.$$

Thus, all the assumptions of Theorem 1 are verified with $m = 1$ and so equation (46) has a solution u such that for any large t

$$u^{(i)}(t) = y^{(i)}(t) + \varepsilon_i(t), \quad i = 0, \dots, 3,$$

where ε_i are functions of bounded variation such that $\lim_{t\to\infty}\varepsilon_i(t) = 0$ and $y(t) = t$, as solution (47) illustrates.

Example 2. Consider the nonlinear equation for $t \geq 1$

$$u^{(3)} + \frac{3}{16}\frac{1}{t^2}u' = \frac{\cos t}{t^5}|u|_\pm^{1/2}. \tag{48}$$

A standard calculation shows that a fundamental system of solutions of the linear equation

$$y^{(3)} + \frac{3}{16}\frac{1}{t^2}y' = 0$$

is given by

$$y_1(t) = 1, \quad y_2(t) = t^{5/4}, \quad y_3 = t^{7/4}.$$

Since

$$\int_1^\infty t^{-2}|\cos t|\mathrm{d}t < \infty,$$

condition (35) is satisfied for $n = 3$, $\lambda = 2^{-1}$, $r(t) = t^{-5}\cos t$, and any $m \in \{0, 1, 2\}$. Hence, from Theorem 2 equation (48) has a solution u_k such that for any large t

$$u_k^{(i)}(t) = y_k^{(i)}(t) + \varepsilon_{k,i}(t), \quad k, i = 0, 1, 2,$$

where $\varepsilon_{k,i}$ are functions of bounded variation such that $\lim_{t\to\infty}\varepsilon_{k,i}(t) = 0$.

Example 3. Consider the nonlinear equation for $t \geq 1$

$$u^{(3)} + \frac{1}{t(2t-1)}u^{(2)} - \frac{1}{t^2(2t-1)}u' = r(t)|u|_\pm^\lambda, \tag{49}$$

where $0 < \lambda < 1$ and r is a continuous function for $t \geq 1$ such that

$$\int_1^\infty t^2|r(t)|\mathrm{d}t < \infty. \tag{50}$$

A standard calculation shows that the corresponding linear equation to (49), that is, the equation

$$y^{(3)} + \frac{1}{t(2t-1)}y^{(2)} - \frac{1}{t^2(2t-1)}y' = 0, \quad t \geq 1,$$

has the fundamental system of solutions consisting of

$$y_1(t) = 1, \quad y_2(t) = t^2, \quad y_3(t) = \int_1^t s\log\left(\frac{2s}{2s-1}\right)\mathrm{d}s.$$

Denoting by a_i, $i = 0, 1, 2$, the functions

$$a_0(t) = 0, \quad a_1(t) = -\frac{1}{t^2(2t-1)}, \quad a_2(t) = \frac{1}{t(2t-1)},$$

it is immediate to verify that (7) holds for all $j = 0, 1, 2$. From (50), also conditions (39) and (41) are valid, and so, in virtue of Corollary 4, there exist three solutions u_i, $i = 1, 2, 3$, of (49) such that

$$\begin{aligned} u_1(t) &= y_1(t)(1 + o(1)) = 1 + o(1), \\ u_2(t) &= y_2(t)(1 + o(1)) = t^2(1 + o(1)), \\ u_3(t) &= y_3(t)(1 + o(1)) = \int_1^t s \log\left(\frac{2s}{2s-1}\right) \mathrm{d}s(1 + o(1)) + o(1). \end{aligned}$$

A similar asymptotic representation holds for the derivatives of u_i', u_i'', $i = 1, 2, 3$. For instance, we have

$$\begin{aligned} u_2'(t) &= 2t(1 + o(1)) + o(t^{-1}) = 2t(1 + o(1)), \\ u_2''(t) &= 2(1 + o(1)) + o(t^{-1}) = 2 + o(1), \end{aligned}$$

and similarly for the first and second derivative of u_1 and u_3.

Open problems.

(**1**) Condition (35) in Theorem 2 is weaker than (10) in Theorem 1. This is due to the fact that it was possible to explicitly calculate the fundamental system of solutions of (5), and from these solutions, to obtain a fundamental system of (2). Since an explicit expression of solutions of (5) is possible also for particular choices of q, like, for example, the Euler equation

$$h'' + \frac{1}{4t^2}h = 0,$$

see, e.g., [19, Chapter 2.1], it should be interesting to study whenever Theorem 2 remains to hold for equations of the type

$$u^{(n)} + \varphi(t)u^{(n-2)} = r(t)|u|_{\pm}^{\lambda},$$

where φ is a continuous function for $t \geq 1$ and asymptotic representations of solutions of the second-order linear equation

$$h'' + \varphi(t)h = 0$$

are known.

(**2**) The study on the proximity of solutions could be extended to the one between delay and neutral equations and the corresponding linear equations without delay or neutral argument, by applying the proximity results obtained in Section 2. More precisely, this study could be accomplished in two steps. First, by considering the possible proximity between the nonlinear equation (1) (or (3)) and the corresponding nonlinear equation with functional argument and later to apply Theorem 1 or Theorem 2. Concerning the first step, observe that the functional argument may produce a loss of proximity. To illustrate this fact, consider the Emden-Fowler equation without deviating argument

$$x''(t) = b(t)|x(t)|_{\pm}^{\lambda}, \tag{51}$$

where b is a continuous function for $t \geq 1$ and the corresponding equation with deviating argument

$$x''(t) = b(t)|x(\tau(t))|_{\pm}^{\lambda}, \quad \tau(t) < t. \tag{52}$$

When $\lambda > 1$ and b is positive, equation (51) always has *Kneser solutions*, that is, solutions x such that $x(t) > 0$, $x'(t) < 0$ for large t. Moreover, if, in addition,

$$\int_1^{\infty} sb(s)\mathrm{d}s < \infty,$$

then (51) does not have Kneser solutions which tend to zero as $t \to \infty$, see, e.g., [13, Section 13.2]. If $\tau(t) < t$, then this result may fail for (52), see, e.g., [15].

In the sublinear case, that is, $0 < \lambda < 1$, it is known that there might exist equations of type (51) without Kneser solutions. For instance, if

$$\liminf_{t\to\infty} t^2 b(t) > 0, \tag{53}$$

then (51) does not have Kneser solution, see [13, Corollary 17.3]. On the other hand, the equation

$$x''(t) = \frac{1}{t^2 \log t}|x(\tau(t))|_{\pm}^{\lambda}, \quad t \geq 2,$$

has Kneser solutions, as [10, Corollary 2] illustrates.

In virtue of these facts, it should be interesting to show if an analogous type of proximity between Kneser solutions arises for equations (1) and

$$w^{(n)} + q(t)w^{(n-2)} = r(t)|w(\tau(t))|^{\lambda}\operatorname{sgn} w(\tau(t)).$$

Similarly, the same question concerns the proximity between Kneser solutions for equations (3) and

$$w^{(n)} + \sum_{j=0}^{n-1} a_j(t)w^{(j)} = r(t)|w(\tau(t))|^{\lambda}\operatorname{sgn} w(\tau(t)),$$

where n is even. Here the meaning of Kneser solution is that the solution x satisfies for large t

$$x(t) > 0, \quad x^{(i)}(t)x^{(i+1)}(t) < 0, \quad i = 0, 1, \ldots, n-1.$$

Acknowledgments: The authors thank the referees for their valuable comments and suggestions, which improve the value of the article.

Funding information: The research of I. Astashova has been supported by RSF (Project 20-11-20272), the research of M. Bartušek and Z. Došlá has been supported by the grant GA20-11846S of the Czech Science Foundation. The research of M. Marini has been partially supported by Gnampa, National Institute for Advanced Mathematics (INdAM) of Italian National Research Council.

Conflict of interest: The authors state no conflict of interest.

References

[1] I. V. Astashova, *On existence of non-oscillatory solutions to quasi-linear differential equations*, Georgian Math. J. **14** (2007), 223–238.
[2] I. V. Astashova, *On the asymptotic behavior at the infinity of solutions to quasi-linear differential equations*, Math. Bohem. **135** (2010), 373–382.
[3] I. Astashova, *On asymptotic equivalence of n-th order nonlinear differential equations*, Tatra Mt. Math. Publ. **63** (2015), 31–38.
[4] I. V. Astashova, *Qualitative properties of solutions to quasilinear ordinary differential equations* (*in Russian*), in: I. V. Astashova (ed.), Qualitative Properties of Solutions to Differential Equations and Related Topics of Spectral Analysis: scientific edition, UNITY-DANA, Moscow, 2012, pp. 22–290.
[5] I. V. Astashova, *On asymptotic classification of solutions to nonlinear regular and singular third- and fourth-order differential equations with power nonlinearity*, Springer Proc. Math. Stat. **164** (2016), 191–203.
[6] M. Bartušek, M. Cecchi, Z. Došlá, and M. Marini, *Asymptotics for higher order differential equations with a middle term*, J. Math. Anal. Appl. **388** (2012), 1130–1140.
[7] M. Bartušek, Z. Došlá, and M. Marini, *Oscillation for higher order differential equations with a middle term*, Bound. Value Probl **2014-48** (2014), 1–18.

[8] M. Cecchi, M. Marini, and G. Villari, *Integral criteria for a classification of solutions to linear differential equations*, J. Diff. Equ. **99** (1992), 391–397.
[9] K. S. Chiu, *Asymptotic equivalence of alternately advanced and delayed differential systems with piecewise constant generalized arguments*, Acta Mathematica Scientia **38B** (2018), 220–236.
[10] Z. Došlá, P. Liška, and M. Marini, *Asymptotic problems for functional differential equations via linearization method*, J. Fixed Point Theory Appl. **21** (2019), 1–16.
[11] I. T. Kiguradze, *On the oscillation to solution to the equation* $d^m/dt^m + a(t)\text{sign } u = 0$, Sb. Math. **65** (1964), 172–187 (Russian).
[12] I. T. Kiguradze, *An oscillation criterion for a class of ordinary differential equations*, Differ. Uravn. **28** (1992), 201–214.
[13] I. T. Kiguradze and T. A. Chanturia, *Asymptotic Properties of Solutions to Nonautonomous Ordinary Differential Equations*, Kluwer Academic Publishers Group, Dordrecht, 1993.
[14] D. Otrocol, *On the asymptotic equivalence of a differential system with maxima*, Rend. Circ. Mat. Palermo **2**-**65** (2016), 387–393.
[15] Ch. G. Philos, *On the existence of nonoscillatory solutions tending to zero at infty for differential equations with positive delays*, Arch. Math. **36** (1981), 168–178.
[16] P. Řehák, *On asymptoticrelationships between two higher order dynamic equations on time scale*, Appl. Math. Lett. **73** (2017), 84–90.
[17] A. Reinfelds and D. Šteinberga, *Dynamical equivalence of quasilinear dynamic equations on time scales*, J. Math. Anal. **7** (2016), 115–120.
[18] I. M. Sobol, *On asymptotic behavior of solutions to linear differential equations*, Dokl. Akad. Nauk. **61** (1948), 219–222 (Russian).
[19] C. A. Swanson, *Comparison and Oscillation Theory of Linear Differential Equations*, Academic Press, New York, 1968.
[20] Z. Zhou and Y. Yan, *On the equivalence of two differential equations by means of reflecting functions coinciding*, J. Appl. Anal. Comput. **8** (2018), 549–557.