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Constant sign and nodal solutions for parametric $(p, 2)$ -equations

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2 -boundary $\partial\Omega$.

We study the following parametric $(p, 2)$ -equation:

$$\begin{cases} -\Delta_p u(z) - \Delta u(z) = \lambda |u(z)|^{p-2} u(z) + f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}, \quad p > 2, \lambda > 0. \quad (P_\lambda)$$

For $1 < q < \infty$, Δ_q denotes the q -Laplace differential operator defined by

$$\Delta_q u = \operatorname{div} \left(|Du|^{q-2} Du \right) \quad \text{for all } u \in W_0^{1,q}(\Omega).$$

When $q = 2$, we have the Laplace differential operator denoted by Δ .

In the right hand side (reaction) of the problem, we have a parametric term $x \mapsto \lambda |x|^{p-2} x$ with $\lambda > 0$ being a parameter and also a perturbation $f(z, x)$ which is a Caratheodory function (that is, for all $x \in \mathbb{R}$, $z \mapsto f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \mapsto f(z, x)$ is continuous).

We do not impose any sign condition on $f(z, \cdot)$ and we assume that for a.a. $z \in \Omega$, $f(z, \cdot)$ is $(p - 1)$ -superlinear near $\pm\infty$. However, we do not assume that it satisfies the usual in such cases Ambrosetti-Rabinowitz condition (the AR-condition for short).

Our aim is to prove multiplicity theorems providing sign information for all the solutions produced. To this end, first we look for constant sign solutions and we prove bifurcation-type results describing in a precise way the changes in the sets of positive and negative solutions respectively as the parameter λ moves in the positive semiaxis $(0, +\infty)$. We also show that there exist extremal constant sign solutions (that is, a smallest positive solution and a biggest negative solution). Then these extremal constant sign solutions are used to generate nodal (that is, sign changing) solutions. By strengthening the conditions on the perturbation $f(z, \cdot)$ and using also tools from the theory of critical groups (Morse theory), we prove a multiplicity theorem for small values of the parameter $\lambda > 0$. So, we show that when the parameter $\lambda > 0$ is small, problem (P_λ) has at least seven nontrivial solutions all with sign information: two positive, two negative and three nodal.

We mention that $(p, 2)$ -equations (that is, equations driven by a p -Laplacian and a Laplacian), arise in problems of mathematical physics (see, for example, Benci-D'Avenia-Fortunato-Pisani [1]). We also mention

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the work of Zhikov [2] who used $(p, 2)$ -equations to describe phenomena in nonlinear elasticity. More precisely, Zhikov introduced models for strongly anisotropic materials in the context of homogenization. For this purpose Zhikov introduces the so-called *double phase* functional

$$J_{p,q}(u) = \int_{\Omega} [|Du|^p + a(z)|Du|^q] dz$$

with $0 \leq a(z) \leq M$ for a.a. $z \in \Omega$, $1 < q < p$, $u \in W_0^{1,p}(\Omega)$. Here the modulating coefficient $a(z)$ dictates the geometry of the composite made of two different materials with hardening exponents p and q respectively.

Recently there have been some existence and multiplicity results for such equations. We mention the works of Aizicovici-Papageorgiou-Staicu [3, 4], Cingolani-Degiovanni [5], Gasiński-Papageorgiou [6, 7], He-Guo-Huang-Lei [8], Papageorgiou-Rădulescu [9, 10], Papageorgiou-Rădulescu-Repovš [11], Sun [12], Sun-Zhang-Su [13]. The multiplicity theorem here is the first one producing seven solutions of nonlinear non-homogeneous equations.

Our approach combines variational methods based on the critical point theory, together with truncation and comparison techniques and Morse theory (critical groups).

2 Mathematical Background

The variational methods which we will use, involve the direct method of the calculus of variations and the mountain pass theorem, which for the convenience of the reader we recall below.

Suppose that X is a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$, we denote the duality brackets for the pair (X^*, X) . Given $\varphi \in C^1(X, \mathbb{R})$, we say that $\varphi(\cdot)$ satisfies the *Cerami condition* (the *C-condition* for short), if the following property holds:

Every sequence $\{u_n\}_{n \geq 1} \subseteq X$ such that

$$\{\varphi(u_n)\}_{n \geq 1} \subseteq \mathbb{R} \text{ is bounded,}$$

$$(1 + \|u_n\|)\varphi'(u_n) \rightarrow 0 \text{ in } X^* \text{ as } n \rightarrow \infty,$$

admits a strongly convergent subsequence.

This compactness-type condition on the functional $\varphi(\cdot)$, leads to a deformation theorem from which one derives the minimax theory of the critical values of φ . One of the first and most important results in this theory, is the so-called *mountain pass theorem*.

Theorem 2.1. *If X is a Banach space, $\varphi \in C^1(X, \mathbb{R})$, it satisfies the C-condition, $u_0, u_1 \in X$, $\|u_1 - u_0\|_X > \rho$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf \{ \varphi(u) : \|u - u_0\|_X = \rho \} = m_\rho$$

and

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \varphi(\gamma(t)) \quad \text{with } \Gamma = \{ \gamma \in C([0, 1], X) : \gamma(0) = u_0, \gamma(1) = u_1 \},$$

then, $c \geq m_\rho$ and c is a critical value of φ (that is, there exists $\hat{u} \in X$ such that $\varphi(\hat{u}) = c$ and $\varphi'(\hat{u}) = 0$).

In what follows for a given $\varphi \in C^1(X, \mathbb{R})$, by K_φ we denote the critical set of φ , that is,

$$K_\varphi = \{ u \in X : \varphi'(u) = 0 \}.$$

The main spaces in the analysis of problem (P_λ) , are the Sobolev spaces $W_0^{1,p}(\Omega)$ and $H_0^1(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \{ u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0 \}$.

We have

$$C_0^1(\overline{\Omega}) \subseteq W_0^{1,p}(\Omega) \subseteq H_0^1(\Omega) \quad (\text{recall that } p > 2)$$

and the space $C_0^1(\bar{\Omega})$ is dense in both $W_0^{1,p}(\Omega)$ and $H_0^1(\Omega)$. By $\|\cdot\|$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. On account of the Poincaré inequality, we have

$$\|u\| = \|Du\|_p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The space $C_0^1(\bar{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ = \{u \in C_0^1(\bar{\Omega}) : u(z) \geq 0 \text{ for all } z \in \bar{\Omega}\}.$$

This cone has a nonempty interior given by

$$\text{int } C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega \text{ and } \frac{\partial u}{\partial n} \Big|_{\partial\Omega} < 0 \right\}.$$

Here $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$ is the normal derivative of $u(\cdot)$, with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

Suppose $f_0 : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Caratheodory function such that

$$|f_0(z, x)| \leq a_0(z) (1 + |x|^{r-1}) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}$$

with $a_0 \in L^\infty(\Omega)$ and

$$1 < r \leq p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N \end{cases} \quad (\text{the critical Sobolev exponent corresponding to } p).$$

We set $F_0(z, x) = \int_0^x f_0(z, s) ds$ and consider the C^1 -functional $\varphi_0 : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \int_\Omega F_0(z, u) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The next result is an outgrowth of the nonlinear regularity theory (see Lieberman [14], Theorem 1). It is a special case of a more general result of Papageorgiou-Rădulescu [15].

Proposition 2.1. *If $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\bar{\Omega})$ -minimizer of φ_0 , that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \text{for all } \|h\|_{C_0^1(\bar{\Omega})} \leq \rho_0,$$

then $u_0 \in C_0^{1,\alpha}(\bar{\Omega}) = C^{1,\alpha}(\bar{\Omega}) \cap C_0^1(\bar{\Omega})$ and it is also a local $W_0^{1,p}(\Omega)$ -minimizer of φ_0 , that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \text{for all } \|h\| \leq \rho_1.$$

This result is more effective when it is combined with the following strong comparison principle, which is a special case of a result of Gasiński-Papageorgiou [16] (Proposition 3.2).

If $h_1, h_2 \in L^\infty(\Omega)$, then we write that $h_1 \prec h_2$ if for all $K \subseteq \Omega$ compact, we have $0 < c_K \leq h_2(z) - h_1(z)$ for a.a. $z \in K$.

Proposition 2.2. *If $\xi, h_1, h_2 \in L^\infty(\Omega)$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$, $h_1 \prec h_2$, and $u \in C_0^1(\bar{\Omega}) \setminus \{0\}$, $v \in \text{int } C_+$, $u \leq v$ satisfy*

$$\begin{aligned} -\Delta_p u(z) - \Delta u(z) + \xi(z)|u(z)|^{p-2}u(z) &= h_1(z), \\ -\Delta_p v(z) - \Delta v(z) + \xi(z)v(z)^{p-1} &= h_2(z) \end{aligned}$$

for a.a. $z \in \Omega$, then $v - u \in \text{int } C_+$.

For $q \in (1, +\infty)$, let $A_q : W_0^{1,q}(\Omega) \rightarrow W^{-1,q'}(\Omega) = W_0^{1,q}(\Omega)^* (\frac{1}{q} + \frac{1}{q'} = 1)$ be the nonlinear map defined by

$$\langle A_q(u), h \rangle = \int_\Omega |Du|^{q-2} (Du, Dh)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in W_0^{1,q}(\Omega).$$

The following proposition recalls the main properties of this map (see, for example, Motreanu-Motreanu-Papageorgiou [17], p. 40).

Proposition 2.3. *The map $A_q(\cdot)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$ (that is, if $u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$ and $\limsup_{n \rightarrow +\infty} \langle A(u_n), u_n - u \rangle \leq 0$, then $u_n \rightarrow u$ in $W_0^{1,p}(\Omega)$).*

If $q = 2$, then $A_2 = A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$.

We will need some basic facts about the spectrum of $(-\Delta, H_0^1(\Omega))$. So, we consider the following linear eigenvalue problem

$$-\Delta u(z) = \widehat{\lambda}u(z) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0. \tag{2.1}$$

We say that $\widehat{\lambda} \in \mathbb{R}$ is an *eigenvalue* of $(-\Delta, H_0^1(\Omega))$, if problem (2.1) admits a nontrivial solution $\widehat{u} \in H_0^1(\Omega)$ known as an *eigenfunction* corresponding to $\widehat{\lambda}$. Via the spectral theorem for compact self-adjoint operators, we show that the spectrum consists of a strictly increasing sequence $\{\widehat{\lambda}_k(2)\}_{k \in \mathbb{N}}$ of eigenvalues and $\widehat{\lambda}_k(2) \rightarrow \infty$. The corresponding sequence $\{\widehat{u}_n(2)\}_{n \in \mathbb{N}} \subseteq H_0^1(\Omega)$ of eigenfunctions of (2.1), forms an orthonormal basis of $H_0^1(\Omega)$ and an orthogonal basis of $L^2(\Omega)$. Standard regularity theory implies that $\{\widehat{u}_n(2)\}_{n \in \mathbb{N}} \subseteq C_0^1(\overline{\Omega})$. By $E(\widehat{\lambda}_k(2))$ we denote the eigenspace corresponding to the eigenvalue $\widehat{\lambda}_k(2)$, $k \in \mathbb{N}$. We have $E(\widehat{\lambda}_k(2)) \subseteq C_0^1(\overline{\Omega})$ and we have the following orthogonal direct sum decomposition

$$H_0^1(\Omega) = \overline{\bigoplus_{k \in \mathbb{N}} E(\widehat{\lambda}_k(2))}.$$

Each eigenspace $E(\widehat{\lambda}_k(2))$ has the so-called *Unique Continuation Property* (UCP for short) which says that, if $u \in E(\widehat{\lambda}_k(2))$ vanishes on a set of positive Lebesgue measure, then $u \equiv 0$.

The eigenvalues $\{\widehat{\lambda}_k(2)\}_{k \in \mathbb{N}}$ have the following properties:

- $\widehat{\lambda}_1(2) > 0$ is simple (that is, $\dim E(\widehat{\lambda}_1(2)) = 1$).

- $$\widehat{\lambda}_1(2) = \inf \left[\frac{\|Du\|_2^2}{\|u\|_2^2} : u \in H_0^1(\Omega), u \neq 0 \right] \tag{2.2}$$

- $$\widehat{\lambda}_m(2) = \sup \left[\frac{\|Du\|_2^2}{\|u\|_2^2} : u \in \bigoplus_{k=1}^m E(\widehat{\lambda}_k), u \neq 0 \right] = \inf \left[\frac{\|Du\|_2^2}{\|u\|_2^2} : u \in \bigoplus_{k \geq m} E(\widehat{\lambda}_k), u \neq 0 \right] \tag{2.3}$$

In (2.2) the infimum is realized on $E(\widehat{\lambda}_1(2))$.

In (2.3) both the supremum and the infimum are realized on $E(\widehat{\lambda}_m(2))$.

The above properties imply that the elements of $E(\widehat{\lambda}_1)$ have constant sign. On the other hand the elements of $E(\widehat{\lambda}_k(2))$, $k \geq 2$, are nodal (that is, sign-changing). Moreover, if by $\widehat{u}_1(2)$ we denote the L^2 -normalized (that is, $\|\widehat{u}_1(2)\|_2 = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1(2)$, then the strong maximum principle implies that $\widehat{u}_1(2) \in \text{int } C_+$.

The following useful inequalities are easy consequences of the above properties.

Proposition 2.4.

- (a) *If $m \in \mathbb{N}$, $\eta \in L^\infty(\Omega)$, $\eta(z) \leq \widehat{\lambda}_m(2)$ for a.a. $z \in \Omega$, $\eta \not\equiv \widehat{\lambda}_m(2)$, then*

$$\|Du\|_2^2 - \int_{\Omega} \eta(z)u^2 \, dz \geq c_1 \|Du\|_2^2$$

for some $c_1 > 0$, all $u \in \overline{\bigoplus_{k \geq m} E(\widehat{\lambda}_k(2))}$.

- (b) *If $m \in \mathbb{N}$, $\eta \in L^\infty(\Omega)$, $\eta(z) \geq \widehat{\lambda}_m(2)$ for a.a. $z \in \Omega$, $\eta \not\equiv \widehat{\lambda}_m(2)$, then*

$$\|Du\|_2^2 - \int_{\Omega} \eta(z)u^2 \, dz \leq -c_2 \|Du\|_2^2$$

for some $c_2 > 0$, all $u \in \bigoplus_{k=1}^m E(\widehat{\lambda}_k(2))$.

We also consider the corresponding nonlinear eigenvalue problem for the p -Laplacian

$$-\Delta_p u(z) = \widehat{\lambda}|u(z)|^{p-2}u(z) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0.$$

This problem has a smallest eigenvalue $\widehat{\lambda}_1(p) > 0$ which is isolated (that is, there exists $\epsilon > 0$ such that $(\widehat{\lambda}_1(p), \widehat{\lambda}_1(p) + \epsilon)$ contains no eigenvalues), simple (that is, if \widehat{u}, \widehat{v} are eigenfunctions corresponding to $\widehat{\lambda}_1(p) > 0$, then $\widehat{u} = \xi\widehat{v}$ for some $\xi \in \mathbb{R} \setminus \{0\}$) and admits the following variational characterization

$$\widehat{\lambda}_1(p) = \inf \left[\frac{\|Du\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega), u \neq 0 \right]. \tag{2.4}$$

The infimum in (2.4) is realized on the corresponding one dimensional eigenspace, the elements of which are in $C_0^1(\overline{\Omega})$ (nonlinear regularity theory, see Lieberman [14]) and have fixed sign. Using (2.4) and these properties, we obtain

Proposition 2.5. *If $\eta \in L^\infty(\Omega)$, $\eta(z) \leq \widehat{\lambda}_1(p)$ for a.a. $z \in \Omega$, $\eta \not\equiv \widehat{\lambda}_1(p)$, then there exists $c_3 > 0$ such that*

$$\|Du\|_p^p - \int_{\Omega} \eta(z)|u|^p \, dz \geq c_3 \|Du\|_p^p \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Next we recall some basic definitions and facts concerning critical groups.

So, let X be a Banach space, $\varphi \in C^1(X, \mathbb{R})$, $c \in \mathbb{R}$. We introduce the following sets

$$\begin{aligned} \varphi^c &= \{x \in X : \varphi(u) \leq c\}, \\ K_\varphi &= \{u \in X : \varphi'(u) = 0\} \quad (\text{the critical set of } \varphi), \\ K_\varphi^c &= \{u \in K_\varphi : \varphi(u) = c\}. \end{aligned}$$

For a topological pair (Y_1, Y_2) such that $Y_2 \subseteq Y_1 \subseteq X$ and every $k \in \mathbb{N}_0$ by $H_k(Y_1, Y_2)$ we denote the k^{th} -relative singular homology group with integer coefficients. Given $u \in K_\varphi^c$ isolated, the critical groups of φ at u , are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap \mathcal{U}, \varphi^c \cap \mathcal{U} \setminus \{u\}),$$

with \mathcal{U} being a neighborhood of u such that $K_\varphi \cap \varphi^c \cap \mathcal{U} = \{u\}$. The excision property of singular homology, implies that the above definition is independent of the particular choice of the neighborhood \mathcal{U} .

Suppose that $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. Then the critical groups of φ at infinity, are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in \mathbb{N}_0.$$

This definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$. Indeed, if $c' < c < \inf \varphi(K_\varphi)$, then by the second deformation theorem (see [18], p. 628), we know that $\varphi^{c'}$ is a strong deformation retract of φ^c . Therefore

$$H_k(X, \varphi^c) = H_k(X, \varphi^{c'}) \quad \text{for all } k \in \mathbb{N}_0$$

(see Motreanu-Motreanu-Papageorgiou [17], p. 145).

Suppose that K_φ is finite. We define the following items:

$$\begin{aligned} M(t, u) &= \sum_{k \in \mathbb{N}_0} \text{rank } C_k(\varphi, u) t^k \quad \text{for all } t \in \mathbb{R}, \text{ all } u \in K_\varphi, \\ P(t, \infty) &= \sum_{k \in \mathbb{N}_0} \text{rank } C_k(\varphi, \infty) t^k \quad \text{for all } t \in \mathbb{R}. \end{aligned}$$

The Morse relation says that

$$\sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1+t)Q(t) \quad \text{for all } t \in \mathbb{R}, \tag{2.5}$$

where $Q(t) = \sum_{k \geq 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

Finally, let us fix our notation. For $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then, for $u \in W_0^{1,p}(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W_0^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

By $|\cdot|_N$ we denote the Lebesgue measure on \mathbb{R}^N and by $|\cdot|$ the norm of \mathbb{R}^N as well as the absolute value in \mathbb{R} . By $(\cdot, \cdot)_{\mathbb{R}^N}$ we denote the inner product in \mathbb{R}^N . Given $u, v \in W_0^{1,p}(\Omega)$, $u \leq v$, then the order interval in $W_0^{1,p}(\Omega)$ determined by u and v is defined by

$$[u, v] = \{y \in W_0^{1,p}(\Omega) : u(z) \leq y(z) \leq v(z) \text{ for a.a. } z \in \Omega\}.$$

By $\text{int}_{C_0^1(\overline{\Omega})}[u, v]$ we denote the interior in the $C_0^1(\overline{\Omega})$ -norm topology of $[u, v] \cap C_0^1(\overline{\Omega})$. By $[u$ we denote the half-line in $W_0^{1,p}(\Omega)$ defined by

$$[u] = \{y \in W_0^{1,p}(\Omega) : u(z) \leq y(z) \text{ for a.a. } z \in \Omega\}.$$

Finally, by $\delta_{k,m}$, $k, m \in \mathbb{N}_0$, we denote the Kronecker symbol, that is,

$$\delta_{k,m} = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{if } k \neq m \end{cases}.$$

3 Constant sign solutions

In this section we produce constant sign solutions and we investigate how the sets of positive and negative solutions of (P_λ) depend on the parameter $\lambda > 0$.

The hypotheses on the perturbation $f(z, x)$ are the following:

H(f): $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Caratheodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

- (i) $|f(z, x)| \leq a(z) (1 + |x|^{r-1})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$ and

$$p < r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N \end{cases} \quad (\text{the critical Sobolev exponent corresponding to } p);$$

- (ii) If $F(z, x) = \int_0^x f(z, s) ds$, then $\lim_{x \rightarrow \pm\infty} \frac{F(z, x)}{|x|^p} = +\infty$ uniformly for a.a. $z \in \Omega$;
- (iii) there exist $\widehat{\eta} > 0$ and $q \in \left((r-p) \max \left\{ \frac{N}{p}, 1 \right\}, p^* \right)$ such that

$$0 < \widehat{\eta} \leq \liminf_{x \rightarrow \pm\infty} \frac{f(z, x)x - pF(z, x)}{|x|^q} \quad \text{uniformly for a.a. } z \in \Omega;$$

- (iv) there exist $m \in \mathbb{N}$, $m \geq 2$, and functions $\vartheta, \widehat{\vartheta} \in L^\infty(\Omega)$ such that

$$\widehat{\lambda}_m(2) \leq \vartheta(z) \leq \widehat{\vartheta}(z) \leq \widehat{\lambda}_{m+1}(2) \quad \text{for a.a. } z \in \Omega,$$

$$\vartheta \not\equiv \widehat{\lambda}_m(2), \quad \widehat{\vartheta} \not\equiv \widehat{\lambda}_{m+1}(2),$$

$$\vartheta(z) \leq \liminf_{x \rightarrow 0} \frac{f(z, x)}{x} \leq \limsup_{x \rightarrow 0} \frac{f(z, x)}{x} \leq \widehat{\vartheta}(z) \quad \text{uniformly for a.a. } z \in \Omega;$$

(v) for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$ the function

$$x \mapsto f(z, x) + \widehat{\xi}_\rho |x|^{p-2}x$$

is nondecreasing on $[-\rho, \rho]$.

Remarks. Hypotheses H(f)(ii), (iii) imply that

$$\lim_{x \rightarrow \pm\infty} \frac{f(z, x)}{|x|^{p-2}x} = +\infty \quad \text{uniformly for a.a. } z \in \Omega.$$

So, the perturbation term is $(p - 1)$ -superlinear. However, we do not use the usual in such cases AR-condition. Recall that the AR-condition says that there exist $q > p$ and $M > 0$ such that

$$0 < qF(z, x) \leq f(z, x)x \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M \quad \text{and} \quad 0 < \operatorname{ess\,inf}_\Omega F(\cdot, \pm M). \tag{3.1}$$

Integrating, we obtain the following weaker condition

$$c_4|x|^q \leq F(z, x) \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M, \text{ with } c_4 > 0. \tag{3.2}$$

From (3.1) and (3.2) it follows that for a.a. $z \in \Omega$, $f(z, \cdot)$ has at least $(q - 1)$ -polynomial growth near $\pm\infty$. So, the AR-condition although very convenient in verifying the C-condition, it is rather restrictive (see the Examples below). For this reason we employ hypothesis H(f)(iii) which is more general. Indeed, suppose that the AR-condition holds. We may assume that $q > (r - p) \max \left\{ \frac{N}{p}, 1 \right\}$. Then

$$\begin{aligned} \frac{f(z, x)x - pF(z, x)}{|x|^q} &= \frac{f(z, x)x - qF(z, x)}{|x|^q} + (q - p) \frac{F(z, x)}{|x|^q} \\ &= (q - p) \frac{F(z, x)}{|x|^q} \quad \text{(see (3.1))} \\ &= (q - p)c_4 > 0 \quad \text{(see (3.2)),} \\ \Rightarrow \liminf_{x \rightarrow \pm\infty} \frac{f(z, x)x - pF(z, x)}{|x|^q} &\geq (q - p)c_4 > 0 \quad \text{uniformly for a.a. } z \in \Omega. \end{aligned}$$

So, hypothesis H(f)(iii) is verified. Near zero, for a.a. $z \in \Omega$, $f(z, \cdot)$ is nonuniformly nonresonant with respect to the spectral interval $[\widehat{\lambda}_m(2), \widehat{\lambda}_{m+1}(2)]$.

Examples. The following functions satisfy hypotheses H(f). For the sake of simplicity, we drop the z -dependence:

$$f_1(x) = \begin{cases} \vartheta x + |x|^{\tau-2}x & \text{if } |x| \leq 1 \\ \vartheta|x|^{\tau-2}x - |x|^{q-2}x & \text{if } |x| > 1 \end{cases},$$

with $\vartheta \in (\widehat{\lambda}_m(2), \widehat{\lambda}_{m+1}(2))$ for some $m \in \mathbb{N}$, $m \geq 2$ and $2 < \tau < \infty$, $p \leq q < r$,

$$f_2(x) = \begin{cases} \vartheta(x - |x|^{\tau-2}x) & \text{if } |x| \leq 1 \\ \vartheta|x|^{p-2}x \ln|x| & \text{if } |x| > 1 \end{cases},$$

with $\vartheta \in (\widehat{\lambda}_m(2), \widehat{\lambda}_{m+1}(2))$ for some $m \in \mathbb{N}$, $m \geq 2$ and $\tau > 2$.

Note that f_1 satisfies the AR-condition, while f_2 does not.

We introduce the following sets:

$$\begin{aligned} \mathcal{L}^+ &= \{ \lambda > 0 : \text{problem } (P_\lambda) \text{ has a positive solution} \}, \\ S_\lambda^+ &= \text{set of positive solutions of } (P_\lambda). \end{aligned}$$

Similarly, we define,

$$\begin{aligned} \mathcal{L}^- &= \{\lambda > 0 : \text{problem } (P_\lambda) \text{ has a negative solution}\}, \\ S_\lambda^- &= \text{set of negative solutions of } (P_\lambda). \end{aligned}$$

We start by establishing the nonemptiness of \mathcal{L}^+ and \mathcal{L}^- and we locate the set S_λ^+ and S_λ^- .

Proposition 3.1. *If hypotheses H(f) hold, then $\mathcal{L}^+, \mathcal{L}^- \neq \emptyset$ and $S_\lambda^+ \subseteq \text{int } C_+, S_\lambda^- \subseteq -\text{int } C_+$.*

Proof. We do the proof for the pair $(\mathcal{L}^+, S_\lambda^+)$, the proof for the pair $(\mathcal{L}^-, S_\lambda^-)$ being similar.

So, we consider the C^1 -functional $\psi_\lambda^+ : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\psi_\lambda^+(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \frac{\lambda}{p} \|u^+\|_p^p - \int_\Omega F(z, u^+) dz, \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Evidently if $\tau \in (1, 2)$, hypothesis H(f)(iv) implies that

$$\lim_{x \rightarrow 0^+} \frac{f(z, x)}{x^{\tau-1}} = 0 \quad \text{uniformly for a.a. } z \in \Omega.$$

So, given $\epsilon > 0$, we can find $c_5 = c_5(\epsilon, \tau) > 0$ such that

$$F(z, x) \leq \epsilon |x|^\tau + c_5 |x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{3.3}$$

Then we have

$$\psi_\lambda^+(u) \geq \frac{1}{p} \|Du^-\|_p^p + \frac{1}{p} (\|Du^+\|_p^p - \lambda \|u^+\|_p^p) - \epsilon c_6 \|u\|^\tau - c_7 \|u\|^r \quad \text{for some } c_6 > 0, c_7 > 0 \text{ (see (3.3)).}$$

If $\lambda \in (0, \widehat{\lambda}_1(p))$, then using Proposition 2.5 we obtain

$$\begin{aligned} \psi_\lambda^+(u) &\geq c_8 \|u\|^p - (\epsilon c_6 \|u\|^\tau + c_7 \|u\|^r) \quad \text{for some } c_8 > 0 \\ &= [c_8 - (\epsilon c_6 \|u\|^{\tau-p} + c_7 \|u\|^{r-p})] \|u\|^p. \end{aligned} \tag{3.4}$$

We consider the function

$$\xi(t) = \epsilon c_6 t^{\tau-p} + c_7 t^{r-p}, \quad t > 0.$$

Evidently $\xi \in C^1(0, +\infty)$. Moreover, since $\tau < 2 < p < r$, we see that

$$\xi(t) \rightarrow +\infty \text{ as } t \rightarrow 0^+ \text{ and as } t \rightarrow +\infty.$$

So, we can find $t_0 \in (0, +\infty)$ such that

$$\begin{aligned} \xi(t_0) &= \inf [\xi(t) : t > 0], \\ \Rightarrow \xi'(t_0) &= 0, \\ \Rightarrow t_0 = t_0(\epsilon) &= \left[\frac{\epsilon c_6 (p - \tau)}{c_7 (r - p)} \right]^{\frac{p-\tau}{r-p}}. \end{aligned}$$

Note that $\xi(t_0) \rightarrow 0^+$ as $\epsilon \rightarrow 0^+$. Therefore we can find $\epsilon_0 > 0$ such that

$$\begin{aligned} \xi(t_0) &< c_8 \quad \text{for all } \epsilon \in (0, \epsilon_0), \\ \Rightarrow \inf [\psi_\lambda^+(u) : \|u\| = t_0] &= m_\lambda^+ > 0 \quad \text{(see (3.4))} \end{aligned} \tag{3.5}$$

Hypothesis H(f)(ii) implies that if $u \in \text{int } C_+$, then

$$\psi_\lambda^+(tu) \rightarrow -\infty \text{ as } t \rightarrow +\infty. \tag{3.6}$$

Claim. For every $\lambda > 0$, the functional ψ_λ^+ satisfies the C-condition.

Let $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that

$$|\psi_\lambda^+(u_n)| \leq M_1 \quad \text{for some } M_1 > 0, \text{ all } n \in \mathbb{N}, \tag{3.7}$$

$$(1 + \|u_n\|)(\psi_\lambda^+)'(u_n) \rightarrow 0 \quad \text{in } W^{-1,p'}(\Omega) \text{ as } n \rightarrow \infty. \tag{3.8}$$

From (3.8) we have

$$\left| \langle A_p(u_n), h \rangle + \langle A(u_n), h \rangle - \lambda \int_\Omega (u_n^+)^{p-1} h \, dz - \int_\Omega f(z, u_n^+) h \, dz \right| \leq \frac{\epsilon_n \|h\|}{1 + \|u_n\|} \tag{3.9}$$

for all $h \in W_0^{1,p}(\Omega)$, with $\epsilon_n \rightarrow 0^+$.

In (3.9) we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} & \|Du_n^-\|_p^p + \|Du_n^-\|_2^2 \leq \epsilon_n \quad \text{for all } n \in \mathbb{N}, \\ \Rightarrow & u_n^- \rightarrow 0 \text{ in } W_0^{1,p}(\Omega) \text{ as } n \rightarrow \infty. \end{aligned} \tag{3.10}$$

From (3.7) and (3.10), we have

$$\|Du_n^+\|_p^p + \frac{p}{2} \|Du_n^+\|_2^2 - \int_\Omega [\lambda(u_n^+)^p + pF(z, u_n^+)] \, dz \leq M_2 \tag{3.11}$$

for some $M_2 > 0$, all $n \in \mathbb{N}$.

Also from (3.9) with $h = u_n^+ \in W_0^{1,p}(\Omega)$, we obtain

$$-\|Du_n^+\|_p^p - \|Du_n^+\|_2^2 + \int_\Omega [\lambda(u_n^+)^p + f(z, u_n^+)u_n^+] \, dz \leq \epsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{3.12}$$

We add (3.11) and (3.12) and obtain

$$\int_\Omega [f(z, u_n^+)u_n^+ - pF(z, u_n^+)] \, dz \leq M_3 \quad \text{for all } M_3 > 0, \text{ all } n \in \mathbb{N}, \text{ (recall } p > 2). \tag{3.13}$$

Hypotheses $H(f)$ (i), $H(f)$ (iii) imply that we can find $\hat{\eta}_0 \in (0, \hat{\eta})$ and $c_9 > 0$ such that

$$\hat{\eta}_0 |x|^q - c_9 \leq f(z, x)x - pF(z, x) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

Using this in (3.13), we obtain that

$$\{u_n^+\}_{n \geq 1} \subseteq L^q(\Omega) \text{ is bounded,} \tag{3.14}$$

First suppose that $N \neq p$. From hypothesis $H(f)$ (iii) it is clear that we can have $q < r < p^*$ (recall that if $N \leq p$, then $p^* = +\infty$). So, we can find $t \in (0, 1)$ such that

$$\frac{1}{r} = \frac{1-t}{q} + \frac{t}{p^*}.$$

Invoking the interpolation inequality (see, for example, Gasiński-Papageorgiou [18], p. 905), we have

$$\begin{aligned} & \|u_n^+\|_r \leq \|u_n^+\|_q^{1-t} \|u_n^+\|_{p^*}^t, \\ \Rightarrow & \|u_n^+\|_r^r \leq c_{10} \|u_n^+\|^{tr} \quad \text{for some } c_{10} > 0, \text{ all } n \in \mathbb{N} \\ & , \quad \text{(see (3.4) and recall that } W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)). \end{aligned} \tag{3.15}$$

In (3.9) let $h = u_n^+ \in W_0^{1,p}(\Omega)$. Then

$$\|Du_n^+\|_p^p + \|Du_n^+\|_2^2 - \int_\Omega [\lambda(u_n^+)^p + f(z, u_n^+)u_n^+] \, dz \leq \epsilon_n \text{ for all } n \in \mathbb{N},$$

$$\begin{aligned} \Rightarrow \|u_n^+\|^p &\leq c_{11} [1 + \|u_n^+\|_r^r] && \text{for some } c_{11} = c_{11}(\lambda) > 0, \text{ all } n \in \mathbb{N} \\ & && \text{(see hypothesis H(f)(i) and recall that } r > p) \\ \Rightarrow \|u_n^+\|^p &\leq c_{12} [1 + \|u_n^+\|^{tr}] && \text{for some } c_{12} > 0, \text{ all } n \in \mathbb{N}, \text{ (see (3.15)).} \end{aligned} \tag{3.16}$$

Hypothesis H(f)(iii) implies that $tr < p$. So, from (3.16) it follows that

$$\begin{aligned} \{u_n^+\}_{n \geq 1} &\subseteq W_0^{1,p}(\Omega) \text{ is bounded,} \\ \Rightarrow \{u_n\}_{n \geq 1} &\subseteq W_0^{1,p}(\Omega) \text{ is bounded (see (3.10)).} \end{aligned} \tag{3.17}$$

Now suppose that $N = p$. In this case $p^* = +\infty$ and $W_0^{1,p} \hookrightarrow L^s(\Omega)$ for all $s \in [1, +\infty)$. Let $s > r > q$ and as before pick $t \in (0, 1)$ such that

$$\begin{aligned} \frac{1}{r} &= \frac{1-t}{q} + \frac{t}{s}, \\ \Rightarrow tr &= \frac{s(r-q)}{s-q}. \end{aligned}$$

We see that

$$\frac{s(r-q)}{s-q} \rightarrow r-q \text{ as } s \rightarrow p^* = +\infty.$$

By hypothesis H(f)(iii) we have

$$\begin{aligned} r - q &< p, \\ \Rightarrow tr = \frac{s(r-q)}{s-q} &< p \text{ for } s > r \text{ big.} \end{aligned}$$

Therefore in this case too, we conclude that (3.17) holds.

Passing to a subsequence if necessary, we have

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \rightarrow u \text{ in } L^r(\Omega). \tag{3.18}$$

In (3.9) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \rightarrow \infty$ and use (3.18). Then

$$\begin{aligned} \lim_{n \rightarrow \infty} [\langle A_p(u_n), u_n - u \rangle + \langle A(u_n), u_n - u \rangle] &= 0, \\ \Rightarrow \limsup_{n \rightarrow \infty} [\langle A_p(u_n), u_n - u \rangle + \langle A(u), u_n - u \rangle] &\leq 0 \text{ (since } A(\cdot) \text{ is monotone)} \\ \Rightarrow \limsup_{n \rightarrow \infty} \langle A_p(u_n), u_n - u \rangle &\leq 0, \\ \Rightarrow u_n \rightarrow u \text{ in } W_0^{1,p}(\Omega) &\text{ (see Proposition 2.3).} \end{aligned}$$

Therefore ψ_λ^+ satisfies the C-condition. This proves the Claim.

Then with $\lambda \in (0, \widehat{\lambda}_1(p))$, from (3.5), (3.6) and the Claim, we see that we can apply Theorem 1 (the mountain pass theorem) and find $u_\lambda \in W_0^{1,p}(\Omega)$ such that

$$u_\lambda \in K_{\psi_\lambda^+} \text{ and } \psi_\lambda^+(0) = 0 < m_\lambda^+ \leq \psi_\lambda^+(u_\lambda) \text{ (see (3.5)).}$$

Therefore $u_\lambda \neq 0$ and we have

$$\langle A_p(u_\lambda), h \rangle + \langle A(u_\lambda), h \rangle = \int_\Omega [\lambda(u_\lambda^+)^{p-1} + f(z, u_\lambda^+)] h \, dz \text{ for all } h \in W_0^{1,p}(\Omega).$$

Choosing $h = -u_\lambda^- \in W_0^{1,p}(\Omega)$, we obtain

$$u_\lambda \geq 0, \quad u_\lambda \neq 0.$$

From (3.9) we have

$$-A_p u_\lambda(z) - \Delta u_\lambda(z) = \lambda u_\lambda(z)^{p-1} + f(z, u_\lambda(z)) \text{ for a.a. } z \in \Omega, \quad u_\lambda|_{\partial\Omega} = 0. \tag{3.19}$$

From (3.19) and Corollary 6.8, p. 208, of Motreanu-Motreanu-Papageorgiou [17], we have that $u_\lambda \in L^\infty(\Omega)$. Then Theorem 1 of Lieberman [14], implies that

$$u_\lambda \in C_+ \setminus \{0\}.$$

Let $\rho = \|u_\lambda\|_\infty$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis $H(f)(\mathbf{v})$. Then from (3.19) we have

$$\begin{aligned} & -\Delta_p u_\lambda(z) - \Delta u_\lambda(z) + \widehat{\xi}_\rho u_\lambda(z)^{p-1} \geq 0 \quad \text{for a.a. } z \in \Omega, \\ \Rightarrow & u_\lambda \in \text{int } C_+ \quad (\text{see Pucci-Serrin [19], pp. 111,120}). \end{aligned}$$

Therefore $(0, \widehat{\lambda}_1(p)) \subseteq \mathcal{L}^+$ and $S_\lambda^+ \subseteq \text{int } C_+$. Similarly we show that $\mathcal{L}^- \neq \emptyset$ and that $S_\lambda^- \subseteq -\text{int } C_+$. □

Next we show that both \mathcal{L}^+ and \mathcal{L}^- are intervals.

Proposition 3.2. *If hypotheses $H(f)$ hold, $\lambda \in \mathcal{L}^+$ (resp. $\lambda \in \mathcal{L}^-$) and $0 < \vartheta < \lambda$, then $\vartheta \in \mathcal{L}^+$ (resp. $\vartheta \in \mathcal{L}^-$).*

Proof. We do the proof for \mathcal{L}^+ , the proof for \mathcal{L}^- being similar.

Let $\lambda \in \mathcal{L}^+$. We can find $u_\lambda \in S_\lambda^+ \subseteq \text{int } C_+$. Then we introduce the following truncation of the reaction in problem (P_ϑ) :

$$e_\vartheta(z, x) = \begin{cases} 0 & \text{if } x < 0 \\ \vartheta x^{p-1} + f(z, x) & \text{if } 0 \leq x \leq u_\lambda(z) \\ \vartheta u_\lambda(z)^{p-1} + f(z, u_\lambda(z)) & \text{if } u_\lambda(z) < x \end{cases} \quad (3.20)$$

This is a Caratheodory function. We set $E_\vartheta(z, x) = \int_0^x e_\vartheta(z, s) ds$ and consider the C^1 -functional $\widehat{\psi}_\vartheta^+ : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\widehat{\psi}_\vartheta^+(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \int_\Omega E_\vartheta(z, u) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (3.20) it is clear that $\widehat{\psi}_\vartheta^+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_\vartheta \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\psi}_\vartheta^+(u_\vartheta) = \inf \left[\widehat{\psi}_\vartheta^+(u) : u \in W_0^{1,p}(\Omega) \right]. \quad (3.21)$$

On account of hypothesis $H(f)(\mathbf{iv})$, we see that given $\epsilon > 0$, we can find $\delta > 0$ such that

$$F(z, x) \geq \frac{1}{2} [\vartheta(z) - \epsilon] x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta. \quad (3.22)$$

Let $u \in E(\widehat{\lambda}_m(2)) \subseteq C_0^1(\overline{\Omega})$ and choose $t \in (0, 1)$ small such that

$$0 \leq tu(z) \leq \delta \quad \text{for all } z \in \overline{\Omega}. \quad (3.23)$$

Then we have

$$\begin{aligned} \widehat{\psi}_\vartheta^+(tu) & \leq \frac{t^p}{p} \|Du\|_p^p + \frac{t^2}{2} \|Du\|_2^2 - \frac{t^2}{2} \int_\Omega \vartheta(z) u^2 dz + \frac{\epsilon}{2} t^2 \|u\|_2^2 \quad (\text{see (3.22), (3.23)}) \\ & = \frac{t^p}{p} \|Du\|_p^p + \frac{t^2}{2} \left[\|Du\|_2^2 - \int_\Omega \vartheta(z) u^2 dz \right] + \frac{\epsilon}{2} t^2 \|u\|_2^2 \\ & \leq \frac{t^p}{p} \|Du\|_p^p + \frac{t^2}{2} (-c_{13} + \epsilon) \|u\|_2^2 \quad \text{for some } c_{13} > 0 \text{ (see Proposition 2.4)}. \end{aligned}$$

Choosing $\epsilon \in (0, c_{13})$, we have that

$$\widehat{\psi}_\vartheta^+(tu) \leq \frac{t^p}{p} \|Du\|_p^p - \frac{t^2}{2} c_{14} \|u\|_2^2.$$

Since $p > 2$, choosing $t \in (0, 1)$ even smaller, we have

$$\begin{aligned} \widehat{\psi}_g^+(tu) &< 0, \\ \Rightarrow \widehat{\psi}_g^+(u_g) &< 0 = \widehat{\psi}_g^+(0) \quad (\text{see (3.21)}), \Rightarrow u_g \neq 0. \end{aligned}$$

From (3.21), we have

$$\begin{aligned} (\widehat{\psi}_g^+)'(u_g) &= 0 \\ \Rightarrow \langle A_p(u_g), h \rangle + \langle A(u_g), h \rangle &= \int_{\Omega} e_g(z, u_g)h \, dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{3.24}$$

In (3.24) we choose $h = -u_g^- \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} \|Du_g^-\|_p^p + \|Du_g^-\|_2^2 &= 0 \quad (\text{see (3.20)}), \\ \Rightarrow u_g &\geq 0, u_g \neq 0. \end{aligned}$$

Also, in (3.24) we choose $h = (u_g - u_\lambda)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} &\langle A_p(u_g), (u_g - u_\lambda)^+ \rangle + \langle A(u_g), (u_g - u_\lambda)^+ \rangle \\ &= \int_{\Omega} [\vartheta u_\lambda^{p-1} + f(z, u_\lambda)] (u_g - u_\lambda)^+ \, dz \quad (\text{see (3.20)}) \\ &\leq \int_{\Omega} [\lambda u_\lambda^{p-1} + f(z, u_\lambda)] (u_g - u_\lambda)^+ \, dz \quad (\text{since } \vartheta < \lambda) \\ &= \langle A_p(u_\lambda), (u_g - u_\lambda)^+ \rangle + \langle A(u_\lambda), (u_g - u_\lambda)^+ \rangle \quad (\text{since } u_\lambda \in S_\lambda), \\ &\Rightarrow u_g \leq u_\lambda. \end{aligned}$$

So, we have proved that

$$u_g \in [0, u_\lambda], u_g \neq 0. \tag{3.25}$$

From (3.24), (3.25) and (3.20), we conclude that

$$\begin{aligned} -\Delta_p u_g(z) - \Delta u_g(z) &= \vartheta u_g(z)^{p-1} + f(z, u_g(z)) \text{ for a.a. } z \in \Omega, \quad u_g|_{\partial\Omega} = 0, \\ \Rightarrow \vartheta &\in \mathcal{L}^+ \text{ and } u_g \in S_\vartheta^+ \subseteq \text{int } C_+. \end{aligned}$$

Similarly for \mathcal{L}^- . □

The following Corollary is a useful byproduct of the above proof.

Corollary 3.1. *If hypotheses H(f) hold, then*

(a) *if $0 < \vartheta < \lambda \in \mathcal{L}^+$ and $u_\lambda \in S_\lambda^+$, then $\vartheta \in \mathcal{L}^+$ and we can find $u_g \in S_\vartheta^+ \subseteq \text{int } C_+$ such that*

$$u_\lambda - u_g \in C_+ \setminus \{0\};$$

(b) *if $0 < \vartheta < \lambda \in \mathcal{L}^-$ and $v_\lambda \in S_\lambda^-$, then $\vartheta \in \mathcal{L}^-$ and we can find $v_g \in S_\vartheta^- \subseteq -\text{int } C_+$ such that*

$$v_g - v_\lambda \in C_+ \setminus \{0\}.$$

We can improve this corollary.

Proposition 3.3. *If hypotheses H(f) hold, then*

(a) *if $0 < \vartheta < \lambda \in \mathcal{L}^+$ and $u_\lambda \in S_\lambda^+$, then $\vartheta \in \mathcal{L}^+$ and we can find $u_\vartheta \in S_\vartheta^+ \subseteq \text{int } C_+$ such that*

$$u_\lambda - u_\vartheta \in \text{int } C_+;$$

(b) *if $0 < \vartheta < \lambda \in \mathcal{L}^-$ and $v_\lambda \in S_\lambda^-$, then $\vartheta \in \mathcal{L}^-$ and we can find $v_\vartheta \in S_\vartheta^- \subseteq -\text{int } C_+$ such that*

$$v_\vartheta - v_\lambda \in \text{int } C_+.$$

Proof.

(a) From Corollary (3.1), we already know that $\vartheta \in \mathcal{L}^+$ and we can find $u_\vartheta \in S_\vartheta^+ \subseteq \text{int } C_+$ such that

$$u_\lambda - u_\vartheta \in C_+ \setminus \{0\}. \tag{3.26}$$

Let $\rho = \|u_\lambda\|_\infty$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis H(f)(v). Then

$$\begin{aligned} & -\Delta_p u_\vartheta - \Delta u_\vartheta + \widehat{\xi}_\rho u_\vartheta^{p-1} \\ = & \vartheta u_\vartheta^{p-1} + f(z, u_\vartheta) + \widehat{\xi}_\rho u_\vartheta^{p-1} \\ = & \lambda u_\vartheta^{p-1} + f(z, u_\vartheta) + \widehat{\xi}_\rho u_\vartheta^{p-1} - (\lambda - \vartheta)u_\vartheta^{p-1} \\ \leq & \lambda u_\lambda^{p-1} + f(z, u_\lambda) + \widehat{\xi}_\rho u_\lambda^{p-1} \quad (\text{see (3.26), hypothesis H(f)(v) and recall that } \vartheta < \lambda) \\ = & -\Delta_p u_\lambda - \Delta u_\lambda + \widehat{\xi}_\rho u_\lambda^{p-1} \quad (\text{since } u_\lambda \in S_\lambda^+). \end{aligned} \tag{3.27}$$

Let

$$\begin{aligned} h_1(z) &= \vartheta u_\vartheta^{p-1} + f(z, u_\vartheta) + \widehat{\xi}_\rho u_\vartheta^{p-1}, \\ h_2(z) &= \lambda u_\lambda^{p-1} + f(z, u_\lambda) + \widehat{\xi}_\rho u_\lambda^{p-1}. \end{aligned}$$

Evidently $h_1, h_2 \in L^\infty(\Omega)$ and we have

$$h_2(z) - h_1(z) \geq (\lambda - \vartheta)u_\vartheta(z)^{p-1} \quad \text{for a.a } z \in \Omega.$$

Since $u_\vartheta \in \text{int } C_+$ we see that $h_1 < h_2$. Invoking Proposition 2.2, from (3.27) we conclude that $u_\lambda - u_\vartheta \in \text{int } C_+$.

(b) The proof is similar, using this time part (b) of Corollary 3.1.

□

We set $\lambda_*^+ = \sup \mathcal{L}^+$ and $\lambda_*^- = \sup \mathcal{L}^-$.

Proposition 3.4. *If hypotheses H(f) hold, then $\lambda_*^+ < +\infty$ and $\lambda_*^- < +\infty$.*

Proof. We do the proof for λ_*^+ , the proof for λ_*^- being similar. On account of hypotheses H(f)(i), (ii), (iii), we can find $\widetilde{\lambda} > 0$ big such that

$$\widetilde{\lambda} x^{p-1} + f(z, x) \geq 0 \quad \text{for a.a } z \in \Omega, \text{ all } x \geq 0. \tag{3.28}$$

Let $\lambda > \widetilde{\lambda}$ and suppose that $\lambda \in \mathcal{L}^+$. We can find $u_\lambda \in S_\lambda \subseteq \text{int } C_+$. So, we have

$$\left. \frac{\partial u_\lambda}{\partial n} \right|_{\partial\Omega} < 0.$$

Therefore we can find $\delta > 0$ such that, if $\partial\Omega_\delta = \{z \in \Omega : d(z, \partial\Omega) = \delta\}$, then

$$\left. \frac{\partial u_\lambda}{\partial n} \right|_{\partial\Omega_\delta} < 0. \tag{3.29}$$

Consider the open set $\Omega_\delta = \{z \in \Omega : d(z, \partial\Omega) > \delta\}$ and set $m_\delta = \min_{\Omega_\delta} u_\lambda > 0$ (recall that $u_\lambda \in \text{int } C_+$). For $\epsilon > 0$, we set $m_\delta^\epsilon = m_\delta + \epsilon$ and for $\rho = \|u_\lambda\|_\infty$ let $\tilde{\xi}_\rho > 0$ be as postulated by hypothesis $H(f)(\mathbf{v})$. We have

$$\begin{aligned} & -\Delta_p m_\delta^\epsilon - \Delta m_\delta^\epsilon + \tilde{\xi}_\rho (m_\delta^\epsilon)^{p-1} \\ & \leq \tilde{\xi}_\rho m_\delta^{p-1} + \mu(\epsilon) \quad \text{with } \mu(\epsilon) \rightarrow 0^+ \text{ as } \epsilon \rightarrow 0^+ \\ & \leq \tilde{\lambda} m_\delta^{p-1} + f(z, m_\delta) + \tilde{\xi}_\rho m_\delta^{p-1} + \mu(\epsilon) \quad (\text{see (3.28)}) \\ & = \lambda m_\delta^{p-1} + f(z, m_\delta) + \tilde{\xi}_\rho m_\delta^{p-1} - (\lambda - \tilde{\lambda}) m_\delta^{p-1} + \mu(\epsilon) \quad (\text{see (3.28)}) \\ & \leq \lambda m_\delta^{p-1} + f(z, m_\delta) + \tilde{\xi}_\rho m_\delta^{p-1} \quad \text{for } \epsilon > 0 \text{ small} \\ & \leq \lambda u_\lambda^{p-1} + f(z, u_\lambda) + \tilde{\xi}_\rho u_\lambda^{p-1} \quad (\text{recall that } m_\delta \leq u_\lambda \text{ on } \overline{\Omega_\delta}) \\ & = -\Delta_p u_\lambda - \Delta u_\lambda + \tilde{\xi}_\rho u_\lambda^{p-1} \quad \text{for a.a. } z \in \Omega_\delta. \end{aligned} \tag{3.30}$$

Then from (3.29), (3.30) and Proposition 2.10 of Papageorgiou-Rădulescu-Repovš [20], we have

$$u_\lambda - m_\delta^\epsilon \in \text{int } C_+(\overline{\Omega_\delta}) \quad \text{for } \epsilon > 0 \text{ small,}$$

which contradicts the definition of m_δ . Therefore $\lambda \notin \mathcal{L}^+$ and so

$$\lambda_*^+ \leq \tilde{\lambda} < +\infty.$$

Similarly we show that $\lambda_*^- < +\infty$. □

Hypotheses $H(f)(\mathbf{i})$, (\mathbf{iv}) , imply that given $\epsilon > 0$, we can find $c_{15} > 0$ such that

$$\lambda|x|^p + f(z, x)x \geq [\vartheta(z) - \epsilon]x^2 - c_{15}|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ all } \lambda > 0. \tag{3.31}$$

This unilateral growth restriction on the reaction of (P_λ) , leads to the following auxiliary $(p, 2)$ -equation:

$$\begin{cases} -\Delta_p u(z) - \Delta u(z) = [\vartheta(z) - \epsilon]u(z) - c_{15}|u(z)|^{r-2}u(z) & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases} \tag{3.32}$$

Proposition 3.5. *For all $\epsilon > 0$ small, problem (3.32) has a unique positive solution $u_\lambda^* \in \text{int } C_+$ and, since (3.32) is odd, $v_\lambda^* = -u_\lambda^* \in -\text{int } C_+$ is the unique solution of (3.32).*

Proof. Consider the C^1 -functional $\sigma : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\sigma(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 + \frac{c_{15}}{r} \|u^+\|_r^r - \frac{1}{2} \int_\Omega [\vartheta(z) - \epsilon](u^+)^2 \, dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Evidently $\sigma(\cdot)$ is coercive (recall that $p > 2$). Also, it is sequentially weakly lower semicontinuous. So, we can find $u_\lambda^* \in W_0^{1,p}(\Omega)$ such that

$$\sigma(u_\lambda^*) = \inf \left[\sigma(u) : u \in W_0^{1,p}(\Omega) \right]. \tag{3.33}$$

As in the proof of Proposition 3.2, for $\epsilon > 0$ small we have

$$\begin{aligned} \sigma(u_\lambda^*) & < 0 = \sigma(0), \\ \Rightarrow u_\lambda^* & \neq 0. \end{aligned}$$

From (3.33) we have

$$\sigma'(u_\lambda^*) = 0,$$

$$\Rightarrow \langle A_p(u_\lambda^*), h \rangle + \langle A(u_\lambda^*), h \rangle = \int_\Omega [\vartheta(z) - \epsilon](u_\lambda^*)^+ h \, dz - \lambda \int_\Omega ((u_\lambda^*)^+)^{r-1} h \, dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \quad (3.34)$$

In (3.34) we choose $h = -(u_\lambda^*)^- \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} & \|D(u_\lambda^*)^-\|_p^p + \|D(u_\lambda^*)^-\|_2^2 = 0, \\ \Rightarrow & u_\lambda^* \geq 0, \quad u_\lambda^* \neq 0. \end{aligned}$$

So, from (3.34) we have that u_λ^* is a positive solution of (3.32) and the nonlinear regularity theory (see [14]) implies that $u_\lambda^* \in C_+ \setminus \{0\}$. We have

$$\begin{aligned} & \Delta_p u_\lambda^* + \Delta u_\lambda^* \leq c_{15} \|u_\lambda^*\|_\infty^{r-p} (u_\lambda^*)^{p-1} \quad \text{for a.a. } z \in \Omega, \\ \Rightarrow & u_\lambda^* \in \text{int } C_+ \quad (\text{see Pucci-Serrin [19], pp. 111, 120}). \end{aligned}$$

Next we show the uniqueness of this positive solution. To this end we consider the integral functional $j : L^1(\Omega) \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p} \|Du^{1/2}\|_p^p + \frac{1}{2} \|Du^{1/2}\|_2^2 & \text{if } u \geq 0, u^{1/2} \in W_0^{1,p}(\Omega) \\ +\infty & \text{otherwise} \end{cases}.$$

Let $\text{dom } j = \{u \in L^1(\Omega) : j(u) < +\infty\}$ (the effective domain of $j(\cdot)$).

From Lemma 1 of Diaz-Saá [21], we have that

$$j(\cdot) \text{ is convex.}$$

Suppose that $u_\lambda^*, \tilde{u}_\lambda^*$ are two positive solutions of (3.32). We have

$$u_\lambda^*, \tilde{u}_\lambda^* \in \text{int } C_+$$

Then, for $h \in C_0^1(\overline{\Omega})$ and for $|t| < 1$ small, we have

$$(u_\lambda^*)^2 + th \in \text{dom } j \quad \text{and} \quad (\tilde{u}_\lambda^*)^2 + th \in \text{dom } j.$$

It is easy to see that $j(\cdot)$ is Gateaux differentiable at $(u_\lambda^*)^2$ and at $(\tilde{u}_\lambda^*)^2$ in the direction h . Moreover, using the chain rule and the nonlinear Green's identity (see Gasiński-Papageorgiou [18], p. 211), we have

$$\begin{aligned} j' \left((u_\lambda^*)^2 \right) (h) &= \frac{1}{2} \int_\Omega \frac{-\Delta_p u_\lambda^* - \Delta u_\lambda^*}{u_\lambda^*} h \, dz \\ j' \left((\tilde{u}_\lambda^*)^2 \right) (h) &= \frac{1}{2} \int_\Omega \frac{-\Delta_p \tilde{u}_\lambda^* - \Delta \tilde{u}_\lambda^*}{\tilde{u}_\lambda^*} h \, dz \end{aligned}$$

for all $h \in C_0^1(\overline{\Omega})$.

The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. Therefore

$$\begin{aligned} 0 & \leq \int_\Omega \left[\frac{-\Delta_p u_\lambda^* - \Delta u_\lambda^*}{u_\lambda^*} - \frac{-\Delta_p \tilde{u}_\lambda^* - \Delta \tilde{u}_\lambda^*}{\tilde{u}_\lambda^*} \right] (u_\lambda^* - \tilde{u}_\lambda^*) \, dz \\ & = \int_\Omega c_{15} \left[(\tilde{u}_\lambda^*)^{r-2} - (u_\lambda^*)^{r-2} \right] (u_\lambda^* - \tilde{u}_\lambda^*) \, dz \leq 0, \end{aligned}$$

$$\Rightarrow u_\lambda^* = \tilde{u}_\lambda^*.$$

This proves the uniqueness of the positive solution of problem (3.32). Since problem (3.32) is odd, it follows that

$$v_\lambda^* = -u_\lambda^* \in -\text{int } C_+,$$

is the unique negative solution of (3.32). □

These solutions provide bounds of the elements of S_λ^+ and of S_λ^- .

Proposition 3.6. *If hypotheses H(f) hold, then*

- (a) $u_\lambda^* \leq u$ for all $u \in S_\lambda^+, \lambda \in \mathcal{L}^+$;
- (b) $v \leq v_\lambda^*$ for all $v \in S_\lambda^-, \lambda \in \mathcal{L}^-$.

Proof.

- (a) Let $\lambda \in \mathcal{L}^+$ and $u \in S_\lambda^+ \subseteq \text{int } C_+$. With $\epsilon > 0$ small as dictated by Proposition 3.5, we introduce the following Caratheodory function:

$$k_+(z, x) = \begin{cases} 0 & \text{if } x < 0 \\ [\vartheta(z) - \epsilon]x - c_{15}x^{r-1} & \text{if } 0 \leq x \leq u(x) \\ [\vartheta(z) - \epsilon]u(z) - c_{15}u(z)^{r-1} & \text{if } u(z) < x \end{cases} \quad (3.35)$$

We set $K_+(z, x) = \int_0^x k_+(z, s) ds$ and consider the C^1 -functional $\tau_+ : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\tau_+(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \int_\Omega K_+(z, u) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Evidently $\tau_+(\cdot)$ is coercive (see (3.35)) and sequentially weakly lower semicontinuous. So, we can find $\hat{u}_\lambda^* \in W_0^{1,p}(\Omega)$ such that

$$\tau_+(\hat{u}_\lambda^*) = \inf \left[\tau_+(u) : u \in W_0^{1,p}(\Omega) \right]. \quad (3.36)$$

As before we have

$$\begin{aligned} \tau_+(\hat{u}_\lambda^*) &< 0 = \tau_+(0) \\ \Rightarrow \hat{u}_\lambda^* &\neq 0. \end{aligned}$$

From (3.36) we have

$$\begin{aligned} \tau'_+(\hat{u}_\lambda^*) &= 0, \\ \Rightarrow \langle A_p(\hat{u}_\lambda^*), h \rangle + \langle A(\hat{u}_\lambda^*), h \rangle &= \int_\Omega k_+(z, \hat{u}_\lambda^*) h dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \end{aligned} \quad (3.37)$$

In (3.37) first we choose $h = -(\hat{u}_\lambda^*)^- \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} \|D(\hat{u}_\lambda^*)^-\|_p^p + \|D(\hat{u}_\lambda^*)^-\|_2^2 &= 0 \quad (\text{see (3.35)}), \\ \Rightarrow \hat{u}_\lambda^* &\geq 0, \hat{u}_\lambda^* \neq 0. \end{aligned}$$

Next in (3.37) we choose $(\hat{u}_\lambda^* - u)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}
 & \langle A_p(\widehat{u}_\lambda^*), (\widehat{u}_\lambda^* - u)^+ \rangle + \langle A(\widehat{u}_\lambda^*), (\widehat{u}_\lambda^* - u)^+ \rangle \\
 &= \int_\Omega \left[(\vartheta(z) - \epsilon)u - c_{15}u^{r-1} \right] (\widehat{u}_\lambda^* - u)^+ \, dz && \text{(see (3.35))} \\
 &\leq \int_\Omega \left[\lambda u^{p-1} + f(z, u) \right] (\widehat{u}_\lambda^* - u)^+ \, dz && \text{(see (3.31))} \\
 &= \langle A_p(u), (\widehat{u}_\lambda^* - u)^+ \rangle + \langle A(u), (\widehat{u}_\lambda^* - u)^+ \rangle && \text{(since } u \in S_\lambda^+), \\
 &\Rightarrow \widehat{u}_\lambda^* \leq u.
 \end{aligned}$$

So, we have proved that

$$\widehat{u}_\lambda^* \in [0, u], \widehat{u}_\lambda^* \neq 0. \tag{3.38}$$

From (3.37) and (3.38) it follows that \widehat{u}_λ^* is a positive solution of problem (3.32). Hence Proposition 3.5 implies that

$$\begin{aligned}
 & \widehat{u}_\lambda^* = u_\lambda^* \in \text{int } C_+, \\
 & \Rightarrow u_\lambda^* \leq u \text{ for all } u \in S_\lambda^+ \text{ (see (3.38)).}
 \end{aligned}$$

(b) Let $\lambda \in \mathcal{L}^-$ and $v \in S_\lambda^-$. We introduce the Caratheodory function $k_-(z, x)$ defined by

$$k_-(z, x) = \begin{cases} [\vartheta(z) - \epsilon]v(z) - c_{15}|v(z)|^{r-2}v(z) & \text{if } x < v(z) \\
 [\vartheta(z) - \epsilon]x - c_{15}|x|^{r-2}x & \text{if } v(z) \leq x \leq 0. \\
 0 & \text{if } 0 < x \end{cases} \tag{3.39}$$

We set $K_-(z, x) = \int_0^x k_-(z, s) \, ds$ and consider the C^1 -functional $\tau_- : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\tau_-(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \int_\Omega K_-(z, u) \, dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Working as in part (a), using this time the functional $\tau_-(\cdot)$ and (3.39) we show that

$$v \leq v_\lambda^* \text{ for all } v \in S_\lambda^-.$$

□

Using these bounds, we can produce extremal constant sign solutions, that is, a smallest positive solution and a biggest negative solution.

Proposition 3.7. *If hypotheses H(f) hold, then*

(a) *for every $\lambda \in \mathcal{L}^+$ problem (P_λ) has a smallest positive solution $\bar{u}_\lambda \in S_\lambda^+ \subseteq \text{int } C_+$, that is,*

$$\bar{u}_\lambda \leq u \text{ for all } u \in S_\lambda^+;$$

(b) *for every $\lambda \in \mathcal{L}^-$ problem (P_λ) has a biggest negative solution $\bar{v}_\lambda \in S_\lambda^- \subseteq -\text{int } C_+$, that is,*

$$v \leq \bar{v}_\lambda \text{ for all } v \in S_\lambda^-.$$

Proof. (a) From Filippakis-Papageorgiou [22], we know that S_λ^+ is downward directed (that is, if $u_1, u_2 \in S_\lambda^+$, then we can find $u \in S_\lambda^+$ such that $u \leq u_1, u \leq u_2$). Hence using Lemma 3.10, p. 178, of Hu-Papageorgiou [23], we can find $\{u_n\}_{n \geq 1} \subseteq S_\lambda^+$ decreasing such that

$$\inf S_\lambda^+ = \inf_{n \geq 1} u_n.$$

We have

$$\langle A_p(u_n), h \rangle + \langle A(u_n), h \rangle = \int_\Omega [\lambda u_n^{p-1} + f(z, u_n)] h \, dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}, \quad (3.40)$$

$$0 \leq u_n \leq u_1 \quad \text{for all } n \in \mathbb{N}. \quad (3.41)$$

In (3.40) we choose $h = u_n \in W_0^{1,p}(\Omega)$. Then on account of (3.41) and hypothesis $H(f)(i)$, we obtain

$$\begin{aligned} \|Du_n\|_p^p + \|Du_n\|_2^2 &\leq c_{16} \quad \text{for some } c_{16} > 0, \text{ all } n \in \mathbb{N}, \\ \Rightarrow \{u_n\}_{n \geq 1} &\subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \end{aligned}$$

So, by passing to a subsequence if necessary, we have

$$u_n \xrightarrow{w} \bar{u}_\lambda \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \rightarrow \bar{u}_\lambda \text{ in } L^p(\Omega). \quad (3.42)$$

If in (3.40) we choose $h = u_n - \bar{u}_\lambda \in W_0^{1,p}(\Omega)$, pass to the limit as $n \rightarrow \infty$, use (3.42) and reason as in the proof of Proposition 3.1 (see the Claim), we obtain

$$u_n \rightarrow \bar{u}_\lambda \text{ in } W_0^{1,p}(\Omega). \quad (3.43)$$

So, if in (3.40) we pass to the limit as $n \rightarrow \infty$ and use (3.43), then

$$\langle A_p(\bar{u}_\lambda), h \rangle + \langle A(\bar{u}_\lambda), h \rangle = \int_\Omega [\lambda \bar{u}_\lambda^{p-1} + f(z, \bar{u}_\lambda)] h \, dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \quad (3.44)$$

From Proposition 3.6, we know that

$$\begin{aligned} u_\lambda^* &\leq u_n \quad \text{for all } n \in \mathbb{N}, \\ \Rightarrow u_\lambda^* &\leq \bar{u}_\lambda \quad (\text{see (3.43)}). \end{aligned} \quad (3.45)$$

From (3.44) and (3.45) we conclude that

$$\bar{u}_\lambda \in S_\lambda^+ \subseteq \text{int } C_+ \quad \text{and} \quad \bar{u}_\lambda = \inf S_\lambda^+.$$

(b) From Filippakis-Papageorgiou [22], we know that S_λ^- is upward directed (that is, if $v_1, v_2 \in S_\lambda^-$, then we can find $v \in S_\lambda^-$ such that $v_1 \leq v, v_2 \leq v$). So, in this case we can find $\{v_n\}_{n \geq 1} \subseteq S_\lambda^-$ increasing such that

$$\sup S_\lambda^- = \sup_{n \geq 1} v_n.$$

Reasoning as in part (a), we obtain

$$\bar{v}_\lambda \in S_\lambda^- \subseteq -\text{int } C_+ \quad \text{and} \quad \bar{v}_\lambda = \sup S_\lambda^-.$$

□

We examine the maps $\lambda \mapsto \bar{u}_\lambda$ from \mathcal{L}^+ into $C_+ \subseteq C_0^1(\bar{\Omega})$ and of $\lambda \mapsto \bar{v}_\lambda$ from \mathcal{L}^- into $-C_+ \subseteq C_0^1(\bar{\Omega})$.

Proposition 3.8. *If hypotheses $H(f)$ hold, then*

(a) *the map $\lambda \mapsto \bar{u}_\lambda$ from \mathcal{L}^+ into C_+ is*

- strictly increasing (that is, if $0 < \vartheta < \lambda \in \mathcal{L}^+$, then $\bar{u}_\lambda - \bar{u}_\vartheta \in \text{int } C_+$);
- left continuous;

(b) the map $\lambda \mapsto \bar{v}_\lambda$ from \mathcal{L}^- into $-C_+$ is

- strictly decreasing (that is, if $0 < \vartheta < \lambda \in \mathcal{L}^-$, then $\bar{u}_\vartheta - \bar{u}_\lambda \in \text{int } C_+$);
- left continuous.

Proof.

(a) From Proposition 3.3(a) we know that we can find $u_\vartheta \in S_\vartheta^+ \subseteq \text{int } C_+$ such that

$$\begin{aligned} \bar{u}_\lambda - u_\vartheta &\in \text{int } C_+, \\ \Rightarrow \bar{u}_\lambda - \bar{u}_\vartheta &\in \text{int } C_+. \end{aligned}$$

Also let $\{\lambda_n\}_{n \geq 1} \subseteq \mathcal{L}^+$ such that $\lambda_n \rightarrow (\lambda^+)^-$. We set $\bar{u}_n = \bar{u}_{\lambda_n} \in S_{\lambda_n}^+ \subseteq \text{int } C_+$ for all $n \in \mathbb{N}$. Then

$$\langle A_p(\bar{u}_n), h \rangle + \langle A(\bar{u}_n), h \rangle = \int_\Omega [\lambda_n(\bar{u}_n)^{p-1} + f(z, \bar{u}_n)] h \, dz \text{ for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}, \quad (3.46)$$

$$0 \leq \bar{u}_n \leq \bar{u}_{\lambda^+} \text{ for all } n \in \mathbb{N} \text{ (from the monotonicity of } \lambda \mapsto \bar{u}_\lambda). \quad (3.47)$$

Then (3.46) and (3.47) imply that

$$\{\bar{u}_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \quad (3.48)$$

From (3.48) and Corollary 8.6, p. 208, of Motreanu-Motreanu-Papageorgiou [17], we know that we can find $c_{17} > 0$ such that

$$\|\bar{u}_n\|_\infty \leq c_{17} \text{ for all } n \in \mathbb{N}. \quad (3.49)$$

Using (3.49) and Theorem 1 of Lieberman [14], we can find $\alpha \in (0, 1)$ and $c_{18} > 0$ such that

$$\bar{u}_n \in C_0^{1,\alpha}(\bar{\Omega}) \text{ and } \|\bar{u}_n\|_{C_0^{1,\alpha}(\bar{\Omega})} \leq c_{18} \text{ for all } n \in \mathbb{N}.$$

The compact embedding of $C_0^{1,\alpha}(\bar{\Omega})$ into $C_0^1(\bar{\Omega})$, implies that at least for a subsequence we have

$$\bar{u}_n \rightarrow \tilde{u}_{\lambda^+} \text{ in } C_0^1(\bar{\Omega}), \quad \tilde{u}_{\lambda^+} \in S_{\lambda^+}^+. \quad (3.50)$$

We claim that $\tilde{u}_{\lambda^+} = \bar{u}_{\lambda^+}$. Arguing by contradiction, suppose that $\tilde{u}_{\lambda^+} \neq \bar{u}_{\lambda^+}$. So, we can find $z_0 \in \Omega$ such that

$$\begin{aligned} \bar{u}_{\lambda^+}(z_0) &< \tilde{u}_{\lambda^+}(z_0) \\ \Rightarrow \bar{u}_{\lambda^+}(z_0) &< \bar{u}_n(z_0) = \bar{u}_{\lambda_n}(z_0) \text{ for all } n \geq n_0, \end{aligned}$$

which contradicts the strict monotonicity of $\lambda \mapsto \bar{u}_\lambda$. Hence $\tilde{u}_{\lambda^+} = \bar{u}_{\lambda^+}$ and for the original sequence we have

$$\begin{aligned} \bar{u}_n &\rightarrow \bar{u}_{\lambda^+} \text{ in } C_0^1(\bar{\Omega}) \text{ as } n \rightarrow \infty, \\ \Rightarrow \lambda &\mapsto \bar{u}_\lambda \text{ is left continuous.} \end{aligned}$$

(b) In this case Proposition 3.5(b) implies that $\lambda \mapsto \bar{v}_\lambda$ is strictly decreasing from \mathcal{L}^- into $C_0^1(\bar{\Omega})$. Also, reasoning as in part (a) and using the maximality of \bar{v}_λ , we establish the left continuity of $\lambda \mapsto \bar{v}_\lambda$ from \mathcal{L}^- into $-C_+$. □

So far we know that

$$\begin{aligned} (0, \lambda_*^+) &\subseteq \mathcal{L}^+ \subseteq (0, \lambda_*^+], \\ (0, \lambda_*^-) &\subseteq \mathcal{L}^- \subseteq (0, \lambda_*^-]. \end{aligned}$$

It is natural to ask whether the critical parameter values λ_*^+ and λ_*^- are admissible. In the next proposition we show that λ_*^+, λ_*^- are not admissible and so

$$\mathcal{L}^+ = (0, \lambda_*^+) \quad \text{and} \quad \mathcal{L}^- = (0, \lambda_*^-).$$

Proposition 3.9. *If hypotheses H(f) hold, then $\lambda_*^+ \notin \mathcal{L}^+$ and $\lambda_*^- \notin \mathcal{L}^-$.*

Proof. We do the proof for λ_*^+ , the proof for λ_*^- being similar.

We argue indirectly. So, suppose that $\lambda_*^+ \in \mathcal{L}^+$. From Proposition 3.7, we know that problem $(P_{\lambda_*^+})$ admits a minimal positive solution $\bar{u}_* = \bar{u}_{\lambda_*^+} \in \text{int } C_+$. Let $\vartheta < \lambda_*^+ < \lambda$. We know that $\bar{u}_* - \bar{u}_\vartheta \in \text{int } C_+$. So, we can define the following Caratheodory function:

$$\widehat{\beta}_\lambda(z, x) = \begin{cases} \lambda \bar{u}_\vartheta(z)^{p-1} + f(z, \bar{u}_\vartheta(z)) & \text{if } x < \bar{u}_\vartheta(z) \\ \lambda x^{p-1} + f(z, x) & \text{if } \bar{u}_\vartheta(z) \leq x \leq \bar{u}_*(z) \\ \lambda \bar{u}_*(z)^{p-1} + f(z, \bar{u}_*(z)) & \text{if } \bar{u}_*(z) < x \end{cases}$$

Let $\widehat{B}_\lambda(z, x) = \int_0^x \widehat{\beta}_\lambda(z, s) \, ds$ and consider the C^1 -functional $\widehat{y}_\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\widehat{y}_\lambda(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \int_\Omega \widehat{B}_\lambda(z, u) \, dz, \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Evidently $\widehat{y}_\lambda(\cdot)$ is coercive and sequentially lower semicontinuous. So, we can find $\widehat{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} \widehat{y}_\lambda(\widehat{u}_\lambda) &= \inf \left[\widehat{y}_\lambda(u) : u \in W_0^{1,p}(\Omega) \right], \\ \Rightarrow \widehat{y}'_\lambda(\widehat{u}_\lambda) &= 0, \\ \Rightarrow \langle A_p(\widehat{u}_\lambda), h \rangle + \langle A(\widehat{u}_\lambda), h \rangle &= \int_\Omega \widehat{\beta}_\lambda(z, \widehat{u}_\lambda) h \, dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \end{aligned}$$

First we choose $h = (\bar{u}_\vartheta - \widehat{u}_\lambda)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} &\langle A_p(\widehat{u}_\lambda), (\bar{u}_\vartheta - \widehat{u}_\lambda)^+ \rangle + \langle A(\widehat{u}_\lambda), (\bar{u}_\vartheta - \widehat{u}_\lambda)^+ \rangle \\ &= \int_\Omega \left[\lambda \bar{u}_\vartheta^{p-1} + f(z, \bar{u}_\vartheta) \right] (\bar{u}_\vartheta - \widehat{u}_\lambda)^+ \, dz \\ &\geq \int_\Omega \left[\vartheta \bar{u}_\vartheta^{p-1} + f(z, \bar{u}_\vartheta) \right] (\bar{u}_\vartheta - \widehat{u}_\lambda)^+ \, dz && \text{(since } \vartheta < \lambda) \\ &= \langle A_p(\bar{u}_\vartheta), (\bar{u}_\vartheta - \widehat{u}_\lambda)^+ \rangle + \langle A(\bar{u}_\vartheta), (\bar{u}_\vartheta - \widehat{u}_\lambda)^+ \rangle && \text{(since } \bar{u}_\vartheta \in S_\vartheta^+), \\ &\Rightarrow \bar{u}_\vartheta \leq \widehat{u}_\lambda. \end{aligned}$$

Similarly, choosing $h = (\widehat{u}_\lambda - \bar{u}_*)^+ \in W_0^{1,p}(\Omega)$, we obtain

$$\widehat{u}_\lambda \leq \bar{u}_*.$$

So, we have proved that

$$\begin{aligned} \widehat{u}_\lambda &\in [\bar{u}_\vartheta, \bar{u}_*], \\ \Rightarrow \lambda &\in \mathcal{L}^+, \text{ a contradiction since } \lambda > \lambda_*^+. \end{aligned}$$

This means that $\lambda_*^+ \notin \mathcal{L}^+$.

Similarly we show that $\lambda_*^- \notin \mathcal{L}^-$. □

Remark. It is worth pointing out that when we have a concave-convex problem (that is, when the parametric term in the reaction, is $\lambda u(z)^{q-1}$ with $1 < q < 2 < p$), then $\lambda_*^+ \in \mathcal{L}^+$ and $\lambda_*^- \in \mathcal{L}^-$ (see Papageorgiou-Rădulescu [24]).

So, we have

$$\mathcal{L}^+ = (0, \lambda_*^+) \quad \text{and} \quad \mathcal{L}^- = (0, \lambda_*^-).$$

Now we show that for all $\lambda \in \mathcal{L}^+$ (resp. all $\lambda \in \mathcal{L}^-$), we have at least two positive (resp. two negative) solutions.

Proposition 3.10. *If hypotheses H(f) hold, then*

- (a) for all $\lambda \in \mathcal{L}^+ = (0, \lambda_*^+)$ problem (P_λ) has at least two positive solutions $u_\lambda, \hat{u}_\lambda \in \text{int } C_+, u_\lambda \leq \hat{u}_\lambda, u_\lambda \neq \hat{u}_\lambda$;
- (b) for all $\lambda \in \mathcal{L}^- = (0, \lambda_*^-)$ problem (P_λ) has at least two negative solutions $v_\lambda, \hat{v}_\lambda \in \text{int } C_+, \hat{v}_\lambda \leq v_\lambda, v_\lambda \neq \hat{v}_\lambda$.

Proof.

- (a) Since $\lambda \in \mathcal{L}^+$, we can find $u_\lambda \in S_\lambda^+ \subseteq \text{int } C_+$. Using $u_\lambda \in \text{int } C_+$ to truncate the reaction of problem (P_λ) , we introduce the Caratheodory function $g_\lambda^+(z, x)$ defined by

$$g_\lambda^+(z, x) = \begin{cases} \lambda u_\lambda(z)^{p-1} + f(z, u_\lambda(z)) & \text{if } x \leq u_\lambda(z) \\ \lambda x^{p-1} + f(z, x) & \text{if } u_\lambda(z) < x \end{cases} \tag{3.51}$$

We set $G_\lambda^+(z, x) = \int_0^x g_\lambda^+(z, s) ds$ and consider the C^1 -functional $\hat{\varphi}_\lambda^+ : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\hat{\varphi}_\lambda^+(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \int_\Omega G_\lambda^+(z, u) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Let $\eta \in (\lambda, \lambda_*^+)$ and $u_\eta \in S_\eta \subseteq \text{int } C_+$ such that $u_\eta - u_\lambda \in \text{int } C_+$. Consider the Caratheodory function

$$\tilde{g}_\lambda^+(z, x) = \begin{cases} g_\lambda^+(z, x) & \text{if } x \leq u_\eta(z) \\ g_\lambda^+(z, u_\eta(z)) & \text{if } u_\eta(z) < x \end{cases} \tag{3.52}$$

We set $\tilde{G}_\lambda^+(z, x) = \int_0^x \tilde{g}_\lambda^+(z, s) ds$ and consider the C^1 -functional $\tilde{\varphi}_\lambda^+ : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\tilde{\varphi}_\lambda^+(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \int_\Omega \tilde{G}_\lambda^+(z, u) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

As before we can check that

$$K_{\hat{\varphi}_\lambda^+} \subseteq [u_\lambda] \cap \text{int } C_+ \quad \text{and} \quad K_{\tilde{\varphi}_\lambda^+} \subseteq [u_\lambda, u_\eta] \cap \text{int } C_+. \tag{3.53}$$

Moreover, since $\tilde{\varphi}_\lambda^+$ is coercive and sequentially weakly lower semicontinuous, we can find $\tilde{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} \tilde{\varphi}_\lambda^+(\tilde{u}_\lambda) &= \inf \left[\tilde{\varphi}_\lambda^+(u) : u \in W_0^{1,p}(\Omega) \right], \\ \Rightarrow \tilde{u}_\lambda &\in K_{\tilde{\varphi}_\lambda^+} \subseteq [u_\lambda, u_\eta] \cap \text{int } C_+ \quad \text{(see (3.53)).} \end{aligned} \tag{3.54}$$

We may assume that $\tilde{u}_\lambda = u_\lambda$ or otherwise we already have a second positive solution of (P_λ) (see (3.51), (3.52)). Note that

$$\tilde{\varphi}_\lambda^+|_{[0, u_\eta]} = \hat{\varphi}_\lambda^+|_{[0, u_\eta]} \quad \text{(see (3.51), (3.52)).} \tag{3.55}$$

Since $u_\eta - u_\lambda \in \text{int } C_+$ and $u_\lambda \in \text{int } C_+$, from (3.55) we infer that

$$\begin{aligned} & u_\lambda \text{ is a local } C_0^1(\bar{\Omega})\text{-minimizer of } \widehat{\varphi}_\lambda^+, \\ \Rightarrow & u_\lambda \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \widehat{\varphi}_\lambda^+ \text{ (see Proposition 2.1).} \end{aligned} \tag{3.56}$$

On account of (3.53) we may assume that

$$K_{\widehat{\varphi}_\lambda^+} \text{ is finite.} \tag{3.57}$$

Otherwise we already have an infinity of positive solutions of problem (P_λ) , all bigger than u_λ and so we are done. Therefore (3.56) and (3.57) imply that there exists $\rho \in (0, 1)$ small such that

$$\widehat{\varphi}_\lambda^+(u_\lambda) < \inf [\widehat{\varphi}_\lambda^+(u) : \|u - u_\lambda\| = \rho] = m_\lambda^+ \tag{3.58}$$

(see Aizicovici-Papageorgiou-Staicu [25], proof of Proposition 29).

Hypothesis $H(f)$ (ii) implies that if $u \in \text{int } C_+$, then

$$\widehat{\varphi}_\lambda^+(tu) \rightarrow -\infty \text{ as } t \rightarrow +\infty. \tag{3.59}$$

Finally as in the proof of Proposition 3.1 (see the Claim), we show that

$$\widehat{\varphi}_\lambda^+(\cdot) \text{ satisfies the C-condition.} \tag{3.60}$$

Then (3.58), (3.59), (3.60) permit the use of Theorem 1 (the mountain pass theorem). So, we can find $\widehat{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\widehat{u}_\lambda \in K_{\widehat{\varphi}_\lambda^+} \subseteq [u_\lambda] \cap \text{int } C_+ \text{ (see (3.53)) and } m_\lambda^+ \leq \widehat{\varphi}_\lambda^+(\widehat{u}_\lambda). \tag{3.61}$$

From (3.58) and (3.61) we conclude that

$$\widehat{u}_\lambda \in \text{int } C_+ \text{ is a solution of } (P_\lambda), \quad u_\lambda \leq \widehat{u}_\lambda, \quad u_\lambda \neq \widehat{u}_\lambda.$$

(b) In this case, let $v_\lambda \in S_\lambda \subseteq -\text{int } C_+$ and consider the Caratheodory function $g_\lambda^-(z, x)$ defined by

$$g_\lambda^-(z, x) = \begin{cases} \lambda|x|^{p-2}x + f(z, x) & \text{if } x \leq v_\lambda(z) \\ \lambda|v_\lambda(z)|^{p-2}v_\lambda(z) + f(z, v_\lambda(z)) & \text{if } x > v_\lambda(z) \end{cases}. \tag{3.62}$$

We set $G_\lambda^-(z, x) = \int_0^x g_\lambda^-(z, s) ds$ and consider the C^1 -functional $\widehat{\varphi}_\lambda^- : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}_\lambda^-(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \int_\Omega G_\lambda^-(z, u) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Working as in part (a) this time using (3.62) and the functional $\widehat{\varphi}_\lambda^-$, we produce a second positive solution $\widehat{v}_\lambda \in -\text{int } C_+$ such that $\widehat{v}_\lambda \leq v_\lambda, v_\lambda \neq \widehat{v}_\lambda$. □

So, summarizing the situation concerning the solutions of constant sign for problem (P_λ) , we can state the following theorem.

Theorem 3.1. *If hypotheses $H(f)$ hold, then*

- (a) *there exists $\lambda_*^+ \in (0, +\infty)$ such that*
 - *for all $\lambda > \lambda_*^+$ problem (P_λ) has no positive solutions;*
 - *for all $\lambda \in (0, \lambda_*^+)$ problem (P_λ) has at least two positive solutions $u_\lambda, \widehat{u}_\lambda \in \text{int } C_+, u_\lambda \leq \widehat{u}_\lambda, u_\lambda \neq \widehat{u}_\lambda$;*

- for all $\lambda \in (0, \lambda^+)$ problem (P_λ) has a smallest positive solution $\bar{u}_\lambda \in \text{int } C_+$ and the map $\lambda \mapsto \bar{u}_\lambda$ from $\mathcal{L}^+ = (0, \lambda^+)$ into C_+ is strictly increasing and left continuous;

(b) there exists $\lambda^- \in (0, +\infty)$ such that

- for all $\lambda > \lambda^-$ problem (P_λ) has no negative solutions;
- for all $\lambda \in (0, \lambda^-)$ problem (P_λ) has at least two negative solutions $v_\lambda, \hat{v} \in -\text{int } C_+, \hat{v}_\lambda \leq v_\lambda, v_\lambda \neq \hat{v}_\lambda$;
- for all $\lambda \in (0, \lambda^-)$ problem (P_λ) has a biggest negative solution $\bar{v}_\lambda \in -\text{int } C_+$ and the map $\lambda \mapsto \bar{v}_\lambda$ from $\mathcal{L}^- = (0, \lambda^-)$ into $-C_+$ is strictly decreasing and left continuous.

4 Nodal solutions

In this section we look for nodal (that is, sign changing) solutions for problem (P_λ) .

To this end, we need to strengthen the conditions on the perturbation $f(z, \cdot)$. The new hypotheses on $f(z, x)$ are the following:

$\underline{H(f)'}: f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Caratheodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega, f(z, \cdot) \in C^1(\mathbb{R})$ and

- (i) $|f'_x(z, x)| \leq a(z)(1 + |x|^{r-2})$ for a.a. $z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ with } a \in L^\infty(\Omega), p < r < p^*$.
- (ii) If $F(z, x) = \int_0^x f(z, s) ds$, then $\lim_{x \rightarrow \pm\infty} \frac{F(z, x)}{|x|^p} = +\infty$ uniformly for a.a. $z \in \Omega$;
- (iii) there exist $\hat{\eta} > 0$ and $q \in \left((r-p) \max \left\{ \frac{N}{p}, 1 \right\}, p^* \right)$ such that

$$0 < \hat{\eta} \leq \liminf_{x \rightarrow \pm\infty} \frac{f(z, x)x - pF(z, x)}{|x|^q} \quad \text{uniformly for a.a. } z \in \Omega;$$

- (iv) there exist $m \in \mathbb{N}, m \geq 2$, such that

$$\begin{aligned} \hat{\lambda}_m(2) &\leq f'_x(z, 0) = \lim_{x \rightarrow 0} \frac{f(z, x)}{x} \leq \hat{\lambda}_{m+1}(2) \quad \text{uniformly for a.a. } z \in \Omega, \\ f'_x(\cdot, 0) &\neq \hat{\lambda}_m(2), \quad f'_x(\cdot, 0) \neq \hat{\lambda}_{m+1}(2). \end{aligned}$$

Remark. Note that in this case hypothesis $\underline{H(f)(v)}$ is automatically satisfied.

Let $\lambda_* = \min\{\lambda^+, \lambda^-\} > 0$. Also, for $\lambda > 0$, let $\varphi_\lambda \in W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ be the energy (Euler) functional for problem (P_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \frac{\lambda}{p} \|u\|_p^p - \int_\Omega F(z, u) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We know that $\varphi_\lambda \in C^2(W_0^{1,p}(\Omega), \mathbb{R})$ for all $\lambda > 0$.

Lemma 4.1. If hypotheses $\underline{H(f)}$ hold and $\lambda > 0$, then $C_k(\varphi_\lambda, 0) = \delta_{k, d_m} \mathbb{Z}$ for all $k \in \mathbb{N}_0$ with $d_m = \bigoplus_{k=1}^m E(\hat{\lambda}_k(2))$.

Proof. Let $\hat{\zeta}_\lambda : H_0^1(\Omega) \rightarrow \mathbb{R}$ be the C^2 -functional defined by

$$\hat{\zeta}_\lambda(u) = \frac{1}{2} \|Du\|_2^2 - \frac{\lambda}{p} \|u\|_p^p - \int_\Omega F(z, u) dz \quad \text{for all } u \in H_0^1(\Omega)$$

We consider the following orthogonal direct sum decomposition of the space $H_0^1(\Omega)$:

$$H_0^1(\Omega) = \overline{H}_m \oplus \widehat{H}_{m+1}, \tag{4.1}$$

with

$$\overline{H}_m = \bigoplus_{k=1}^m E(\widehat{\lambda}_k(2)) \quad \text{and} \quad \widehat{H}_{m+1} = \overline{\bigoplus_{k \geq m+1} E(\widehat{\lambda}_k(2))}.$$

Hypothesis H(f)(iv) implies that given $\epsilon > 0$, we can find $\delta > 0$ such that

$$\frac{1}{2}[\vartheta(z) - \epsilon]x^2 \leq F(z, x) \leq \frac{1}{2}[\widehat{\vartheta}(z) + \epsilon]x^2 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta. \tag{4.2}$$

The subspace \overline{H}_m is finite dimensional. So, all norms on \overline{H}_m are equivalent. Therefore, we can find $\rho_1 \in (0, 1)$ small such that

$$u \in \overline{H}_m, \|u\|_{H_0^1(\Omega)} \leq \rho_1 \quad \Rightarrow \quad |u(z)| \leq \delta \text{ for all } z \in \overline{\Omega} \text{ (see (4.2)).} \tag{4.3}$$

Therefore for $u \in \overline{H}_m$ with $\|u\|_{H_0^1(\Omega)} \leq \rho_1$, we have

$$\begin{aligned} \widehat{\zeta}_\lambda(u) &\leq \frac{1}{2}\|Du\|_2^2 - \frac{1}{2} \int_{\Omega} \vartheta(z)u^2 \, dz + \frac{\epsilon}{2}\|u\|_{H_0^1(\Omega)}^2 && \text{(see (4.3))} \\ &\leq \frac{1}{2}[-c_2 + \epsilon]\|u\|_{H_0^1(\Omega)}^2 && \text{(see Proposition 2.4(b)).} \end{aligned}$$

Choosing $\epsilon \in (0, c_2)$, we obtain

$$\widehat{\zeta}_\lambda(u) \leq 0 \quad \text{for all } u \in \overline{H}_m \text{ with } \|u\|_{H_0^1(\Omega)} \leq \rho_1. \tag{4.4}$$

On the other hand from (4.2) and hypothesis H(f)(i), we have

$$F(z, x) \leq \frac{1}{2}[\widehat{\vartheta}(z) + \epsilon]x^2 + c_{19}|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R} \tag{4.5}$$

with $c_{19} > 0$. For $u \in \widehat{H}_{m+1}$ we have

$$\begin{aligned} \widehat{\zeta}_\lambda(u) &\geq \frac{1}{2}\|Du\|_2^2 - \frac{\lambda}{p}\|u\|_p^p - \frac{1}{2} \int_{\Omega} \widehat{\vartheta}(z)u^2 \, dz - \frac{\epsilon}{2}\|u\|_{H_0^1(\Omega)}^2 - c_{19}\|u\|_r^r && \text{(see (4.5))} \\ &\geq \frac{1}{2}[c_1 - \epsilon]\|u\|_{H_0^1(\Omega)}^2 - c_{20} \left[\lambda\|u\|_{H_0^1(\Omega)}^p + \|u\|_{H_0^1(\Omega)}^r \right] && \text{for some } c_{20} > 0. \end{aligned}$$

Choosing $\epsilon \in (0, c_1)$ and assuming that $\|u\|_{H_0^1(\Omega)} \leq 1$, we have

$$\widehat{\zeta}_\lambda(u) \geq c_{21}\|u\|_{H_0^1(\Omega)}^2 - c_{22}\|u\|_{H_0^1(\Omega)}^p \quad \text{for all } u \in H_0^1(\Omega) \text{ and with } c_{21} > 0, c_{22} = c_{22}(\lambda) > 0.$$

Since $p > 2$, we can find $\rho_2 \in (0, 1)$ small such that

$$\widehat{\zeta}_\lambda(u) > 0 \quad \text{for all } u \in \widehat{H}_{m+1}, 0 < \|u\|_{H_0^1(\Omega)} \leq \rho_2. \tag{4.6}$$

Let $\rho = \min\{\rho_1, \rho_2\} > 0$. From (4.4) and (4.6) it follows that $\widehat{\zeta}_\lambda(\cdot)$ has a local linking at the origin with respect to the decomposition (4.1). Since $\widehat{\zeta}_\lambda \in C^2(H_0^1(\Omega), \mathbb{R})$, we can apply Proposition 2.3 of Su [26] and infer that

$$C_k(\widehat{\zeta}_\lambda, 0) = \delta_{k, d_m} \mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{4.7}$$

Let $\zeta_\lambda = \widehat{\zeta}_\lambda|_{W_0^{1,p}(\Omega)}$. Since $W_0^{1,p}(\Omega)$ is dense in $H_0^1(\Omega)$, from (4.7) we have

$$\begin{aligned} C_k(\zeta_\lambda, 0) &= C_k(\widehat{\zeta}_\lambda, 0) \quad \text{for all } k \in \mathbb{N}_0 \text{ (see [10]),} \\ \Rightarrow C_k(\zeta_\lambda, 0) &= \delta_{k, d_m} \mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \text{ (see (4.7)).} \end{aligned} \tag{4.8}$$

Note that

$$|\varphi_\lambda(u) - \zeta_\lambda(u)| = \frac{1}{p} \|u\|^p \tag{4.9}$$

and

$$\begin{aligned} |\langle \varphi'_\lambda(u) - \zeta'_\lambda(u), h \rangle| &= |\langle A_p(u), h \rangle| \leq \|Du\|_p^{p-1} \|h\| \\ \Rightarrow \|\varphi'_\lambda(u) - \zeta'_\lambda(u)\|_* &\leq \|u\|^{p-1}. \end{aligned} \tag{4.10}$$

From (4.9), (4.10) and the C^1 -continuity of critical groups (see Gasiński-Papageorgiou [27], Theorem 5.126, p. 836), we have

$$\begin{aligned} C_k(\zeta_\lambda, 0) &= C_k(\varphi_\lambda, 0) \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow C_k(\varphi_\lambda, 0) &= \delta_{k,d_m} \mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \text{ (see (4.8)).} \end{aligned}$$

□

We can use this lemma to produce multiple nodal solutions.

Proposition 4.1. *If hypotheses $H(f)'$ hold and $\lambda \in (0, \lambda^*)$, then problem (P_λ) admits at least three nodal solutions*

$$y_0, \hat{y}, \tilde{y} \in C_0^1(\bar{\Omega}).$$

Proof. According to Proposition 3.7, we have two extremal constant sign solutions

$$\bar{u}_\lambda \in \text{int } C_+ \quad \text{and} \quad \bar{v}_\lambda \in -\text{int } C_+.$$

We consider the Caratheodory function $w_\lambda(z, x)$ defined by

$$w_\lambda(z, x) = \begin{cases} \lambda |\bar{v}_\lambda(z)|^{p-2} \bar{v}_\lambda(z) + f(z, \bar{v}_\lambda(z)) & \text{if } x < \bar{v}_\lambda(z) \\ \lambda |x|^{p-2} x + f(z, x) & \text{if } \bar{v}_\lambda(z) \leq x \leq \bar{u}_\lambda(z) \\ \lambda \bar{u}_\lambda(z)^{p-1} + f(z, \bar{u}_\lambda(z)) & \text{if } \bar{u}_\lambda(z) < x \end{cases} \tag{4.11}$$

We set $W_\lambda(z, x) = \int_0^x w_\lambda(z, s) ds$ and consider the C^1 -functional $\hat{\tau}_\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\hat{\tau}_\lambda(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \int_\Omega W_\lambda(z, u) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Also, let $\hat{\tau}_\lambda^\pm$ be the positive and negative truncations of $\hat{\tau}_\lambda$, that is,

$$\hat{\tau}_\lambda^\pm(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{2} \|Du\|_2^2 - \int_\Omega W_\lambda(z, \pm u^\pm) dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

As before, using (4.11) we can show that

$$K_{\hat{\tau}_\lambda} \subseteq [\bar{v}_\lambda, \bar{u}_\lambda] \cap C_0^1(\bar{\Omega}), \quad K_{\hat{\tau}_\lambda^+} \subseteq [0, \bar{u}_\lambda] \cap C_+, \quad K_{\hat{\tau}_\lambda^-} \subseteq [\bar{v}_\lambda, 0] \cap (-C_+).$$

The extremality of \bar{u}_λ and \bar{v}_λ implies that

$$K_{\hat{\tau}_\lambda} \subseteq [\bar{v}_\lambda, \bar{u}_\lambda] \cap C_0^1(\bar{\Omega}), \quad K_{\hat{\tau}_\lambda^+} = \{0, \bar{u}_\lambda\}, \quad K_{\hat{\tau}_\lambda^-} \subseteq \{\bar{v}_\lambda, 0\}. \tag{4.12}$$

On account of (4.12) we see that we may assume that

$$K_{\hat{\tau}_\lambda} \text{ is finite.} \tag{4.13}$$

Otherwise from (4.11) and the extremality of \bar{u}_λ and \bar{v}_λ , we see that we already have an infinity of smooth nodal solutions.

Claim. $\bar{u}_\lambda \in \text{int } C_+$ and $\bar{v}_\lambda \in -\text{int } C_+$ are local minimizers of $\widehat{\tau}_\lambda$.

Evidently $\widehat{\tau}_\lambda^+$ is coercive (see (4.11)) and sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\tau}_\lambda^+(\tilde{u}_\lambda) = \inf \left[\widehat{\tau}_\lambda^+ : u \in W_0^{1,p}(\Omega) \right]. \tag{4.14}$$

As in the proof of Proposition 3.2, exploiting hypothesis H(f)(iv) we see that

$$\begin{aligned} \widehat{\tau}_\lambda^+(\tilde{u}_\lambda) < 0 &= \widehat{\tau}_\lambda^+(0), \\ \Rightarrow \tilde{u}_\lambda &\neq 0. \end{aligned} \tag{4.15}$$

From (4.14) we have

$$\begin{aligned} \tilde{u}_\lambda &\in K_{\widehat{\tau}_\lambda^+} = \{0, \bar{u}_\lambda\} \quad (\text{see (4.12)}) \\ \Rightarrow \tilde{u}_\lambda &= \bar{u}_\lambda \in \text{int } C_+ \quad (\text{see (4.15)}) \end{aligned}$$

Note that

$$\widehat{\tau}_\lambda^+|_{C_+} = \widehat{\tau}_\lambda|_{C_+}.$$

So, it follows that

$$\begin{aligned} \bar{u}_\lambda \in \text{int } C_+ &\text{ is a local } C_0^1(\bar{\Omega})\text{-minimizer of } \widehat{\tau}_\lambda, \\ \Rightarrow \bar{u}_\lambda \in \text{int } C_+ &\text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \widehat{\tau}_\lambda \text{ (see Proposition 2.1).} \end{aligned}$$

Similarly for $\bar{v}_\lambda \in -\text{int } C_+$, using this time the functional $\widehat{\tau}_\lambda^-$.

This proves the Claim.

Without any loss of generality, we assume that

$$\widehat{\tau}_\lambda(\bar{v}_\lambda) \leq \widehat{\tau}_\lambda(\bar{u}_\lambda).$$

The reasoning is similar if the opposite inequality holds. From (4.13) and the Claim it follows that there exists $\rho \in (0, 1)$ small such that

$$\widehat{\tau}_\lambda(\bar{v}_\lambda) \leq \widehat{\tau}_\lambda(\bar{u}_\lambda) < \inf \left[\widehat{\tau}_\lambda(u) : \|u - \bar{u}_\lambda\| = \rho \right] = \widehat{m}_\lambda, \quad \|\bar{v}_\lambda - \bar{u}_\lambda\| > \rho. \tag{4.16}$$

The functional $\widehat{\tau}_\lambda$ is coercive, hence

$$\widehat{\tau}_\lambda \text{ satisfies the C-condition.} \tag{4.17}$$

Then (4.16) and (4.17) permit the use of Theorem 2.1 (the mountain pass theorem). So, there exists $y_0 \in W_0^{1,p}(\Omega)$ such that

$$y_0 \in K_{\widehat{\tau}_\lambda} \subseteq [\bar{v}_\lambda, \bar{u}_\lambda] \cap C_0^1(\bar{\Omega}) \text{ (see (4.12)), } \widehat{m}_\lambda \leq \widehat{\tau}_\lambda(y_0) \text{ (see (4.16)).} \tag{4.18}$$

From (4.16) and (4.18) we see that

$$y_0 \notin \{\bar{u}_\lambda, \bar{v}_\lambda\}. \tag{4.19}$$

We consider the homotopy

$$\widehat{h}(t, u) = (1 - t)\widehat{\tau}_\lambda(u) + t\varphi_\lambda(u) \quad \text{for all } (t, u) \in [0, 1] \times W_0^{1,p}(\Omega).$$

Suppose we could find $\{t_n\}_{n \geq 1} \subseteq [0, 1]$ and $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$t_n \rightarrow t \text{ in } [0, 1], \quad u_n \rightarrow 0 \text{ in } W_0^{1,p}(\Omega), \quad \widehat{h}'_u(t_n, u_n) = 0 \text{ for all } n \in \mathbb{N}. \tag{4.20}$$

From the equality in (4.20), we have

$$\begin{aligned} \langle A_p(u_n), h \rangle + \langle A(u_n), h \rangle &= (1 - t_n) \int_\Omega w_\lambda(t, u_n) h \, dz + t_n \int_\Omega \lambda |u_n|^{p-2} u_n h \, dz + t_n \int_\Omega f(z, u_n) h \, dz \\ &\text{for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \end{aligned} \tag{4.21}$$

In (4.21) we choose $h = u_n \in W_0^{1,p}(\Omega)$ and we infer that

$$\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

Invoking Corollary 6.8, p. 208, of Motreanu-Motreanu-Papageorgiou [17], we see that we can find $\alpha \in (0, 1)$ and $c_{23} > 0$ such that

$$u_n \in C_0^{1,\alpha}(\bar{\Omega}) \quad \text{and} \quad \|u_n\|_{C_0^{1,\alpha}(\bar{\Omega})} \leq c_{23} \quad \text{for all } n \in \mathbb{N}. \tag{4.22}$$

From (4.20) and the compact embedding of $C_0^{1,\alpha}(\bar{\Omega})$ into $C_0^1(\bar{\Omega})$, we have

$$\begin{aligned} &u_n \rightarrow 0 \text{ in } C_0^1(\bar{\Omega}), \\ \Rightarrow &u_n \in [\bar{v}_\lambda, \bar{u}_\lambda] \quad \text{for all } n \geq n_0, \\ \Rightarrow &\{u_n\}_{n \geq n_0} \subseteq K_{\bar{\tau}_\lambda} \quad \text{(see (4.11)).} \end{aligned}$$

This contradicts (4.13). Therefore (4.20) can not occur and so from the homotopy invariance of critical groups (see Gasiński-Papageorgiou [27], Theorem 5.125, p. 836), we have that

$$\begin{aligned} &C_k(\hat{\tau}_\lambda, 0) = C_k(\varphi_\lambda, 0) \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow &C_k(\hat{\tau}_\lambda, 0) = \delta_{k,d_m} \mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \end{aligned} \tag{4.23}$$

Recall that y_0 is a critical point of $\hat{\tau}_\lambda$ of mountain pass type. Therefore

$$C_1(\hat{\tau}_\lambda, y_0) \neq 0 \tag{4.24}$$

(see Motreanu-Motreanu-Papageorgiou [17], Proposition 6.100, p. 176).

Comparing (4.23) and (4.24), we infer that

$$y_0 \notin \{0, \bar{u}_\lambda, \bar{v}_\lambda\} \quad \text{(see (3.61)).}$$

Then (4.18), (4.11) and the extremality of \bar{u}_λ and \bar{v}_λ , imply that $y_0 \in C_0^1(\bar{\Omega})$ is a nodal solution of (P_λ) .

Let $a : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be defined by

$$a(y) = |y|^{p-2}y + y \quad \text{for all } y \in \mathbb{R}^N.$$

Note that $a \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ (recall that $p > 2$) and

$$\operatorname{div} a(Du) = \Delta_p u + \Delta u \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We have

$$\begin{aligned} \nabla a(y) &= |y|^{p-2} \left[I + \frac{y \otimes y}{|y|^2} \right] + I \quad \text{for all } y \in \mathbb{R}^n \\ \Rightarrow \quad (\nabla a(y)\xi, \xi)_{\mathbb{R}^N} &\geq |\xi|^2 \quad \text{for all } y, \xi \in \mathbb{R}^N. \end{aligned}$$

So, applying the tangency principle of Pucci-Serrin [19] (Theorem 2.5.2, p. 35), we obtain

$$\bar{v}_\lambda(z) < y_0(z) < \bar{u}_\lambda(z) \quad \text{for all } z \in \Omega. \tag{4.25}$$

Let $\rho = \max \{ \|\bar{u}_\lambda\|_\infty, \|\bar{v}_\lambda\|_\infty \}$. The differentiability of $f(z, \cdot)$ and hypothesis $H(f)'(\mathbf{i})$ imply that we can find $\widehat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function

$$x \mapsto f(z, x) + \widehat{\xi}_\rho |x|^{p-2}x$$

is nondecreasing on $[-\rho, \rho]$. Then we have

$$\begin{aligned} &-\Delta_p y_0(z) - \Delta y_0(z) + \widehat{\xi}_\rho |y_0(z)|^{p-2}y_0(z) \\ &= \lambda |y_0(z)|^{p-2}y_0(z) + f(z, y_0(z)) + \widehat{\xi}_\rho |y_0(z)|^{p-2}y_0(z) \\ &\leq \lambda \bar{u}_\lambda(z)^{p-1} + f(z, \bar{u}_\lambda(z)) + \widehat{\xi}_\rho \bar{u}_\lambda(z)^{p-1} \\ &= -\Delta_p \bar{u}_\lambda(z) - \Delta \bar{u}_\lambda(z) + \widehat{\xi}_\rho \bar{u}_\lambda(z)^{p-1} \quad \text{for a.a. } z \in \Omega. \end{aligned} \tag{4.26}$$

We set

$$\begin{aligned} h_1(z) &= \lambda |y_0(z)|^{p-2} y_0(z) + f(z, y_0(z)) + \widehat{\xi}_\rho |y_0(z)|^{p-2} y_0(z), \\ h_2(z) &= \lambda \bar{u}_\lambda(z)^{p-1} + f(z, \bar{u}_\lambda(z)) + \widehat{\xi}_\rho \bar{u}_\lambda(z)^{p-1}. \end{aligned}$$

Evidently $h_1, h_2 \in L^\infty(\Omega)$ and we have

$$\begin{aligned} \lambda \left[\bar{u}_\lambda(z)^{p-1} - |y_0(z)|^{p-2} y_0(z) \right] &\leq h_2(z) - h_1(z) \quad \text{for a.a. } z \in \Omega, \\ \Rightarrow h_1 &\prec h_2 \quad (\text{see (4.25)}). \end{aligned}$$

Then from (4.26) and invoking Proposition 2.2, we infer that

$$\bar{u}_\lambda - y_0 \in \text{int } C_+.$$

In a similar fashion, we show that

$$\begin{aligned} y_0 - \bar{v}_\lambda &\in \text{int } C_+, \\ \Rightarrow y_0 &\in \text{int}_{C_0^1(\bar{\Omega})} [\bar{v}_\lambda, \bar{u}_\lambda]. \end{aligned} \tag{4.27}$$

Consider the homotopy

$$\tilde{h}(t, u) = (1 - t)\widehat{\tau}_\lambda(u) + t\varphi_\lambda(u) \quad \text{for all } (t, u) \in [0, 1] \times W_0^{1,p}(\Omega).$$

Suppose we could find $\{t_n\}_{n \geq 1} \subseteq [0, 1]$ and $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$t_n \rightarrow t \text{ in } [0, 1], \quad u_n \rightarrow y_0 \text{ in } W_0^{1,p}(\Omega), \quad \tilde{h}'_u(t_n, u_n) = 0 \text{ for all } n \in \mathbb{N}. \tag{4.28}$$

Then reasoning as before, via the nonlinear regularity theory, we obtain

$$\begin{aligned} u_n &\rightarrow y_0 \text{ in } C_0^1(\bar{\Omega}) \text{ as } n \rightarrow \infty, \\ \Rightarrow u_n &\in [\bar{v}_\lambda, \bar{u}_\lambda] \text{ for all } n \geq n_0 \text{ (see (4.27))} \\ \Rightarrow \{u_n\}_{n \geq n_0} &\subseteq K_{\widehat{\tau}_\lambda} \text{ (see (4.11)),} \end{aligned}$$

which contradicts (4.13). So, (4.28) can not be true and we have

$$C_k(\widehat{\tau}_\lambda, y_0) = C_k(\varphi_\lambda, y_0) \quad \text{for all } k \in \mathbb{N}_0, \tag{4.29}$$

$$\Rightarrow C_1(\varphi_\lambda, y_0) \neq 0 \quad (\text{see (4.24)}). \tag{4.30}$$

But $\varphi_\lambda \in C^2(W_0^{1,p}(\Omega), \mathbb{R})$. So, from (4.29) and Proposition 3.5, Claim 3, in Papageorgiou-Rădulescu [9], we have

$$C_k(\varphi_\lambda, y_0) = \delta_{k,1} \mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0, \tag{4.31}$$

$$\Rightarrow C_k(\widehat{\tau}_\lambda, y_0) = \delta_{k,1} \mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0 \text{ (see (4.29))}. \tag{4.32}$$

From the Claim in the beginning of the proof, we know that \bar{u}_λ and \bar{v}_λ are local minimizers of $\widehat{\tau}_\lambda$. Hence

$$C_k(\widehat{\tau}_\lambda, \bar{u}_\lambda) = C_k(\widehat{\tau}_\lambda, \bar{v}_\lambda) = \delta_{k,0} \mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{4.33}$$

From (4.23) we have

$$C_k(\widehat{\tau}_\lambda, 0) = \delta_{k,d_m} \mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{4.34}$$

We know that $\widehat{\tau}_\lambda$ is coercive (see (4.11)). Therefore

$$C_k(\widehat{\tau}_\lambda, \infty) = \delta_{k,0} \mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{4.35}$$

Suppose that $K_{\widehat{\tau}_\lambda} = \{0, \bar{u}_\lambda, \bar{v}_\lambda, y_0\}$. Then using (4.34), (4.33), (4.31), (4.35) and the Morse relation with $t = -1$ (see (2.5)), we obtain

$$\begin{aligned} & (-1)^{d_m} + 2(-1)^0 + (-1)^1 = (-1)^0, \\ \Rightarrow & (-1)^{d_m} = 0, \quad \text{a contradiction.} \end{aligned}$$

So, there exists $\widehat{y} \in K_{\widehat{\tau}_\lambda}$, $\widehat{y} \notin \{0, \bar{u}_\lambda, \bar{v}_\lambda, y_0\}$. From (4.12) it follows that $\widehat{y} \in C_0^1(\overline{\Omega})$ is nodal. Moreover, as for y_0 , using Proposition 2.2, we show that

$$\widehat{y} \in \text{int}_{C_0^1(\overline{\Omega})} [\bar{v}_\lambda, \bar{u}_\lambda]. \quad (4.36)$$

Finally, from Proposition 10 of He-Guo-Huang-Lei [8], we know that (P_λ) has a nodal solution $\widetilde{y} \in C_0^1(\overline{\Omega})$ such that

$$\begin{aligned} & \widetilde{y} \notin \text{int}_{C_0^1(\overline{\Omega})} [\bar{v}_\lambda, \bar{u}_\lambda], \\ \Rightarrow & \widetilde{y} \in C_0^1(\overline{\Omega}) \quad \text{is the third nodal solution of } (P_\lambda). \end{aligned}$$

□

So, we can state the following multiplicity theorem for problem (P_λ) .

Theorem 4.2. *If hypotheses $H(f)'$ hold, then there exists $\lambda_* > 0$ such that for all $\lambda \in (0, \lambda_*)$ problem (P_λ) has at least seven nontrivial solutions*

$$\begin{aligned} & u_\lambda, \widehat{u}_\lambda \in \text{int } C_+, \quad u_\lambda \leq \widehat{u}_\lambda, \quad u_\lambda \neq \widehat{u}_\lambda, \\ & v_\lambda, \widehat{v}_\lambda \in -\text{int } C_+, \quad \widehat{v}_\lambda \leq v_\lambda, \quad v_\lambda \neq \widehat{v}_\lambda, \\ & y_0, \widehat{y}, \widetilde{y} \in C_0^1(\overline{\Omega}) \quad \text{nodal with } y_0, \widehat{y} \in \text{int}_{C_0^1(\overline{\Omega})} [\bar{v}_\lambda, \bar{u}_\lambda]. \end{aligned}$$

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