Adv. Nonlinear Anal. 2021; 10: 400–419 DE GRUYTER

Research article

Sihua Liang, Patrizia Pucci, and Binlin Zhang*

Multiple solutions for critical Choquard-Kirchhoff type equations

https://doi.org/10.1515/anona-2020-0119
Received April 27, 2020; accepted June 26, 2020.

Abstract: In this article, we investigate multiplicity results for Choquard-Kirchhoff type equations, with Hardy-Littlewood-Sobolev critical exponents,

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u=\alpha k(x)|u|^{q-2}u+\beta\left(\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}dy\right)|u|^{2^*_\mu-2}u,\quad x\in\mathbb{R}^N,$$

where $a>0$, $b\geq 0$, $0<\mu<N$, $N\geq 3$, α and β are positive real parameters, $2^*_\mu=(2N-\mu)/(N-2)$ is the critical exponent in the sense of Hardy-Littlewood-Sobolev inequality, $k\in L^r(\mathbb{R}^N)$, with $r=2^*/(2^*-q)$ if $1<q<2^*$ and $r=\infty$ if $q\geq 2^*$. According to the different range of q, we discuss the multiplicity of solutions to the above equation, using variational methods under suitable conditions. In order to overcome the lack of compactness, we appeal to the concentration compactness principle in the Choquard-type setting.

Keywords: Kirchhoff equation; Hardy-Littlewood-Sobolev critical exponent; Choquard nonlinearity; Concentraction compactness principle

MSC: 35A15, 35J60, 35J20, 35B33

1 Introduction and main results

In this paper, we consider the following Kirchhoff-type equation with Hardy-Littlewood-Sobolev critical nonlinearity in $\mathbb{R}^N$:

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u=\alpha k(x)|u|^{q-2}u+\beta\left(\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}dy\right)|u|^{2^*_\mu-2}u,\tag{1.1}$$

where $a>0$, $b\geq 0$, $0<\mu<N$, $N\geq 3$, α and β are positive real parameters, $2^*_\mu=(2N-\mu)/(N-2)$ is the critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality, $k\in L^r(\mathbb{R}^N)$, with $r=2^*/(2^*-q)$ if $1<q<2^*$ and $r=\infty$ if $q\geq 2^*$.

The paper was motivated by some works appeared in recent years. On one hand, the following Choquard or nonlinear Schrödinger-Newton equation

$$-\Delta u+V(x)u=(\mathcal{K}_\mu\star u^2)u+\lambda f(x,u)\quad\text{in }\mathbb{R}^N,\tag{1.2}$$

Sihua Liang, College of Mathematics, Changchun Normal University, Changchun, 130032, P.R. China; College of Mathematics and Informatics, Fujian Normal University, Qishan Campus, Fuzhou 350108, P.R. China,
E-mail: liangsihua@163.com
Patrizia Pucci, Dipartimento di Matematica e Informatica, Università degli Studi di Perugia, via Vanvitelli 1, 06123 Perugia, Italy,
E-mail: patrizia.pucci@unipg.it
***Corresponding Author: Binlin Zhang,** College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, 266590, P.R. China, E-mail: zhangbinlin2012@163.com

was studied by Pekar [41] in the framework of quantum mechanics. Subsequently, it was adopted as an approximation of the Hartree-Fock theory in [27]. Recently, Penrose [38] settled it as a model of the self-gravitational collapse of a quantum mechanical wave function. The first existence and symmetry results of solutions to (1.2) go back to the works of Lieb [27] and Lions [30]. Equations of type (1.2) have been extensively studied, see e.g. [3, 15, 16, 18, 20, 27, 34–36, 43] for the study of Choquard-type equations. In the fractional Laplacian framework, we refer to the recent papers [32, 40, 45].

On the other hand, existence of solutions for Kirchhoff-type problems involving the critical Sobolev exponent has been considered by many authors. In [10], Chen, Kuo and Wu studied the following Kirchhoff-type problem

$$-M(\|\nabla u\|_{L^2}^2)\Delta u = \lambda f(x)|u|^{q-2}u + g(x)|u|^{p-2}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $M(t) = a + bt$, $a, b > 0$ and f and g are continuous real valued sign changing functions. In [10] the authors prove existence and multiplicity of solutions by using the classical Nehari manifold method. The literature on Kirchhoff-type problems and related elliptic problems is very interesting and quite large, here we just list a few, for example, see [2, 12, 13, 24–26, 33, 37, 39, 47, 48] for the recent existence results.

Motivated by the above works, especially by the ideas of [11, 19, 21], in this paper we study the multiplicity of solutions for the Kirchhoff-type equations (1.1), with Hardy-Littlewood-Sobolev critical nonlinearities. There is no doubt that we encounter serious difficulties because of the lack of compactness. To overcome the challenge we use the second concentration compactness principle and the concentration compactness principle at infinity in order to prove the $(PS)_c$ condition at special levels c.

The equation (1.1) is variational, so that the (weak) solutions of (1.1) are just the critical points of the underlying functional $J_{\alpha,\beta}$ in $D^{1,2}(\mathbb{R}^N)$. The first two multiplicity results cover the cases $1 < q < 2$ and $q = 2$.

Theorem 1.1. *Let $0 < \mu < 4$ and $1 < q < 2$. Suppose that $\Omega := \left\{x \in \mathbb{R}^N : k(x) > 0\right\}$ is an open subset of $\mathbb{R}^N$ and that $0 < |\Omega| < \infty$. Then,*
(*i*) *for each $\beta > 0$ there exists $\overline{\Lambda} > 0$ such that if $\alpha \in (0, \overline{\Lambda})$ equation (1.1) has a sequence of nontrivial solutions $(u_n)_n$, with $J_{\alpha,\beta}(u_n) \le 0$ and $u_n \to 0$ as $n \to \infty$;*
(*ii*) *for each $\alpha > 0$ there exists $\underline{\Lambda} > 0$ such that if $\beta \in (0, \underline{\Lambda})$ equation (1.1) has a sequence of nontrivial solutions $(u_n)_n$, with $J_{\alpha,\beta}(u_n) \le 0$ and $u_n \to 0$ as $n \to \infty$.*

Theorem 1.2. *Let $0 < \mu < 4$, $q = 2$ and $\beta = 1$. Then, there exists a positive constant $a^\star$ such that for each $a > a^\star$ and $\alpha \in (0, aS\|k\|_r^{-1})$ equation (1.1) has at least n pairs of nontrivial solutions.*

In [45] Wang and Xiang obtain, in the fractional setting, the existence of at least two nontrivial solutions, when $2 < q < 2^\star$, $N > \mu \ge 4$. For the Laplacian counterpart of Theorem 1.1 in [45] their result can be stated as follows.

Theorem 1.3. *Let $N > \mu \ge 4$, $2 < q < 2^\star$, $\beta = 1$, $k \ge 0$ and $k \not\equiv 0$ in $\mathbb{R}^N$ be satisfied. If either $\mu = 4$, $a > 0$ and $b > 4S_{H,L}^{-1}$ or $\mu > 4$, $a > 0$ and*

$$b > (2_\mu^\star - 1)\left(a(2 - 2_\mu^\star)\right)^{-\frac{2-2_\mu^\star}{2_\mu^\star - 1}}\left(4S_{H,L}^{-1}\right)^{\frac{1}{2_\mu^\star-1}} := b^\star, \tag{1.3}$$

then there exists $\alpha_\star$ such that equation (1.1) admits at least two nontrivial solutions in $D^{1,2}(\mathbb{R}^N)$ for all $\alpha > \alpha_\star$.

In the following, we are interested in looking for more solutions in the case $2 < q < 2^\star$. To this end, we shall employ the genus theory to obtain multiplicity of solutions. Regrettably, we have to restrict ourselves to the special case $N = 3$ and $4 < q < 2^\star := 6$. More precisely, we obtain the following result.

Theorem 1.4. *Assume that $4 < q < 6$, $0 < \mu < 2$, $\alpha = \beta$ and $k \in L^\infty(\mathbb{R}^3)$, with $0 < k_\star \le k(x) \le k^\star$ in $\mathbb{R}^3$. Then, there exists $\beta^\star > 1$ such that if $\beta > \beta^\star$*
(*i*) *equation (1.1) has at least one nontrivial solution u_β and $u_\beta \to 0$ in $D^{1,2}(\mathbb{R}^3)$ as $\beta \to \infty$;*

(*ii*) *equation* (1.1) *has at least m pairs of nontrivial solutions* $u_{\beta,i}$, $u_{\beta,-i}$, $i = 1, 2, \cdots, m$, *and* $u_{\beta,i} \to 0$ *in* $D^{1,2}(\mathbb{R}^3)$ *as* $\beta \to \infty$, *for all* $i = 1, 2, \cdots, m$.

Remark 1.1. Theorems 1.3 and 1.4 leave some gaps. Indeed, existence of solutions for (1.1) is not covered in this paper, when either $2 < q \leq 4$ and $N = 3,\ 4$, or $2^\star \leq q \leq 4$. However, the approaches used in this paper do not seem to be applicable in the above cases. Thus, these missing values will be studied in future work.

The paper is organized as follows. In Section 2, we recall some preliminaries and set up the underlying functional $J_{\alpha,\beta}$ associated to (1.1). In Section 3, we prove the Palais-Smale condition at some special energy levels. In Section 4, we introduce a truncation argument for the functional $J_{\alpha,\beta}$ and prove Theorem 1.1 by using the Kajikiya new version of the symmetric mountain pass theorem. In Section 5, existence and multiplicity of nontrivial solutions for (1.1) is proved when $q = 2$. Section 6 deals with the existence of two nontrivial solutions for (1.1) when $2 < q < 2^\star$ and $\beta = 1$, that is with the proof of Theorem 1.3. Finally, Section 7 is devoted to the proof of Theorem 1.4, that is to the proof of existence and multiplicity of solutions for (1.1) when $N = 3$, $4 < q < 6$ and $\alpha = \beta$.

2 Preliminaries

Here and in what follows, $\|\cdot\|_{\mathfrak{p}}$ denotes the canonical $L^{\mathfrak{p}}(\mathbb{R}^N)$ norm for any exponent $\mathfrak{p} > 1$. First, let us recall the Hardy-Littlewood-Sobolev inequality, see [28, Theorem 4.3].

Proposition 2.1. *Let* $p, \mathfrak{p} > 1$ *and* $0 < \mu < N$, *with* $1/p + 1/\mathfrak{p} + \mu/N = 2$. *Then, there exists a sharp constant* $C(p, \mathfrak{p}, \mu, N)$ *such that*

$$\iint\limits_{\mathbb{R}^{2N}} \frac{f(x)h(y)}{|x-y|^\mu}\,dxdy \leq C(t, \tau, \mu, N)\|f\|_p\|h\|_{\mathfrak{p}}$$

for all $f \in L^p(\mathbb{R}^N)$ *and* $h \in L^{\mathfrak{p}}(\mathbb{R}^N)$.

If $p = \mathfrak{p} = 2N/(2N-\mu)$, *then*

$$C(p, \mathfrak{p}, \mu, N) = C(N, \mu) = \pi^{\frac{\mu}{2}} \frac{\Gamma(\frac{N}{2} - \frac{\mu}{2})}{\Gamma(N - \frac{\mu}{2})} \left\{ \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{\mu}{2})} \right\}^{\frac{\mu}{N}-1}.$$

Equality holds in (2.1) *if and only if* $f \equiv (\text{constant})h$, *where*

$$h(x) = A\left(\gamma^2 + |x - x_0|^2\right)^{(2N-\mu)/2}, \quad x \in \mathbb{R}^N,$$

for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ *and* $x_0 \in \mathbb{R}^N$.

Let us introduce $D^{1,2}(\mathbb{R}^N)$ as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\| = (\int_{\mathbb{R}^N} |\nabla u|^2 dx)^{1/2}$. Then, the best constant for the embedding of $D^{1,2}(\mathbb{R}^N)$ into $L^{2^\star}(\mathbb{R}^N)$ is S, defined by

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \left\{ \int\limits_{\mathbb{R}^N} |\nabla u|^2 dx \ : \ \int\limits_{\mathbb{R}^N} |u|^{2^\star} dx = 1 \right\}.$$

Obviously, $S > 0$, see [44]. By the Hardy-Littlewood-Sobolev inequality, the integral

$$\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)|^p |u(y)|^p}{|x-y|^\mu}\,dxdy$$

is well defined in $D^{1,2}(\mathbb{R}^N)$ if $|u|^p \in L^{\mathfrak{p}}(\mathbb{R}^N)$ for $\mathfrak{p} > 1$ such that $(2/\mathfrak{p}) + (\mu/N) = 2$, that is $\mathfrak{p} = 2N/(2N-\mu)$. Hence, in $D^{1,2}(\mathbb{R}^N)$ we must have

$$p = \frac{2^\star}{\mathfrak{p}} = \frac{2N-\mu}{N-2} := 2^\star_\mu.$$

The exponent 2_μ^* is called the (upper) critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality. In particular,

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy \le C(N,\mu)\|u\|_{2^*}^{2\cdot 2_\mu^*} \tag{2.1}$$

for all $u \in D^{1,2}(\mathbb{R}^N)$. Hence, we set

$$S_{H,L} = \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx \; : \; \iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy = 1 \right\} \tag{2.2}$$

and clearly $S_{H,L} > 0$. For more details on $S_{H,L}$, we refer to the following result.

Lemma 2.1. (see [16, Lemma 1.2]) *The constant $S_{H,L}$ defined in* (2.2) *is achieved if and only if*

$$u(x) = C\left(\frac{l}{l^2 + |x - x_0|^2}\right)^{\frac{N-2}{2}}$$

where $C > 0$ is a fixed constant, $x_0 \in \mathbb{R}^N$ and $l \in \mathbb{R}^+$ are parameters. Moreover, $S = S_{H,L}C(N,\mu)^{\frac{N-2}{2N-\mu}}$.

Lemma 2.2. (see [16, Lemma 2.3]) *Let $N \ge 3$ and $0 < \mu < N$. Then*

$$\|u\|_\star := \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy\right)^{\frac{1}{2\cdot 2_\mu^*}}$$

defines a norm on $L^{2^*}(\mathbb{R}^N)$.

The energy functional associated to (1.1) is $J_{\alpha,\beta} : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\begin{aligned} J_{\alpha,\beta}(u) &= \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 - \frac{\alpha}{q}\int_{\mathbb{R}^N} k(x)|u|^q\, dx \\ &\quad - \frac{\beta}{2\cdot 2_\mu^*}\iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy \\ &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\alpha}{q}\|u\|_{k,q}^q - \frac{\beta}{2\cdot 2_\mu^*}\|u\|_\star^{2\cdot 2_\mu^*}. \end{aligned} \tag{2.3}$$

The Hardy-Littlewood-Sobolev inequality (2.1) gives

$$\|u\|_\star \le C(N,\mu)^{2_\mu^*/2}\|u\|_{2^*}$$

for all $u \in D^{1,2}(\mathbb{R}^N)$. Consequently, the functional $J_{\alpha,\beta}$ is of class $C^1(D^{1,2}(\mathbb{R}^N))$. Moreover,

$$\begin{aligned} \langle J'_{\alpha,\beta}(u), v\rangle &= a\int_{\mathbb{R}^N} \nabla u\cdot\nabla v dx + b\int_{\mathbb{R}^N} |\nabla u|^2 dx \int_{\mathbb{R}^N} \nabla u\cdot\nabla v dx - \alpha\int_{\mathbb{R}^N} k(x)|u|^{q-2}uv\, dx \\ &\quad - \beta\iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*-2}u(y)v(y)}{|x-y|^\mu} dxdy \end{aligned}$$

for all $u, v \in D^{1,2}(\mathbb{R}^N)$. This means that (weak) solutions of (1.1) are exactly the critical points of the functional $J_{\alpha,\beta}$ in $D^{1,2}(\mathbb{R}^N)$.

In order to prove that the $(PS)_c$ condition holds, we use the second concentration compactness principle and the concentration compactness principle at infinity. Now, we recall the concentration compactness principle for studying the critical Choquard equation [17] due to Lions in [29].

Lemma 2.3. *Let $(u_n)_n$ be a bounded sequence in $D^{1,2}(\mathbb{R}^N)$ converging weakly and a.e. to some u as $n \to \infty$ and such that $|u_n|^{2^*}dx \overset{*}{\rightharpoonup} \zeta$ and $|\nabla u_n|^2 dx \overset{*}{\rightharpoonup} \omega$ in the sense of measures, where ζ and ω are bounded nonnegative Radon measures on $\mathbb{R}^N$. Assume moreover that*

$$\left(\int_{\mathbb{R}^N} \frac{|u_n(y)|^{2^*_\mu}}{|x-y|^\mu} dy \right) |u_n(x)|^{2^*_\mu} dx \overset{*}{\rightharpoonup} \nu$$

in the sense of measure, where ν is a bounded nonnegative Radon measure on $\mathbb{R}^N$. Then, there exists a (at most countable) set of distinct points $\{z_i\}_{i\in I} \subseteq \mathbb{R}^N$ and nonnegative numbers $\{\nu_i\}_{i\in I}$, $\{\zeta_i\}_{i\in I}$ and $\{\omega_i\}_{i\in I}$ such that

$$\nu = \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy \right) |u(x)|^{2^*_\mu} dx + \sum_{i\in I} \delta_{z_i} \nu_i, \qquad \sum_{i\in I} \nu_i^{\frac{1}{2^*_\mu}} < \infty,$$

$$\omega \geq |\nabla u|^2 dx + \sum_{i\in I} \delta_{z_i} \omega_i, \qquad \zeta \geq |u|^{2^*} dx + \sum_{i\in I} \delta_{z_i} \zeta_i,$$

where δ_x is the Dirac function of mass 1 concentrated at $x \in \mathbb{R}^N$. Finally, for all $i \in I$

$$S_{H,L} \nu_i^{\frac{1}{2^*_\mu}} \leq \omega_i, \qquad \nu_i^{\frac{N}{2N-\mu}} \leq C(N,\mu)^{\frac{N}{2N-\mu}} \zeta_i.$$

However, roughly speaking, the second concentration compactness principle, stated in Lemma 2.3, is only concerned with a possible concentration of a weakly convergent sequence at finite points and it does not provide any information about the loss of mass of a sequence at infinity. The next concentration-compactness principle at infinity was developed by Chabrowski [8], Bianchi, Chabrowski, Szulkin [6], Ben-Naoum, Troestler, Willem [5] and provides some quantitative information about the loss of mass of a sequence at infinity.

Lemma 2.4. *Let $(u_n)_n \subset D^{1,2}(\mathbb{R}^N)$ be a sequence as in Lemma 3.1 and define*

$$\omega_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |\nabla u_n|^2 dx, \quad \zeta_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |u_n|^{2^*} dx.$$

Then $S\zeta_\infty^{\frac{2}{2^*}} \leq \omega_\infty$ and

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx = \omega_\infty + \int_{\mathbb{R}^N} d\omega, \qquad \limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2^*} dx = \zeta_\infty + \int_{\mathbb{R}^N} d\zeta.$$

The next result is the concentration compactness principle at infinity for the critical Choquard equation, as proved by Gao *et al.* in [17].

Lemma 2.5. *Let $(u_n)_n \subset D^{1,2}(\mathbb{R}^N)$ be such that $u_n \rightharpoonup u$ weakly in $D^{1,2}(\mathbb{R}^N)$ and $u_n \rightharpoonup u$ a.e. in $\mathbb{R}^N$. Let ω, ζ, and ν be the bounded nonnegative Radon measures, while let ω_∞ and ζ_∞ be the numbers given as in Lemmas 2.3 and 2.4. Assume that*

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} \left(\int_{\mathbb{R}^N} \frac{|u_n(y)|^{2^*_\mu}}{|x-y|^\mu} dy \right) |u_n(x)|^{2^*_\mu} dx.$$

Then there exists a nonnegative number ν_∞ satisfying the relations

$$\limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)|^{2^*_\mu} |u_n(y)|^{2^*_\mu}}{|x-y|^\mu} dydx = \nu_\infty + \int_{\mathbb{R}^N} d\nu,$$

$$C(N,\mu)^{\frac{2N}{\mu-2N}} \nu_\infty^{\frac{2N}{2N-\mu}} \leq \zeta_\infty \left(\int_{\mathbb{R}^N} d\zeta + \zeta_\infty \right), \qquad S_{H,L}^2 \nu_\infty^{\frac{2}{2^*_\mu}} \leq \omega_\infty \left(\int_{\mathbb{R}^N} d\omega + \omega_\infty \right).$$

3 The Palais–Smale condition

In this section, we use the second concentration compactness principle and concentration compactness principle at infinity to prove that the $(PS)_c$ condition holds, when $c < 0$ and $1 < q < 2$. We recall in passing that throughout the paper α and β in (1.1) are positive real parameters, without further mentioning.

Lemma 3.1. *Suppose that $0 < \mu < 4$ and $1 < q < 2$. Then any $(PS)_c$ sequence $(u_n)_n$ of $J_{\alpha,\beta}$ is bounded in $D^{1,2}(\mathbb{R}^N)$.*

Proof. Let $(u_n)_n$ be a sequence in $D^{1,2}(\mathbb{R}^N)$ such that as $n \to \infty$

$$J_{\alpha,\beta}(u_n) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\alpha}{q}\|u\|_{k,q}^q - \frac{\beta}{2 \cdot 2_\mu^*}\|u\|_*^{2\cdot 2_\mu^*} = c + o(1), \tag{3.1}$$

$$\begin{aligned}\langle J'_{\alpha,\beta}(u_n), v\rangle &= a\int_{\mathbb{R}^N} \nabla u_n \cdot \nabla v dx + b\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \int_{\mathbb{R}^N} \nabla u_n \cdot \nabla v dx \\ &\quad - \alpha\int_{\mathbb{R}^N} k(x)|u_n|^{q-2}u_n v dx - \beta\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*-2}u_n(y)v(y)}{|x-y|^\mu} dxdy \\ &= o(1)\|u_n\|.\end{aligned} \tag{3.2}$$

Using the Hölder inequality and the Sobolev embedding theorem, we get for all $u \in D^{1,2}(\mathbb{R}^N)$

$$\|u\|_{k,q}^q = \int_{\mathbb{R}^N} k(x)|u|^q \, dx \le S^{-\frac{q}{2}}\|k\|_r\|u\|^q. \tag{3.3}$$

Thus, (3.1), (3.2) and (3.3) give as $n \to \infty$

$$\begin{aligned} c + o(1)\|u_n\| &= J_{\alpha,\beta}(u_n) - \frac{1}{2 \cdot 2_\mu^*}\langle J'_{\alpha,\beta}(u_n), u_n\rangle \\ &\ge \left(\frac{1}{2} - \frac{1}{2 \cdot 2_\mu^*}\right) a\|u_n\|^2 + \left(\frac{1}{4} - \frac{1}{2 \cdot 2_\mu^*}\right) b\|u_n\|^4 - \left(\frac{1}{q} - \frac{1}{2 \cdot 2_\mu^*}\right)\alpha\|u_n\|_{k,q}^q \\ &\ge \left(\frac{1}{2} - \frac{1}{2 \cdot 2_\mu^*}\right) a\|u_n\|^2 + \left(\frac{1}{4} - \frac{1}{2 \cdot 2_\mu^*}\right) b\|u_n\|^4 - \left(\frac{1}{q} - \frac{1}{2 \cdot 2_\mu^*}\right)\alpha S^{-\frac{q}{2}}\|k\|_r\|u_n\|^q. \end{aligned}$$

This implies at once that $(u_n)_n$ is bounded in $D^{1,2}(\mathbb{R}^N)$, since $0 < \mu < 4$ gives $2 \cdot 2_\mu^* > 4$ and since $1 < q < 2$. □

Lemma 3.2. *Let $c < 0$, $0 < \mu < 4$ and $1 < q < 2$. The next two properties hold.*
(i) For each $\beta > 0$ there exists $\overline{\Lambda} > 0$ such that $J_{\alpha,\beta}$ satisfies the $(PS)_c$ condition for all $\alpha \in (0, \overline{\Lambda})$.
(ii) For each $\alpha > 0$ there exists $\underline{\Lambda} > 0$ such that $J_{\alpha,\beta}$ satisfies the $(PS)_c$ condition for any $\beta \in (0, \underline{\Lambda})$.

Proof. Let $c < 0$ and let $(u_n)_n$ be a $(PS)_c$ sequence of $J_{\alpha,\beta}$ in $D^{1,2}(\mathbb{R}^N)$. Lemma 3.1 yields that $(u_n)_n$ is bounded in $D^{1,2}(\mathbb{R}^N)$. Thus, there exists $u \in D^{1,2}(\mathbb{R}^N)$ such that up to a subsequence $u_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^N)$, $u_n \rightharpoonup u$ in $L^{2^*}(\mathbb{R}^N)$, $u_n \to u$ in $L^p_{\text{loc}}(\mathbb{R}^N)$ for all $p \in [1, 2^*)$, $u_n \to u$ a.e in $\mathbb{R}^N$, and there exists $h_R \in L^p(B_R(0))$ such that $|u_n| \le h_R$ a.e in $B_R(0)$ for all n and all $R > 0$, with $p \in [1, 2^*)$. Furthermore, by Proposition 1.202 of [14] there exist bounded nonnegative Radon measures ω, ζ and ν such that as $n \to \infty$

$$|\nabla u_n|^2 dx \overset{*}{\rightharpoonup} \omega, \quad |u_n|^{2^*} dx \overset{*}{\rightharpoonup} \zeta, \quad \left(\int_{\mathbb{R}^N} \frac{|u_n(y)|^{2_\mu^*}}{|x-y|^\mu} dy\right)|u_n|^{2_\mu^*} dx \overset{*}{\rightharpoonup} \nu$$

in the sense of measure. Hence, by Lemma 2.3, there exist a at most countable set I, a sequence of points $\{z_i\}_{i\in I} \subset \mathbb{R}^N$ and families of nonnegative numbers $\{\nu_i : i \in I\}$, $\{\omega_i : i \in I\}$ and $\{\zeta_i : i \in I\}$ such that

$$\nu = \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu} dy\right)|u|^{2_\mu^*} dx + \sum_{i\in I} \nu_i \delta_{z_i},$$

$$\omega \geq |\nabla u|^2 dx + \sum_{i\in I} \omega_i \delta_{z_i}, \qquad \zeta \geq |u|^{2^*} dx + \sum_{i\in I} x_i \delta_{z_i},$$

$$S_{H,L} \nu_i^{\frac{1}{2^*_\mu}} \leq \omega_i \quad \text{and} \quad \nu_i \leq C(N,\mu)\zeta_i^{\frac{2N-\mu}{N}} \quad \text{for all } i \in I,$$

where δ_{z_i} is the Dirac function at z_i.

Fix a test function $\varphi \in C_0^\infty(\mathbb{R}^N)$, such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in the closed ball $B_1(0)$, while $\varphi \equiv 0$ in $\mathbb{R}^N \setminus B_2(0)$ and $\|\nabla\varphi\|_\infty \leq 2$. Take $\varepsilon > 0$ and put $\varphi_{\varepsilon,i}(x) = \varphi(2(x - z_i)/\varepsilon)$, $x \in \mathbb{R}^N$, for any fixed $i \in I$, where $\{z_i\}_{\in I}$ is introduced above. Observe that as $n \to \infty$

$$\left| \int_{\mathbb{R}^N} k(x)|u_n|^q \varphi_{\varepsilon,i} dx \right| \leq \int_{B_\varepsilon(z_i)} |k(x)| \cdot |u_n|^q dx \leq \|k\|_r \left(\int_{B_\varepsilon(z_i)} |u_n|^{2^*} dx \right)^{\frac{q}{2^*}}$$

$$\to \|k\|_r \left(\int_{B_\varepsilon(z_i)} |u|^{2^*} dx \right)^{\frac{q}{2^*}}.$$

Therefore, as $\varepsilon \to 0$ we finally get

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \int_{\mathbb{R}^N} k(x)|u_n|^q \varphi_{\varepsilon,i}\, dx = 0.$$

On the other hand, the Hölder inequality yields

$$\begin{aligned}
\limsup_{n\to\infty} \left| \int_{\mathbb{R}^N} u_n \nabla u_n \cdot \nabla\varphi_{\varepsilon,i} dx \right| &\leq \limsup_{n\to\infty} \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |u_n \nabla\varphi_{\varepsilon,i}|^2 dx \right)^{\frac{1}{2}} \\
&\leq C \left(\int_{B_{2\varepsilon}(z_i)} |u|^2 |\nabla\varphi_{\varepsilon,i}|^2 dx \right)^{\frac{1}{2}} \\
&\leq C \left(\int_{B_{2\varepsilon}(z_i)} |\nabla\varphi_{\varepsilon,i}|^N dx \right)^{\frac{1}{N}} \left(\int_{B_{2\varepsilon}(z_i)} |u|^{2^*} dx \right)^{\frac{1}{2^*}} \\
&\leq C_\varphi \left(\int_{B_{2\varepsilon}(z_i)} |u|^{2^*} dx \right)^{\frac{1}{2^*}} \to 0
\end{aligned}$$

as $\varepsilon \to 0$, where $C = \sup_n \|u_n\|$ and $C_\varphi = C \left(\int_{B_2(0)} |\nabla\varphi|^N dy \right)^{1/N}$. Therefore

$$\begin{aligned}
0 = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \langle J'_{\alpha,\beta}(u_n), \varphi_{\varepsilon,i} u_n \rangle &= \lim_{\varepsilon\to 0} \lim_{n\to\infty} \Bigg\{ \left(a + b\|u_n\|^2\right) \int_{\mathbb{R}^N} \nabla u_n \cdot \nabla(\varphi_{\varepsilon,i} u_n)\, dx \\
&\quad - \alpha \int_{\mathbb{R}^N} k(x)|u_n|^q \varphi_{\varepsilon,i}\, dx - \beta \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)|^{2^*_\mu} |u_n(y)|^{2^*_\mu} \varphi_{\varepsilon,i}(y)}{|x-y|^\mu}\, dxdy \Bigg\} \\
&= \lim_{\varepsilon\to 0} \lim_{n\to\infty} \Bigg\{ \left(a + b\|u_n\|^2\right) \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 \varphi_{\varepsilon,i} + u_n \nabla u_n \cdot \nabla\varphi_{\varepsilon,i} \right) dx \\
&\quad - \alpha \int_{\mathbb{R}^N} k(x)|u_n|^q \varphi_{\varepsilon,i}\, dx - \beta \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)|^{2^*_\mu} |u_n(y)|^{2^*_\mu} \varphi_{\varepsilon,i}(y)}{|x-y|^\mu}\, dxdy \Bigg\}
\end{aligned}$$

$$\geq \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left\{a\int_{\mathbb{R}^N}|\nabla u_n|^2\varphi_{\varepsilon,i}\,dx-\beta\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)|^{2^*_\mu}|u_n(y)|^{2^*_\mu}\varphi_{\varepsilon,i}(y)}{|x-y|^\mu}\,dxdy\right\}$$
$$\geq \lim_{\varepsilon\to 0}\lim_{n\to\infty}\left\{a\int_{\mathbb{R}^N}\varphi_{\varepsilon,i}\,d\omega-\beta\int_{\mathbb{R}^N}\varphi_{\varepsilon,i}\,d\nu\right\}$$
$$\geq a\omega_i-\beta\nu_i.$$

Therefore, $a\omega_i \leq \beta\nu_i$. Combining this with Lemma 2.3, we obtain that either

$$\omega_i \geq \left(a\beta^{-1}S_{H,L}^{2^*_\mu}\right)^{\frac{1}{2^*_\mu-1}} \quad \text{or} \quad \omega_i = 0. \tag{3.4}$$

We claim that the first case can never occur. Otherwise, there exists $i_0 \in I$ such that

$$\omega_{i_0} \geq \left(a\beta^{-1}S_{H,L}^{2^*_\mu}\right)^{\frac{1}{2^*_\mu-1}}.$$

Now, (3.3), the Hölder inequality, the Sobolev embedding and the Young inequality imply that

$$\begin{aligned}
\alpha\int_{\mathbb{R}^N}k(x)|u|^q\,dx \leq \alpha\|k\|_r S^{-\frac{q}{2}}\|u\|^q &= \left(\left[\left(\frac12-\frac{1}{2\cdot 2^*_\mu}\right)\frac{a}{q}\left(\frac1q-\frac{1}{2\cdot 2^*_\mu}\right)^{-1}\right]^{\frac{q}{2}}\|u\|^q\right)\\
&\quad\times\left(\left[\left(\frac12-\frac{1}{2\cdot 2^*_\mu}\right)\frac{a}{q}\left(\frac1q-\frac{1}{2\cdot 2^*_\mu}\right)^{-1}\right]^{\frac{-q}{2}}\alpha\|k\|_r S^{-\frac{q}{2}}\right)\\
&\leq \left(\frac12-\frac{1}{2\cdot 2^*_\mu}\right)\frac{a}{2}\left(\frac1q-\frac{1}{2\cdot 2^*_\mu}\right)^{-1}\|u\|^2\\
&\quad+\frac{2-q}{2}\left[\left(\frac12-\frac{1}{2\cdot 2^*_\mu}\right)^{-1}\frac{q}{aS}\left(\frac1q-\frac{1}{2\cdot 2^*_\mu}\right)\right]^{\frac{q}{2-q}}\|k\|_r^{\frac{2}{2-q}}\alpha^{\frac{2}{2-q}}.
\end{aligned}\tag{3.5}$$

According to this fact, we have

$$\begin{aligned}
0 > c &= \lim_{n\to\infty}\left(J_{\alpha,\beta}(u_n)-\frac{1}{2\cdot 2^*_\mu}\langle J'_{\alpha,\beta}(u_n),u_n\rangle\right)\\
&\geq \lim_{n\to\infty}\left\{\left(\frac12-\frac{1}{2\cdot 2^*_\mu}\right)a\|u_n\|^2+\left(\frac14-\frac{1}{2\cdot 2^*_\mu}\right)b\|u_n\|^4-\left(\frac1q-\frac{1}{2\cdot 2^*_\mu}\right)\alpha\int_\Omega k(x)|u_n|^q dx\right\}\\
&\geq \left(\frac12-\frac{1}{2\cdot 2^*_\mu}\right)a\left(\|u\|^2+\sum_{i\in I}w_i\right)-\left(\frac1q-\frac{1}{2\cdot 2^*_\mu}\right)\alpha\int_\Omega k(x)|u|^q dx\\
&\geq \left(\frac12-\frac{1}{2\cdot 2^*_\mu}\right)\frac{a}{2}w_{i_0}-\frac{2-q}{2}\left[\left(\frac12-\frac{1}{2\cdot 2^*_\mu}\right)^{-1}\frac{q}{aS}\left(\frac1q-\frac{1}{2\cdot 2^*_\mu}\right)\right]^{\frac{q}{2-q}}\|k\|_r^{\frac{2}{2-q}}\alpha^{\frac{2}{2-q}}\\
&\geq \left(\frac14-\frac{1}{4\cdot 2^*_\mu}\right)(aS_{H,L})^{\frac{2^*_\mu}{2^*_\mu-1}}\beta^{-\frac{1}{2^*_\mu-1}}-\frac{2-q}{2}\left[\left(\frac12-\frac{1}{2\cdot 2^*_\mu}\right)^{-1}\frac{q}{aS}\left(\frac1q-\frac{1}{2\cdot 2^*_\mu}\right)\right]^{\frac{q}{2-q}}\|k\|_r^{\frac{2}{2-q}}\alpha^{\frac{2}{2-q}}.
\end{aligned}\tag{3.6}$$

Thus, for any $\beta > 0$, we choose $\alpha_1 > 0$ so small that for every $\alpha \in (0, \alpha_1)$ the right-hand side of (3.6) is greater than zero, which is an obvious contradiction.

Similarly, if $\alpha > 0$ is given, we take $\beta_1 > 0$ so small that for every $\beta \in (0, \beta_1)$ again the right-hand side of (3.6) is greater than zero. This gives the required contradiction. Consequently, $\omega_i = 0$ for all $i \in I$ in (3.4).

To obtain the possible concentration of mass at infinity, similarly, we define a cut off function ψ_R in $C^\infty(\mathbb{R}^N)$ such that $\psi_R = 0$ in $B_R(0)$, $\psi_R = 1$ in $\mathbb{R}^N \setminus B_{R+1}(0)$, and $|\nabla\psi_R| \leq 2/R$ in $\mathbb{R}^N$. On the one hand, the Hardy-Littlewood-Sobolev and the Hölder inequalities give

$$\nu_\infty = \lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2^*_\mu}}{|x-y|^\mu}\,dy\right)|u_n(x)|^{2^*_\mu}\psi_R(y)dx$$

$$\leq C(N,\mu)\lim_{R\to\infty}\lim_{n\to\infty}\|u_n\|_{2^*}^{2^*_\mu}\left(\int_{\mathbb{R}^N}|u_n(x)|^{2^*}\psi_R(y)dx\right)^{\frac{2^*_\mu}{2^*}}$$
$$\leq \hat{C}\zeta_\infty^{\frac{2^*_\mu}{2^*}}.$$

On the other hand, the fact that $\langle J'_{\alpha,\beta}(u_n), u_n\psi_R\rangle \to 0$ implies that

$$0 = \lim_{R\to\infty}\lim_{n\to\infty}\langle J'_{\alpha,\beta}(u_n), \psi_R u_n\rangle = \lim_{R\to\infty}\lim_{n\to\infty}\left\{\left(a+b\|u_n\|^2\right)\int_{\mathbb{R}^N}\nabla u_n\cdot\nabla(\psi_R u_n)\,dx\right.$$
$$\left.-\alpha\int_{\mathbb{R}^N}k(x)|u_n|^q\psi_R\,dx-\beta\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)|^{2^*_\mu}|u_n(y)|^{2^*_\mu}\psi_R(y)}{|x-y|^\mu}\,dxdy\right\}$$
$$\geq \lim_{R\to\infty}\lim_{n\to\infty}\left\{a\int_{\mathbb{R}^N}\left(|\nabla u_n|^2\psi_R+u_n\nabla u_n.\nabla\psi_R\right)\,dx-\beta\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)|^{2^*_\mu}|u_n(y)|^{2^*_\mu}\psi_R(y)}{|x-y|^\mu}\,dxdy\right\}$$
$$\geq \lim_{R\to\infty}\lim_{n\to\infty}\left\{a\int_{\mathbb{R}^N}|\nabla u_n|^2\psi_R\,dx-\beta\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)|^{2^*_\mu}|u_n(y)|^{2^*_\mu}\psi_R(y)}{|x-y|^\mu}\,dxdy\right\}$$
$$\geq a\omega_\infty-\hat{C}\beta\zeta_\infty^{\frac{2^*_\mu}{2^*}}.$$

Therefore $a\omega_\infty \leq \hat{C}\beta\zeta_\infty^{\frac{2^*_\mu}{2^*}}$. Combining this with the Lemma 2.4, we obtain that either

$$\omega_\infty \geq \left(aS^{\frac{2^*_\mu}{2}}\hat{C}^{-1}\beta^{-1}\right)^{\frac{2}{2^*_\mu-2}} \quad\text{or}\quad \omega_\infty = 0. \tag{3.7}$$

Therefore, as in (3.5) and (3.6), we have

$$0 > c \geq \left(\frac{1}{4}-\frac{1}{4\cdot 2^*_\mu}\right)(aS)^{\frac{2^*_\mu}{2^*_\mu-2}}\hat{C}^{-\frac{2}{2^*_\mu-2}}\beta^{-\frac{2}{2^*_\mu-2}}$$
$$-\frac{2-q}{2}\left[\left(\frac{1}{2}-\frac{1}{2\cdot 2^*_\mu}\right)^{-1}\frac{q}{aS}\left(\frac{1}{q}-\frac{1}{2\cdot 2^*_\mu}\right)\right]^{\frac{q}{2-q}}\|k\|_r^{\frac{2}{2-q}}\alpha^{\frac{2}{2-q}}. \tag{3.8}$$

Thus, for any $\beta > 0$, we choose $\alpha_2 > 0$ so small that for every $\alpha \in (0, \alpha_2)$ the right-hand side of (3.8) is greater than zero, which is a contradiction.

Similarly, if $\alpha > 0$ is given, we select $\beta_2 > 0$ so small that for every $\beta \in (0, \beta_2)$ the right-hand side of (3.8) is greater than zero. This gives the required contradiction. Therefore, $\omega_\infty = 0$ in (3.7).

From the arguments above, put

$$\overline{\Lambda} = \min\{\alpha_1, \alpha_2\} \quad\text{and}\quad \underline{\Lambda} = \min\{\beta_1, \beta_2\}.$$

Then, for any $c < 0$ and $\beta > 0$ we have

$$\omega_i = 0 \text{ for all } i \in I \quad\text{and}\quad \omega_\infty = 0$$

for all $\alpha \in (0, \overline{\Lambda})$.

Similarly, for any $c < 0$ and $\alpha > 0$ we again have

$$\omega_i = 0 \text{ for all } i \in I \quad\text{and}\quad \omega_\infty = 0$$

for any $\beta \in (0, \overline{\Lambda})$.

Hence as $n \to \infty$

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)|^{2^*_\mu}|u_n(y)|^{2^*_\mu}}{|x-y|^\mu}\,dxdy \to \iint_{\mathbb{R}^{2N}} \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,dxdy$$

$$\int_{\mathbb{R}^N} k(x)(|u_n|^q - |u|^q)dx \le \|k\|_r \||u_n|^q - |u|^q\|_{\frac{2^*_\mu}{q}} \to 0.$$

Since $(\|u_n\|)_n$ is bounded and $J'_{\alpha,\beta}(u) = 0$, the weak lower semicontinuity of the norm and the Brézis-Lieb lemma yield as $n \to \infty$

$$\begin{aligned} o(1) = \langle J'_{\alpha,\beta}(u_n), u_n\rangle &= a\|u_n\|^2 + b\|u_n\|^4 - \alpha\|u_n\|^q_{k,q} - \beta\|u_n\|_\star^{2\cdot 2^*_\mu} \\ &\ge a\left(\|u_n\|^2 - \|u\|^2\right) + a\|u\|^2 + b\|u\|^4 - \alpha\|u\|^q_{k,q} - \beta\|u\|_\star^{2\cdot 2^*_\mu} + o(1) \\ &= a\|u_n - u\|^2 + o(1). \end{aligned}$$

Thus $(u_n)_n$ strongly converges to u in $D^{1,2}(\mathbb{R}^N)$. This completes the proof. □

4 Proof of Theorem 1.1

In this section, we prove the existence of infinitely many solutions of (1.1) which tend to zero and we assume, without further mentioning, that all the assumptions of Theorem 1.1 hold. To this aim, we apply a new version of the symmetric mountain pass lemma, due to Kajikiya in [21, Theorem 1].

Lemma 4.1. *Let E be an infinite-dimensional Banach space and $J \in C^1(E)$. Suppose that the following properties hold.*
(J_1) J is even, bounded from below in E, $J(0) = 0$ and J satisfies the local Palais-Smale condition.
(J_2) For each $n \in \mathbb{N}$ there exists $A_n \in \Sigma_n$ such that $\sup_{u\in A_n} J(u) < 0$, where

$$\Sigma_n := \{A : A \subset E \text{ is closed symmetric}, 0 \notin A,\ \gamma(A) \ge n\}$$

and $\gamma(A)$ is a genus of A.

Then J admits a sequence of critical points $(u_n)_n$ such that $J(u_n) \le 0$, $u_n \neq 0$ for each n and $(u_n)_n$ converges to zero as $n \to \infty$.

To obtain infinitely many solutions of (1.1), we need some technical lemmas. Let $J_{\alpha,\beta}$ be the functional defined in (2.3). Then, by (3.3) and the Hardy-Littlewood-Sobolev inequality

$$\begin{aligned} J_{\alpha,\beta}(u) &\ge \frac{a}{2}\|u\|^2 - \alpha\|k\|_r S^{-\frac{q}{2}}\|u\|^q - \frac{S^{-1}_{H,L}}{2\cdot 2^*_\mu}\beta\|u\|^{2\cdot 2^*_\mu} \\ &= l_1\|u\|^2 - \alpha l_2\|u\|^q - \beta l_3\|u\|^{2\cdot 2^*_\mu}. \end{aligned}$$

Define

$$h(t) = l_1 t^2 - \alpha l_2 t^q - \beta l_3 t^{2\cdot 2^*_\mu}, \quad t \in \mathbb{R}^+_0.$$

Then, for any given parameter $\alpha > 0$ there exists $\overline{\beta} > 0$ so small that for every $\beta \in (0, \overline{\beta})$ there exist t_0, t_1, with $0 < t_0 < t_1$, such that $h < 0$ in $(0, t_0)$, $h > 0$ in (t_0, t_1) and $h(t) < 0$ for all $t > t_1$.

Similarly, for any fixed number $\beta > 0$ we choose $\overline{\alpha} > 0$ so small that for every $\alpha \in (0, \overline{\alpha})$ there exist $t_0^\star$, $t_1^\star$, with $0 < t_0^\star < t_1^\star$, such that $h < 0$ in $(0, t_0^\star)$, $h > 0$ in $(t_0^\star, t_1^\star)$ and $h(t) < 0$ for all $t > t_1^\star$.

Clearly, $h(t_0) = 0 = h(t_1)$ and $h(t_0^\star) = 0 = h(t_1^\star)$. Following the same idea as in [19], we consider the truncated functional $\widetilde{J}_{\alpha,\beta}$ of $J_{\alpha,\beta}$, defined for all $u \in D^{1,2}(\mathbb{R}^N)$ by

$$\widetilde{J}_{\alpha,\beta}(u) := \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\alpha}{q}\psi(u)\|u\|^q_{k,q} - \frac{\beta}{2\cdot 2^*_\mu}\psi(u)\|u\|_\star^{2\cdot 2^*_\mu}, \tag{4.1}$$

where $\psi(u) = \tau(\|u\|)$ and $\tau : \mathbb{R}_0^+ \to [0,1]$ is a non-increasing C^∞ function such that $\tau(t) = 1$ if $t \in [0, t_0]$ and $\tau(t) = 0$ if $t \geq t_1$. It is clear that $\widetilde{J}_{\alpha,\beta} \in C^1\left(D^{1,2}(\mathbb{R}^N)\right)$ and $\widetilde{J}_{\alpha,\beta}$ is bounded from below in $D^{1,2}(\mathbb{R}^N)$.

From the above arguments, recalling that all the assumptions of Theorem 1.1 hold, we have the next result.

Lemma 4.2. *Let $\widetilde{J}_{\alpha,\beta}$ be the functional introduced in* (4.1) *The following properties hold.*
(i) If $\widetilde{J}_{\alpha,\beta}(u) < 0$, then $\|u\| \leq t_0$ and $\widetilde{J}_{\alpha,\beta}(u) = J_{\alpha,\beta}(u)$.
(ii) Let $c < 0$. Then, for any $\beta > 0$ there exists $\overline{\Lambda} > 0$ such that $\widetilde{J}_{\alpha,\beta}$ satisfies the $(PS)_c$ condition for all $\alpha \in (0, \overline{\Lambda})$.
(iii) Let $c < 0$. Then, for any $\alpha > 0$ there exists $\underline{\Lambda} > 0$ such that $\widetilde{J}_{\alpha,\beta}$ satisfies the $(PS)_c$ condition for all $\beta \in (0, \underline{\Lambda})$.

Proof of Theorem 1.1. Clearly, $\widetilde{J}_{\alpha,\beta}(0) = 0$, $\widetilde{J}_{\alpha,\beta}$ is of class $C^1\left(D^{1,2}(\mathbb{R}^N)\right)$, even, coercive and bounded frow below in $D^{1,2}(\mathbb{R}^N)$. Furthermore, $\widetilde{J}_{\alpha,\beta}$ satisfies the $(PS)_c$ condition in $D^{1,2}(\mathbb{R}^N)$, with $c < 0$, by Lemma 4.2.

For any $n \in N$, we take n disjoint open sets X_i such that $\sqcup_{i=1}^n X_i \subset \Omega$, where Ω is the nonempty open set introduced in the statement of Theorem 1.1. For each $i = 1, 2, \cdots, n$, take $u_i \in \left(D^{1,2}(\mathbb{R}^N) \cap C_0^\infty(X_i)\right) \setminus \{0\}$, with $\|u_i\| = 1$. Put $E_n = \operatorname{span}\{u_1, u_2, \cdots, u_n\}$.

Thus, for any $u \in E_n$, with $\|u\| = \rho$, we have

$$\begin{aligned}\widetilde{J}_{\alpha,\beta}(u) &\leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\alpha}{q}\int_\Omega k(x)|u|^q dx - \frac{\beta}{2\cdot 2_\mu^*}\|u\|_\star^{2\cdot 2_\mu^*} \\ &\leq \frac{a}{2}\rho^2 + \frac{b}{4}\rho^4 - C_1\rho^q - C_2\rho^{2\cdot 2_\mu^*},\end{aligned}$$

where C_1 and C_2 are some positive constants, since all the norms are equivalent in the finite dimensional space E_n. Hence, $\widetilde{J}_{\alpha,\beta}(u) < 0$ provided that $\rho > 0$ is sufficiently small, being $1 < q < 2$. Therefore,

$$\{u \in E_n : \|u\| = \rho\} \subset \left\{u \in E_n : \widetilde{J}_{\alpha,\beta}(u) < 0\right\}.$$

As proved in the book [9] by Chang

$$\gamma\left(\{u \in E_n \,:\, \|u\| = \rho\}\right) = n.$$

Hence by the monotonicity of the genus γ, see Krasnoselskii [23], we get

$$\gamma\left(\left\{u \in E_n \,:\, \widetilde{J}_{\alpha,\beta}(u) < 0\right\}\right) \geq n.$$

Choosing $A_n = \{u \in E_n \,:\, \widetilde{J}_{\alpha,\beta}(u) < 0\}$, we have $A_n \in \sum_n$ and $\sup_{u \in A_n} \widetilde{J}_{\alpha,\beta}(u) < 0$. Therefore, all the assumptions of Lemma 4.1 are satisfied, since $D^{1,2}(\mathbb{R}^N)$ is a real infinite Hilbert space. Thus, there exists a sequence $(u_n)_n$ in $D^{1,2}(\mathbb{R}^N)$ such that

$$\widetilde{J}_{\alpha,\beta}(u_n) \leq 0, \quad u_n \neq 0, \quad \widetilde{J}'_{\alpha,\beta}(u_n) = 0 \text{ for each } n \quad \text{and} \quad \|u_n\| \to 0 \text{ as } n \to \infty.$$

Combining with Lemma 4.2 and taking n so large that $\|u_n\| \leq \rho$ is small enough, then these infinitely many nontrivial functions u_n are solutions of (1.1). □

5 Proof of Theorem 1.2

In this section we study (1.1), when $q = 2$, $0 < \mu < 4$ and $\beta = 1$, and shall apply the mountain pass theorem for even functionals, in order to obtain a multiplicity result for (1.1). Actually, here (1.1) reduces to

$$-(a + b\|u\|^2)\Delta u = \alpha k(x)u + \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu} dy\right)|u|^{2_\mu^*-2}u \quad \text{in } \mathbb{R}^N. \tag{5.1}$$

Clearly, the associated functional J_α to (5.1) is

$$J_\alpha(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\alpha}{2}\|u\|_{k,2}^2 - \frac{1}{2\cdot 2_\mu^*}\|u\|_\star^{2\cdot 2_\mu^*}.$$

Lemma 5.1. *Let $\alpha \in \left(0,\ aS\|k\|_r^{-1}\right)$ and let $(u_n)_n$ be a $(PS)_c$ sequence for J_α in $D^{1,2}(\mathbb{R}^N)$, with*

$$c < c^\star, \quad c^\star := \frac{1}{4}(aS_{H,L})^{\frac{2N-\mu}{N-\mu+2}}.$$

Then $(u_n)_n$ contains a strongly convergent subsequence.

Proof. The Hölder inequality and the Sobolev embedding theorem imply that

$$\|u\|_{k,2}^2 \le S^{-1}\|k\|_r\|u\|^2 \tag{5.2}$$

for each $u \in D^{1,2}(\mathbb{R}^N)$.

Fix a $(PS)_c$ sequence $(u_n)_n$ for J_α in $D^{1,2}(\mathbb{R}^N)$ at level $c < c^\star$. By the facts that $\alpha \in \left(0,\ aS\|k\|_{L^r}^{-1}\right)$, $0 < \mu < 4$ and by (5.2), proceeding as in proof of Lemma 3.2, in place of (3.6) we get

$$\begin{aligned}
c^\star > c &= \lim_{n\to\infty} J_\alpha(u_n) - \frac{1}{4}\langle J_\alpha'(u_n), u_n\rangle \\
&\ge \left\{\frac{a}{4}w_{i_0} + \left(\frac{1}{2}-\frac{1}{4}\right)\left(a - \alpha S^{-1}\|k\|_r\right)\|u\|^2 + \left(\frac{1}{4} - \frac{1}{2\cdot 2_\mu^\star}\right)v_{i_0}\right\} \\
&\ge \frac{1}{4}aw_{i_0} \ge \frac{1}{4}(aS_{H,L})^{\frac{2N-\mu}{N-\mu+2}} = c^\star,
\end{aligned}$$

which is impossible. Therefore, the compactness of the Palais-Smale sequence follows as in the proof of Lemma 3.2. □

Now, let us recall a version of the mountain pass theorem for even functionals, which is the main tool for proving Theorem 1.2. For its proof readers are referred to [42].

Proposition 5.1. *Let X be an infinite dimensional Banach space, with $X = V \oplus Y$, where V is finite dimensional. Let $J \in C^1(X)$ be an even functional such that $J(0) = 0$ and satisfying the following conditions.*
(I_1) There exist positive constants $\varrho, \rho > 0$ such that $J(u) \ge \varrho$ for all $u \in \partial B_\rho(0) \cap Y$.
(I_2) There exists $c^\star > 0$ such that J satisfies the $(PS)_c$ condition for all $c \in (0, c^\star)$.
(I_3) For each finite dimensional subspace $\widehat{X} \subset X$ there exists $R = R(\widehat{X})$ such that $J(u) \le 0$ for all $u \in \widehat{X}\backslash B_R(0)$.

Suppose that V is k dimensional and $V = \operatorname{span}\{e_1, e_2, \ldots, e_k\}$. For $n \ge k$, inductively choose $e_{n+1} \notin X_n := \operatorname{span}\{e_1, e_2, \ldots, e_n\}$. Let $R_n = R(X_n)$ and $D_n = B_{R_n}(0) \cap X_n$. Define

$$\begin{aligned}
G_n &:= \left\{h \in C(D_n, X) \,:\, h \text{ is odd and } h(u) = u \text{ for all } u \in \partial B_{R_n}(0) \cap X_n\right\}, \\
\Gamma_j &:= \left\{h\left(\overline{D_n\backslash E}\right) \,:\, h \in G_n,\ n \ge j,\ E \in \Sigma_{n-j} \text{ and } \gamma(E) \le n-j\right\}, \\
\Sigma_n &:= \{E \,:\, E \subset X \text{ is closed symmetric}, 0 \notin E, \gamma(E) \ge n\}
\end{aligned} \tag{5.3}$$

For each $j \in \mathbb{N}$, let

$$c_j := \inf_{K\in\Gamma_j}\max_{u\in K} J(u).$$

Then, $0 < \varrho \le c_j \le c_{j+1}$ for $j > k$, and if $j > k$ and $c_j < c^\star$, then c_j is a critical value of J. Moreover, if $c_j = c_{j+1} = \ldots = c_{j+l} = c < c^\star$ for $j > k$, then $\gamma(K_c) \ge l+1$, where

$$K_c := \left\{u \in E :\ J(u) = c \text{ and } J'(u) = 0\right\}.$$

From now on we assume that all the assumptions of Theorem 1.2 hold, without further mentioning.

Lemma 5.2. *For any $\alpha \in (0, aS\|k\|_r^{-1})$, then the functional J_α satisfies conditions $(I_1) - (I_3)$.*

Proof. First, the fact that $\alpha \in (0,\ aS\|k\|_r^{-1})$, the definitions of S and $S_{H,L}$ yield

$$J_\alpha(u) \ge \frac{1}{2}(a - \alpha S^{-1}\|k\|_r)\|u\|^2 - \frac{S_{H,L}^{-1}}{2\cdot 2_\mu^\star}\|u\|^{2\cdot 2_\mu^\star}.$$

Since $2 < 2 \cdot 2^\star_\mu$, there exists $\varrho > 0$ such that $J_\alpha(u) \geq \varrho$ for all $u \in D^{1,2}(\mathbb{R}^N)$, with $\|u\| = \rho$, where ρ is chosen sufficiently small. Thus, J_α satisfies (I_1).

Since $\alpha \in (0,\ aS\|k\|_r^{-1})$, a direct consequence of Lemma 5.1 implies that J_α satisfies (I_2), with

$$c^\star = (aS_{H,L})^{\frac{2N-\mu}{N-\mu+2}}/4.$$

Let E be a finite dimensional subspace of $D^{1,2}(\mathbb{R}^N)$. Thus, for any $u \in E$, with $\|u\|$ large enough, by Lemma 2.2, we have

$$\begin{aligned} J_\alpha(u) &\leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{\alpha}{2}\|u\|^2_{k,2} - \frac{1}{2 \cdot 2^\star_\mu}\|u\|_\star^{2\cdot 2^\star_\mu} \\ &\leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{\alpha}{2}c_1\|u\|^2 - \frac{1}{2 \cdot 2^\star_\mu}c_2\|u\|^{2\cdot 2^\star_\mu}, \end{aligned}$$

for some positive constants $c_1, c_2 > 0$, since all the norms on finite dimensional space are equivalent. Since $4 < 2 \cdot 2^\star_\mu$, we conclude that $J_\alpha(u) < 0$ for all $u \in E$, with $\|u\| \geq R$, where R is chosen large enough. Consequently, J_α verifies (I_3), as stated. □

Lemma 5.3. *There exists a sequence $(M_n)_n \subset \mathbb{R}^+$, independent of α, such that $M_n \leq M_{n+1}$ for all n and for any $\alpha > 0$*

$$c_n^\alpha := \inf_{K \in \Gamma_n} \max_{u \in K} J_\alpha(u) < M_n,$$

where Γ_n is defined in (5.3).

Proof. The proof is similar to that presented in [46, Lemma 5]. From the definition of c_n^α and the fact that $k \geq 0$, $k \not\equiv 0$ in $\mathbb{R}^N$, we deduce that

$$\begin{aligned} c_n^\alpha &= \inf_{K \in \Gamma_n} \max_{u \in K} \left\{ \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\alpha}{2}\|u\|^2_{k,2} - \frac{1}{2 \cdot 2^\star_\mu}\|u\|_\star^{2\cdot 2^\star_\mu} \right\} \\ &< \inf_{K \in \Gamma_n} \max_{u \in K} \left\{ \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{2 \cdot 2^\star_\mu}\|u\|_\star^{2\cdot 2^\star_\mu} \right\} := M_n. \end{aligned}$$

Then, $M_n < \infty$ and $M_n \leq M_{n+1}$ by the definition of Γ_n. □

Proof of Theorem 1.2. According to Lemma 5.3, let us choose $a^\star > 0$ so large that for any $a > a^\star$, we have

$$\sup_n M_n < \frac{1}{4}(aS_{H,L})^{\frac{2N-\mu}{N-\mu+2}} = c^\star.$$

Therefore

$$c_n^\alpha < M_n < \frac{1}{4}(aS_{H,L})^{\frac{2N-\mu}{N-\mu+2}}.$$

Thus, for all $\alpha \in (0, aS\|k\|_r^{-1})$ and $a > a^\star$, we get

$$0 < c_1^\alpha \leq c_2^\alpha \leq \cdots \leq c_n^\alpha < M_n < c^\star.$$

An application of Proposition 5.1 guarantees that the levels $c_1^\alpha \leq c_2^\alpha \leq \cdots \leq c_n^\alpha$ are critical values of J_α. Thus, if $c_1^\alpha < c_2^\alpha < \cdots < c_n^\alpha$, then the functional J_α has at least n critical points. Now, if $c_j^\alpha = c_{j+1}^\alpha$ for some $j = 1, 2, \cdots, k-1$, again Proposition 5.1 implies that $K_{c_j^\alpha}$ is an infinite set, see [42, Chapter 7], and so in this case, (5.1) has infinitely many solutions. Consequently, (5.1) has at least n pairs of solutions in $D^{1,2}(\mathbb{R}^N)$, as stated. □

6 Proof of Theorem 1.3

In this section we require that all the assumptions of Theorem 1.3 are satisfied. Thus, (1.1) becomes

$$-(a + b\|u\|^2)\Delta u = \alpha k(x)|u|^{q-2}u + \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{2^\star_\mu}}{|x-y|^\mu}dy\right)|u|^{2^\star_\mu - 2}u, \quad x \in \mathbb{R}^N. \tag{6.1}$$

This case was investigated in [45, Theorem 1.1] in the fractional Laplacian context. For the convenience of the reader, we present a concise treatment. The aim of this section is to obtain two nontrivial solutions of (6.1). The first is a least energy solution and the latter is a mountain pass solution. To begin with, let us introduce the functional $\mathcal{J}_\alpha$ associated to (6.1)

$$\mathcal{J}_\alpha(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\alpha}{q}\|u\|_{k,q}^q - \frac{1}{2\cdot 2_\mu^*}\|u\|_\star^{2\cdot 2_\mu^*}$$

for all $u \in D^{1,2}(\mathbb{R}^N)$. Since $2 < q < 2^\star$, $4 \le \mu < N$ and $k \in L^r(\mathbb{R}^N)$, with $r = 2^\star/(2^\star - q)$, the Hardy-Littlehood-Sobolev inequality and the Sobolev inequality, show that $\mathcal{J}_\alpha$ is well-defined and of class $C^1\left(D^{1,2}(\mathbb{R}^N)\right)$. Next, we give a compactness result, which is crucial to prove Theorem 1.3.

Lemma 6.1. *Assume that $2 < q < 2^\star$. If either $\mu = 4$, $a > 0$ and $b > 4S_{H,L}^{-1}$ or $\mu > 4$, $a > 0$ and $b > b^\star$, with $b^\star$ given in* (1.3). *Then, the functional $\mathcal{J}_\alpha$ satisfies the $(PS)_c$ condition in $D^{1,2}(\mathbb{R}^N)$ for all $\alpha > 0$, provided that $c < 0$.*

Proof. Let $\alpha > 0$ and let $(u_n)_n$ be a $(PS)_c$ sequence of $\mathcal{J}_\alpha$ in $D^{1,2}(\mathbb{R}^N)$ at any level $c < 0$.

By Lemma 2.1 of [7], in the subcase $s = 1$ and $p = 2$, the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N, k)$ is compact. Therefore,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} k(x)|u_n|^q dx = \int_{\mathbb{R}^N} k(x)|u|^q dx.$$

Moreover, we easily deduce that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} k(x)\left[|u_n|^{q-2}u_n - |u|^{q-2}u\right](u_n - u)dx = 0. \tag{6.2}$$

Put $w_n = u_n - u$ for all n. Without loss of generality, we assume that $\lim_{n\to\infty}\|w_n\| = \ell$. Theorem 2.3 of [40] in the subcase $s = 1$ and $p = 2$, see also [16], yields

$$\|w_n\|_\star^{2\cdot 2_\mu^*} = \|u_n\|_\star^{2\cdot 2_\mu^*} - \|u\|_\star^{2\cdot 2_\mu^*} + o(1).$$

Since $(u_n)_n$ is a $(PS)_c$ sequence, by the boundedness of $(u_n)_n$, we have thanks to (6.2)

$$\begin{aligned}
o(1) &= \langle \mathcal{J}_\alpha'(u_n) - \mathcal{J}_\alpha'(u), u_n - u\rangle \\
&= \left(a + b\|u_n\|^2\right)\int_{\mathbb{R}^N} \nabla u_n \nabla(u_n - u)dx - \left(a + b\|u\|^2\right)\int_{\mathbb{R}^N} \nabla u \nabla(u_n - u)dx \\
&\quad - \alpha \int_{\mathbb{R}^N} k(x)\left[|u_n|^{q-2}u_n - |u|^{q-2}u\right](u_n - u)dx \\
&\quad - \iint_{\mathbb{R}^{2N}} \left[\frac{|u_n(y)|^{2_\mu^*}|u_n(x)|^{2_\mu^*-2}u_n}{|x-y|^\mu} - \frac{|u(y)|^{2_\mu^*}|u(x)|^{2_\mu^*-2}u}{|x-y|^\mu}\right](u_n - u)dxdy \\
&= \left(a + b\|u_n\|^2\right)\left[\int_{\mathbb{R}^N} \nabla u_n \nabla(u_n - u)dx - \int_{\mathbb{R}^N} \nabla u \nabla(u_n - u)dx\right] - \|u_n - u\|_\star^{2\cdot 2_\mu^*} + o(1).
\end{aligned} \tag{6.3}$$

In (6.3) we have used the weak convergence of $(u_n)_n$ in $D^{1,2}(\mathbb{R}^N)$, which implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \nabla u \nabla(u_n - u)dx = 0.$$

Now, (6.3) yields as $n \to \infty$

$$\left(a + b\|u_n\|^2\right)\left[\int_{\mathbb{R}^N} \nabla u_n \nabla(u_n - u)dx - \int_{\mathbb{R}^N} \nabla u \nabla(u_n - u)dx\right] - \|u_n - u\|_\star^{2\cdot 2_\mu^*} = o(1).$$

Thus, as $n \to \infty$

$$\left(a + b\|u_n - u\|^2 + b\|u\|^2\right)\left[\int\limits_{\mathbb{R}^N} \nabla u_n \nabla(u_n - u)dx - \int\limits_{\mathbb{R}^N} \nabla u \nabla(u_n - u)dx\right] - \|u_n - u\|_\star^{2\cdot 2_\mu^\star} = o(1).$$

Let us now recall the following well-known inequality, see [22]: for any $p \geq 2$ there holds

$$\left(|s|^{p-2}s - |t|^{p-2}t\right)(s - t) \geq \frac{1}{2^p}|s - t|^p \tag{6.4}$$

for all $s, t \in \mathbb{R}$. From the inequality (6.4) and the definition of $S_{H,L}$, we get as $n \to \infty$

$$\left(a + b\|u_n - u\|^2 + b\|u\|^2\right)\frac{1}{4}\|u_n - u\|^2 \leq S_{H,L}^{-1}\|u_n - u\|^{2\cdot 2_\mu^\star} + o(1).$$

Letting $n \to \infty$, we have

$$a\ell^2 + b\ell^4 + \ell^2\|u\|^2 \leq 4S_{H,L}^{-1}\ell^{2\cdot 2_\mu^\star},$$

which implies that

$$a\ell^2 + b\ell^4 \leq 4S_{H,L}^{-1}\ell^{2\cdot 2_\mu^\star}. \tag{6.5}$$

When $\mu = 4$ and $4S_{H,L}^{-1} < b$, it follows from (6.5) that $\ell = 0$, since $2 \cdot 2_\mu^\star = 4$. Thus, $u_n \to u$ in $D^{1,2}(\mathbb{R}^N)$. When $\mu > 4$, it follows from (6.5) and the Young inequality that

$$\begin{aligned} a\ell^2 + b\ell^4 \leq\ & \frac{1}{\frac{2}{4-2\cdot 2_\mu^\star}}\left(\ell^{4-2\cdot 2_\mu^\star}\right)^{\frac{2}{4-2\cdot 2_\mu^\star}}\left[\left(\frac{a(4-2\cdot 2_\mu^\star)}{2}\right)^{\frac{2}{4-2\cdot 2_\mu^\star}}\right]^{\frac{4-2\cdot 2_\mu^\star}{2}} \\ & + \frac{1}{\frac{2}{2\cdot 2_\mu^\star-2}}\left(\frac{a(4-2\cdot 2_\mu^\star)}{2}\right)^{-\frac{4-2\cdot 2_\mu^\star}{2\cdot 2_\mu^\star-2}}\left(4S_{H,L}^{-1}\right)^{\frac{2}{2\cdot 2_\mu^\star-2}}\left(\ell^{4\cdot 2_\mu^\star-4}\right)^{\frac{2}{2\cdot 2_\mu^\star-2}} \\ \leq\ & a\ell^2 + (2_\mu^\star - 1)\left(a(2-2_\mu^\star)\right)^{-\frac{2-2_\mu^\star}{2_\mu^\star-1}}\left(4S_{H,L}^{-1}\right)^{\frac{1}{2_\mu^\star-1}}\ell^4 \\ =\ & a\ell^2 + b^\star, \end{aligned}$$

where $b^\star$ is given in (1.3). Therefore, $(b - b^\star)\ell^4 \leq 0$. Hence, assumption (1.3) implies that $\ell = 0$. In conclusion, $u_n \to u$ in $D^{1,2}(\mathbb{R}^N)$ in both cases, as required. □

Proof of Theorem 1.3. First, we show that (6.1) has a nontrivial least energy solution. Clearly,

$$m := \inf_{u\in D^{1,2}(\mathbb{R}^N)} \mathcal{I}_\alpha(u)$$

is well-defined. Now we claim that there exists $\alpha_\star > 0$ such that $m < 0$ for all $\alpha > \alpha_\star$. Indeed, fix a function $v \in D^{1,2}(\mathbb{R}^N)$, with $\|v\| = 1$ and $\|v\|_{k,q} > 0$, which is possible since $k \geq 0$ and $k \not\equiv 0$ in $\mathbb{R}^N$. Then,

$$\mathcal{I}_\alpha(v) = \frac{a}{2} + \frac{b}{4} - \frac{\alpha}{q}\|v\|_{k,q}^q - \frac{1}{2\cdot 2_\mu^\star}\|v\|_\star^{2\cdot 2_\mu^\star} \leq \frac{a}{2} + \frac{b}{4} - \frac{\alpha}{q}\|v\|_{k,q}^q < 0,$$

for all $\alpha > \alpha^\star$, with $\alpha^\star = q\left(\frac{a}{2} + \frac{b}{4}\right)/\|v\|_{k,q}^q$. This proves the claim.

Hence, by Lemma 6.1 and [31, Theorem 4.4], there exists $u_1 \in D^{1,2}(\mathbb{R}^N)$ such that $\mathcal{I}_\alpha(u_1) = m$ and $\mathcal{I}'_\alpha(u_1) = 0$. Therefore, u_1 is a nontrivial least energy solution of (6.1), with $\mathcal{I}_\alpha(u_1) < 0$.

Now we prove that (6.1) has a mountain pass solution. We deduce from (2.2) that

$$\mathcal{I}_\alpha(u) \geq \left[\frac{a}{2} + \frac{b}{4}\|u\|^2 - \alpha\|k\|_r S^{-\frac{q}{2}}\|u\|^{q-2} - \frac{S_{H,L}^{-1}}{2\cdot 2_\mu^\star}\|u\|^{2\cdot 2_\mu^\star-2}\right]\|u\|^2$$

for all $u \in D^{1,2}(\mathbb{R}^N)$. Since $2 < q < 2^*$, there exists $\rho > 0$ small enough and $\varrho > 0$ such that $\mathcal{I}_\alpha(u) > \varrho$ for all $u \in D^{1,2}(\mathbb{R}^N)$, with $\|u\| = \rho$. Define

$$c = \inf_{\xi \in \Xi} \max_{t \in [0,1]} \mathcal{I}_\alpha(\xi(t)),$$

where $\Xi = \{\xi \in C([0,1], D^{1,2}(\mathbb{R}^N)) \,:\, \xi(0) = 0,\ \xi(1) = u_1\}$. Then $c > 0$. Lemma 6.1 yields that $\mathcal{I}_\alpha$ satisfies the assumptions of the mountain pass lemma, see [1, Theorem 2.1]. Hence, there exists $u_2 \in D^{1,2}(\mathbb{R}^N)$ such that $\mathcal{I}_\alpha(u_2) = c > 0$ and $\mathcal{I}'_\alpha(u_2) = 0$. Thus, u_2 is a nontrivial solution of (6.1), independent of u_1. □

7 Proof of Theorem 1.4

In this section we assume, without further mentioning, that all the hypotheses of Theorem 1.4 hold in order to prove multiplicity results for Kirchhoff-type equations with Hardy-Littlewood-Sobolev critical nonlinearity in $\mathbb{R}^3$. Being $\alpha = \beta$, then (1.1) becomes

$$-(a + b\|u\|^2)\Delta u = \beta k(x)|u|^{q-2}u + \beta \left(\int_{\mathbb{R}^3} \frac{|u(y)|^{6-\mu}}{|x-y|^\mu} dy \right) |u|^{4-\mu}u, \quad x \in \mathbb{R}^3, \tag{7.1}$$

where $\beta > 1$, $0 < \mu < 2$, $4 < q < 2^\star_\mu := 6 - \mu$ and $0 < k_\star \le k(x) \le k^\star$ in $\mathbb{R}^3$.

The associated functional J_β to (7.1) is

$$J_\beta(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\beta}{2}\|u\|^q_{k,q} - \frac{\beta}{2(6-\mu)}\|u\|_\star^{2(6-\mu)}$$

for all $u \in D^{1,2}(\mathbb{R}^3)$. Let us first show that J_β has a mountain pass geometry in $D^{1,2}(\mathbb{R}^3)$.

Lemma 7.1. *Let $\beta \in (0,\ aS\|k\|_r^{-1})$. Then J_β satisfies the following conditions.*
(*i*) *There exists $\kappa_\beta, \rho_\beta > 0$ such that $J_\beta(u) \ge \kappa_\beta$ for all $u \in D^{1,2}(\mathbb{R}^3)$, with $\|u\| = \rho_\beta$.*
(*ii*) *There exists $e \in D^{1,2}(\mathbb{R}^3)$ such that $J_\beta(e) < 0$ and $\|e\| > \rho_\beta$.*

Proof. (*i*) The fact that $\beta \in (0,\ aS\|k\|_r^{-1})$, the definitions of S and $S_{H,L}$ give

$$J_\beta(u) \ge \frac{1}{2}(a - \beta S^{-1}\|k\|_{L^r})\|u\|^2 - \frac{S_{H,L}^{-1}}{2(6-\mu)}\|u\|^{2(6-\mu)}.$$

Since $4 < 2(6-\mu)$, we can choose $\kappa_\beta, \rho_\beta > 0$ such that $J_\beta(u) \ge \kappa_\beta$ for all $u \in D^{1,2}(\mathbb{R}^3)$, with $\|u\| = \rho_\beta$.

Let $\varphi \in C_0^\infty(\mathbb{R}^3)$, with $\|\varphi\| > 0$, then as $t \to \infty$

$$J_\beta(t\varphi) \le \frac{a}{2}t^2\|\varphi\|^2 + \frac{b}{4}t^4\|\varphi\|^4 - \frac{1}{2(6-\mu)}t^{2(6-\mu)}\|\varphi\|_\star^{2(6-\mu)} \to -\infty.$$

Hence we choose $t_0 > 0$ so large that $e := t_0\varphi$ verifies (*ii*). □

First, we recall that

$$\inf\{\|\phi\| \,:\, \phi \in C_0^\infty(\mathbb{R}^3,\ \|\phi\|_q = 1\} = 0.$$

For any $\delta \in (0,1)$ there exists $\phi_\delta \in C_0^\infty(\mathbb{R}^3)$, with $\|\phi_\delta\|_q = 1$, supp $\phi_\delta \subset B_{r_\delta}(0)$ and $\|\phi_\delta\|^2 \le \delta$. Set

$$e_\beta(x) = \phi_\delta(\beta^{\frac{1}{5-\mu}}x), \quad x \in \mathbb{R}^3. \tag{7.2}$$

Then we have, for $t \ge 0$,

$$J_\beta(te_\beta) \le \frac{a}{2}t^2\|e_\beta\|^2 + \frac{b}{4}t^4\|e_\beta\|^4 - \frac{k_\star}{q}\beta t^q\|e_\beta\|_q^q$$

$$
\begin{aligned}
&= \frac{a}{2}t^2\beta^{-\frac{1}{5-\mu}}\|\phi_\delta\|^2 + \frac{b}{4}t^4\beta^{-\frac{2}{5-\mu}}\|\phi_\delta\|^4 - \frac{k_\star}{q}\beta t^q\beta^{\frac{-3}{5-\mu}}\|\phi_\delta\|_q^q \\
&= \beta^{-\frac{1}{5-\mu}}\left[\frac{a}{2}t^2\|\phi_\delta\|^2 + \frac{b}{4}t^4\beta^{-\frac{1}{5-\mu}}\|\phi_\delta\|^4 - \frac{k_\star}{q}t^q\beta^{\frac{3-\mu}{5-\mu}}\|\phi_\delta\|_q^q\right] \\
&\le \beta^{-\frac{1}{5-\mu}}\left[\frac{a}{2}t^2\|\phi_\delta\|^2 + \frac{b}{4}t^4\|\phi_\delta\|^4 - \frac{k_\star}{q}t^q\|\phi_\delta\|_q^q\right] \\
&= \beta^{-\frac{1}{5-\mu}}\Psi(t\phi_\delta),
\end{aligned}
\tag{7.3}
$$

since $0 < \mu < 2$ implies that $(3-\mu)/(5-\mu) > 0$, where

$$\Psi_\beta(\phi) := \frac{a}{2}\|\phi\|^2 + \frac{b}{4}\|\phi\|^4 - \frac{k_\star}{q}\|\phi\|_q^q.$$

Since $q > 4$, there exists a finite positive number $t_0 \in \mathbb{R}^+$ such that

$$
\begin{aligned}
\max_{t\ge 0}\Psi_\beta(t\phi_\delta) &= \frac{at_0^2}{2}\|\phi_\delta\|^2 + \frac{t_0^4 b}{4}\|\phi_\delta\|^4 - \frac{k_\star}{q}t_0^q\|u\|_q^q \\
&\le \frac{at_0^2}{2}\|\phi_\delta\|^2 + \frac{t_0^4 b}{4}\|\phi_\delta\|^4 \le \frac{at_0^2}{2}\delta + \frac{t_0^4 b}{4}\delta^2 \\
&\le T^\star\delta, \quad \text{where } T^\star := \frac{at_0^2}{2} + \frac{t_0^4 b}{4}.
\end{aligned}
$$

Therefore,

$$\max_{t\ge 0} J_\beta(t\phi_\delta) \le \beta^{-\frac{1}{2_\mu^\star - 1}}T^\star\delta. \tag{7.4}$$

Lemma 7.2. *Let $4 < q < 6$ and $(u_n)_n$ be a $(PS)_c$ sequence for J_β, with $c < L\beta^{-\frac{1}{5-\mu}}$, where*

$$L := \min\left\{\left(\frac{1}{2}-\frac{1}{q}\right)(aS_{H,L})^{\frac{6-\mu}{5-\mu}},\ \left(\frac{1}{2}-\frac{1}{q}\right)a\left(aS^{\frac{6-\mu}{2}}\hat{C}^{-1}\right)^{\frac{2}{4-\mu}}\right\}. \tag{7.5}$$

Then $(u_n)_n$ contains a strongly convergent subsequence in $D^{1,2}(\mathbb{R}^3)$.

Proof. Let $(u_n)_n$ be a $(PS)_c$ sequence for J_β, as in the statement. Then, it is easy to see that $(u_n)_n$ is bounded in $D^{1,2}(\mathbb{R}^3)$. Next, using the same arguments up to (3.4) as in the proof of Lemma 3.2, we have

$$
\begin{aligned}
c &= \lim_{n\to\infty}\left(J_\beta(u_n) - \frac{1}{q}\langle J_\beta'(u_n), u_n\rangle\right) \\
&\ge \lim_{n\to\infty}\left\{\left(\frac{1}{2}-\frac{1}{q}\right)a\|u_n\|^2 + \left(\frac{1}{4}-\frac{1}{q}\right)b\|u_n\|^4\right\} + \left(\frac{1}{q} - \frac{1}{2\cdot 2_\mu^\star}\right)\beta\|u\|_\star^{2(6-\mu)} \\
&\ge \left(\frac{1}{2}-\frac{1}{q}\right)(aS_{H,L})^{\frac{6-\mu}{5-\mu}}\beta^{-\frac{1}{5-\mu}}.
\end{aligned}
\tag{7.6}
$$

Similarly, it follows from (3.7) that

$$c \ge \left(\frac{1}{2}-\frac{1}{q}\right)(aS)^{\frac{6-\mu}{4-\mu}}\hat{C}^{-\frac{2}{4-\mu}}\beta^{-\frac{2}{4-\mu}}. \tag{7.7}$$

Therefore, the compactness of the Palais-Smale sequence holds, since $\beta > 1$ and $0 < \mu < 2$. □

Proof of Theorem 1.4 (*i*). Fix $\delta \in (0,1)$. Then, Lemma 7.1 implies that J_β possesses a $(PS)_{c_\beta}$ sequence, with $c_\beta \ge \kappa_\beta > 0$, where

$$c_\beta := \inf_{\gamma\in\Gamma_\beta}\max_{t\in[0,1]} J_\beta(\gamma(t)),$$

where

$$\Gamma_\beta := \left\{\gamma \in C\big([0,1], D^{1,2}(\mathbb{R}^3)\big) \ :\ \gamma(0) = 0 \quad\text{and}\quad \gamma(1) = e_\beta\right\}.$$

Thus, (7.4) gives that

$$0 < \kappa_\beta \le c_\beta \le T^\star \delta \beta^{-\frac{1}{5-\mu}}.$$

Furthermore, Lemma 7.2 guarantees that J_β satisfies the $(PS)_{c_\beta}$ condition. Hence, there is u_β in $D^{1,2}(\mathbb{R}^3)$ such that $J'_\beta(u_\beta) = 0$ and $J_\beta(u_\beta) = c_\beta$. Moreover, it is well-known that such a mountain pass solution is a least energy solution of (7.1).

Because u_β is a critical point of J_β, for any $\iota \in [q, 6-\mu]$,

$$\begin{aligned} T^\star \delta \beta^{-\frac{1}{5-\mu}} &\ge J_\beta(u_\beta) = J_\beta(u_\beta) - \frac{1}{\iota} J'_\beta(u_\beta) u_\beta \\ &= \left(\frac{1}{2} - \frac{1}{\iota}\right) a\|u_\beta\|^2 + \left(\frac{1}{4} - \frac{1}{\iota}\right) b\|u_\beta\|^4 + \left(\frac{1}{\iota} - \frac{1}{q}\right) \beta \int_{\mathbb{R}^3} k(x)|u_\beta|^q dx \\ &\quad + \left(\frac{1}{\iota} - \frac{1}{2\cdot 2^\star_\mu}\right) \beta \|u_\beta\|_\star^{2(6-\mu)}. \end{aligned}$$

Taking $\iota = q$, we obtain the estimates $\|u_\beta\| \to 0$ as $\beta \to \infty$. This completes the proof of part (*i*). □

For any $m^\star \in \mathbb{N}$ we choose $m^\star$ functions $\phi^i_\delta \in C_0^\infty(\mathbb{R}^3)$ such that $\operatorname{supp} \phi^i_\delta \cap \operatorname{supp} \phi^k_\delta = \emptyset$, for $i \neq k$, $\|\phi^i_\delta\|_q = 1$ and $\|\phi^i_\delta\|^2 < \delta$. Let $r^{m^\star}_\delta > 0$ be such that $\operatorname{supp} \phi^i_\delta \subset B^i_{r_\delta}(0)$ for $i = 1, 2, \cdots, m^\star$. Set

$$e^i_\beta(x) = \phi^i_\delta(\beta^{\frac{1}{5-\mu}} x) \quad x \in \mathbb{R}^3,\ i = 1, 2, \cdots, m^\star \tag{7.8}$$

and $H^{m^\star}_{\beta\delta} = \operatorname{span}\{e^1_\beta, e^2_\beta, \cdots, e^{m^\star}_\beta\}$. Arguing as in (7.4) and (7.6), we obtain for each $u = \sum_{i=1}^{m^\star} c_i e^i_\beta \in H^{m^\star}_{\beta\delta}$ that

$$J_\beta(c_i e^i_\beta) \le \beta^{-\frac{1}{5-\mu}} \Psi(|c_i| e^i_\beta).$$

Proceeding as in case (*i*) above, we get that

$$\max_{u \in H^{m^\star}_{\beta\delta}} J_\beta(u) \le m^\star T^\star \delta \beta^{-\frac{1}{5-\mu}}. \tag{7.9}$$

Lemma 7.3. *For any $m^\star \in \mathbb{N}$ and $\beta > 0$ there exists an $m^\star$-dimensional subspace $F_{\beta m^\star}$ such that*

$$\max_{u \in F_{\beta m^\star}} J_\beta(u) \le L\beta^{-\frac{1}{5-\mu}},$$

where $L > 0$ is given in (7.5).

Proof. Choose $\delta \in (0, 1)$ so small that $m^\star T^\star \delta \le L$. Taking $F_{\beta m^\star} = H^{m^\star}_{\beta\delta}$, then from (7.9) we know that the conclusion of Lemma 7.3 holds. □

Proof of Theorem 1.4 (*ii*)**.** Denote the set of all symmetric (in the sense that $-Z = Z$) and closed subsets of $D^{1,2}(\mathbb{R}^3)$ by Σ. For each $Z \in \Sigma$. Let $\operatorname{gen}(Z)$ be the Krasnoselkski genus and

$$j(Z) := \min_{\varsigma \in \Gamma_{m^\star}} \operatorname{gen}(\varsigma(Z) \cap \partial B_{\rho_\beta}),$$

where $\Gamma_{m^\star}$ is the set of all odd homeomorphisms $\varsigma \in C(E, E)$ and ρ_β is the number given in Lemma 7.1. Then j is a version of Benci's pseudoindex (see [4]). Let

$$c_{\beta_i} := \inf_{j(Z) \ge i} \sup_{u \in Z} J_\beta(u), \quad 1 \le i \le m^\star.$$

Since $J_\beta(u) \ge \kappa_\beta$ for all $u \in \partial B^+_{\rho_\beta}$ and since $j(F_{\beta m^\star}) = \dim F_{\beta m^\star} = m^\star$,

$$\kappa_\beta \le c_{\beta_1} \le \cdots \le c_{\lambda m^\star} \le \sup_{u \in H_{\beta_{m^\star}}} J_\beta(u) \le L\beta^{-\frac{1}{5-\mu}}.$$

It follows from Lemma 7.2 that J_β satisfies the $(PS)_c$ condition at all levels $c < L\beta^{-\frac{1}{5-\mu}}$. By the usual critical point theory, all $c_{\beta i}$ are critical levels and J_β has at least $m^\star$ pairs of nontrivial critical points which tend to zero as $\beta \to \infty$. □

Acknowledgments: S. Liang would like to thank Professor S. Peng for several useful and valuable discussions during his visit at the *Central China Normal University*, as visiting scholar.

S. Liang was supported by the Foundation for China Postdoctoral Science Foundation (Grant no. 2019M662220), Natural Science Foundation of Jilin Province, Research Foundation during the 13th Five-Year Plan Period of Department of Education of Jilin Province, China (JJKH20181161KJ), Natural Science Foundation of Changchun Normal University (No. 2017-09).

P. Pucci is a member of the *Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni* (GNAMPA) of the *Istituto Nazionale di Alta Matematica* (INdAM). P. Pucci was partly supported by of the *Fondo Ricerca di Base di Ateneo – Esercizio 2017–2019* of the University of Perugia, named *PDEs and Nonlinear Analysis*.

B. Zhang was supported by the National Natural Science Foundation of China (No. 11871199), the Heilongjiang Province Postdoctoral Startup Foundation (LBH-Q18109), and the Cultivation Project of Young and Innovative Talents in Universities of Shandong Province.

Statement: Prof. Binlin Zhang and Prof. Patrizia Pucci were an Editors of the ANONA although had no involvement in the final decision.

References

[1] A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349–381.

[2] C.O. Alves, F.J. Correa, G.M. Figueiredo, On a class of nonlocal elliptic problems with critical growth, Differ. Equ. Appl. 2 (2010) 409–417.

[3] C.O. Alves, V. Rădulescu, L.S. Tavares, Generalized Choquard equations driven by nonhomogeneous operators, Mediterr. J. Math. 16 (2019) 1–24.

[4] V. Benci, On critical point theory of indefinite functionals in the presence of symmetries, Trans. Amer. Math. Soc. 274 (1982) 533–572.

[5] A. K. Ben-Naoum, C. Troestler, M. Willem, Extrema problems with critical Sobolev exponents on unbounded domains, Nonlinear Anal. 26 (1996) 823–833.

[6] G. Bianchi, J. Chabrowski, A. Szulkin, On symmetric solutions of an elliptic equation with a nonlinearity involving critical Sobolev exponent, Nonlinear Anal. 25 (1995) 41–59.

[7] M. Caponi, P. Pucci, Existence theorems for entire solutions of stationary Kirchhoff fractional p-Laplacian equations, Ann. Mat. Pura Appl. 195 (2016) 2099–2129.

[8] J. Chabrowski, Concentration–compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents, Calc. Var. Partial Differential Equations, 3 (1995) 493–512.

[9] K. Chang, Critical Point Theory and Applications, Shanghai: Shanghai Scientific and Technology Press; 1986.

[10] C. Chen, Y. Kuo, T.F. Wu, The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions, J. Differential Equations, 250 (2011) 1876–1908.

[11] J. Chen, S. Li, On multiple solutions of a singular quasi-linear equation on unbounded domain, J. Math. Anal. Appl. 275 (2002) 733–746.

[12] G.M. Figueiredo, Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument, J. Math. Anal. Appl. 401 (2013) 706–713.

[13] G.M. Figueiredo, J.R. Junior, Multiplicity of solutions for a Kirchhoff equation with subcritical or critical growth, Differ. Integral Equ. 25 (2012) 853–868.

[14] I. Fonseca, G. Leoni, Modern methods in the calculus of variations: L^p spaces, Springer Monographs in Mathematics, Springer, New York, 2007, xiv+599 pp.

[15] F. Gao, M. Yang, On nonlocal Choquard equations with Hardy–Littlewood–Sobolev critical exponents, J. Math. Anal. Appl. 448 (2017) 1006–1041.

[16] F. Gao, M. Yang, On the Brézis–Nirenberg type critical problem for the nonlinear Choquard equation, Sci. China Math. 61 (2018) 1219–1242.

[17] F. Gao, E.D. da Silva, M. Yang, J. Zhou, Existence of solutions for critical Choquard equations via the concentration compactness method, Proc. Roy. Soc. Edinburgh Sect. A 150 (2020) 921–954.

[18] F. Gao, M. Yang, C.A. Santos, J. Zhou, Infinitely many solutions for a class of critical Choquard equation with zero mass, Topol. Methods Nonlinear Anal. 54 (2019) 219–232.

[19] J. Garcia Azorero, I. Peral, Hardy inequalities and some critical elliptic and parabolic problems, J. Differential Equations 144 (1998) 441–476.
[20] J. Giacomoni, T. Mukherjee, K. Sreenadh, Doubly nonlocal system with Hardy-Littlewood-Sobolev critical nonlinearity, J. Math. Anal. Appl. 467 (2018) 638–672.
[21] R. Kajikiya, A critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations, J. Funct. Anal. 225 (2005) 352–370.
[22] S. Kichenassamy, L. Véron, Singular solutions of the p-Laplace equation, Math. Ann. 275 (1985) 599–615.
[23] M.A. Krasnoselskii, Topological Methods in the Theory of Nonlinear Integral Equations, New York: MacMillan; 1964.
[24] S. Liang, S. Shi, Soliton solutions to Kirchhoff type problems involving the critical growth in $\mathbb{R}^N$, Nonlinear Anal. 81 (2013) 31–41.
[25] S. Liang, J. Zhang, Existence of solutions for Kirchhoff type problems with critical nonlinearity in $\mathbb{R}^3$, Nonlinear Anal. Real World Appl. 17 (2014) 126–136.
[26] S. Liang, J. Zhang, Multiplicity of solutions for the noncooperative Schrödinger–Kirchhoff system involving the fractional p–Laplacian in $\mathbb{R}^N$, Z. Angew. Math. Phys. (2017) Art. 63, 18 pp.
[27] E.H. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation, Stud. Appl. Math. 57 (1976/1977) 93–105.
[28] E.H. Lieb, M. Loss, Analysis, 2nd edition, Graduate Studies in Mathematics, vol. 14. AMS, Providence, Rhode island, 2001.
[29] P.L. Lions, The concentration compactness principle in the calculus of variations. The locally compact case. Part I and II, Ann. Inst. H. Poincaré Anal. Non. Lineaire. 1 (1984) 109–145 and 223–283.
[30] P.L. Lions, The Choquard equation and related questions, Nonlinear Anal. 4 (1980) 1063–1072.
[31] J. Mawhin, M. Willem, Critical Point Theory and Hamiltonian Systems, Springer, New York (1989).
[32] X. Mingqi, V. Rădulescu, B. Zhang, A critical fractional Choquard-Kirchhoff problem with magnetic field, Commun. Contemp. Math. 21 (2019), 1850004.
[33] A. Mohammed, V. Rădulescu, A. Vitolo, Blow-up solutions for fully nonlinear equations: existence, asymptotic estimates and uniqueness, Adv. Nonlinear Anal. 9 (2020) 39–64.
[34] V. Moroz, J. Van Schaftingen, Existence of groundstates for a class of nonlinear Choquard equations, Trans. Amer. Math. Soc. 367 (2015) 6557–6579.
[35] V. Moroz, J. Van Schaftingen, Groundstates of nonlinear Choquard equations: Hardy-Littlewood-Sobolev critical exponent, Commun. Contemp. Math. 17 (2015) 1550005.
[36] V. Moroz, J. Van Schaftingen, Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics, J. Funct. Anal. 265 (2013) 153–184.
[37] N.S. Papageorgiou, V. Rădulescu, D. Repovš, Ground state and nodal solutions for a class of double phase problems, Z. Angew. Math. Phys. 71 (2020) 1–15.
[38] R. Penrose, Quantum computation, entanglement and state reduction, Philos. Trans. Roy. Soc. 356 (1998) 1–13.
[39] P. Pucci, V. Rădulescu, Progress in nonlinear Kirchhoff problems, Nonlinear Anal, 186 (2019) 1–5.
[40] P. Pucci, M. Xiang, B. Zhang, Existence results for Schrödinger-Choquard-Kirchhoff equations involving the fractional p–Laplacian, Adv. Calc. Var. 12 (2019) 253–275.
[41] S. Pekar, Untersuchung über die Elektronentheorie der Kristalle, Akademie Verlag, Berlin, 1954.
[42] P.H. Rabinowitz, Minimax methods in critical point theory with applications to diferential equations, in: CBMS Reg. Conf. Series in Math, Vol. 65, 1984.
[43] Z. Shen, F. Gao, M. Yang, Multiple solutions for nonhomogeneous Choquard equation involving Hardy-Littlewood-Sobolev critical exponent, Z. Angew. Math. Phys. 68 (2017) Art. 61, 25 pp.
[44] G. Talenti, Best constants in Sobolev inequality, Annali di Mat. 110 (1976) 353–372.
[45] F. Wang, M. Xiang, Multiplicity of solutions for a class of fractional Choquard–Kirchhoff equations involving critical nonlinearity, Anal. Math. Phys. 9 (2019) 1–16.
[46] Z.H. Wei, X.M. Wu, A multiplicity result for quasilinear elliptic equations involving critical Sobolev exponents, Nonlinear Anal. 18 (1992) 559–567.
[47] M. Xiang, B. Zhang, V. Rădulescu, Superlinear Schrödinger-Kirchhoff type problems involving the fractional p-Laplacian and critical exponent, Adv. Nonlinear Anal. 9 (2020) 690–709.
[48] M. Xiang, B. Zhang, V. Rădulescu, Multiplicity of solutions for a class of quasilinear Kirchhoff system involving the fractional p-Laplacian, Nonlinearity, 29 (2016) 3186–3205.