Kupenko and Manzo *Boundary Value Problems* 2014, 2014:72
http://www.boundaryvalueproblems.com/content/2014/1/72

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

On optimality conditions for optimal control problem in coefficients for Δ_p-Laplacian

Olha P Kupenko[1,2] and Rosanna Manzo[3*]

*Correspondence: rmanzo@unisa.it
[3]Dipartimento di Ingegneria dell'Informazione, Ingegneria Elettrica e Matematica Applicata, Università degli Studi di Salerno, Via Giovanni Paolo II, 132, Fisciano, 84084, Italy
Full list of author information is available at the end of the article

Abstract

In this paper we study an optimal control problem for a nonlinear monotone Dirichlet problem where the control is taken as $L^\infty(\Omega)$-coefficient of Δ_p-Laplacian. Given a cost function, the objective is to derive first-order optimality conditions and provide their substantiation. We propose some ideas and new results concerning the differentiability properties of the Lagrange functional associated with the considered control problem. The obtained adjoint boundary value problem is not coercive and, hence, it may admit infinitely many solutions. That is why we concentrate not only on deriving the adjoint system, but also, following the well-known Hardy-Poincaré Inequality, on a formulation of sufficient conditions which would guarantee the uniqueness of the adjoint state to the optimal pair.

MSC: 35J70; 49J20; 49J45; 93C73

Keywords: nonlinear Dirichlet problem; control in coefficients; optimality conditions

1 Introduction

The aim of this paper is to derive a first-order optimality system for a nonlinear Dirichlet optimal control problem where the control is taken as an L^∞-coefficient in a nonlinear state equation. The optimal control problem we consider in this paper is to minimize the discrepancy $\|y - y_d\|^p_{W_0^{1,p}(\Omega)}$, where $p \geq 2$, Ω is an open bounded Lipschitz domain in $\mathbb{R}^N$ with $N \geq 2$, $y_d \in W_0^{1,p}(\Omega)$ is a given distribution, and y is the solution of a nonlinear Dirichlet problem by choosing an appropriate coefficient $u \in L^\infty(\Omega)$ of Δ_p-Laplacian. Namely, we consider the following minimization problem:

$$\text{Minimize } \left\{ I(u,y) = \int_\Omega \left|\nabla y(x) - \nabla y_d(x)\right|^p dx \right\} \tag{1.1}$$

subject to the constraints

$$u \in \mathfrak{A}_{\text{ad}} \subset L^\infty(\Omega) \cap BV(\Omega), \quad y \in W_0^{1,p}(\Omega), \tag{1.2}$$

$$-\operatorname{div}\left(u|\nabla y|^{p-2}\nabla y\right) = f \quad \text{in } \Omega, \tag{1.3}$$

$$y = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

where $\mathfrak{A}_{\text{ad}}$ is a class of admissible controls and $f \in W^{-1,q}(\Omega)$, $q = p/(p-1)$.

Optimal control in coefficients for partial differential equations is a classical subject initiated by Lurie [1, 2], Lions [3], Zolezzi [4]. Tartar and Murat [5–7] showed examples of the non-existence for such problems (see *e.g.* [8] also for the historical development). Since the range of OCPs in coefficients is very wide, including as well optimal shape design problems, optimization of certain evolution systems, some problems originating in mechanics and others, this topic has been widely studied by many authors. In particular, it leads to the possibility to optimize material properties what is extremely important for material sciences. The crucial point is to give the right interpretation of the optimal coefficients in the context of applications (see, for instance, [9, 10]). Usually this aspect is closely related with the structural assumptions that have to be considered during the optimization process in terms of constraints. One way of doing so is via proper parametrization of the material, respectively, the coefficients, using mixtures, represented by characteristic functions. This has been pursued by Allaire [11] and many other authors in recent years. Another restriction can be realized via regularity of the coefficients and hard constraints. This procedure has been followed first by Casas [12] for a scalar problem, as one of the first papers in that direction, and later by Haslinger *et al.* [13] in the context of what has come to be known as *Free Material Optimization* (FMO). However, most of the results and methods rely on linear PDEs, while only very few articles deal with nonlinear problems, see Kogut [14] and Kogut and Leugering [15]. Another point of interest is degeneration in the coefficients which is typically avoided by assuming lower bounds on the coefficients. However, degeneration occurs genuinely in topology optimization, damage and crack problems. In Kogut and Leugering [16–18] and in Kupenko and Manzo [19] this problem has been considered in the context of linear problems (see also [20]). The nonlinear case was considered in [21–24]. In this article, we extend our results to scalar nonlinear problems, where degeneration occurs already with respect to the states.

Another important point, arising after the solvability of the optimization problem had been proved, is the question as regards optimality conditions. The classical approach to deriving such conditions is based on the Lagrange principle. However, in the case when the control is considered in the coefficients of the main part of the state equation, the classical adjoint system often cannot be directly constructed due to the lack of differential properties of the solution to the boundary value problem with respect to control variables. It was the main reason why Serovajskiy has proposed the concept of the so-called quasi-adjoint system [25] and showed that optimality conditions for the linear elliptic control problem in coefficients can be derived, provided the mapping $u \mapsto \psi_\varepsilon(u)$ possesses the weakened continuity property. However, the verification of this property is not easy matter even for linear systems. In the case of quasi-linear or nonlinear state equations, we are faced with another problem - the Lagrange functional to the indicated problem is not Gâteaux differentiable at the origin. To overcome this difficulty, Casas and Fernández introduced the special family of perturbed optimal control problems and derived the optimality conditions passing to the limit in optimality conditions for approximating control problems. In order to apply this approach to optimal control problem (1.1)-(1.4) it would suffice to assume the following extra conditions: $p > N/2$ and $\mathfrak{A}_{\mathrm{ad}} \subset C^1(\Omega)$ that look rather restrictive from physical point of view. The second option coming from the approach of Casas and Fernández is the fact that the linear elliptic equation for the adjoint state is not coercive in general and, hence, the adjoint boundary value problem may admit infinitely many solutions. As a result, the attainability of some solutions is rather a questionable matter. That is why in this paper we concentrate not only on deriving of the adjoint system, but also on formulation of sufficient conditions which would guarantee the uniqueness of the adjoint state to the optimal pair.

The paper is organized as follows. In Section 2 we give some preliminaries and prescribe the class of admissible controls to problem (1.1)-(1.4). In Section 3 we analyze the solvability properties of optimal control problem (1.1)-(1.4), using the monotonicity of generalized Δ_p-Laplacian (see for comparison [26, 27]). The aim of Section 4 is to give a collection of preliminary results concerning the differentiability properties of the Lagrange functional associated with problem (1.1)-(1.4)

$$\Lambda(u,y,\lambda) = I(u,y) + a_u(y,\lambda) - \langle f,\lambda\rangle_{W_0^{1,p}(\Omega)},$$

where

$$a_u(y,\lambda) = \int_\Omega u(x)|\nabla y|^{p-2}(\nabla y,\nabla\lambda)_{\mathbb{R}^N}\,dx$$

and show that it admits the Gâteaux derivative with respect to the so-called non-degenerate directions $h \in W_0^{1,p}(\Omega)$ at the point y.

In Section 5 we discuss the formal approach in deriving first-order optimality conditions for optimal control problem (1.1)-(1.4). In order to derive an optimality system, we apply the Lagrange principle. It is well known that the proof of this principle is different for different classes of optimal control problem (see, for instance, [10, 12, 23, 28–30]). The complexity of this procedure significantly depends on the form of the extremal problem under consideration. The procedure is rather simple if the controllable system is described by a linear well-posed controllable boundary value problem, but it becomes much more complicated if the controllable system is either ill-posed or nonlinear and singular.

With that in mind, we introduce the notion of a quasi-adjoint state ψ_ε to an optimal solution $y_0 \in W_0^{1,p}(\Omega)$ as a solution of the following Dirichlet boundary value problem for degenerate linear elliptic equation:

$$\begin{aligned} &-\operatorname{div}\left(u_\theta|\nabla y_\theta|^{p-2}\left[I + (p-2)\frac{\nabla y_\theta}{|\nabla y_\theta|}\otimes\frac{\nabla y_\theta}{|\nabla y_\theta|}\right]\nabla\psi_\theta\right) \\ &\quad = p\operatorname{div}\big(|\nabla y_\theta - \nabla y_d|^{p-2}(\nabla y_\theta - \nabla y_d)\big) \quad \text{in } \Omega, \psi_\theta \in W_0^{1,p}(\Omega), \end{aligned} \tag{1.5}$$

where the degeneration occurs in a natural way with respect to the states.

This concept was proposed for linear problems by Serovajskiy [25], where it was shown that an optimality system for the optimal control problems in coefficients can be recovered in an explicit form if the mapping $\mathfrak{A}_{\mathrm{ad}} \ni u \mapsto \psi_\varepsilon(u)$ possesses the so-called weakened continuity property. However, it should be stressed that the fulfilment of this property is not proved for the case of Δ_p-Laplacian with $p > 2$ and, thus, should be considered as some extra hypothesis. Moreover, from a practical point of view, the verification of the weakened continuity property for quasi-adjoint states is not an easy matter, in general. That is why, in order to derive optimality conditions in the framework of more appropriate assumptions, we provide in Section 6 the analysis of the well-posedness of variational problem (1.5) and describe the asymptotic behavior of its solutions as parameter θ tends to zero. However, in contrast to Casas and Fernandez [29], we do not apply a perturbation of the differential operator that removes the singularity at the origin.

In particular, following the well-known Hardy-Poincaré Inequality, we show that the sequence of quasi-adjoint states $\{\psi_{\varepsilon_\theta,\theta}\}_{\theta\to 0}$ to $y_0 \in W_0^{1,p}(\Omega)$ can be defined in a unique way (as unique solutions for (1.5)) and is bounded in $W_0^{1,p}(\Omega)$ provided, for given distributions $f \in W^{-1,q}(\Omega)$ and $y_d \in W_0^{1,p}(\Omega)$ with $q = \frac{p}{p-1}$ and $p \geq 2$, an optimal solution $y_0 = y(u_0)$ to the nonlinear Dirichlet boundary value problem (1.3)-(1.4) satisfies the properties

$$\nabla \ln|\nabla y_0| \in L^2(\Omega;\mathbb{R}^N), \qquad \nabla(|\nabla y_0|) \in L^p(\Omega;\mathbb{R}^N) \quad \text{and} \quad \frac{|\nabla y_d|}{|\nabla y_0|} \in L^\infty(\Omega),$$

$$-\widehat{C}(\Omega) \leq V(x) \leq \frac{2\lambda}{L}\sum_{i=1}^{L}\frac{1}{|x-x_i|^2} \quad \text{a.e. in } \Omega,$$

where

$$V(x) = (2-p)\operatorname{div}\big(A(u_0,y_0)\nabla \ln|\nabla y_0|\big) - \frac{(p-2)^2}{2}\big(\nabla \ln|\nabla y_0|, A(u_0,y_0)\nabla \ln|\nabla y_0|\big)_{\mathbb{R}^N},$$

$$A(u_0,y_0) = u_0\left[I + (p-2)\frac{\nabla y_0}{|\nabla y_0|} \otimes \frac{\nabla y_0}{|\nabla y_0|}\right],$$

for some positive constants $\widehat{C}(\Omega) > 0$, $\lambda < \lambda^* := (N-2)^2/4$, and some collection of points $\{x_1, x_2, \ldots, x_L\} \subset \Omega$.

Note that the fulfilment of this assumption is feasible if the matrix $|\nabla y|^{p-2}A(u,y)$ has a non-degenerate spectrum for each $\mathcal{U} \in U_{\mathrm{ad}}$ (for the details, we refer to [17]). The main argument given in Sections 6-7 is to look for the solutions $\psi_\theta \in W_0^{1,p}(\Omega)$ of the quasi-adjoint problem (1.5) in the form $\psi_\theta = |\nabla y_\theta|^{(2-p)/2} z_\theta$, where $z_\theta \in H_0^1(\Omega)$. As a result, we show that each of the variational problems for the corresponding quasi-adjoint states has a unique solution, and these solutions form a weakly convergent sequence $\{\psi_\theta\}_{\theta\to 0}$ in $W_0^{1,p}(\Omega)$. This property suffices in order to establish that the optimality system for problem (1.1)-(1.4) remains valid even if the matrix $|\nabla y|^{p-2}A(u,y)$ has a degenerate spectrum.

2 Notation and preliminaries

Throughout the paper Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$. The space $\mathcal{D}'(\Omega)$ of distributions in Ω is the dual of the space $C_0^\infty(\Omega)$. For real numbers $2 \leq p < +\infty$, and $1 < q < +\infty$ such that $1/p + 1/q = 1$, the space $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the Sobolev space $W^{1,p}(\Omega)$, while $W^{-1,q}(\Omega)$ is the space of distributions of the form $f = f_0 + \sum_j D_j f_j$, with $f_0, f_1, \ldots, f_n \in L^q(\Omega)$ (*i.e.* $W^{-1,q}(\Omega)$ is the dual space of $W_0^{1,p}(\Omega)$). As a norm in the space $W_0^{1,p}(\Omega)$ we can take the following one:

$$\|y\|_{W_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla y|_{\mathbb{R}^N}^p\right)^{1/p}.$$

Let χ_E be the characteristic function of a set $E \subset \mathbb{R}^N$ and let $|E|$ be its N-dimensional Lebesgue measure.

For any vector field $\mathbf{v} \in L^q(\Omega;\mathbb{R}^N)$, the divergence is an element of the space $W^{-1,q}(\Omega)$ defined by the formula

$$\langle \operatorname{div} v, \varphi\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} = -\int_\Omega (v, \nabla\varphi)_{\mathbb{R}^N}\,dx, \quad \forall \varphi \in W_0^{1,p}(\Omega), \tag{2.1}$$

where $\langle \cdot, \cdot \rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)}$ denotes the duality pairing between $W^{-1,q}(\Omega)$ and $W_0^{1,p}(\Omega)$, and $(\cdot, \cdot)_{\mathbb{R}^N}$ denotes the scalar product of two vectors in $\mathbb{R}^N$.

Functions with bounded variations. Let $f : \Omega \to \mathbb{R}$ be a function of $L^1(\Omega)$. Define

$$TV(f) := \int_\Omega |Df| = \sup\left\{ \int_\Omega f(\nabla, \varphi)_{\mathbb{R}^N}\, dx : \varphi \in C_0^1(\Omega; \mathbb{R}^N), |\varphi(x)| \le 1 \text{ for } x \in \Omega \right\},$$

where $(\nabla, \varphi)_{\mathbb{R}^N} = \sum_{i=1}^N \frac{\partial \varphi_i}{\partial x_i}$.

According to the Radon-Nikodym Theorem, if $TV(f) < +\infty$ then the distribution Df is a measure and there exist a vector-valued function $\nabla f \in L^1(\Omega; \mathbb{R}^N)$ and a measure $D_s f$, singular with respect to the N-dimensional Lebesgue measure $\mathcal{L}^N \lfloor \Omega$ restricted to Ω, such that $Df = \nabla f \mathcal{L}^N \lfloor \Omega + D_s f$.

Definition 2.1 A function $f \in L^1(\Omega)$ is said to have a bounded variation in Ω if $TV(f) < +\infty$. By $BV(\Omega)$ we denote the space of all functions in $L^1(\Omega)$ with bounded variation, *i.e.* $BV(\Omega) = \{f \in L^1(\Omega) : TV(f) < +\infty\}$.

Under the norm $\|f\|_{BV(\Omega)} = \|f\|_{L^1(\Omega)} + TV(f)$, $BV(\Omega)$ is a Banach space. For our further analysis, we need the following properties of BV-functions (see [31]).

Proposition 2.2

(i) *Let $\{f_k\}_{k=1}^\infty$ be a sequence in $BV(\Omega)$ strongly converging to some f in $L^1(\Omega)$ and satisfying condition* $\sup_{k\in\mathbb{N}} TV(f_k) < +\infty$. *Then*

$$f \in BV(\Omega) \quad \textit{and} \quad TV(f) \le \liminf_{k\to\infty} TV(f_k);$$

(ii) *for every $f \in BV(\Omega) \cap L^r(\Omega)$, $r \in [1, +\infty)$, there exists a sequence $\{f_k\}_{k=1}^\infty \subset C^\infty(\Omega)$ such that*

$$\lim_{k\to\infty} \int_\Omega |f - f_k|^r\, dx = 0 \quad \textit{and} \quad \lim_{k\to\infty} TV(f_k) = TV(f);$$

(iii) *for every bounded sequence $\{f_k\}_{k=1}^\infty \subset BV(\Omega)$ there exist a subsequence, still denoted by f_k, and a function $f \in BV(\Omega)$ such that $f_k \to f$ in $L^1(\Omega)$.*

Admissible Controls and Generalized p-Laplacian. Let α, β, γ, and m be given positive constants such that $0 < \alpha \le \beta < +\infty$ and $\alpha|\Omega| \le m \le \beta|\Omega|$. We define the class of admissible controls $\mathfrak{A}_{\mathrm{ad}}$ as follows:

$$\mathfrak{A}_{\mathrm{ad}} = \left\{ u \in BV(\Omega) \cap L^\infty(\Omega) | TV(u) \le \gamma, \|u\|_{L^1(\Omega)} = m, \alpha \le u(x) \le \beta \text{ a.e. in } \Omega \right\}. \tag{2.2}$$

It is clear that $\mathfrak{A}_{\mathrm{ad}}$ is a nonempty convex subset of $L^1(\Omega)$ with empty topological interior.

We say that a nonlinear operator $\Delta_p : \mathfrak{A}_{\mathrm{ad}} \times W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ is the generalized p-Laplacian if it has a representation

$$\Delta_p(u, y) = -\operatorname{div}\big(u(x)|\nabla y|^{p-2}\nabla y\big), \quad \text{where } |\nabla y|^{p-2} := |\nabla y|_{\mathbb{R}^N}^{p-2} = \left(\sum_{i=1}^N \left| \frac{\partial y}{\partial x_j} \right|^2 \right)^{\frac{p-2}{2}},$$

or via the pairing

$$\langle \Delta_p(u,y), v\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} = \int_\Omega u(x)|\nabla y|^{p-2}(\nabla y, \nabla v)_{\mathbb{R}^N}\,dx, \quad \forall v \in W_0^{1,p}(\Omega).$$

It is easy to see that for every admissible control $u \in \mathfrak{A}_{\mathrm{ad}}$, the operator $\Delta_p(u,\cdot)$ turns out to be coercive, *i.e.*

$$\langle \Delta_p(u,y), y\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} \ge \alpha \|y\|^p_{W_0^{1,p}(\Omega)},$$

and demi-continuous, where by the demi-continuity property we mean the fulfilment of the implication: $y_k \to y_0$ strongly in $W_0^{1,p}(\Omega)$ implies that $\Delta_p(u,y_k) \rightharpoonup \Delta_p(u,y_0)$ weakly in $W^{-1,q}(\Omega)$ (see [26, 32]). Moreover, p-Laplacian $\Delta_p(u,y)$ is a strictly monotone operator for each $u \in \mathfrak{A}_{\mathrm{ad}}$. Indeed, having applied the trick

$$\begin{aligned}
\int_\Omega |\nabla y|^{p-1}|\nabla v|u(x)\,dx &\le \left(\int_\Omega \left(|\nabla y|^{p-1}u^{\frac{p-1}{p}}\right)^{\frac{p}{p-1}}dx\right)^{(p-1)/p}\left(\int_\Omega \left(|\nabla v|u^{\frac{1}{p}}\right)^p dx\right)^{1/p}\\
&= \left(\int_\Omega |\nabla y|^p u\,dx\right)^{(p-1)/p}\left(\int_\Omega |\nabla v|^p u\,dx\right)^{1/p}\\
&=: \|y\|^{p-1}_{W_0^{1,p}(\Omega;u\,dx)}\|v\|_{W_0^{1,p}(\Omega;u\,dx)},
\end{aligned}$$

it is easy to check the validity of the following estimate:

$$\begin{aligned}
&\langle \Delta_p(u,y) - \Delta_p(u,v), y - v\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)}\\
&\quad= \int_\Omega u(x)\left(|\nabla y|^{p-2}\nabla y - |\nabla v|^{p-2}\nabla v, \nabla y - \nabla v\right)_{\mathbb{R}^N}dx \quad \text{(by the Cauchy Inequality)}\\
&\quad\ge \int_\Omega |\nabla y|^{p-2}\left(|\nabla y|^2 - |\nabla v||\nabla y|\right)u(x)\,dx\\
&\qquad- \int_\Omega |\nabla v|^{p-2}\left(|\nabla v||\nabla y| - |\nabla v|^2\right)u(x)\,dx \quad \text{(by the Hölder Inequality)}\\
&\quad\ge \|y\|^p_{W_0^{1,p}(\Omega;u\,dx)} + \|v\|^p_{W_0^{1,p}(\Omega;u\,dx)}\\
&\qquad- \|y\|^{p-1}_{W_0^{1,p}(\Omega;u\,dx)}\|v\|_{W_0^{1,p}(\Omega;u\,dx)} - \|v\|^{p-1}_{W_0^{1,p}(\Omega;u\,dx)}\|y\|_{W_0^{1,p}(\Omega;u\,dx)}\\
&\quad= \left(\|y\|^{p-1}_{W_0^{1,p}(\Omega;u\,dx)} - \|v\|^{p-1}_{W_0^{1,p}(\Omega;u\,dx)}\right)\left(\|y\|_{W_0^{1,p}(\Omega;u\,dx)} - \|v\|_{W_0^{1,p}(\Omega;u\,dx)}\right)\\
&\quad\ge 2^{2-p}\left|\|y\|_{W_0^{1,p}(\Omega;u\,dx)} - \|v\|_{W_0^{1,p}(\Omega;u\,dx)}\right|^p, \quad \forall y, v \in W_0^{1,p}(\Omega).
\end{aligned}$$

As a result, since $\beta^{-1}\|y\|^p_{W_0^{1,p}(\Omega;u\,dx)} \le \|y\|^p_{W_0^{1,p}(\Omega)} \le \alpha^{-1}\|y\|^p_{W_0^{1,p}(\Omega;u\,dx)}$, it follows that

$$\langle \Delta_p(u,y) - \Delta_p(u,v), y - v\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} > 0 \quad \text{for all } y, v \in W_0^{1,p}(\Omega), y \ne v.$$

Then by well-known existence results for nonlinear elliptic equations with strictly monotone demi-continuous coercive operators (see [32, 33]), one can conclude: for every $u \in \mathfrak{A}_{\mathrm{ad}}$ and $f \in W^{-1,q}(\Omega)$, the nonlinear Dirichlet boundary value problem

$$\Delta_p(u,y) = f \quad \text{in } \Omega, y \in W_0^{1,p}(\Omega), \tag{2.3}$$

admits a unique weak solution in $W_0^{1,p}(\Omega)$. Let us recall that a function y is the weak solution of (2.3) if

$$y \in W_0^{1,p}(\Omega), \tag{2.4}$$

$$\int_\Omega u(x)|\nabla y|^{p-2}(\nabla y, \nabla v)_{\mathbb{R}^N}\, dx = \langle f, v\rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)}, \quad \forall v \in W_0^{1,p}(\Omega). \tag{2.5}$$

3 Setting of the optimal control problem

We consider the following optimal control problem:

$$\text{Minimize } \left\{ I(u,y) = \int_\Omega \left|\nabla y(x) - \nabla y_d(x)\right|^p dx \right\}, \tag{3.1}$$

subject to the constraints

$$\int_\Omega u(x)|\nabla y|^{p-2}(\nabla y, \nabla v)_{\mathbb{R}^N}\, dx = \langle f, v\rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)}, \quad \forall v \in W_0^{1,p}(\Omega), \tag{3.2}$$

$$u \in \mathfrak{A}_{\mathrm{ad}} \subset L^\infty(\Omega), \quad y \in W_0^{1,p}(\Omega), \tag{3.3}$$

where $f \in W^{-1,q}(\Omega)$ and $y_d \in W_0^{1,p}(\Omega)$ are given distributions.

Hereinafter, $\Xi \subset L^\infty(\Omega) \times W_0^{1,p}(\Omega)$ denotes the set of all admissible pairs (u,y) to optimal control problem (3.1)-(3.3).

Remark 3.1 As was mentioned in the previous section, the characteristic feature of optimal control problem (3.1)-(3.3) is the fact that the set of admissible controls $\mathfrak{A}_{\mathrm{ad}}$ is a convex set with an empty topological interior. As we will see later on, this circumstance entails some technical difficulties in the substantiation of optimality conditions for the given problem.

Let τ be the topology on the set $L^1(\Omega) \times W_0^{1,p}(\Omega)$ which we define as a product of the strong topology of $L^1(\Omega)$ and the weak topology of $W_0^{1,p}(\Omega)$. Further we make use of the following results, which play a key role for the solvability of optimal control problem (3.1)-(3.3) (see [26, 32] and [15, 21] for comparison).

Proposition 3.1 *For any $u \in \mathfrak{A}_{\mathrm{ad}}$ and $f \in W^{-1,q}(\Omega)$, a weak solution $y \in W_0^{1,p}(\Omega)$ to the variational problem* (3.2)-(3.3) *satisfies the estimate*

$$\|y\|_{W_0^{1,p}(\Omega)} \le \alpha^{-1/(p-1)} \|f\|_{W^{-1,q}(\Omega)}^{1/(p-1)}. \tag{3.4}$$

Proof To prove the proposition it is enough to put in equality (3.2) as a test function the element y and then use the properties of the class $\mathfrak{A}_{\mathrm{ad}}$ on the left-hand side of the relation and the Cauchy-Bunjakowsky Inequality on the right-hand side. Indeed, we get

$$\begin{aligned}\alpha\|y\|^p_{W_0^{1,p}(\Omega)} &\le \int_\Omega u(x)|\nabla y|^{p-2}(\nabla y,\nabla y)_{\mathbb{R}^N}\,dx\\ &= \langle f,y\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} \le \|f\|_{W^{-1,q}(\Omega)}\|y\|_{W_0^{1,p}(\Omega)}.\end{aligned}$$

Now to get the desired estimate we divide each part of the obtain relation into $\|y\|_{W_0^{1,p}(\Omega)}$ and raise each side to the power $1/(p-1)$. □

Proposition 3.2 *If $\{u_k\}_{k\in\mathbb{N}}\subset\mathfrak{A}_{\mathrm{ad}}$ and $u_k\to u$ in $L^1(\Omega)$, then $u_k\to u$ in $L^r(\Omega)$ for any $r\in[1,+\infty)$ and $u_k\overset{*}{\to}u$ in $L^\infty(\Omega)$.*

Proof Since $u_k\to u$ in $L^1(\Omega)$ and

$$\int_\Omega u_k\,dx = m,\quad TV(u_k)\le\gamma,\ \text{and}\ \alpha\le u_k\le\beta\ \text{a.e. in}\ \Omega,\forall k\in\mathbb{N},$$

by Proposition 2.2(i) it follows that

$$TV(u)\le\gamma,\quad \int_\Omega u\,dx = m,\ \text{and}\ \alpha\le u\le\beta\ \text{a.e. in}\ \Omega.$$

Hence, $u\in\mathfrak{A}_{\mathrm{ad}}$. Moreover, for any $r\in[1,+\infty)$, the estimate

$$\|u_k-u\|^r_{L^r(\Omega)}\le \operatorname*{vrai\,sup}_{x\in\Omega}\bigl|u_k(x)-u(x)\bigr|^{r-1}\|u_k-u\|_{L^1(\Omega)}\le(\beta-\alpha)^{r-1}\|u_k-u\|_{L^1(\Omega)}$$

implies that $u_k\to u$ in $L^r(\Omega)$.

To end the proof, it is enough to note that strong convergence $u_k\to u$ in $L^1(\Omega)$ implies, up to a subsequence, convergence $u_k(x)\to u(x)$ almost everywhere in Ω. Hence, by the Lebesgue Theorem, we have

$$\int_\Omega (u_k-u)\varphi\,dx\to 0,\quad \forall\varphi\in L^1(\Omega),$$

that is $u_k\overset{*}{\to}u$ in $L^\infty(\Omega)$. Since this conclusion is true for any weakly-$*$ convergent subsequence of $\{u_k\}_{k\in\mathbb{N}}$, it follows that u is the weak-$*$ limit for the whole sequence $\{u_k\}_{k\in\mathbb{N}}$. □

Proposition 3.3 *$\mathfrak{A}_{\mathrm{ad}}$ is a sequentially compact subset of $L^r(\Omega)$ for any $r\in[1,+\infty)$, and it is a sequentially weakly-$*$ compact subset of $L^\infty(\Omega)$.*

Proof Let $\{u_k\}_{k\in\mathbb{N}}$ be any sequence of $\mathfrak{A}_{\mathrm{ad}}$. Then $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $BV(\Omega)\cap L^\infty(\Omega)$. As a result, the statement immediately follows from Propositions 3.2 and 2.2(iii). □

Proposition 3.4 *For every $f\in W^{-1,q}(\Omega)$ the set Ξ is sequentially compact, i.e. for each sequence $\{(u_k,y_k)\in\Xi\}_{k\in\mathbb{N}}$ it can be found a subsequence $\{(u_{k_n},y_{k_n})\in\Xi\}_{n\in\mathbb{N}}$ such that $u_{k_n}\to u_0$ in $L^1(\Omega)$, $y_{k_n}\to y_0$ in $W_0^{1,p}(\Omega)$, where $(u_0,y_0)\in\Xi$, that is, y_0 is a weak solution to the Dirichlet boundary value problem* (3.2) *with $u=u_0$.*

Proof Let $\{(u_k, y_k)\}_{k\in\mathbb{N}} \subset \Xi$ be an arbitrary sequence of admissible pairs. Then Proposition 3.3 and a priori estimate (3.4) lead to the existence of a subsequence, still denoted by $\{(u_k, y_k)\}_{k\in\mathbb{N}}$ and a pair $(u, y) \in \mathfrak{A}_{\mathrm{ad}} \times W_0^{1,p}(\Omega)$ such that

$$y_k \to y \text{ weakly in } W_0^{1,p}(\Omega), \tag{3.5}$$

$$u_k \to u \text{ strongly in } L^r(\Omega), \forall r \in [1, +\infty) \text{ and weakly-}* \text{ in } L^\infty(\Omega). \tag{3.6}$$

Taking into account the inequality (see [34])

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta\right)_{\mathbb{R}^N} \geq 2^{2-p}|\xi - \eta|^p \quad \forall \xi, \eta \in \mathbb{R}^N,$$

and definition of the class of admissible controls $\mathfrak{A}_{\mathrm{ad}}$, we conclude: there exists a constant $C > 0$ independent of $k \in \mathbb{N}$ such that

$$\begin{aligned} C\int_\Omega |\nabla y_k - \nabla y|^p\,dx &\leq \int_\Omega u_k\left(|\nabla y_k|^{p-2}\nabla y_k - |\nabla y|^{p-2}\nabla y, \nabla y_k - \nabla y\right)_{\mathbb{R}^N} dx \\ &= \langle f, y_k - y\rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)} \\ &\quad + \int_\Omega (u - u_k)\left(|\nabla y|^{p-2}\nabla y, \nabla y_k - \nabla y\right)_{\mathbb{R}^N} dx \\ &\quad - \int_\Omega u\left(|\nabla y|^{p-2}\nabla y, \nabla y_k - \nabla y\right)_{\mathbb{R}^N} dx = I_1 + I_2 - I_3. \end{aligned} \tag{3.7}$$

Since $u \leq \beta$, $u|\nabla y|^{p-2}\nabla y \in L^q(\Omega; \mathbb{R}^N)$, and $\nabla y_k \rightharpoonup \nabla y$ in $L^p(\Omega; \mathbb{R}^N)$, it follows that

$$\begin{aligned} &I_1 := \langle f, y_k - y\rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)} \overset{\text{by (3.5)}}{\longrightarrow} 0 \quad \text{as } k \to \infty, \\ &I_2 := \int_\Omega u\left(|\nabla y|^{p-2}\nabla y, \nabla y_k - \nabla y\right)_{\mathbb{R}^N} dx \overset{\text{by (3.5)}}{\longrightarrow} 0 \quad \text{as } k \to \infty, \\ &(u - u_k)|\nabla y|^{p-2}\nabla y \overset{\text{by (3.6)}}{\longrightarrow} 0 \text{ strongly in } L^q\left(\Omega; \mathbb{R}^N\right), \text{and, therefore,} \\ &I_2 := \int_\Omega (u - u_k)\left(|\nabla y|^{p-2}\nabla y, \nabla y_k - \nabla y\right)_{\mathbb{R}^N} dx \overset{\text{by (3.5)}}{\longrightarrow} 0 \quad \text{as } k \to \infty. \end{aligned}$$

Hence, passing to the limit in (3.7) as $k \to \infty$, we arrive at a conclusion (see [26]):

$$y_k \to y \text{ strongly in } W_0^{1,p}(\Omega), \text{ and } |\nabla y_k|^{p-2}\nabla y_k \rightharpoonup |\nabla y|^{p-2}\nabla y \text{ in } L^q\left(\Omega; \mathbb{R}^N\right).$$

As a result, we finally have

$$\begin{aligned} \langle f, v\rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)} &= \lim_{k\to\infty} \int_\Omega u_k(x)|\nabla y_k|^{p-2}(\nabla y_k, \nabla v)_{\mathbb{R}^N}\,dx \\ &= \int_\Omega u(x)|\nabla y|^{p-2}(\nabla y, \nabla v)_{\mathbb{R}^N}\,dx, \quad \forall v \in W_0^{1,p}(\Omega), \end{aligned}$$

that is, the limit pair (u, y) is an admissible to optimal control problem (3.1)-(3.3). The proof is complete. □

Taking into account Propositions 3.1-3.3, in a similar manner to [15, 21], it is easy to conclude the following existence result.

Theorem 3.5 *The optimal control problem* (3.1)-(3.3) *admits at least one solution*

$$\begin{aligned}&(u^{\mathrm{opt}}, y^{\mathrm{opt}}) \in \Xi \subset [BV(\Omega) \cap L^\infty(\Omega)] \times W_0^{1,p}(\Omega),\\ &I(u^{\mathrm{opt}}, y^{\mathrm{opt}}) = \inf_{(u,y)\in\Xi} I(u,y).\end{aligned}$$

4 Auxiliary results

The main goal of this paper is to derive the optimality conditions for optimal control problem (3.1)-(3.3). However, we deal with the case when we cannot apply the well-known classical approach (see, for instance, [35, 36]), since for a given distribution $f \in W^{-1,q}(\Omega)$ the mapping $u \mapsto y(u)$ is not Fréchet differentiable on the class of admissible controls, in general, and the class $\mathfrak{A}_{\mathrm{ad}}$ has an empty topological interior. With that in mind, we apply the so-called differentiation concept on convex sets and introduce the notion of a quasi-adjoint state ψ_ε to an optimal solution $y_0 \in W_0^{1,p}(\Omega)$ that was proposed for linear problems by Serovajskiy [25].

To begin with, we discuss the differentiable properties of the Lagrange functional associated with problem (3.1)-(3.3). Since (3.3) can be seen as a constraint, we define the Lagrangian as follows:

$$\Lambda(u,y,\mu) = I(u,y) + a_u(y,\mu) - \langle f, \mu\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)}, \tag{4.1}$$

where $\mu \in W_0^{1,p}(\Omega)$ is a Lagrange multiplier and

$$a_u(y,\mu) = \langle -\Delta_p(u,y), \mu\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} = \int_\Omega u(x)\big(|\nabla y|^{p-2}\nabla y, \nabla\mu\big)_{\mathbb{R}^N}\,dx.$$

For given $h \in W_0^{1,p}(\Omega)$ and $\theta \in [0,1]$, let us consider the following sets:

$$\Omega_0 = \{x \in \Omega : |\nabla y| > 0\}, \qquad \Omega_{0,\theta} = \{x \in \Omega : |\nabla y + \theta\nabla h| > 0\}.$$

Clearly, we cannot claim that $\chi_{\Omega_{0,\theta}} \to \chi_{\Omega_0}$ in $L^r(\Omega)$ for some $1 \le r < \infty$, because convergence of the sequence $\{\nabla y + \theta\nabla h\}_{\theta\to+0}$ to ∇y does not imply, in general, the χ-convergence of subsets $\{\Omega_{0,\theta}\}_{\theta\to+0}$ to Ω_0 as $\theta \to +0$ [15, p.218]. Indeed, let $h \in W_0^{1,p}(\Omega)$ be such that $|\nabla h(x)| > 0$ almost everywhere in Ω and $\nabla y = 0$. Then $\Omega_0 = \emptyset$ whereas $\Omega_{0,\theta} = \Omega$ for all positive θ small enough. Hence, in this case the convergence $\chi_{\Omega_{0,\theta}} \to \chi_{\Omega_0}$ fails. In view of this, we make use of the following notion (see [23]).

Definition 4.1 We say that an element $y \in W_0^{1,p}(\Omega)$ is a regular point for the Lagrangian (4.1) if for each $v \in W_0^{1,p}(\Omega)$ the direction $h = v - y$ is non-degenerate in the following sense:

$$\chi_{\Omega_{0,\theta}} \to \chi_{\Omega_0} \quad \text{in } L^1(\Omega). \tag{4.2}$$

We have the following result.

Proposition 4.2 *Assume that y is an element of $W_0^{1,p}(\Omega)$ such that the set*

$$S_0 = \{x \in \Omega : |\nabla y(x)| = 0\} \tag{4.3}$$

has zero Lebesgue measure. Then $y \in W_0^{1,p}(\Omega)$ is a regular point of the Lagrangian (4.1) *in the sense of Definition* 4.1.

Proof Let h be a given element of $W_0^{1,p}(\Omega)$. If $x \in \Omega_0$, then, by definition, $|\nabla y(x)| > 0$. Thus, there is a value $\theta_0 \in (0,1]$ such that $|\nabla y(x) + \theta \nabla h(x)| > 0$ for all $\theta \in [0,\theta_0]$. It is worth to note that this pointwise inequality makes a sense if only $x \in \Omega$ is a Lebesgue point of both ∇y and ∇h. However, the Lebesgue Differentiation Theorem states that, given any $g \in L^1(\Omega)$, almost every $x \in \Omega$ is a Lebesgue point. Hence, almost all Lebesgue points of ∇y are the Lebesgue points of $\nabla y + \theta \nabla h$ for θ small enough, and $\chi_{\Omega_{0,\theta}}(x) = \chi_{\Omega_0}(x) = 1$ for all $\theta \in [0,\theta_0]$. Since the set S_0 has zero Lebesgue measure, it follows that $|\Omega \setminus \Omega_0| = 0$ and $|\Omega \setminus \Omega_{0,\theta}| = 0$ for $\theta \in [0,1]$ small enough. Therefore, $\chi_{\Omega_{0,\theta}} \to \chi_{\Omega_0}$ almost everywhere in Ω and hence, $\chi_{\Omega_{0,\theta}} \to \chi_{\Omega_0}$ strongly in $L^1(\Omega)$. □

Remark 4.1 It is worth noting that due to the results of Manfredi (see [37]), the assumptions of Proposition 4.2 appears natural and it is not a restrictive supposition in practice. Indeed, following [37], we can ensure that the set $S_0 := \{x \in \Omega : \nabla y = 0\}$ for non-constant solutions of the p-Laplace equation (a p-harmonic function) has zero Lebesgue measure. Moreover, it is also easy to observe that if y and v in $W_0^{1,p}(\Omega)$ are two regular points of the functional $\Lambda(u,y,\lambda)$, then there exists a positive number $\alpha \in \mathbb{R}$ ($\alpha \neq 0$) such that each point of the segment $[y,\alpha v] = \{y + t(\alpha v - y) : \forall t \in [0,1]\} \subset W_0^{1,p}(\Omega)$ is also regular for $\Lambda(u,y,\lambda)$.

We are now ready to study the differentiability properties of the Lagrangian $\Lambda(u,y,\lambda)$. We begin with the following result.

Lemma 4.3 *Let $u \in \mathfrak{A}_{\mathrm{ad}}$ be a given element, and let $y \in W_0^{1,p}(\Omega)$ be a regular point of the Lagrangian* (4.1). *Then the mapping*

$$W_0^{1,p}(\Omega) \ni y \mapsto \Delta_p(u,y) = -\operatorname{div}\big(u(x)|\nabla y|^{p-2}\nabla y\big) \in W^{-1,q}(\Omega), \quad p \geq 2$$

is Gâteaux differentiable at y and its Gâteaux derivative

$$\big(-\Delta_p(u,y)\big)'_G \in \mathcal{L}\big(W_0^{1,p}(\Omega), W^{-1,q}(\Omega)\big)$$

exists and takes the form

$$\begin{aligned}\big(-\Delta_p(u,y)\big)'_G[h] = {} & -\operatorname{div}\big(u(x)|\nabla y|^{p-2}\nabla h\big) \\ & - (p-2)\operatorname{div}\big(u(x)|\nabla y|^{p-4}(\nabla y, \nabla h)_{\mathbb{R}^N}\nabla y\big).\end{aligned} \tag{4.4}$$

Proof Let $y \in W_0^{1,p}(\Omega)$ be a regular point for the Lagrangian (4.1) and let $h \in W_0^{1,p}(\Omega)$ be an arbitrary distribution. Following the definition of the Gâteaux derivative, we have to deduce the following equality:

$$\lim_{\lambda\to+0}\bigg\| \frac{\Delta_p(u,y+\lambda h)-\Delta_p(u,y)}{\lambda} - (p-2)\operatorname{div}\big(u(x)|\nabla y|^{p-4}(\nabla y,\nabla h)_{\mathbb{R}^N}\nabla y\big) \\ - \operatorname{div}\big(u|\nabla y|^{p-2}\nabla h\big)\bigg\|_{L^q(\Omega;\mathbb{R}^N)} = 0.$$

With that in mind, let us consider the vector-valued function

$$g(\lambda) := |\nabla y+\lambda\nabla h|^{p-2}(\nabla y+\lambda\nabla h)$$

for which the Taylor expansion with the remainder term in the Lagrange form leads to the relation

$$\big|g(\lambda)-g(0)\big| \le \lambda\big|g'(\theta)\big|, \quad \theta\in(0,\lambda),$$

where $g(0) = |\nabla y|^{p-2}\nabla y$ and

$$\begin{aligned} g'(\theta) &= |\nabla y+\theta\nabla h|^{p-2}\nabla h \\ &\quad + (p-2)|\nabla y+\theta\nabla h|^{p-2}\big(\theta|\nabla h|^2+(\nabla y,\nabla h)_{\mathbb{R}^N}\big)(\nabla y+\theta\nabla h)\frac{1}{|\nabla y+\theta\nabla h|^2} \\ &= |\nabla y+\theta\nabla h|^{p-2}\nabla h + (p-2)|\nabla y+\theta\nabla h|^{p-2}\frac{(\nabla y+\theta\nabla h,\nabla h)_{\mathbb{R}^N}}{|\nabla y+\theta\nabla h|}\frac{\nabla y+\theta\nabla h}{|\nabla y+\theta\nabla h|}. \end{aligned}$$

Let $\delta>0$ be an arbitrary value. Let us consider the following decomposition:

$$\Omega = S_0\cup\Omega'_\delta\cup\Omega''_\delta,$$

where the set S_0 is defined by (4.3), and Ω'_δ and Ω''_δ are measurable subsets of Ω such that

$$\Omega'_\delta = \big\{x\in\Omega : \big|\nabla y(x)\big|\ge\delta\big\}, \qquad \Omega''_\delta = \big\{x\in\Omega : 0<\big|\nabla y(x)\big|<\delta\big\}.$$

Following [34, p.598] (see also [38]), we have: for any $\varepsilon>0$ there exists a positive value $\delta_0>0$ such that

$$\left\|\frac{g'(\theta)}{\lambda} - (p-2)|\nabla y|^{p-4}(\nabla y,\nabla h)_{\mathbb{R}^N}\nabla y - |\nabla y|^{p-2}\nabla h\right\|_{L^q(\Omega'_\delta;\mathbb{R}^N)} < \frac{\varepsilon}{2}, \tag{4.5}$$

$$\left\|\frac{g'(\theta)}{\lambda} - (p-2)|\nabla y|^{p-4}(\nabla y,\nabla h)_{\mathbb{R}^N}\nabla y - |\nabla y|^{p-2}\nabla h\right\|_{L^q(\Omega''_\delta;\mathbb{R}^N)} < \frac{\varepsilon}{2} \tag{4.6}$$

for all $\delta\in(0,\delta_0)$, $\theta\in(0,\lambda)$, and $\lambda>0$ small enough. Since $|S_0|=0$ by the initial assumptions, it follows from (4.5)-(4.6) that the vector-valued function $|\nabla y|^{p-2}\nabla y$ is Gâteaux differentiable. Hence, the operator $\Delta_p(u,y) = -\operatorname{div}(u(x)|\nabla y|^{p-2}\nabla y)$ is Gâteaux differentiable for any regular point $y\in W_0^{1,p}(\Omega)$ and for any admissible control $u\in\mathfrak{A}_{\mathrm{ad}}$, and its Gâteaux derivative takes the form (4.4). □

Since Gâteaux differentiability of the operator $y \mapsto \Delta_p(u, y)$ implies the existence of Gâteaux derivative for the functional

$$\varphi(y) = \langle -\Delta_p(u,y), \mu\rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)} = \int_\Omega u(x)\big(|\nabla y|^{p-2}\nabla y, \nabla \mu\big)_{\mathbb{R}^N}\, dx,$$
$$\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$$

such that

$$\langle \varphi'_G(y), h\rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)} = \big\langle \big(-\Delta_p(u,y)\big)'_G[h], \mu\big\rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)}, \quad \forall \mu \in W_0^{1,p}(\Omega),$$

we arrive at the following obvious consequence of Lemma 4.3.

Corollary 4.4 *Let $u \in \mathfrak{A}_{\mathrm{ad}}$ and $\lambda \in W_0^{1,p}(\Omega)$ be given elements, and let $y \in W_0^{1,p}(\Omega)$ be a regular point of the Lagrangian* (4.1). *Then the mapping*

$$W_0^{1,p}(\Omega) \ni y \mapsto \Lambda(u, y, \mu) = I(u,y) + a_u(y, \mu) - \langle f, \mu\rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)} \in \mathbb{R}$$

is Gâteaux differentiable at y and its Gâteaux derivative,

$$\Lambda'_G(u, y, \mu) \in W^{-1,q}(\Omega),$$

exists and takes the form

$$\begin{aligned}
&\langle \Lambda'_G(u,y,\mu), h\rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)} \\
&\quad = p\int_\Omega |\nabla y - \nabla y_d|^{p-2}(\nabla y - \nabla y_d, \nabla h)_{\mathbb{R}^N}\, dx + \int_\Omega \big(u(x)|\nabla y|^{p-2}\nabla \mu, \nabla h\big)_{\mathbb{R}^N}\, dx \\
&\qquad + (p-2)\int_\Omega u(x)|\nabla y|^{p-4}(\nabla y, \nabla \mu)_{\mathbb{R}^N}(\nabla y, \nabla h)_{\mathbb{R}^N}\, dx.
\end{aligned} \tag{4.7}$$

Remark 4.2 In view of the equality

$$(\nabla y, \nabla \mu)_{\mathbb{R}^N}\nabla y = [\nabla y \otimes \nabla y]\nabla \mu,$$

the last term in (4.7) can be rewritten as follows:

$$(p-2)\int_\Omega u(x)|\nabla y|^{p-4}\big([\nabla y \otimes \nabla y]\nabla \mu, \nabla h\big)_{\mathbb{R}^N}\, dx.$$

Before deriving the optimality conditions, we need the following auxiliary result.

Lemma 4.5 *Let $u \in \mathfrak{A}_{\mathrm{ad}}$, $y \in W_0^{1,p}(\Omega)$, and $v \in W_0^{1,p}(\Omega)$ be given distributions. Assume that each point of the segment $[y, v] = \{y + \alpha(v - y) : \forall \alpha \in [0,1]\} \subset W_0^{1,p}(\Omega)$ is regular for the mapping $v \to \Lambda(u, v, \mu)$. Then there exists a positive value $\varepsilon \in (0,1)$ such that*

$$\begin{aligned}
&\Lambda(u, v, \mu) - \Lambda(u, y, \mu) \\
&\quad = \langle \Lambda'_G(u, y + \varepsilon h, \mu), h\rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)}
\end{aligned}$$

$$
\begin{aligned}
&= p\int_\Omega |\nabla y + \varepsilon\nabla h - \nabla y_d|^{p-2}(\nabla y + \varepsilon\nabla h - \nabla y_d, \nabla h)_{\mathbb{R}^N}\,dx \\
&\quad + \int_\Omega u(x)|\nabla y + \varepsilon\nabla h|^{p-2}(\nabla\mu, \nabla h)_{\mathbb{R}^N}\,dx \\
&\quad + (p-2)\int_\Omega u(x)|\nabla y + \varepsilon\nabla h|^{p-4} \\
&\quad \times \bigl(\bigl[(\nabla y + \varepsilon\nabla h)\otimes(\nabla y + \varepsilon\nabla h)\bigr]\nabla\mu, \nabla h\bigr)_{\mathbb{R}^N}\,dx \qquad (4.8)
\end{aligned}
$$

with $h = v - y$.

Proof For given u, μ, y_d, y, and v, let us consider the scalar function $\varphi(t) = \Lambda(u, y + t(v - y), \mu)$. Since by Corollary 4.4, the functional $\Lambda(u, \cdot, \mu)$ is Gâteaux differentiable at each point of the segment $[y, v]$, it follows that the function $\varphi = \varphi(t)$ is differentiable on $(0,1)$ and

$$
\varphi'(t) = \bigl\langle \Lambda'_G\bigl(u, y + t(v - y), \lambda\bigr), v - y\bigr\rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)}, \quad \forall t \in (0,1).
$$

To conclude the proof, it remains to take into account (4.7) and apply the Mean Value Theorem:

$$
\varphi(1) - \varphi(0) = \varphi'(\varepsilon) \quad \text{for some } \varepsilon \in (0,1). \qquad \square
$$

5 Formalism of the quasi-adjoint technique

We begin with the following assumption:

(H1) The distribution $f \in W^{-1,q}(\Omega)$ is such that, for each admissible control $u \in \mathfrak{A}_{\mathrm{ad}}$, the corresponding weak solution $y = y(u)$ of the nonlinear Dirichlet boundary value problem (2.3) is a regular point of the mapping $y \mapsto \Lambda(u, y, \lambda)$.

Let $(u_0, y_0) \in \Xi$ be an optimal pair for problem (3.1)-(3.3). Then

$$
\Delta\Lambda = \Lambda(u, y, \lambda) - \Lambda(u_0, y_0, \lambda) \geq 0, \quad \forall (u, y) \in \Xi, \forall \lambda \in W_0^{1,p}(\Omega). \qquad (5.1)
$$

Hence, splitting it in two terms, we obtain

$$
\begin{aligned}
\Lambda(u, y, \lambda) - \Lambda(u_0, y_0, \lambda) &= \Lambda(u, y, \lambda) - \Lambda(u, y_0, \lambda) + \Lambda(u, y_0, \lambda) - \Lambda(u_0, y_0, \lambda) \\
&= \Delta_y\Lambda(u, y_0, \lambda) + \Lambda(u - u_0, y_0, \lambda) \geq 0, \qquad (5.2)
\end{aligned}
$$

for all $\lambda \in W_0^{1,p}(\Omega)$ and $u \in \mathfrak{A}_{\mathrm{ad}}$ such that $(u - u_0) \in \mathfrak{A}_{\mathrm{ad}}$.

Since the set of admissible controls $\mathfrak{A}_{\mathrm{ad}} \subset L^1(\Omega)$ has an empty topological interior, we justify the choice of perturbation for an optimal control as follows:

$$
u_\theta := u_0 + \theta(\widehat{u} - u_0),
$$

where $(\widehat{u}, \widehat{y}) \in \Xi$ is an arbitrary admissible pair, and $\theta \in [0,1]$. Then due to Hypothesis (H1) and Remark 4.1, we can suppose that each point of the segment $[y_0, y_\theta] \subset W_0^{1,p}(\Omega)$ is regular for the mapping $v \to \Lambda(u, v, \lambda)$, where $y_\theta := y(u_\theta) = y(u_0 + \theta(\widehat{u} - u_0))$ is the corresponding solution of boundary value problem (3.2)-(3.3). Hence, by Lemma 4.5, there exists a value $\varepsilon_\theta \in (0,1)$ such that condition (5.1) can be represented as follows:

$$\begin{aligned}
\Delta\Lambda &= \Lambda(u_\theta, y_\theta, \lambda) - \Lambda(u_0, y_0, \lambda) \\
&= \big\langle \Lambda'_G\big(u_\theta, y_0 + \varepsilon_\theta(y_\theta - y_0), \lambda\big), y_\theta - y_0 \big\rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)} + \Lambda(u_\theta - u_0, y_0, \lambda) \\
&= \big\langle \Lambda'_G\big(u_\theta, y_0 + \varepsilon_\theta(y_\theta - y_0), \lambda\big), y_\theta - y_0 \big\rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)} \\
&\quad + \Lambda\big(\theta(\widehat{u} - u_0), y_0, \lambda\big) \ge 0.
\end{aligned} \tag{5.3}$$

Using (4.7), we obtain

$$\begin{aligned}
\Delta\Lambda &= p \int_\Omega |\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d|^{p-2} (\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d, \nabla y_\theta - \nabla y_0)_{\mathbb{R}^N}\, dx \\
&\quad + \int_\Omega u_\theta |\nabla y_{\varepsilon_\theta,\theta}|^{p-2} (\nabla\lambda, \nabla y_\theta - \nabla y_0)_{\mathbb{R}^N}\, dx \\
&\quad + (p-2) \int_\Omega u_\theta |\nabla y_{\varepsilon_\theta,\theta}|^{p-4} \big([\nabla y_{\varepsilon_\theta,\theta} \otimes \nabla y_{\varepsilon_\theta,\theta}]\nabla\lambda, \nabla y_\theta - \nabla y_0\big)_{\mathbb{R}^N}\, dx \\
&\quad + \theta \int_\Omega (\widehat{u} - u_0)\big(|\nabla y_0|^{p-2}\nabla y_0, \nabla\lambda\big)_{\mathbb{R}^N}\, dx \ge 0, \quad \forall \widehat{u} \in \mathfrak{A}_{\mathrm{ad}},
\end{aligned} \tag{5.4}$$

where $y_{\varepsilon_\theta,\theta} = y_0 + \varepsilon_\theta(y_\theta - y_0)$.

Now we introduce the concept of quasi-adjoint states that was first considered for linear problems by Serovajskiy [25].

Definition 5.1 We say that, for given $\theta \in [0,1]$ and $\widehat{u} \in \mathfrak{A}_{\mathrm{ad}}$, a distribution $\psi_{\varepsilon_\theta,\theta}$ is the quasi-adjoint state to $y_0 \in W_0^{1,p}(\Omega)$ if $\psi_{\varepsilon_\theta,\theta}$ satisfies the following integral identity:

$$\begin{aligned}
&\int_\Omega u_\theta |\nabla y_{\varepsilon_\theta,\theta}|^{p-2} \left(\left[I + (p-2) \frac{\nabla y_{\varepsilon_\theta,\theta}}{|\nabla y_{\varepsilon_\theta,\theta}|} \otimes \frac{\nabla y_{\varepsilon_\theta,\theta}}{|\nabla y_{\varepsilon_\theta,\theta}|} \right] \nabla \psi_{\varepsilon_\theta,\theta}, \nabla\varphi \right)_{\mathbb{R}^N} dx \\
&\quad + p \int_\Omega |\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d|^{p-2} (\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d, \nabla\varphi)_{\mathbb{R}^N}\, dx = 0, \quad \forall \varphi \in W_0^{1,p}(\Omega)
\end{aligned} \tag{5.5}$$

or in the operator form

$$\begin{aligned}
&-\operatorname{div}\left(u_\theta |\nabla y_{\varepsilon_\theta,\theta}|^{p-2} \left[I + (p-2) \frac{\nabla y_{\varepsilon_\theta,\theta}}{|\nabla y_{\varepsilon_\theta,\theta}|} \otimes \frac{\nabla y_{\varepsilon_\theta,\theta}}{|\nabla y_{\varepsilon_\theta,\theta}|} \right] \nabla \psi_{\varepsilon_\theta,\theta} \right) \\
&\quad = p \operatorname{div}\big(|\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d|^{p-2} (\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d) \big) \quad \text{in } \mathcal{D}'(\Omega).
\end{aligned} \tag{5.6}$$

Here, $I \in \mathcal{L}(\mathbb{R}^N; \mathbb{R}^N)$ is the identity matrix, $y_\theta := y(u_\theta) = y(u_0 + \theta(\widehat{u} - u_0))$ is the solution of problem (3.2)-(3.3), $y_{\varepsilon_\theta,\theta} = y_0 + \varepsilon_\theta(y_\theta - y_0)$, and $\varepsilon_\theta = \varepsilon(u_\theta) \in (0,1)$ is a constant coming from equality (5.3).

Let us assume, for a moment, that quasi-adjoint state $\psi_{\varepsilon_\theta,\theta}$ to $y_0 \in W_0^{1,p}(\Omega)$ is a distribution of $W_0^{1,p}(\Omega)$. Then we can define the element λ in (5.4) as the quasi-adjoint state, that is, we can set $\lambda = \psi_{\varepsilon_\theta,\theta}$. As a result, the increment of Lagrangian (5.4) can be simplified to the form

$$\int_\Omega (\widehat{u} - u_0)\big(|\nabla y_0|^{p-2}\nabla y_0, \nabla\psi_{\varepsilon_\theta,\theta}\big)_{\mathbb{R}^N}\, dx \ge 0, \quad \forall \widehat{u} \in \mathfrak{A}_{\mathrm{ad}}. \tag{5.7}$$

Thus, in order to derive the necessary optimality conditions and provide their substantial analysis, it remains to pass to the limit in (5.5)-(5.7) as $\theta \to +0$, and to show that the sequence of quasi-adjoint states $\{\psi_{\varepsilon,\theta}\}_{\theta\to 0}$ is defined in a unique way through relation (5.5) and it is compact with respect to the weak topology of $W_0^{1,p}(\Omega)$.

6 The Hardy Inequality and asymptotic behavior of quasi-adjoint states

The main goal of this section is to study the well-posedness of variational problem (5.5) and describe the asymptotic behavior of its solutions as parameter θ tends to zero.

We begin with the following evident consequence of Proposition 3.4.

Lemma 6.1 *Assume that $u_k, u \in \mathfrak{A}_{\mathrm{ad}}$ and $u_k \to u$ strongly in $L^\infty(\Omega)$. Then, for the corresponding solutions of boundary value problem* (3.2)-(3.3), *we have strong convergence $y_k = y(u_k) \to y = y(u)$ in $W_0^{1,p}(\Omega)$.*

Since, by the initial suppositions, $u_\theta \to u_0$ in $L^\infty(\Omega)$ as $\theta \to 0$, we immediately arrive at the following consequence of Lemma 6.1.

Corollary 6.2

(i) $y_\theta \to y_0$ *in* $W_0^{1,p}(\Omega)$ *as* $\theta \to 0$;

(ii) $y_{\varepsilon_\theta,\theta} \to y_0$ *in* $W_0^{1,p}(\Omega)$ *as* $\theta \to 0$.

Further, we note that

(A_1) if $r \in (1,p]$, where $p \geq 2$, and $f, g \in L^p(\Omega)$, then

$$\left| |f(x)|^r - |g(x)|^r \right| \leq r\left(|f(x)| + |g(x)|\right)^{r-1}|f(x) - g(x)|, \quad \text{a.e. in } \Omega, \tag{6.1}$$

and, hence, by the Hölder Inequality,

$$\begin{aligned} \left| \|f\|^r_{L^r(\Omega)} - \|g\|^r_{L^r(\Omega)} \right| &= \left| \int_\Omega |f|^r\,dx - \int_\Omega |g|^r\,dx \right| \leq \int_\Omega \left| |f|^r - |g|^r \right| dx \\ &\leq \|f-g\|_{L^p(\Omega)} \left(\int_\Omega \left(|f| + |g|\right)^{(r-1)p/(p-1)}\,dx \right)^{(p-1)/p} \\ &\leq \|f-g\|_{L^p(\Omega)} \left\| |f| + |g| \right\|^{r-1}_{L^p(\Omega)} |\Omega|^{(p-r)/p}. \end{aligned} \tag{6.2}$$

(A_2) If $r \in [0,1]$ and $f, g \in L^p(\Omega)$ then

$$\left| |f(x)|^r - |g(x)|^r \right| \leq |f(x) - g(x)|^r, \quad \text{a.e. in } \Omega, \tag{6.3}$$

and, therefore,

$$\left| \|f\|^r_{L^r(\Omega)} - \|g\|^r_{L^r(\Omega)} \right| \leq \int_\Omega |f-g|^r\,dx \leq \|f-g\|^r_{L^p(\Omega)} |\Omega|^{(p-r)/p}. \tag{6.4}$$

Then, in view of Corollary 6.2 and estimates (6.2) and (6.4), we can give the following obvious conclusion.

Corollary 6.3 *For any $r \in [0,p]$ with $p \geq 2$, we have $|\nabla y_{\varepsilon_\theta,\theta}|^r \to |\nabla y_0|^r$ in $L^1(\Omega)$ as $\theta \to 0$.*

Our next intention is to study the variational problem (5.5). With that in mind, we rewrite it in the form

$$-\operatorname{div}(\rho_\theta A_\theta \nabla \psi_{\varepsilon_\theta,\theta}) = f_\theta \quad \text{in } \Omega, \tag{6.5}$$

where

$$\rho_\theta := |\nabla y_{\varepsilon_\theta,\theta}|^{p-2}, \tag{6.6}$$

$$A_\theta := u_\theta[I + C_\theta] = u_\theta\left[I + (p-2)\frac{\nabla y_{\varepsilon_\theta,\theta}}{|\nabla y_{\varepsilon_\theta,\theta}|} \otimes \frac{\nabla y_{\varepsilon_\theta,\theta}}{|\nabla y_{\varepsilon_\theta,\theta}|}\right], \tag{6.7}$$

$$f_\theta := p \operatorname{div}\big(|\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d|^{p-2}(\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d)\big). \tag{6.8}$$

To begin with, we make use of the following observation.

Proposition 6.4 *Let* $(u_0, y_0) \in \Xi$ *be an optimal pair for problem* (3.1)-(3.3). *Then, for any* $\theta \in [0,1]$ *and* $\widehat{u} \in \mathfrak{A}_{\mathrm{ad}}$, *we have*:

(1) $A_\theta \in L^\infty(\Omega; \mathbb{S}^N_{\mathrm{sym}})$, *where* $\mathbb{S}^N_{\mathrm{sym}}$ *denotes the set of all* $N \times N$ *symmetric matrices, which are obviously determined by* $N(N+1)/2$ *scalars.*

(2) $\alpha|\xi|^2 \le (\xi, A_\theta \xi)_{\mathbb{R}^N} \le \beta[1 + (p-2)2^{N-1}]|\xi|^2$ *for all* $\xi \in \mathbb{R}^N$.

Proof The first property is obvious. To prove the second one, it enough to take into account the definition of the class of admissible controls $\mathfrak{A}_{\mathrm{ad}}$ and the following chain of estimates:

$$\begin{aligned}\alpha|\xi|^2 &\le (\xi, u_\theta I\xi)_{\mathbb{R}^N} \le (\xi, u_\theta I\xi)_{\mathbb{R}^N} + (\xi, u_\theta C_\theta \xi)_{\mathbb{R}^N} \\ &= (\xi, u_\theta I\xi)_{\mathbb{R}^N} + (p-2)u_\theta\left(\frac{\nabla y_{\varepsilon_\theta,\theta}}{|\nabla y_{\varepsilon_\theta,\theta}|}, \xi\right)^2_{\mathbb{R}^N} \\ &\le \beta|\xi|^2 + (p-2)\beta(\xi_1 + \cdots + \xi_N)^2 \le \beta\big[1 + (p-2)2^{N-1}\big]|\xi|^2.\end{aligned}$$

□

Proposition 6.5

(a) $\rho_\theta := |\nabla y_{\varepsilon_\theta,\theta}|^{p-2} \to |\nabla y_0|^{p-2} =: \rho_0$ *in* $L^1(\Omega)$ *as* $\theta \to 0$;

(b) $A_\theta \to A_0$ *in* $L^r(\Omega; \mathbb{S}^N_{\mathrm{sym}})$ *for any* $r \in [1, +\infty)$ *and* $A_\theta \stackrel{*}{\to} A_0$ *in* $L^\infty(\Omega; \mathbb{S}^N_{\mathrm{sym}})$ *as* $\theta \to 0$;

(c) $\rho_\theta A_\theta \to \rho_0 A_0 = u_0[I + (p-2)\frac{\nabla y_0}{|\nabla y_0|} \otimes \frac{\nabla y_0}{|\nabla y_0|}]$ *in* $L^1(\Omega; \mathbb{R}^N)$ *as* $\theta \to 0$.

Proof Validity of assertions (a)-(b) immediately follows from Corollary 6.3 and Propositions 3.2 and 6.4. The strong convergence property in (c) is a direct consequence of the Lebesgue Convergence Theorem. □

The following results are crucial for our further analysis.

Lemma 6.6 *Assume that, for given* $\theta \in [0,1]$ *and* $\widehat{u} \in \mathfrak{A}_{\mathrm{ad}}$, $\nabla \ln|\nabla y_{\varepsilon_\theta,\theta}| \in L^2(\Omega; \mathbb{R}^N)$. *Then each element* ψ *of* $W_0^{1,p}(\Omega)$ *can be represented in a unique way as follows*:

$$\psi = |\nabla y_{\varepsilon_\theta,\theta}|^{(2-p)/2} z_\theta, \quad \text{where } z_\theta \in W_0^{1,1}(\Omega) \cap L^2(\Omega). \tag{6.9}$$

Proof Let us fix an element $\psi \in W_0^{1,p}(\Omega)$. Then for $z_\theta := |\nabla y_{\varepsilon_\theta,\theta}|^{(p-2)/2}\psi$, by the Hölder Inequality with $p' = p/2$ and $q' = p/(p-2)$, we have

$$\begin{aligned}\|z_\theta\|^2_{L^2(\Omega)} &= \int_\Omega |\nabla y_{\varepsilon_\theta,\theta}|^{p-2}\psi^2\,dx \le \|\nabla y_{\varepsilon_\theta,\theta}\|^{p-2}_{L^p(\Omega;\mathbb{R}^N)}\|\psi\|^2_{L^p(\Omega)} \\ &\le C\|y_{\varepsilon_\theta,\theta}\|^{p-2}_{W_0^{1,p}(\Omega)}\|\psi\|^2_{W_0^{1,p}(\Omega)} < \infty,\end{aligned}$$

where the constant C comes from the Poincaré Inequality. Using the evident equality

$$\nabla z_\theta = \left(\frac{1}{2}\sqrt{\rho}\psi\nabla\ln\rho + \sqrt{\rho}\nabla\psi\right)\Big|_{\rho=|\nabla y_{\varepsilon_\theta,\theta}|^{p-2}},$$

and applying the Hölder Inequality with exponents $p' = 2(p-1)/(p-2)$ and $q' = 2(p-1)/p$ for the first term and with $p' = p/2$ and $q' = p/(p-2)$ for the second one, we get

$$\begin{aligned}\|\nabla z_\theta\|_{L^1(\Omega;\mathbb{R}^N)} \le{}& \frac{p-2}{2}\int_\Omega |\nabla y_{\varepsilon_\theta,\theta}|^{(p-2)/2}|\psi|\big|\nabla\ln|\nabla y_{\varepsilon_\theta,\theta}|\big|\,dx \\ &+ |\Omega|^{1/2}\left(\int_\Omega |\nabla y_{\varepsilon_\theta,\theta}|^{p-2}|\nabla\psi|^2\,dx\right)^{1/2} \\ \le{}& \frac{p-2}{2}\|\psi\|_{L^p(\Omega)}\left(\int_\Omega |\nabla y_{\varepsilon_\theta,\theta}|^{\frac{(p-2)p}{2(p-1)}}\big|\nabla\ln|\nabla y_{\varepsilon_\theta,\theta}|\big|^{\frac{p}{p-1}}\,dx\right)^{(p-1)/p} \\ &+ |\Omega|^{1/2}\|\nabla y_{\varepsilon_\theta,\theta}\|^{(p-2)/2}_{L^p(\Omega;\mathbb{R}^N)}\|\nabla\psi\|_{L^p(\Omega;\mathbb{R}^N)} \\ \le{}& \frac{p-2}{2}\|\psi\|_{L^p(\Omega)}\|\nabla y_{\varepsilon_\theta,\theta}\|^{(p-2)/2}_{L^p(\Omega;\mathbb{R}^N)}\big\|\nabla\ln|\nabla y_{\varepsilon_\theta,\theta}|\big\|_{L^2(\Omega;\mathbb{R}^N)} \\ &+ |\Omega|^{1/2}\|y_{\varepsilon_\theta,\theta}\|^{(p-2)/2}_{W_0^{1,p}(\Omega)}\|\psi\|_{W_0^{1,p}(\Omega)} \\ \le{}& \left(\frac{p-2}{2}C\big\|\nabla\ln|\nabla y_{\varepsilon_\theta,\theta}|\big\|_{L^2(\Omega;\mathbb{R}^N)} + |\Omega|^{1/2}\right)\|y_{\varepsilon_\theta,\theta}\|^{(p-2)/2}_{W_0^{1,p}(\Omega)}\|\psi\|_{W_0^{1,p}(\Omega)} \\ <{}& \infty.\end{aligned}$$

Thus, $z_\theta \in W^{1,1}(\Omega)\cap L^2(\Omega)$. Since the element $z_\theta := |\nabla y_{\varepsilon_\theta,\theta}|^{(p-2)/2}\psi$ inherits the trace properties along $\partial\Omega$ from its parent element ψ, we finally obtain $z_\theta \in W_0^{1,1}(\Omega)\cap L^2(\Omega)$. The proof is complete. □

As an obvious consequence of this result and continuity of the embedding of Sobolev spaces $H_0^1(\Omega) \hookrightarrow W_0^{1,1}(\Omega)$, we can give the following conclusion.

Corollary 6.7 *If, for given* $\theta \in [0,1]$ *and* $\widehat{u} \in \mathfrak{A}_{\mathrm{ad}}$, $\nabla\ln|\nabla y_{\varepsilon_\theta,\theta}| \in L^2(\Omega;\mathbb{R}^N)$, *then there exists a dense subset* $D(y_{\varepsilon_\theta,\theta})$ *of* $H_0^1(\Omega)$ *such that*

$$|\nabla y_{\varepsilon_\theta,\theta}|^{(2-p)/2}z \in W_0^{1,p}(\Omega), \quad \forall z \in D(y_{\varepsilon_\theta,\theta}). \tag{6.10}$$

Taking this fact into account, it is plausible to introduce the following linear mapping:

$$\mathfrak{F} : D(y_{\varepsilon_\theta,\theta}) \subset H_0^1(\Omega) \to W_0^{1,p}(\Omega), \quad \text{where } \mathfrak{F}z = |\nabla y_{\varepsilon_\theta,\theta}|^{(2-p)/2}z. \tag{6.11}$$

Since domain $D(y_{\varepsilon_\theta,\theta})$ of $\mathfrak{F}$ is dense in Banach space $H_0^1(\Omega)$, it follows that for $\mathfrak{F}$, as for a densely defined operator, there exists an adjoint operator

$$\mathfrak{F}^* : D(\mathfrak{F}^*) \subset W^{-1,q}(\Omega) \to H^{-1}(\Omega)$$

such that

$$\langle \mathfrak{F}^* v, z \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \langle v, \mathfrak{F} z \rangle_{W^{-1,q}(\Omega), W_0^{1,p}(\Omega)}, \quad \forall z \in D(y_{\varepsilon_\theta,\theta}) \text{ and } \forall v \in D(\mathfrak{F}^*),$$

where

$$\begin{aligned} D(\mathfrak{F}^*) = \{ v \in W^{-1,q}(\Omega) \mid & \text{ there exists } C > 0 \text{ such that for all } z \in D(y_{\varepsilon_\theta,\theta}) \\ & \left| \langle v, \mathfrak{F} z \rangle_{W^{-1,q}(\Omega), W_0^{1,p}(\Omega)} \right| \le C \|z\|_{H_0^1(\Omega)} \}. \end{aligned}$$

Notice that, in general, the adjoint operator $\mathfrak{F}^*$ is not densely defined.

Let us consider the following linear operator:

$$\mathcal{A}_\theta \psi := -\operatorname{div}(\rho_\theta A_\theta \nabla \psi), \quad \forall \psi \in W_0^{1,p}(\Omega),$$

where ρ_θ and A_θ are defined by (6.6)-(6.7). By Proposition 6.4, we have

$$\begin{aligned} \|\rho_\theta A_\theta \nabla \psi\|^q_{L^q(\Omega;\mathbb{R}^N)} &\le \|A_\theta\|^q_{L^\infty(\Omega;\mathbb{S}^N_{\mathrm{sym}})} \int_\Omega |\nabla y_{\varepsilon_\theta,\theta}|^{(p-2)p/(p-1)} |\nabla \psi|^{p/(p-1)}\, dx \quad \begin{pmatrix} p' = \frac{p-1}{p-2} \\ q' = p-1 \end{pmatrix} \\ &\le \|A_\theta\|^q_{L^\infty(\Omega;\mathbb{S}^N_{\mathrm{sym}})} \|y_{\varepsilon_\theta,\theta}\|^{p(p-2)/(p-1)}_{W_0^{1,p}(\Omega)} \|\psi\|^{p/(p-1)}_{W_0^{1,p}(\Omega)}. \end{aligned}$$

Hence, $\mathcal{A}_\theta \psi : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ and this operator is obviously monotone,

$$\begin{aligned} &\langle \mathcal{A}_\theta(\psi - \phi), \psi - \phi \rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)} \\ &\quad = \int_\Omega |\nabla y_{\varepsilon_\theta,\theta}|^{p-2} \big(A_\theta(\nabla\psi - \nabla\phi), \nabla\psi - \nabla\phi \big)_{\mathbb{R}^N} \ge 0, \end{aligned}$$

and demi-continuous. However, because of the multiplier $\rho_\theta = |\nabla y_{\varepsilon_\theta,\theta}|^{p-2}$, this operator can lose the coercivity property.

Let λ be a positive constant such that $\lambda < \lambda^* := (N-2)^2/4$. Let $\{x_1, x_2, \ldots, x_L\} \subset \Omega$ be a given collection of points. Let $u \in \mathfrak{A}_{\mathrm{ad}}$ be a given control. We define a nonempty subset $\mathfrak{M}(\Omega)_u \subset W_0^{1,p}(\Omega)$ as follows: $y \in \mathfrak{M}_u(\Omega)$ if and only if

$$-\widehat{C}(\Omega) \le V_{u,y}(x) \le \frac{2\lambda}{L} \sum_{i=1}^{L} \frac{1}{|x - x_i|^2} \quad \text{a.e. in } \Omega, \tag{6.12}$$

for some positive constant $\widehat{C}(\Omega) > 0$, where

$$V_{u,y}(x) = (2-p)\operatorname{div}\big(A(u,y)\nabla \ln|\nabla y|\big) - \frac{(p-2)^2}{2} \big(\nabla \ln|\nabla y|, A(u,y)\nabla \ln|\nabla y|\big)_{\mathbb{R}^N}, \tag{6.13}$$

$$A(u,y) = u\left[I + (p-2)\frac{\nabla y}{|\nabla y|} \otimes \frac{\nabla y}{|\nabla y|} \right]. \tag{6.14}$$

We are now in a position to give an important property of the operator $\mathcal{A}_\theta\psi : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$.

Lemma 6.8 *Assume that, for given* $\widehat{u} \in \mathfrak{A}_{\mathrm{ad}}$, $\theta \in [0,1]$, *and* $\varepsilon_\theta \in (0,1)$, *the distribution* $y_{\varepsilon_\theta,\theta} \in W_0^{1,p}(\Omega)$ *is such that*

$$\nabla \ln|\nabla y_{\varepsilon_\theta,\theta}| \in L^2(\Omega;\mathbb{R}^N) \quad \text{and} \quad y_{\varepsilon_\theta,\theta} \in \mathfrak{M}_{u_\theta}(\Omega) \subset W_0^{1,p}(\Omega).$$

Then

$$\langle \mathcal{A}_\theta(\mathfrak{F}z), \mathfrak{F}v\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} = \langle B_\theta(z), v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \tag{6.15}$$

where

$$B_\theta(z) = -\operatorname{div}(A_\theta \nabla z) - \frac{1}{2} V_\theta(x) z, \tag{6.16}$$

$$\begin{aligned} V_\theta(x) &= (2-p)\operatorname{div}\bigl(A_\theta \nabla \ln|\nabla y_{\varepsilon_\theta,\theta}|\bigr) \\ &\quad - \frac{(p-2)^2}{2}\bigl(\nabla \ln|\nabla y_{\varepsilon_\theta,\theta}|, A_\theta \nabla \ln|\nabla y_{\varepsilon_\theta,\theta}|\bigr)_{\mathbb{R}^N}, \end{aligned} \tag{6.17}$$

and the linear operator B_θ *defines an isomorphism from* $H_0^1(\Omega)$ *into its dual* $H^{-1}(\Omega)$.

Proof Let v and z be arbitrary elements of $D(y_{\varepsilon_\theta,\theta}) \subset H_0^1(\Omega)$. Then by Corollary 6.7, we have $\mathfrak{F}z, \mathfrak{F}v \in W_0^{1,p}(\Omega)$. Further, following the definition of the operator $\mathfrak{F}$, we can provide the following chain of transformations:

$$\begin{aligned} \mathcal{A}_\theta(\mathfrak{F}z) &= -\operatorname{div}\bigl(\rho_\theta A_\theta \nabla(\mathfrak{F}z)\bigr) = -\operatorname{div}\left(\rho_\theta A_\theta \nabla\left(\frac{z}{\sqrt{\rho_\theta}}\right)\right) \\ &= -\operatorname{div}\left(-\frac{1}{2}\rho_\theta^{-1/2} z A_\theta \nabla\rho_\theta + \rho_\theta^{1/2} A_\theta \nabla z\right) \\ &= -\frac{1}{4}\rho_\theta^{-3/2} z(\nabla\rho_\theta, A_\theta\nabla\rho_\theta)_{\mathbb{R}^N} + \frac{1}{2}\rho^{-1/2}(\nabla z, A_\theta\nabla\rho_\theta)_{\mathbb{R}^N} \\ &\quad + \frac{1}{2}\rho_\theta^{-1/2} z \operatorname{div}(A_\theta\nabla\rho_\theta) - \frac{1}{2}\rho_\theta^{-1/2}(\nabla\rho_\theta, A_\theta\nabla z)_{\mathbb{R}^N} - \rho_\theta^{1/2}\operatorname{div}(A_\theta\nabla z) \\ &= -\rho_\theta^{1/2}\operatorname{div}(A_\theta\nabla z) - \frac{1}{2}\rho_\theta^{1/2}\left[\frac{1}{2}\left(\frac{\nabla\rho_\theta}{\rho_\theta}, A_\theta\frac{\nabla\rho_\theta}{\rho_\theta}\right)_{\mathbb{R}^N} - \frac{1}{\rho_\theta}\operatorname{div}(A_\theta\nabla\rho_\theta)\right] z \\ &= \rho_\theta^{1/2}\left(-\operatorname{div}(A_\theta\nabla z) - \frac{1}{2}\left[\frac{1}{2}(\nabla\ln\rho_\theta, A_\theta\nabla\ln\rho_\theta)_{\mathbb{R}^N} - \frac{1}{\rho_\theta}\operatorname{div}(\rho_\theta A_\theta\nabla\ln\rho_\theta)\right] z\right) \\ &= \rho_\theta^{1/2}\left(-\operatorname{div}(A_\theta\nabla z) - \frac{1}{2}V_\theta(x) z\right). \end{aligned}$$

Hence,

$$\begin{aligned} &\langle \mathcal{A}_\theta(\mathfrak{F}z), \mathfrak{F}v\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} \\ &\quad = \left\langle -\operatorname{div}\left(\rho_\theta A_\theta\nabla\left(\frac{z}{\sqrt{\rho_\theta}}\right)\right), \frac{v}{\sqrt{\rho_\theta}}\right\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} \end{aligned}$$

$$= \left\langle \rho_\theta^{1/2}\left(-\operatorname{div}(A_\theta \nabla z) - \frac{1}{2} V_\theta(x) z\right), \frac{v}{\sqrt{\rho_\theta}} \right\rangle_{W^{-1,q}(\Omega); W_0^{1,p}(\Omega)}$$
$$= \left\langle -\operatorname{div}(A_\theta \nabla z) - \frac{1}{2} V_\theta(x) z, v \right\rangle_{H^{-1}(\Omega); H_0^1(\Omega)} = \langle B_\theta(z), v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.$$

To conclude the proof it remains to show that operator $B_\theta := -\operatorname{div}(A_\theta \nabla) - \frac{1}{2} V_\theta(x)$ defines an isomorphism from $H_0^1(\Omega)$ into its dual $H^{-1}(\Omega)$. With that in mind, we make use of the following version of the Hardy-Poincaré Inequality: for a given internal point $x^* \in \Omega$ there exists a constant $\widehat{C}(\Omega) > 0$ such that for every $v \in H_0^1(\Omega)$

$$\int_\Omega \left[|\nabla v|_{\mathbb{R}^N}^2 - \lambda_* \frac{v^2}{|x - x^*|_{\mathbb{R}^N}^2} \right] dx \geq \widehat{C}(\Omega) \int_\Omega v^2\, dx,$$

where $\lambda^* := (N-2)^2/4$ and $N \geq 2$.

According to this result and the fact that $y_{\varepsilon_\theta,\theta} \in \mathfrak{M}_{u_\theta}(\Omega)$, we have

$$-\widehat{C}(\Omega) \leq V_\theta(x) \leq \frac{2\lambda}{L} \sum_{i=1}^{L} \frac{1}{|x - x_i|^2} < \frac{(N-2)^2}{2L} \sum_{i=1}^{L} \frac{1}{|x - x_i|^2} \quad \text{a.e. in } \Omega \tag{6.18}$$

and, therefore,

$$\begin{aligned}
&\left(1 + \frac{\widehat{C}(\Omega)}{2C}\right) \|v\|_{H_0^1(\Omega)}^2 \\
&\quad \geq \int_\Omega \left[|\nabla v|^2 + \frac{\widehat{C}(\Omega)}{2} v^2 \right] dx \\
&\quad \geq \int_\Omega \left[|\nabla v|^2 - \frac{\lambda}{L} \left(\sum_{i=1}^{L} \frac{1}{|x - x_i^*|} \right) v^2 \right] dx \\
&\quad = \left(1 - \frac{\lambda}{\lambda_*}\right) \int_\Omega |\nabla v|^2\, dx + \frac{\lambda}{\lambda_*} \int_\Omega \left[|\nabla v|^2 - \frac{\lambda_*}{L} \left(\sum_{i=1}^{L} \frac{1}{|x - x_i^*|} \right) v^2 \right] dx \\
&\quad \geq \left(1 - \frac{\lambda}{\lambda_*}\right) \int_\Omega |\nabla v|^2\, dx + \frac{\lambda \widehat{C}(\Omega)}{\lambda_*} \int_\Omega v^2\, dx \geq \left(1 - \frac{\lambda}{\lambda_*}\right) \|v\|_{H_0^1(\Omega)}^2.
\end{aligned} \tag{6.19}$$

Thus, in view of (6.18)-(6.19),

$$\big\| [v] \big\|_\theta^2 := \int_\Omega \left[|\nabla v|^2 - \frac{1}{2} V_\theta(x) v^2 \right] dx = \int_\Omega \left[(\nabla v, \nabla v)_{\mathbb{R}^N} - \frac{1}{2} V_\theta(x) v^2 \right] dx$$

is equivalent to the standard norm of $H_0^1(\Omega)$, and therefore, the operator B_θ given by (6.16) defines an isomorphism from $H_0^1(\Omega)$ into its dual $H^{-1}(\Omega)$. □

The next step of our analysis is to show that, for every $\widehat{u} \in \mathfrak{A}_{\mathrm{ad}}$, $\theta \in [0,1]$, and $\varepsilon_\theta \in (0,1)$, the quasi-adjoint state $\psi_{\varepsilon_\theta,\theta}$ to $y_0 \in W_0^{1,p}(\Omega)$ can be defined as a unique solution to the Dirichlet boundary value problem (6.5). With that in mind, we make use of the following hypothesis.

(H2) For a given distribution $f \in W^{-1,q}(\Omega)$ with $q = \frac{p}{p-1}$ and $p \geq 2$, the weak solutions $y(u)$ of the nonlinear Dirichlet boundary value problem (2.3) satisfy the property:

$y(u) \in \mathfrak{M}_w(\Omega)$ and $\nabla \ln|\nabla y(u)| \in L^2(\Omega;\mathbb{R}^N)$ for every pair of admissible controls $u, w \in \mathfrak{A}_{\mathrm{ad}}$.

Lemma 6.9 *Assume that the Hypothesis* (H2) *is valid. Then the Dirichlet boundary value problem* (6.5) *has a unique solution* $\psi_{\varepsilon_\theta,\theta} \in W_0^{1,p}(\Omega)$ *for every* $\widehat{u} \in \mathfrak{A}_{\mathrm{ad}}$, $\theta \in [0,1]$, *and* $\varepsilon_\theta \in (0,1)$.

Proof Let $(u_0, y_0) \in \Xi$ be an optimal pair to problem (3.1)-(3.3). Let $y_\theta := y(u_\theta) = y(u_0 + \theta(\widehat{u} - u_0))$ be the solution of problem (3.2)-(3.3) for given $\widehat{u} \in \mathfrak{A}_{\mathrm{ad}}$, $\theta \in [0,1]$, and let $y_{\varepsilon_\theta,\theta} = y_0 + \varepsilon_\theta(y_\theta - y_0)$, where ε_θ takes an arbitrary value in $(0,1)$. Since, by the initial assumptions, $f_\theta := p\operatorname{div}(|\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d|^{p-2}(\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d)) \in W^{-1,q}(\Omega)$, it follows that

$$\left\langle -\operatorname{div}\left(|\nabla y_{\varepsilon_\theta,\theta}|^{p-2} A_\theta \nabla \psi\right) - f_\theta, \phi\right\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} = 0,$$

for all $\phi \in \mathfrak{F}(D(y_{\varepsilon_\theta,\theta})) = W_0^{1,p}(\Omega)$. Then Lemma 6.8 implies that

$$\begin{aligned}
&\left\langle -\operatorname{div}\left(|\nabla y_{\varepsilon_\theta,\theta}|^{p-2} A_\theta \nabla \psi\right), \phi\right\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} \\
&\quad = \left\langle \mathcal{A}_\theta(\mathfrak{F}z), \mathfrak{F}v\right\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} \\
&\quad = \left\langle -\operatorname{div}(A_\theta \nabla z) - \frac{1}{2} V_\theta(x) z, v\right\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} \\
&\quad = \left\langle B_\theta(z), v\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \\
&\langle f_\theta, \phi\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} = -p\int_\Omega |\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d|^{p-2}(\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d, \nabla\phi)_{\mathbb{R}^N}\,dx \\
&\quad = p\left\langle \operatorname{div}\left(|\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d|^{p-2}(\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d)\right), \phi\right\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} \\
&\quad = \langle f_\theta, \mathfrak{F}v\rangle_{W^{-1,q}(\Omega);W_0^{1,p}(\Omega)} = \left\langle \mathfrak{F}^* f_\theta, v\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}
\end{aligned}$$

provided $\phi \in \mathfrak{F}(D(y_{\varepsilon_\theta,\theta}))$. By the Hardy-Poincaré Inequality (see (6.19)), the expression

$$\int_\Omega \left[(\nabla v, \nabla z)_{\mathbb{R}^N} - \frac{1}{2} V_\theta(x) v z\right] dx$$

can be considered as a scalar product in $H_0^1(\Omega)$. Then, in view of the Riesz Representation Theorem, we conclude to the existence of a unique element $z_\theta \in H_0^1(\Omega)$ such that

$$\left\langle B_\theta(z_\theta), v\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \left\langle \mathfrak{F}^* f_\theta, v\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega).$$

As a result, we have: $\psi_{\varepsilon_\theta,\theta} := \mathfrak{F}z_\theta = |\nabla y_{\varepsilon_\theta,\theta}|^{(p-2)/2} z_\theta$ is a unique solution to the Dirichlet boundary value problem (2.3). Moreover, by Corollary 6.7, we finally get $\psi_{\varepsilon_\theta,\theta} \in W_0^{1,p}(\Omega)$. □

In view of Lemma 6.9, it makes a sense to accept the following hypothesis.

(H3) For given distributions $f \in W^{-1,q}(\Omega)$ and $y_d \in W_0^{1,p}(\Omega)$ with $q = \frac{p}{p-1}$ and $p \geq 2$, the weak solution $y(u)$ to the nonlinear Dirichlet boundary value problem (2.3) satisfies the properties

$$\nabla\big(|\nabla y(u)|\big) \in L^p\big(\Omega;\mathbb{R}^N\big) \quad \text{and} \quad \frac{|\nabla y_d|}{|\nabla y(u)|} \in L^\infty(\Omega), \quad \forall u \in \mathfrak{A}_{\mathrm{ad}}.$$

Lemma 6.10 *Assume that Hypotheses* (H2)-(H3) *are valid. Then there exists a constant* $C^* > 0$ *independent of* θ *such that* $\sup_{\theta\to 0} \|\psi_{\varepsilon_\theta,\theta}\|_{W_0^{1,p}(\Omega)} \le C^*$, *i.e. the sequence of quasi-adjoint states* $\{\psi_{\varepsilon_\theta,\theta}\}_{\theta\to 0}$ *to* $y_0 \in W_0^{1,p}(\Omega)$ *is relatively compact with respect to the weak convergence of* $W_0^{1,p}(\Omega)$.

Proof Due to Hypothesis (H2) and Lemma 6.9, the sequence of quasi-adjoint states $\{\psi_{\varepsilon_\theta,\theta}\}_{\theta\to 0}$ to $y_0 \in W_0^{1,p}(\Omega)$ can be defined in a unique way $\psi_{\varepsilon_\theta,\theta} = |\nabla y_{\varepsilon_\theta,\theta}|^{(2-p)/2} z_\theta$ for all $\theta \in [0,1]$, where z_θ is the solution of the following Dirichlet problem:

$$B_\theta z_\theta = p\mathfrak{F}^*\big[\operatorname{div}\big(|\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d|^{p-2}(\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d)\big)\big], \quad z_\theta \in H_0^1(\Omega). \tag{6.20}$$

Therefore, in order to prove this lemma, it is enough to show that the sequence $\{z_\theta\}_{\theta\to 0}$ is uniformly bounded in $H_0^1(\Omega)$. To this end, we note that the integral identity (6.20) leads to the corresponding energy equality

$$\int_\Omega \left[|\nabla z_\theta|^2 - \frac{1}{2} V_\theta(x) z_\theta^2 \right] dx = \langle \mathfrak{F}^* f_\theta, z_\theta \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \tag{6.21}$$

Since

$$\begin{aligned}
&\big|\langle \mathfrak{F}^* f_\theta, z_\theta \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}\big| \\
&\quad \le p \int_\Omega |\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d|^{p-1} \big|\nabla\big(z_\theta |\nabla y_{\varepsilon_\theta,\theta}|^{(2-p)/2}\big)\big|\, dx \\
&\quad \le p \int_\Omega \frac{|\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d|^{p-1}}{|\nabla y_{\varepsilon_\theta,\theta}|^{(p-1)}} \left(\frac{p-2}{2} |z_\theta| \big|\nabla \ln |\nabla y_{\varepsilon_\theta,\theta}|\big| + |\nabla z_\theta| \right) \frac{|\nabla y_{\varepsilon_\theta,\theta}|^{p-1}}{|\nabla y_{\varepsilon_\theta,\theta}|^{(p-2)/2}}\, dx \\
&\quad \overset{\text{by (H3)}}{\le} p \left(\operatorname*{vrai\,sup}_{x\in\Omega} \left| \frac{\nabla y_{\varepsilon_\theta,\theta}}{|\nabla y_{\varepsilon_\theta,\theta}|} - \frac{\nabla y_d}{|\nabla y_{\varepsilon_\theta,\theta}|} \right| \right)^{p-1} \\
&\qquad \times \int_\Omega \left(\frac{p-2}{2} |z_\theta| |\nabla y_{\varepsilon_\theta,\theta}|^{p/2} \big|\nabla \ln |\nabla y_{\varepsilon_\theta,\theta}|\big| + |\nabla y_{\varepsilon_\theta,\theta}|^{p/2} |\nabla z_\theta| \right) dx \\
&\quad \le \frac{p(p-2)}{2} \left(1 + \left\| \frac{|\nabla y_d|}{|\nabla y_{\varepsilon_\theta,\theta}|} \right\|_{L^\infty(\Omega)} \right)^p \|z_\theta\|_{L^2(\Omega)} \left(\int_\Omega |\nabla y_{\varepsilon_\theta,\theta}|^p \big|\nabla \ln |\nabla y_{\varepsilon_\theta,\theta}|\big|^2\, dx \right)^{1/2} \\
&\qquad + p \left(1 + \left\| \frac{|\nabla y_d|}{|\nabla y_{\varepsilon_\theta,\theta}|} \right\|_{L^\infty(\Omega)} \right)^p \|\nabla y_{\varepsilon_\theta,\theta}\|_{L^p(\Omega;\mathbb{R}^N)} \|\nabla z_\theta\|_{L^2(\Omega;\mathbb{R}^N)},
\end{aligned}$$

and

$$\|z_\theta\|_{L^2(\Omega)} \le C\|\nabla z_\theta\|_{L^2(\Omega;\mathbb{R}^N)} \quad \text{by the Poincaré Inequality,}$$

$$\left\| \frac{|\nabla y_d|}{|\nabla y_{\varepsilon_\theta,\theta}|} \right\|_{L^\infty(\Omega)} < +\infty \quad \text{by Hypothesis (H3)}_2,$$

$$\begin{aligned}
&\int_\Omega |\nabla y_{\varepsilon_\theta,\theta}|^p \big|\nabla \ln|\nabla y_{\varepsilon_\theta,\theta}|\big|^2\,dx \\
&\quad = \int_\Omega |\nabla y_{\varepsilon_\theta,\theta}|^{p-2}\big|\nabla\big(|\nabla y_{\varepsilon_\theta,\theta}|\big)\big|^2\,dx \quad \big(\text{by Hypothesis (H3)}_1\big) \\
&\quad \le \|y_{\varepsilon_\theta,\theta}\|^{p-2}_{W_0^{1,p}(\Omega)} \big\|\nabla\big(|\nabla y_{\varepsilon_\theta,\theta}|\big)\big\|^2_{L^p(\Omega;\mathbb{R}^N)} < +\infty,
\end{aligned}$$

it follows that there exists a constant C_0 independent of θ such that

$$\big|\langle \mathfrak{F}^* f_\theta, z_\theta\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\big| \le C_0\|z_\theta\|_{H_0^1(\Omega)}.$$

As a result, in view of (6.19), the energy equality (6.21) immediately leads to the estimate.

$$\left(1 - \frac{\lambda}{\lambda_*}\right)\|z_\theta\|_{H_0^1(\Omega)} \le C_1 + C_2, \quad \forall\theta \in [0,1].$$

This concludes the proof. □

7 Optimality conditions

We are now in a position to derive the first-order optimality conditions for optimal control problem (3.1)-(3.3).

Theorem 7.1 *Let us suppose that $f \in W^{-1,q}(\Omega)$, $y_d \in W_0^{1,p}(\Omega)$, and $\mathfrak{A}_{\mathrm{ad}} \ne \emptyset$ are given with $p \ge 2$. Let $(u_0, y_0) \in L^\infty(\Omega) \times W_0^{1,p}(\Omega)$ be an optimal pair to problem* (3.1)-(3.3). *Let $\psi_{\varepsilon_\theta,\theta}$ be a quasi-adjoint state to $y_0 \in W_0^{1,p}(\Omega)$ defined for each $\varepsilon_\theta \in (0,1)$ and $\theta \in [0,1]$ by* (5.5). *Then Hypotheses* (H1)-(H3) *imply the existence of an element $\psi \in W_0^{1,p}(\Omega)$ such that* (*within a subsequence*) *$\psi_{\varepsilon_\theta,\theta} \rightharpoonup \psi$ in $W_0^{1,p}(\Omega)$ as $\theta \to 0$, and*

$$\int_\Omega (u - u_0)\big(|\nabla y_0|^{p-2}\nabla y_0, \nabla\psi\big)_{\mathbb{R}^N}\,dx \ge 0, \quad \forall u \in \mathfrak{A}_{\mathrm{ad}}, \tag{7.1}$$

$$-\operatorname{div}\big(u_0(x)|\nabla y_0|^{p-2}\nabla y_0\big) = f \quad \text{in } \mathcal{D}'(\Omega), \tag{7.2}$$

$$\begin{aligned}
&-\operatorname{div}\left(u_0|\nabla y_0|^{p-2}\left[I + (p-2)\frac{\nabla y_0}{|\nabla y_0|} \otimes \frac{\nabla y_0}{|\nabla y_0|}\right]\nabla\psi\right) \\
&\quad = p\operatorname{div}\big(|\nabla y_0 - \nabla y_d|^{p-2}(\nabla y_0 - \nabla y_d)\big) \quad \text{in } \mathcal{D}'(\Omega).
\end{aligned} \tag{7.3}$$

Proof Let $(\widehat{\mathcal{U}}, \widehat{y}) \in \Xi$ be an admissible pair. Let $y_\theta := y(u_\theta) = y(u_0 + \theta(\widehat{u} - u_0))$ be the solution of problem (3.2)-(3.3) for given $\widehat{u} \in \mathfrak{A}_{\mathrm{ad}}$ and $\theta \in [0,1]$. Then, as was shown at the end of Section 5, there exists a value $\varepsilon_\theta \in [0,1]$ such that the increment of Lagrangian $\Delta\Lambda = \Lambda(u_\theta, y_\theta, \lambda) - \Lambda(u_0, y_0, \lambda)$ can be simplified to the form (5.7), provided the element λ has been defined as the quasi-adjoint state, that is, $\lambda = \psi_{\varepsilon_\theta,\theta}$. By Lemmas 6.9-6.10, the sequence of quasi-adjoint states $\{\psi_{\varepsilon_\theta,\theta}\}_{\theta\to 0}$ to $y_0 \in W_0^{1,p}(\Omega)$ can be defined in a unique way and is bounded in $W_0^{1,p}(\Omega)$. Hence, there exists an element $\psi \in W_0^{1,p}(\Omega)$ such that, up to a subsequence of $\{\psi_{\varepsilon_\theta,\theta}\}_{\theta\to 0}$, we have $\psi_{\varepsilon_\theta,\theta} \rightharpoonup \psi$ in $W_0^{1,p}(\Omega)$ as $\theta \to 0$. It remains to pass to the limit in (5.5), (5.7) as $\theta \to +0$ and to show that in the limit we will arrive at the relations (7.3) and (7.1), respectively. To this end, we note that

(A_1) by the initial suppositions, $u_\theta \to u_0$ in $L^\infty(\Omega)$ as $\theta \to 0$;

(A_2) by Corollary 6.2, $y_\theta \to y_0$ in $W_0^{1,p}(\Omega)$ as $\theta \to 0$, and, hence, $y_{\varepsilon_\theta,\theta} := y_0 + \varepsilon_\theta(y_\theta - y_0) \to y_0$ in $W_0^{1,p}(\Omega)$ as $\theta \to 0$;

(A_3) up to a subsequence

$$\begin{aligned} A_\theta &:= u_\theta\left(I + (p-2)\frac{\nabla y_{\varepsilon_\theta,\theta}}{|\nabla y_{\varepsilon_\theta,\theta}|} \otimes \frac{\nabla y_{\varepsilon_\theta,\theta}}{|\nabla y_{\varepsilon_\theta,\theta}|}\right) \\ &\to u_0\left(I + (p-2)\frac{\nabla y_0}{|\nabla y_0|} \otimes \frac{\nabla y_0}{|\nabla y_0|}\right) =: A_0 \quad \text{in } L^\infty\big(\Omega; \mathbb{S}^N_{\mathrm{sym}}\big) \end{aligned}$$

by the strong convergence of $u_\theta \to u_0$ in $L^\infty(\Omega)$ and convergence $y_{\varepsilon_\theta,\theta} \to y_0$ almost everywhere in Ω;

(A_4) if $p > 3$, then

$$\big||a|^{p-2} - |b|^{p-2}\big| \le (p-2)\big(|a| + |b|\big)^{p-3}|a-b|, \quad \forall a, b \in \mathbb{R}; \tag{7.4}$$

(A_5) if $2 \le p \le 3$, then

$$\big||a|^{p-2} - |b|^{p-2}\big| \le |a-b|^{p-2}, \quad \forall a, b \in \mathbb{R}. \tag{7.5}$$

Then the limit passage in (5.7) immediately leads to (7.1). Therefore, in order to end the proof, it remains to establish the validity of relation (7.3). With that in mind, we rewrite (5.5) as follows:

$$I_1^\theta + pI_2^\theta = 0,$$

where

$$\begin{aligned} I_1^\theta &= \int_\Omega |\nabla y_{\varepsilon_\theta,\theta}|^{p-2}\left(u_\theta\left[I + (p-2)\frac{\nabla y_{\varepsilon_\theta,\theta}}{|\nabla y_{\varepsilon_\theta,\theta}|} \otimes \frac{\nabla y_{\varepsilon_\theta,\theta}}{|\nabla y_{\varepsilon_\theta,\theta}|}\right]\nabla \psi_{\varepsilon_\theta,\theta}, \nabla\varphi\right)_{\mathbb{R}^N} dx \\ &= \int_\Omega |\nabla y_{\varepsilon_\theta,\theta}|^{p-2}(A_\theta \nabla\psi_{\varepsilon_\theta,\theta}, \nabla\varphi)_{\mathbb{R}^N}\, dx, \\ I_2^\theta &= \int_\Omega |\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d|^{p-2}(\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d, \nabla\varphi)_{\mathbb{R}^N}\, dx, \end{aligned}$$

and φ is an arbitrary element of $W_0^{1,p}(\Omega)$.

Since

$$\begin{aligned} I_1^\theta &= \int_\Omega \big(|\nabla y_{\varepsilon_\theta,\theta}|^{p-2} - |\nabla y_0|^{p-2}\big)(A_\theta \nabla\psi_{\varepsilon_\theta,\theta}, \nabla\varphi)_{\mathbb{R}^N}\, dx \\ &\quad + \int_\Omega |\nabla y_0|^{p-2}\big((A_\theta - A_0)\nabla\psi_{\varepsilon_\theta,\theta}, \nabla\varphi\big)_{\mathbb{R}^N}\, dx \\ &\quad + \int_\Omega |\nabla y_0|^{p-2}\big(A_0(\nabla\psi_{\varepsilon_\theta,\theta} - \nabla\psi), \nabla\varphi\big)_{\mathbb{R}^N}\, dx \\ &\quad + \int_\Omega |\nabla y_0|^{p-2}(A_0 \nabla\psi, \nabla\varphi)_{\mathbb{R}^N}\, dx = J_{1,1}^\theta + J_{1,2}^\theta + J_{1,3}^\theta + J_{1,4}, \end{aligned}$$

let us show that $\lim_{\theta\to 0} J_{1,j}^\theta = 0$ ($j = 1, 2, 3$), and, hence, $I_1^\theta \to J_{1,4}$ as $\theta \to +0$.

Using the Hölder Inequality and estimate (2) in Proposition 6.4, we have

$$
\begin{aligned}
|J_{1,1}^{\theta}| &\le \int_{\Omega} \big| |\nabla y_{\varepsilon_\theta,\theta}|^{p-2} - |\nabla y_0|^{p-2} \big| \|A_\theta\|_{\mathbb{S}^N_{\mathrm{sym}}} |\nabla \psi_{\varepsilon_\theta,\theta}| |\nabla \varphi| \, dx \\
&\le \beta \big[1 + (p-2)2^{N-1}\big] \int_{\Omega} \big| |\nabla y_{\varepsilon_\theta,\theta}|^{p-2} - |\nabla y_0|^{p-2} \big| |\nabla \psi_{\varepsilon_\theta,\theta}| |\nabla \varphi| \, dx. \qquad (7.6)
\end{aligned}
$$

Therefore, if $p > 3$, then, by (A_4) and Hölder's Inequality with Hölder conjugates $p' = \frac{p}{p-2} > 1$ and $q' = \frac{p}{2}$, we can estimate (7.6) as follows:

$$
\begin{aligned}
|J_{1,1}^{\theta}| &\le \beta \big[1 + (p-2)2^{N-1}\big](p-2) \\
&\quad \times \int_{\Omega} \big(|\nabla y_{\varepsilon_\theta,\theta}| + |\nabla y_0|\big)^{p-3} |\nabla y_{\varepsilon_\theta,\theta} - \nabla y_0| |\nabla \psi_{\varepsilon_\theta,\theta}| |\nabla \varphi| \, dx \\
&\le c_1 \left(\int_{\Omega} \big(|\nabla y_{\varepsilon_\theta,\theta}| + |\nabla y_0|\big)^{\frac{p(p-3)}{p-2}} |\nabla y_{\varepsilon_\theta,\theta} - \nabla y_0|^{\frac{p}{p-2}} \, dx \right)^{\frac{p-2}{p}} \\
&\quad \times \left(\int_{\Omega} |\nabla \psi_{\varepsilon_\theta,\theta}|^{\frac{p}{2}} |\nabla \varphi|^{\frac{p}{2}} \, dx \right)^{\frac{2}{p}} \quad \left(\begin{array}{c} \text{by Hölder's Inequality with} \\ p' = \frac{p-2}{p-3}, q' = (p-2) \end{array} \right) \\
&\le c_1 \left(\int_{\Omega} \big(|\nabla y_{\varepsilon_\theta,\theta}| + |\nabla y_0|\big)^{p} \, dx \right)^{\frac{p-3}{p}} \left(\int_{\Omega} |\nabla y_{\varepsilon_\theta,\theta} - \nabla y_0|^{p} \, dx \right)^{\frac{1}{p}} \\
&\quad \times \|\nabla \psi_{\varepsilon_\theta,\theta}\|_{L^p(\Omega;\mathbb{R}^N)} \|\nabla \varphi\|_{L^p(\Omega;\mathbb{R}^N)}. \qquad (7.7)
\end{aligned}
$$

Since $\sup_{\theta\in[0,1]} \|\nabla \psi_{\varepsilon_\theta,\theta}\|_{L^p(\Omega;\mathbb{R}^N)} < +\infty$ by Lemma 6.10, and $y_{\varepsilon_\theta,\theta} \to y_0$ in $W_0^{1,p}(\Omega)$ (see the condition (A_2)), it follows that

$$
\sup_{\theta\in[0,1]} \int_{\Omega} \big(|\nabla y_{\varepsilon_\theta,\theta}| + |\nabla y_0|\big)^{p} \, dx < +\infty, \qquad \|\nabla y_{\varepsilon_\theta,\theta} - \nabla y_0\|_{L^p(\Omega;\mathbb{R}^N)} \stackrel{\theta\to 0}{\to} 0.
$$

Thus, in view of estimate (7.7), we conclude: $\lim_{\theta\to 0} J_{1,1}^{\theta} = 0$.

As for the case $2 \le p \le 3$, the inequality (7.6) and condition (A_5) lead to the estimate

$$
\big|J_{1,1}^{\theta}\big| \le c_1 \int_{\Omega} |\nabla y_{\varepsilon_\theta,\theta} - \nabla y_0|^{p-2} |\nabla \psi_{\varepsilon_\theta,\theta}| |\nabla \varphi| \, dx.
$$

Further it remains to repeat the trick like in (7.7). As a result, we obtain

$$
\big|J_{1,1}^{\theta}\big| \le c_1 \left(\int_{\Omega} |\nabla y_{\varepsilon_\theta,\theta} - \nabla y_0|^{p} \, dx \right)^{\frac{p-2}{p}} \|\nabla \psi_{\varepsilon_\theta,\theta}\|_{L^p(\Omega;\mathbb{R}^N)} \|\nabla \varphi\|_{L^p(\Omega;\mathbb{R}^N)}.
$$

Therefore, having applied the arguments given before, we can conclude: if $2 \le p \le 3$, then $\lim_{\theta\to 0} J_{1,1}^{\theta} = 0$.

As for the term $J_{1,2}^{\theta}$, we have

$$
\begin{aligned}
\big|J_{1,2}^{\theta}\big| &\le \int_{\Omega} |\nabla y_0|^{p-2} \|A_\theta - A_0\|_{\mathbb{S}^N_{\mathrm{sym}}} |\nabla \psi_{\varepsilon_\theta,\theta}| |\nabla \varphi| \, dx \\
&\le c_2 \|A_\theta - A_0\|_{L^\infty(\Omega;\mathbb{S}^N_{\mathrm{sym}})} \|\nabla y_0\|^{p-2}_{L^p(\Omega;\mathbb{R}^N)} \|\nabla \psi_{\varepsilon_\theta,\theta}\|_{L^p(\Omega;\mathbb{R}^N)} \|\nabla \varphi\|_{L^p(\Omega;\mathbb{R}^N)}
\end{aligned}
$$

$$\le c_2 \sup_{\theta\in[0,1]} \|\psi_{\varepsilon_\theta,\theta}\|_{W_0^{1,p}(\Omega)} \|\nabla y_0\|^{p-2}_{W_0^{1,p}(\Omega)} \|\varphi\|_{W_0^{1,p}(\Omega)} \|A_\theta - A_0\|_{L^\infty(\Omega;\mathbb{S}^N_{\mathrm{sym}})}$$

$$\le c_3 \|A_\theta - A_0\|_{L^\infty(\Omega;\mathbb{S}^N_{\mathrm{sym}})} \stackrel{\text{by }(\mathrm{A}_3)}{\to} 0 \quad \text{as } \theta \to 0. \tag{7.8}$$

To clarify the asymptotic behavior of the term $J^\theta_{1,3}$ as θ tends to zero, we note that

$$J^\theta_{1,3} = \int_\Omega |\nabla y_0|^{p-2} (\nabla \psi_{\varepsilon_\theta,\theta} - \nabla\psi, A_0 \nabla\varphi)_{\mathbb{R}^N}\, dx.$$

Since $\nabla\varphi \in L^p(\Omega;\mathbb{R}^N)$, $A_0 \in L^\infty(\Omega;\mathbb{S}^N_{\mathrm{sym}})$, and $\nabla y_0 \in L^p(\Omega;\mathbb{R}^N)$, it follows that the inclusion $A_0|\nabla y_0|^{p-2}\nabla\varphi \in L^q(\Omega;\mathbb{R}^N)$ holds true with $q = p/(p-1)$. Hence, the condition $\lim_{\theta\to 0} J^\theta_{1,3} = 0$ is ensured by the weak convergence $\nabla\psi_{\varepsilon_\theta,\theta} \rightharpoonup \nabla\psi$ in $L^p(\Omega;\mathbb{R}^N)$.

Thus, summing up the results given above, we finally obtain

$$\lim_{\theta\to 0} I^\theta_1 = \lim_{\theta\to 0} \left(\sum_{j=1}^3 J^\theta_{1,j} + J_{1,4} \right) = \int_\Omega |\nabla y_0|^{p-2} (A_0 \nabla\psi, \nabla\varphi)_{\mathbb{R}^N}\, dx. \tag{7.9}$$

As for the term

$$I^\theta_2 := \int_\Omega |\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d|^{p-2} (\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d, \nabla\varphi)_{\mathbb{R}^N}\, dx,$$

we see that

$$|\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d|^{p-2} (\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d) \in L^q(\Omega;\mathbb{R}^N) \quad \text{with } q = \frac{p}{p-1}, \forall\theta \in [0,1].$$

Hence, strong convergence $y_\theta \to y_0$ in $W_0^{1,p}(\Omega)$ implies strong convergence

$$\begin{aligned} &|\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d|^{p-2} (\nabla y_{\varepsilon_\theta,\theta} - \nabla y_d) \\ &\quad \to |\nabla y_0 - \nabla y_d|^{p-2} (\nabla y_0 - \nabla y_d) \quad \text{in } L^q(\Omega;\mathbb{R}^N). \end{aligned} \tag{7.10}$$

As a result, we finally get

$$\lim_{\theta\to 0} I^\theta_2 = \int_\Omega |\nabla y_0 - \nabla y_d|^{p-2} (\nabla y_0 - \nabla y_d, \nabla\varphi)_{\mathbb{R}^N}\, dx, \quad \forall\varphi \in W_0^{1,p}(\Omega). \tag{7.11}$$

Thus, combining relations (7.9)-(7.11), it is easy to see that the limit passage in (5.5) leads to the variational statement of the Dirichlet boundary value problem (7.3). The proof is complete. □

8 Conclusions

In this paper the optimal control problem (3.1)-(3.3) for a nonlinear monotone elliptic equation with homogeneous Dirichlet conditions and $L^\infty(\Omega)$-control in coefficients of Δ_p-Laplacian has been studied. Having defined the class of admissible control in form (2.2), we have proved solvability of the considered problem. After that, using Lagrange principle, the concept of quasi-adjoint system and the well-known Hardy-Poincaré Inequality, we have derived the corresponding optimality system and formulated sufficient conditions under which the degenerate adjoint boundary value problem admits a unique solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of System Analysis and Control, National Mining University, Karl Marx av., 19, Dnipropetrovsk, 49005, Ukraine. [2]Institute of Applied and System Analysis, National Technical University of Ukraine 'Kiev Polytechnical Institute', Peremogy av., 37, build. 35, Kiev, 03056, Ukraine. [3]Dipartimento di Ingegneria dell'Informazione, Ingegneria Elettrica e Matematica Applicata, Università degli Studi di Salerno, Via Giovanni Paolo II, 132, Fisciano, 84084, Italy.

Received: 9 January 2014 Accepted: 11 March 2014 Published: 28 Mar 2014

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10.1186/1687-2770-2014-72
Cite this article as: Kupenko and Manzo: **On optimality conditions for optimal control problem in coefficients for Δ_p-Laplacian.** *Boundary Value Problems* 2014, 2014:72