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Study on the nonlinear vibration of embedded carbon nanotube via the Hamiltonian-based method

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Abstract

This article mainly studies the vibration of the carbon nanotubes embedded in elastic medium. A new novel method called the Hamiltonian-based method is applied to determine the frequency property of the nonlinear vibration. Finally, the effectiveness and reliability of the proposed method is verified through the numerical results. The obtained results in this work are expected to be helpful for the study of the nonlinear vibration.

Keywords

Nonlinear vibration, Hamiltonian-based method, variational principle, carbon nanotube, residual equation

Introduction

The carbon nanotube (CNT) has attracted wide attention since it is first discovered by Japanese scientist Iijima[1] in 1991. The vibration of the CNT is usually used to directly or indirectly measure the elastic modulus or other mechanical behaviors of the CNTs.[2–5] Therefore, the study of the vibration characteristics of the CNT is important. In this study, we will investigate the vibration of CNT embedded in the elastic medium, which can be governed as[6,7]

$$\frac{d^2\mu}{dt^2} + \left(\frac{\pi^4 EI}{l^4\rho A} + \frac{k}{\rho A}\right)\mu + \frac{\pi^4 E}{4l^4\rho}\mu^3 = 0 \tag{1.1}$$

where l is the length of the CNT, ρ represents the density, E indicates Young's modulus, and A and I mean the cross-sectional area and cross-sectional inertia moment, respectively. There are many methods available for obtaining the frequency property of equation (1.1) such as the incremental harmonic balanced method,[6] variational iteration method,[8] Fourier series and Stokes' transformation,[9] homotopy perturbation method,[10,11] and He's frequency formulation.[12] In this study, we will use a new method called the Hamiltonian-based method to determine the frequency property. The overall structure of this study is arranged as follows. The variational principle of the problem is constructed and its Hamiltonian is obtained in *The variational principle and the Hamiltonian*. In *Hamiltonian-based method*, the Hamiltonian-based method is adopted to solve the problem. In *Results and discussion*, the correctness of the Hamiltonian-based method is verified via a comparison with He's frequency formulation through the numerical results. And the conclusion is presented in *Conclusion and future recommendation*.

The variational principle and the Hamiltonian

The variational principle of equation (1.1) can be easily established via the semi-inverse method,[13–22] which reads as

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$$J(\mu)=\int\left\{\frac{1}{2}\left(\frac{\partial\mu}{\partial t}\right)^2-\left[\frac{1}{2}\left(\frac{\pi^4EI}{l^4\rho A}+\frac{k}{\rho A}\right)\mu^2+\frac{\pi^4E}{16l^4\rho}\mu^4\right]\right\}dt. \tag{2.1}$$

Where He–Weierstrass function can be obtained as[23]

$$\begin{aligned}E(t,\Xi,\Xi',\zeta)&=\frac{1}{2}\zeta^2-\left[\frac{1}{2}\left(\frac{\pi^4EI}{l^4\rho A}+\frac{k}{\rho A}\right)\mu^2+\frac{\pi^4E}{16l^4\rho}\mu^4\right]\\&-\left\{\frac{1}{2}\left(\frac{d\mu}{dt}\right)^2-\left[\frac{1}{2}\left(\frac{\pi^4EI}{l^4\rho A}+\frac{k}{\rho A}\right)\mu^2+\frac{\pi^4E}{16l^4\rho}\mu^4\right]\right\}-\left(\zeta-\frac{d\mu}{dt}\right)\frac{d\mu}{dt}\\&=\frac{1}{2}\zeta^2-\frac{1}{2}\left(\frac{d\mu}{dt}\right)^2-\left(\zeta-\frac{d\mu}{dt}\right)\frac{d\mu}{dt}\end{aligned} \tag{2.2}$$

where the variable ζ that defined as

$$\zeta=\frac{d\mu}{dt} \tag{2.3}$$

From equation (2.2), it can be proved that

$$E(t,\mu,\mu',\zeta)=0 \quad \text{and} \quad \frac{\partial^2E}{\partial\zeta^2}>0 \tag{2.4}$$

which shows that equation (2.1) is a minimal variational principle.

In equation (2.1), $K=1/2(\partial\mu/\partial t)^2$ is the kinetic energy, and $P=1/2(\pi^4EI/l^4\rho A+k/\rho A)\mu^2+\pi^4E/16l^4\rho\mu^4$ represents the potential energy. It is well known that the total energy is conserved in the whole process of the vibration. So the Hamiltonian can be obtained as[24]

$$\mathrm{H}=K+P=\frac{1}{2}\left(\frac{\partial\mu}{\partial t}\right)^2+\frac{1}{2}\left(\frac{\pi^4EI}{l^4\rho A}+\frac{k}{\rho A}\right)\mu^2+\frac{\pi^4\mu}{16l^4\rho}\mu^4=\text{constant}=\mathrm{H}_0 \tag{2.5}$$

Equation (2.5) can be simplified as

$$\frac{1}{2}\left(\frac{d\mu}{dt}\right)^2+\alpha\mu^2+\beta\mu^4=\mathrm{H}_0 \tag{2.6}$$

where $\alpha=1/2(\pi^4EI/l^4\rho A+k/\rho A)$, $\beta=\pi^4E/16l^4\rho$.

Hamiltonian-based method

In this section, we will apply the Hamiltonian-based method to obtain the solution of equation (1.1). As is known to all, the kinetic energy and the potential energy are changed during the oscillation process, but the total energy will keep unchanged for a conservative oscillator. The Hamiltonian-based method is based on this and the variational theory, so it can present a more accurate solution compared with the VIM or HPM. Here the solution of equation (1.1) is assumed with the following form

$$\mu(t)=M\cos(\varpi t) \tag{3.1}$$

With the initial conditions of equation (3.1), we can determine the Hamiltonian constant as

$$\mathrm{H}_0=\alpha M^2+\beta M^4 \tag{3.2}$$

Substituting equation (3.2) into equation (2.6), there is

$$\frac{1}{2}\left(\frac{d\mu}{dt}\right)^2+\alpha\mu^2+\beta\mu^4-\alpha M^2-\beta M^4=0 \tag{3.3}$$

Taking equation (3.1) into equation (3.3) leads to the residual equation,[24] which is

$$R = \frac{1}{2}\varpi^2 M^2 \sin^2(\varpi t) + \alpha M^2 \cos^2(\varpi t) + \beta M^4 \cos^4(\varpi t) - \alpha M^2 - \beta M^4 \tag{3.4}$$

Now we define two average residuals

$$\tilde{R}_1 = \frac{4}{T_1}\int_0^{T_1/4} R_1 \cos(\varpi_1 t) dt \tag{3.5}$$

$$\tilde{R}_2 = \frac{4}{T_2}\int_0^{T_2/4} R_2 \cos(\varpi_2 t) dt \tag{3.6}$$

So the frequency–amplitude formulation can be obtained as follows[24]

$$\varpi^2 = \frac{\varpi_2^2 \tilde{R}_1 - \varpi_1^2 \tilde{R}_2}{\tilde{R}_1 - \tilde{R}_2} \tag{3.7}$$

Here we select two arbitrary frequencies as $\varpi_1 = 1$ and $\varpi_2 = 2$, and then we get the two residual equations as follows

$$R_1 = \frac{1}{2}M^2 \sin^2(t) + \alpha M^2 \cos^2(t) + \beta M^4 \cos^4(t) - \alpha M^2 - \beta M^4 \tag{3.8}$$

$$R_2 = 2M^2 \sin^2(2t) + \alpha M^2 \cos^2(2t) + \beta M^4 \cos^4(2t) - \alpha M^2 - \beta M^4 \tag{3.9}$$

So the two average residuals are obtained as

$$\tilde{R}_1 = \frac{4}{T_1}\int_0^{T_1/4} R_1 \cos t dt = -\frac{M^2\left(10\alpha + 14M^2\beta - 5\right)}{15\pi} \tag{3.10}$$

$$\tilde{R}_2 = \frac{4}{T_2}\int_0^{T_2/4} R_2 \cos(2t) dt = -\frac{2M^2\left(5\alpha + 7M^2\beta - 10\right)}{15\pi} \tag{3.11}$$

According to equation (3.7), there is

$$\varpi = \sqrt{\frac{\varpi_2^2 \tilde{R}_1 - \varpi_1^2 \tilde{R}_2}{\tilde{R}_1 - \tilde{R}_2}} = \sqrt{2\alpha + \frac{14}{5}\beta M^2} \tag{3.12}$$

With this, we get the solution of equation (1.1) as

$$\mu(t) = M \cos\left(\sqrt{2\alpha + \frac{14}{5}\beta M^2}\, t\right) \tag{3.13}$$

That is

$$\mu(t) = M \cos\left(\sqrt{\frac{\pi^4 EI}{l^4 \rho A} + \frac{k}{\rho A} + \frac{7M^2\pi^4 E}{40 l^4 \rho}}\, t\right) \tag{3.14}$$

Results and discussion

Recently, He's frequency formulation, which is first proposed by Chinese mathematician Ji-Huan He, has been widely used to solve the nonlinear vibrations arising in three-dimensional printing technology,[25] micro-electromechanical,[26] N/MEMS,[27] and so on .[28,29] By using He's frequency formulation, we can get the frequency–amplitude formulation of equation (1.1) as

$$\varpi^2 = \frac{\pi^4 EI}{l^4 \rho A} + \frac{k}{\rho A} + \frac{3M^2 \pi^4 E}{16 l^4 \rho} \tag{4.1}$$

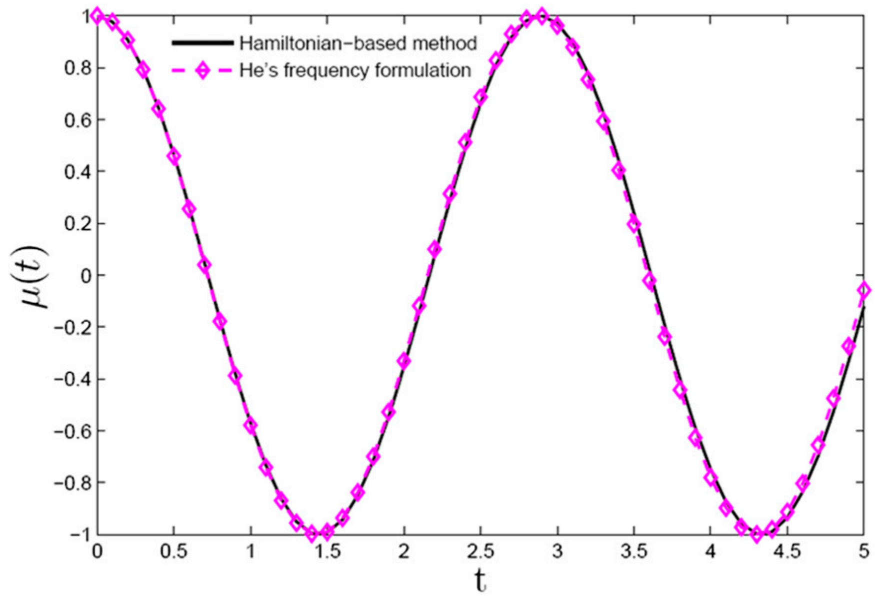

Figure 1. Comparison between the Hamiltonian-based method and He's frequency formulation.

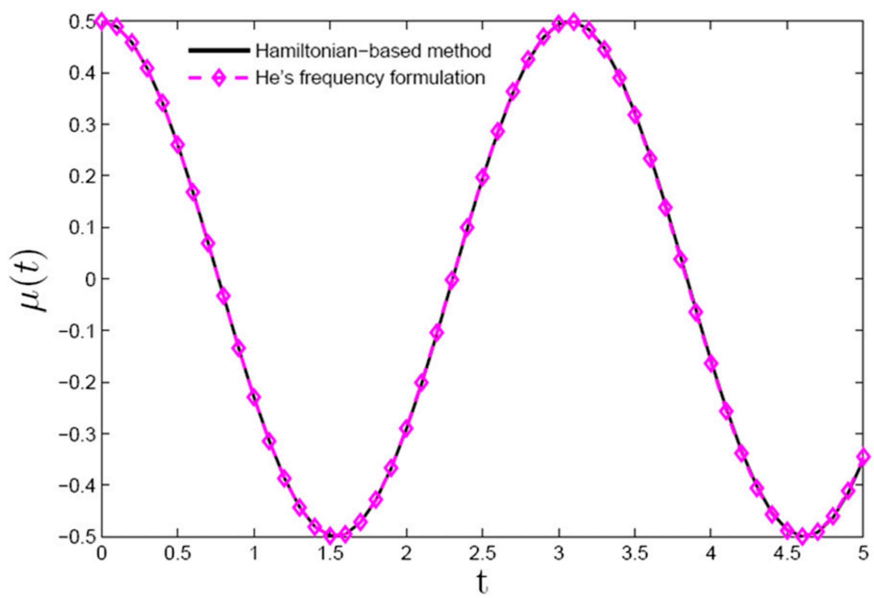

Figure 2. Comparison between the Hamiltonian-based method and He's frequency formulation.

Then the solution of equation (1.1) can be obtained as

$$\mu(t) = M\cos\left(\sqrt{\frac{\pi^4 EI}{l^4\rho A} + \frac{k}{\rho A} + \frac{3M^2\pi^4 E}{16l^4\rho}}t\right) \tag{4.2}$$

which has a good agreement with equation (3.14).

For $E = 1, \rho = 1, l = \pi, M = 1, I = 1, A = 1$, and $k = 1$, we plot a comparison between the Hamiltonian-based method and He's frequency formulation in Figure 1, which shows a good match between the two methods.

When we select $E = 1, \rho = 1, l = \pi, M = \frac{1}{2}, I = 1, A = 1$, and $k = 1$, the comparison of the two methods is plotted in Figure 2, which also has a good agreement.

Conclusion and future recommendation

In this study, a novel and effective method called the Hamiltonian-based method is used to study the nonlinear vibration of the CNT embedded in elastic medium. The comparison between our proposed method and He's frequency formulation shows a good agreement, which strongly proves the correctness of the Hamiltonian-based method. The finding in this study is expected to open up new horizons for the study of the nonlinear vibration.

Recently, the fractal and fractional calculus have been widely used to model many complex problems arising in circuit,[30,31] physics,[32–35] filter,[36–38] biomedical science,[39] and so on.[40–46] The proposed method in this work is also extended to the fractal cases.

Declaration of conflicting interests

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References

1. Iijima S. Helical microtubules of graphitic carbon. *Nature* 1991; 354: 56–58.
2. Qian D, Wagner GJ, LIU WK, et al. Mechanics of carbon nanotubes. *Appl Mech Rev* 2002; 55(6): 495–533.
3. Wong EW, Sheehan PE, and Lieber CM. Nanobeam mechanics: elasticity, strength, and toughness of nanorods and nanotubes. *Science* 1997; 277: 1971–1975.
4. Baughman RH, zakhidov AA, and de heer WA. Carbon nanotubes–the route toward applications. *Science* 2002; 297: 787–792.
5. Choi WB, bae E, kang D, et al. Aligned carbon nanotubes for nanoelectronics. *Nanotechnology* 2004; 15: 512–516.
6. Fu YM, Hong JW, and Wang XQ. Analysis of nonlinear vibration for embedded carbon nanotubes. *J Sound Vibration* 2006; 296(4–5): 746–756.
7. Besseghier A, Heireche H, Bousahla AA, et al. Nonlinear vibration properties of a zigzag single-walled carbon nanotube embedded in a polymer matrix. *Adv nano Res* 2015; 3(1): 029.
8. Hemmatnezhad M and Aminikhah H. Application of VIM to the nonlinear vibrations of multiwalled carbon nanotubes. *Open Appl Maths J* 2010; 4(1): 18–23.
9. Yayli MÖ. Free vibration analysis of a single-walled carbon nanotube embedded in an elastic matrix under rotational restraints. *Micro Nano Lett* 2018; 13(2): 202–206.
10. He J-H and El-Dib YO. Homotopy perturbation method for Fangzhu oscillator. *J Math Chem* 2020; 58(10): 2245–2253.
11. El-Dib YO and Elgazery NS. Effect of fractional derivative properties on the periodic solution of the nonlinear oscillations. *Fractals* 2020; 28(4): 2050095.
12. Qie N, Hou W-F, and He J-H. The fastest insight into the large amplitude vibration of a string. *Rep Mech Eng* 2020; 2(1): 1–5, DOI: 10.31181/rme200102001q.
13. He J-H. Semi-inverse method of establishing generalized variational principles for fluid mechanics with emphasis on turbomachinery aerodynamics. *Int J Turbo Jet Engines* 1997; 14(1): 23–28.
14. Wang K-J, Zhu H-W, Liu X-L, et al. Constructions of new abundant traveling wave solutions for system of the ion sound and Langmuir waves by the variational direct method. *Results Phys* 2021; 26: 104375.
15. Wang K-J and Wang G-D. Variational principle and approximate solution for the fractal generalized benjamin-bona-mahony-burgers equation in fluid mechanics. *Fractals* 2021; 29(3): 2150075.
16. Wang KJ and Wang GD. He's variational method for the time-space fractional nonlinear Drinfeld-Sokolov-Wilson system. In: *Mathematical Methods in the Applied Sciences*. New Jersey, USA: Wiley Online Library, 2020. DOI: 10.1002/mma.7200.
17. Wang KJ and Wang GD. Solitary and periodic wave solutions of the generalized fourth-order Boussinesq equation via He's variational methods. *Math Methods Appl Sci* 2021; 44(7): 5617–5625.

18. Wang K-J and Wang G-D. Periodic solution of the (2 + 1)-dimensional nonlinear electrical transmission line equation via variational method. *Results Phys* 2021; 20(20): 103666.
19. He JH. Variational principle for the generalized KdV-burgers equation with fractal derivatives for shallow water waves. *J Appl Comput Mech* 2020; 6(4): 735–740.
20. Wang KJ and Wang KL. Variational principles for fractal Whitham-Broer-Kaup Equations in Shallow Water. *Fractals* 2021; 29(2): 2150028.
21. Wang KL. Variational principle for nonlinear oscillator arising in a fractal nano/microelectromechanical system. *Math Methods Appl Sci* 2020. DOI: 10.1002/mma.6726.
22. Wang KJ. Variational principle and approximate solution for the generalized Burgers-Huxley equation with fractal derivative. *Fractals* 2021; 29(2): 2150044.
23. He JH, Qie N, He CH, et al. On a strong minimum condition of a fractal variational principle. *Appl Maths Lett* 2021; 119: 107199.
24. He J-H, Hou W-F, Qie N, et al. Hamiltonian-based frequency-amplitude formulation for nonlinear oscillators. *Facta Universitatis-Series Mech Eng* 2021. DOI: 10.22190/FUME201205002.
25. Zuo Y. A gecko-like fractal receptor of a three-dimensional printing technology: a fractal oscillator. *J Math Chem* 2021; 59: 735–744. DOI: 10.1007/s10910-021-01212-y.
26. Tian D, He C-H, and He J-H. Fractal pull-in stability theory for microelectromechanical systems. *Front Phys* 2021; 9: 606011.
27. Tian D, Ain QT, Anjum N, et al. Fractal N/MEMS: from pull-in instability to pull-in stability. *Fractals* 2020; 29(2): 2150030.
28. Wang K.. He's frequency formulation for fractal nonlinear oscillator arising in a microgravity space. *Numer Methods Partial Differential Equations* 2020; 37(2): 1374–1384.
29. He J-H. On the frequency-amplitude formulation for nonlinear oscillators with general initial conditions. *Int J Appl Comput Maths* 2021; 7: 111. DOI: 10.1007/s40819-021-01046-x.
30. Wang K-J, Sun H-C, and Fei Z. The transient analysis for zero-input response of fractal RC circuit based on local fractional derivative. *Alexandria Eng J* 2020; 59(6): 4669–4675.
31. Wang KJ, Sun HC, and Fei Z. A fractal RC circuit model for the current flowing in porous media. *Therm Sci* 2021; 25(2B): 1477–1481.
32. Liu J-G, Yang X-J, Feng Y-Y, et al. On integrability of the higher dimensional time fractional KdV-type equation. *J Geometry Phys* 2021; 160: 104000.
33. Wang KJ, Wang GD, and Zhu HW. A new perspective on the study of the fractal coupled Boussinesq-Burger equation in shallow water. *Fractals* 2021; 29(5): 2150122. DOI: 10.1142/S0218348X2150122X.
34. Wang KJ and Wang GD. Variational principle, solitary and periodic wave solutions of the fractal modified equal width equation in plasma physics. *Fractals* 2021; 29(5): 2150115. DOI: 10.1142/S0218348X21501152.
35. Liu J-G, Yang X-J, Geng L-L, et al. Group analysis of the time fractional (3+1)-dimensional KdV-type equation. *Fractals* 2021. DOI: 10.1142/S0218348X21501693.
36. Wang K-J, Sun H-C, and Cui Q-C. The fractional Sallen-Key filter described by local fractional derivative. *IEEE Access* 2020; 8: 166377–166383.
37. Wang KJ and Li C-L. A a-order R-L high-pass filter modeled by local fractional derivative. *Alexandria Eng J* 2020; 59(5): 3244–3248.
38. Wang KJ. On a High-pass filter described by local fractional derivative. *Fractals* 2020; 28(3): 2050031.
39. Wang K-J. A new fractional nonlinear singular heat conduction model for the human head considering the effect of febrifuge. *The Eur Phys J Plus* 2020; 135: 871.
40. Wang KL. A novel perspective for the fractal Schrödinger equation. *Fractals* 2021; 29(4): 2150093. DOI: 10.1142/S0218348X21500936.
41. Wang KL and Wei CF. New analytical approach for fractal Kmodel. *Fractals*, 2021. DOI: 10.1142/S0218348X21501164.
42. Wang KL. A new fractal transform frequency formulation for fractal nonlinear oscillators. *Fractals* 2021; 29(3): 2150062.
43. He CH, Liu C, and Gepreel KA. Low frequency property of a fractal vibration model for a concrete beam. *Fractals* 2022. DOI: 10.1142/S0218348X21501176.
44. He C-H, He J-H, and Sedighi HM. Fangzhu (方诸): An ancient Chinese nanotechnology for water collection from air: history, mathematical insight, promises, and challenges. *Math Methods Appl Sci* 2020. DOI: 10.1002/mma.6384.
45. He CH, Liu C, He JH, et al. Passive atmospheric water harvesting utilizing an ancient Chinese inklab and its possible applications in modern architecture. *Facta Universitatis Ser Mech Eng* 2021. DOI: 10.22190/FUME201203001H.
46. He JH, Kou SJ, He CH, et al. Fractal oscillation and its frequency-amplitude property. *Fractals*, 2021; 29(4): 2150105. DOI: 10.1142/S0218348X2150105X