Long-Term Statistics and Extreme Waves of Sea Storms

FRANCESCO FEDELE

School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, Georgia

FELICE ARENA

Department of Mechanics and Materials, Mediterranea University, Reggio Calabria, Italy

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ABSTRACT

A stochastic model of sea storms for describing long-term statistics of extreme wave events is presented. The formulation generalizes Boccotti's equivalent triangular storm model by describing an actual storm history in the form of a generic power law. The latter permits the derivation of analytical solutions for the return periods of extreme wave events and associated statistical properties. Lastly, the relative validity of the new model and its predictions is assessed by analyzing wave measurements retrieved from two NOAA National Oceanographic Data Center (NODC) buoys in the Atlantic and Pacific Oceans.

1. Introduction

Stochastic modeling of time series of the significant wave height H_s recorded at a given ocean site is the principal focus of statistical methods employed in the long-term prediction of extreme wave events during sea storms (Krogstad 1985; Prevosto et al. 2000; Boccotti 2000). The reviews of several methods used for this can be found in the work of Isaacson and Mackenzie (1981), Guedes Soares (1989), and Goda (1999). In these methods, the effects of the sea state on the short-term scales of $T_s \sim$ 1–3 h are accumulated to predict the wave conditions on the long-term time scales of $T_l \sim$ yr. In doing so, it is reasonable to assume that with T_s, the sea state is a homogenous and stationary stochastic field whose properties are fully characterized by the directional spectrum of the sea surface and associated moments. As a result, wave parameters such as H_s, mean periods, and mean wavelengths can be easily estimated from either the time series of surface elevations or the associated spectrum. With T_l, we then have a succession of storms where each storm, according to Boccotti (2000), is identified as a nonstationary sequence of sea states in which H_s exceeds 1.5 times the mean annual significant wave height, say, H_{sm} at a given site, and it does not fall below that level during an interval of time longer than 12 h (see also Arena 2004).

Given a succession of storm events in time, Boccotti (1986, 2000) proposed the equivalent triangular storm (ETS) model to predict the return period of extreme wave events. In this model, a storm is described in time as a triangle of height a indicative of storm intensity and base b as a measure of duration. The statistical equivalence is achieved by requiring that a equal the actual maximum H_s in the storm, and b is chosen so that the maximum expected wave height during the storm is the same as that of the triangular storm (Borgman 1970, 1973). It is then assumed that a and b are realizations of two random variables, say, A and B, respectively. Then, the storm peak probability density function (pdf) $p_A(a) = \Pr[A \in (a, a + da)]$ is not fitted directly to the observed storm peak data via ad hoc regressions, but it follows analytically by requiring that the average times spent by the equivalent and actual storm sequences above any threshold h be identical. So, the significant wave height history or the actual storm sequence is stochastically equivalent to a succession of random triangle storms. This type of equivalence defines the probabilistic structure of the ETS model, which depends on wave data only via the observed significant wave height exceedance $P(h) = \Pr(H_s > h)$ and the conditional average duration $\overline{b}(a) = \overline{B|A = a}$, both estimated via regression. Then, the estimates of wave extremes and their associated statistics

Corresponding author address: Felice Arena, Department of Mechanics and Materials, Mediterranea University, Loc. Feo di Vito, 89123 Reggio Calabria, Italy.
E-mail: arena@unirc.it

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simply follow from the density p_A with no need for data fitting.

In particular, Boccotti (2000) derived an analytical solution for the return period $R(H_s > h)$ of a storm during which the maximum H_s exceeds h as that of a triangle storm whose a is above h. Arena (2004) extended this result to account for seasonal effects and wave directionality. Boccotti (1986, 2000) also derived the return period $R(H_{\max} > H)$ of a storm where the maximum individual wave height $H_{\max}$ exceeds a fixed threshold H. Recently, Arena (2004) and Arena and Pavone (2006) exploited the ETS model to define the return period $R(C_{\max} > C)$ of a storm during which the largest nonlinear crest height $C_{\max}$ exceeds a fixed threshold C. Further, Arena and Pavone (2009) derived the solution of the return periods $R_N(R_{\geq N})$ of a storm during which exactly N waves (at least N waves) exceed a given threshold. These analytical results have been compared against buoy measurements, demonstrating their relevance to the design of coastal and offshore structures (Arena and Pavone 2006, 2009).

In this paper, we extend and generalize Boccotti's triangular storm model to include other possible and plausibly more realistic descriptions of the temporal history of the significant wave heights observed at a fixed point during a sea storm. In particular, we describe an actual storm in time t in the form of a generic power law $|t - t_0|^{\lambda}$, where λ is a shape parameter and t_0 is the time of the storm peak. Boccotti's ETS model is recovered for $\lambda = 1$. For the ''generalized'' model, the associated storm peak density p_A stems from the analytical solution of a Volterra integral equation of the first kind. We then derive new analytical expressions for both $R(H_s > h)$ and $R(H_{\max} > H)$ as function of p_A and perform a sensitivity analysis with respect to λ to assess the deviations from the ETS predictions. Further, we present some statistical properties of the largest waves in storms. Finally, we apply the generalized model to the wave data gathered by two National Oceanic and Atmospheric Administration (NOAA) National Oceanographic Data Center (NODC) buoys moored off the Georgia and California coasts.

2. Equivalent power storm (EPS) model

Consider a time interval τ during which $N(\tau)$ storm events occur at a site. The exceedance probability of the $H_{\max}$ in a particular storm is approximated by (Borgman 1973)

$$P(H_{\max} > H) = 1 - \exp\left\{\int_0^D \frac{\ln\{1 - P[H|H_s = h(t)]\}}{\overline{T}[h(t)]}\, dt\right\}, \tag{1}$$

where D is the storm duration, $h(t)$ is the significant wave height history, $\overline{T}$ is the mean zero upcrossing wave period (Boccotti 2000), and $P(H|H_s = h)$ is the exceedance probability of the crest-to-trough wave height H, given $H_s = h$. The latter is of the form (Boccotti 1981, 1997, 2000)

$$P(H|H_s = h) = \exp\left[-\frac{4H^2}{H_s^2(1 + \psi^*)}\right], \tag{2}$$

where $\psi^* = |\psi(T^*)|/\psi(0)$, with T^* representing the abscissa of the first absolute minimum of the surface elevation covariance $\psi(T)$. In the narrowband limit, $\psi^* \to 1$ and (2) reduce to the Rayleigh law (Rice 1944, 1945; Longuet-Higgins 1952). The maximum wave height expected during the actual storm follows from

$$\overline{H}_{\max} = \int_0^{\infty} P(H_{\max} > H)\, dH. \tag{3}$$

Now, assume that each actual storm can be described as an EPS whose significant wave height h varies in t according to the power law

$$h(t) = a\left[1 - \left(\frac{2|t - t_0|}{b}\right)^{\lambda}\right], \quad t_0 - \frac{b}{2} \leq t \leq t_0 + \frac{b}{2}, \tag{4}$$

where b is the storm duration, a is the peak amplitude at t_0, and λ $(0 < \lambda < \infty)$ is a shape parameter (see Fig. 1). The EPS model has one degree of freedom in λ to better represent the actual storm peak. Given (4), the probability that $H_{\max} > H$ follows from (1) as

$$P(H_{\max} > H; a, b) = 1 - \exp\left\{\frac{b}{\lambda a}\int_0^a \frac{\ln[1 - P(H/H_s = h)]}{\overline{T}(h)}\left(1 - \frac{h}{a}\right)^{1/\lambda - 1} dh\right\}. \tag{5}$$

Then, the expected $\overline{H}_{\max}$ of the equivalent storm can be computed from (3). The estimation of the EPS parameters follows easily by equating a to the maximum significant wave height and b corresponds to the expected $\overline{H}_{\max}$ of the actual storm.

As an example, Fig. 2 shows the equivalent model (4) estimated for some actual storms for different values of λ. It is seen that the EPS approximation has smooth peaks for $\lambda \geq 1$ and sharp cusps if $0 < \lambda < 1$. Peaks become smoothly sharper as λ decreases. For $\lambda = 1$, the

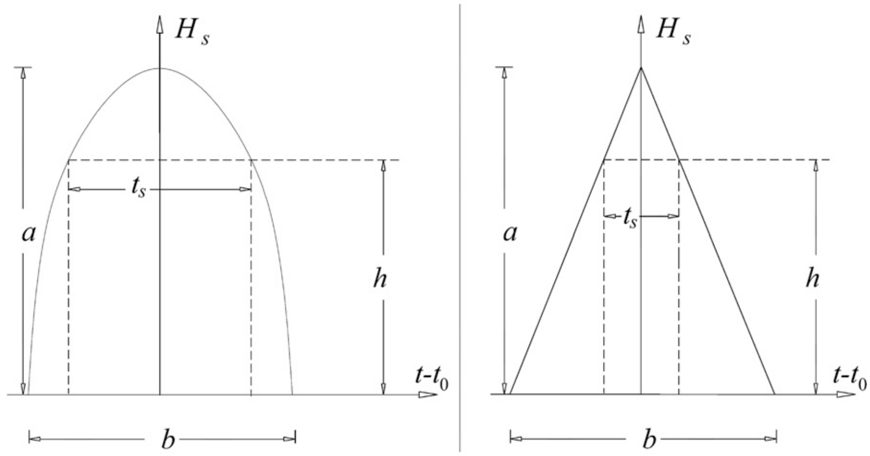

FIG. 1. Reference frame: (left) equivalent parabolic storm and (right) equivalent triangular storm.

ETS model of Boccotti with linear cusps is recovered. The analysis of the data suggests the presence of a correlation trend between duration b and peak a. Small values of b—say, 20–40 h—in the ETS model are typical of brief and very severe storms, whereas large values of b—say, 100–200 h—for the same model tend to characterize storms with either a persistent single peak or multiple peaks.

Note that it is not generally guaranteed a priori that the $P(H_{\max} > H)$ in an actual storm is the same as that of the associated EPS, as we require just the first-order statistical moments of the two distributions to be identical. In practice they are so and also slightly sensitive to the parameter λ, as seen in Fig. 3 for the "storm of the century" from Fig. 2a (Cardone et al. 1996). The same figure also shows that the ETS model slightly overestimates the observed $P(H_{\max} > H)$ of the actual storm in the range of low probabilities. More significantly, the EPS model allows us to improve on the ETS predictions by choosing the parameter λ that will best fit the tail of the exceedance distribution describing the maximum wave height in a storm. For the storms analyzed in this paper, the optimal λ turns out to be roughly 0.75, as shown in Fig. 3. The analysis of oceanic data later on will show that an EPS model with an optimized λ provides long-term predictions of wave extremes slightly more conservative than those based on ETS models.

a. Distribution of storm peak

The stochastic modeling of the EPS approximation for an actual sea storm sequence, as that shown in Fig. 4, proceeds by assuming that the height a and base b of the equivalent storm are values of the random variables A and B, respectively. Thus, we introduce the joint pdf $p_{A,B}(a, b) = p_A(a)p_{B|A}(b|a)$ of A and B and define $p_{A,B}(a, b)\, db\, da$ as the fraction of equivalent storms having a duration in $(b, b + db)$ and peak amplitude in $(a, a + da)$. The pdf of A follows from

$$p_A(a) = \int_0^\infty p_{A,B}(a,b)\, db, \tag{6}$$

and the conditional average duration of B, given $A = a$, is

$$\overline{b}(a) = \overline{B|A=a} = \int_0^\infty b p_{B|A}(b|a)\, db. \tag{7}$$

This and the exceedance probability $P(h) = P(H_s > h)$ are the only two quantities in the EPS model that are estimated from data via regression, as it will be shown later on. The storm peak density $p_A(a)$ is not fitted directly from the observed storm peak data via ad hoc regressions, but it is stemmed in an analytical form as function of both $P(h)$ and $\overline{b}(a)$ by invoking the stochastic equivalence between the sequence of actual storms and that of the equivalent storms (see Fig. 4). This is accomplished by imposing the condition that the average time during which H_s is above h is the same in both the actual and equivalent storm sequences. For the actual storm sequence, the average time T_R within the interval τ during which H_s stays above h is given by

$$T_R(h) = \tau P(h). \tag{8}$$

To derive the average time $T_{\rm EPS}$ during which H_s is above h in the equivalent storm sequence, we first consider

$$dN(a, b) = N(\tau) p_A(a) p_{B|A}(b, a)\, db\, da \tag{9}$$

as the number of equivalent storms in the random sequence having a duration B in $(b, b + db)$ and peak

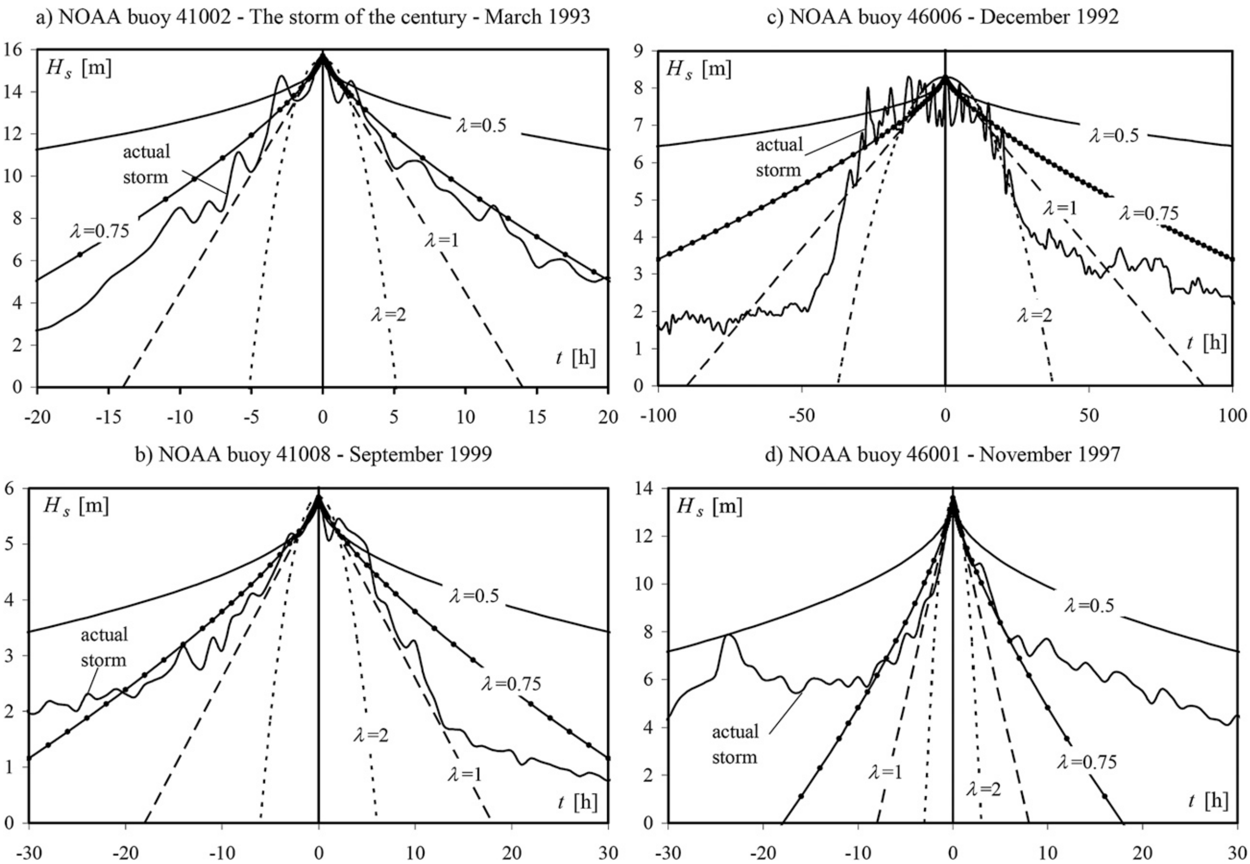

FIG. 2. Sea storms recorded by NOAA NODC buoys. For each actual storm, the EPS models are plotted for different values of λ.

amplitude A in $(a, a + da)$, and we recall that $N(\tau)$ is the total number of storms in τ. It follows then that

$$T_{\mathrm{EPS}}(h) = \int_{a=h}^{\infty}\int_{b=0}^{\infty} t_s(h, a, b)\, dN(a,b), \tag{10}$$

where, from (4)

$$t_s(h, a, b) = b\left(1 - \frac{h}{a}\right)^{1/\lambda} \tag{11}$$

is the time interval during which H_s stays above h in an equivalent storm with a and b (see Fig. 1). Thus, using (7), (9), and (11), (10) can be simplified further as

$$T_{\mathrm{EPS}}(h) = N(\tau)\int_h^{\infty} \overline{b}(a)\left(1 - \frac{h}{a}\right)^{1/\lambda}(a)\, da. \tag{12}$$

We can now require that $T_{\mathrm{EPS}}(h) = T_R(h)$ for any h, and this will lead to the following Volterra integral equation of the first kind:

$$\tau P(h) = N(\tau)\int_h^{\infty} \overline{b}(a)\left(1 - \frac{h}{a}\right)^{1/\lambda} p_A(a)\, da. \tag{13}$$

Solving (13) for p_A yields (see appendix)

$$p_A(a) = \frac{\tau}{N(\tau)}\frac{a}{\overline{b}(a)}G(\lambda,\ a). \tag{14}$$

In the applications to follow, we will assume that $P(h)$ is given by the Weibull distribution (cf. Battjes 1972; Isaacson and Mackenzie 1981; Ochi 1998)

$$P(h) = \exp\left[-\left(\frac{h - h_l}{w}\right)^u\right], \quad h \ge h_l, \tag{15}$$

where the parameters h_l, w, and u can be estimated iteratively (Goda 1999).

In general, (15) does not describe the observed overall distributions of H_s well, but it does fit the tails fairly accurately (Ferreira and Guedes Soares 2000). Thus, it is preferable to fit (15) to relatively large values of the observational data in the low-probability region (Guedes Soares 1989; Boccotti 2000) and to use a lognormal fit for the high-probability region, as suggested by Haver (1985).

We point out that the EPS model depends on the measured data only via the observed $P(h)$ and $\overline{b}(a)$, as in

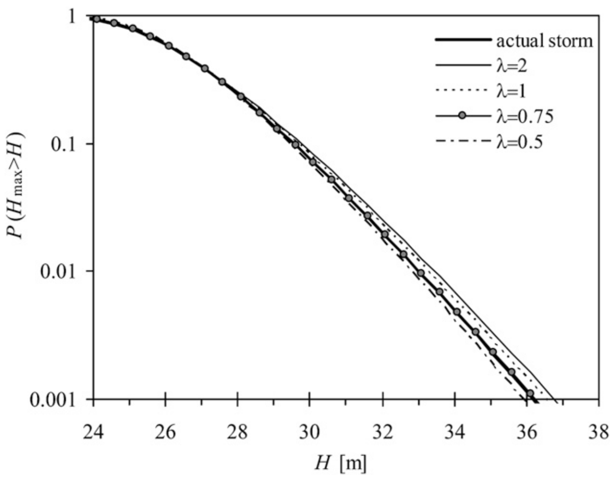

FIG. 3. Comparison among the exceedance probabilities $P(H_{\max} > H)$ calculated for the actual storm of the century in Fig. 2 and for the associated equivalent storms with different values of λ. The best fit of the observed distribution's tail is attained for $\lambda = 0.75$.

the ETS formulation. Then, the density p_A satisfies the Volterra's integral equation of first kind (13), which imposes the stochastic equivalence between the equivalent and actual storm sequences for an arbitrary $\lambda > 0$. As a result, the EPS model is defined in a probabilistic setting and no more data fitting is required to estimate wave extremes and develop their associated statistics. Indeed, in the following we shall derive analytical expressions of the return periods $R(H_s > h)$ and $R(H_{\max} > H)$ explicitly as a function of the peak distribution (14) as well as investigate some statistical properties of large waves in storms using probabilistic principles.

b. Return period $R(H_s > h)$

The return period $R(H_s > h)$ of an actual storm where the maximum H_s exceeds h is the same as the return period of an equivalent power storm whose peak A exceeds h. Thus,

$$R(H_s > h) = \frac{\tau}{N(\tau; A > h)}, \tag{16}$$

where $N(\tau; A > h)$ represents the average number of equivalent storms whose peak A exceeds h during τ. Integrating (9) over all the possible b and $a > h$ yields

$$N(\tau; A > h) = \int_{a=h}^{\infty}\int_{b=0}^{\infty} dN(a, b) = N(\tau)\int_{h}^{\infty} p_A(a)\, da \tag{17}$$

and together with (14), (16) reduces to

$$R(H_s > h) = \frac{1}{\displaystyle\int_h^{\infty} \frac{a}{\overline{b}(a)} G(\lambda, a)\, da}. \tag{18}$$

The mean persistence of H_s above h is given by the general expression (Boccotti 2000)

$$\overline{D}(h) = R(H_s > h)P(H_s > h). \tag{19}$$

For $\lambda = 1$, (18) reduces to the expressions valid for ETS models, as to be expected (cf. Boccotti 2000).

c. Return period $R(H_{max} > H)$

Consider the number $N_w(H)$ of equivalent storms where the largest wave occurs with a crest-to-trough height $H_{\max}$ greater than H. Then, the return period $R(H_{\max} > H)$ of an actual storm is defined as that of an equivalent storm whose maximum $H_{\max}$ exceeds H. Thus,

$$R(H_{\max} > H) = \frac{\tau}{N_w(H)}, \tag{20}$$

where

$$N_w(H) = \int_{H_{\max}=H}^{\infty}\int_{a=0}^{\infty}\int_{b=0}^{\infty} dN_w(H, a, b). \tag{21}$$

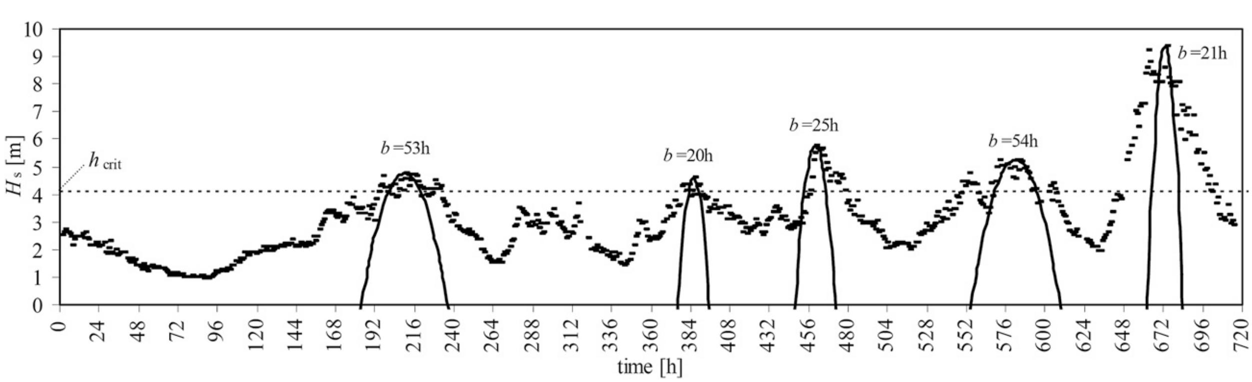

FIG. 4. Significant wave height time series recorded by NOAA NODC 46006 buoy located in the Pacific Ocean during the month of October 2000. The sequence of equivalent parabolic storms is also plotted ($\lambda = 2$). Dotted line shows the storm threshold $h_{\mathrm{crit}} = 1.5H_{\mathrm{sm}}$.

In the preceding, $dN_w(H, a, b)$ is the number of equivalent storms in the sequence of random storms with peak amplitude A in $(a, a + da)$ and duration B in $(b, b + db)$, and whose maximum wave occurs with a height H_{max} in $(H, H + dH)$. More explicitly,

$$dN_w(H, a, b) = dN(a,b)p(H_{max} = H; a, b)\, dH, \qquad (22)$$

where $p(H_{max} = H; a, b)$ is the pdf of H_{max} that follows from (5) and dN is given by (9). Thus, (20) reduces to

$$R(H_{max} > H) = \frac{\tau/N(\tau)}{\int_h^\infty p_A(a)P[H_{max} > H;\, a,\, \overline{b}(a)]\, da}. \qquad (23)$$

Using (14), this is simplified further to

$$R(H_{max} > H) = \frac{1}{\int_h^\infty \frac{a}{\overline{b}(a)} G(\lambda, a)P[H_{max} > H;\, a,\, \overline{b}(a)]\, da}, \qquad (24)$$

which generalizes the solution appropriate to the ETS model (Boccotti 2000; Arena and Pavone 2006, 2009).

3. Extreme waves in sea storms

Given λ and R, consider now the wave with the largest crest-to-trough height H_{max} greater than H that follows from (24). What is the most probable value of the significant wave height peak A of the storm during which that wave occurred?

This question has some practical interest, and it can be addressed in the context of EPS models without data fitting, by simply using probabilistic principles as follows. We first integrate (22) over b to obtain the number $dN_w(H, a)$ of equivalent storms whose largest wave height H_{max} is greater than H, and the peak intensity A in $[a, a + da]$ as

$$dN_w(H, a) = \int_{H_{max}=H}^\infty \int_{b=0}^\infty dN_w(H, a, b). \qquad (25)$$

Then, given $F = \{H_{max} > H\}$, the conditional probability that the extreme event occurs in an equivalent storm whose peak intensity A is in $[a, a + da]$ follows from (25) as

$$p_{A|F}(a; H)\, da = \frac{dN_w(H, a)}{\int_{a=0}^\infty dN_w(H, a)}. \qquad (26)$$

By virtue of (22), the preceding can be rewritten as

$$p_{A|F}(a; H) = \frac{p_A(a)P[H_{max} > H;\, a,\, \overline{b}(a)]}{\int_0^\infty p_A(a)P[H_{max} > H;\, a,\, \overline{b}(a)]\, da}. \qquad (27)$$

This pdf is characterized by its conditional mean $\mu_{A|F}(H)$ and standard deviation $\sigma_{A|F}(H)$, which are both functions of the given height H. If the coefficient of variation $\gamma = \sigma_{A|F}/\mu_{A|F} \ll 1$, then an exceptional wave event most probably occurs during a storm whose maximum significant wave height, that is, the storm peak A, is very close to $\mu_{A|F}$. In the applications, we will show that such theoretical predictions based on EPS models are approximately satisfied in actual storm data.

4. Analysis of storm data

Hereafter, we will apply the EPS model to elaborate some wave measurements retrieved by the NOAA buoys 41008 and 46006 moored off the Georgia and California coasts, respectively. Operational since 1997, buoy 41008 is at 40 n mi southeast of Savannah, Georgia, in a water depth of 18 m. Buoy 46006 has been operational since 1977 at 600 n mi west of Eureka, California, in a water depth of 4023 m.

Given that a sequence of actual storms occurred at either buoy locations, the long-term wave statistics are uniquely defined by the following:

1) the distribution $P(H_s > h)$ of H_s at the site,
2) the conditional average base $\overline{b}(a)$, and
3) the pdf $p_A(a)$ of A.

The first two items above are readily estimated from data, whereas p_A follows from the analytical solution (14) of the Volterra integral Eq. (13). For example, Fig. 5 shows that $P(H_s > h)$ is well represented by the Weibull law (15) at both buoy locations. The corresponding distributional parameters u, w, and h_l are given in Table 1. For each storm, we also computed the $\overline{H}_{max}$ as function of a, as in Fig. 6. Further, for λ given, the conditional duration $\overline{b}(a)$ of a location is described by

$$\overline{b}(a) = K_1 \ln(a/a') + K_2, \qquad (28)$$

where K_1 and K_2 are regression parameters and a' is set equal to 1 m. In particular, consider the significant wave height series recorded by buoy 46006 during the period 2000–07. Figure 7 shows the regression (28) estimated from the values of a and b observed during the actual storms for different shapes of the equivalent storm, including λ = 0.5 (cusp), 1 (triangular), 2 (parabolic), and 3 (cubic). The corresponding regression parameters are given in Table 2. It is seen that $\overline{b}(a)$ tends to increase as λ

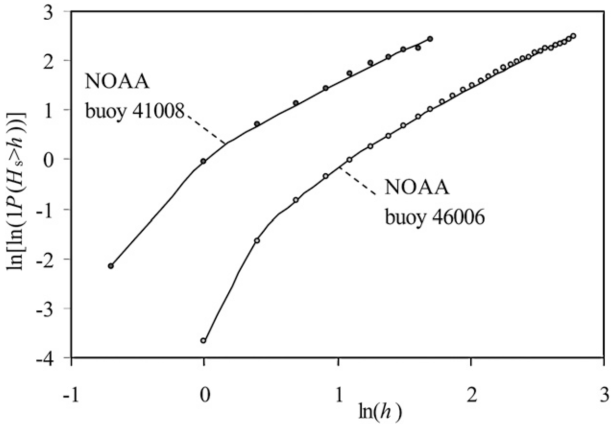

FIG. 5. Weibull plot of the probabilities of exceedance $P(H_s > h)$ estimated for both NOAA buoys 41008 and 46006 in the Atlantic and Pacific Oceans, respectively. Continuous lines are associated with the theoretical Weibull distributions with parameters given in Table 1.

TABLE 1. Parameters of the Weibull distribution (15).

Buoy location	u	w (m)	h_l (m)
NDBC 41008	1.169	0.619	0.40
NDBC 46006	1.310	2.120	0.80

decreases. In particular, the duration of a parabolic-type storm ($\lambda > 1$) is in general smaller than that of a cusp-type storm ($\lambda < 1$). This trend is confirmed in Fig. 8, where we plotted for each buoy the mean duration

$$b_m(\lambda) = \int_0^\infty p_A(a)\overline{b}(a)\,da, \tag{29}$$

estimated from the actual storms recorded as a function of λ. Note that b_m follows the same trend at both buoys despite that one buoy is located in the Pacific Ocean and the other in the Atlantic Ocean. Further, b_m slightly changes for values of λ above $\lambda_c = 0.7$. For λ values smaller than λ_c, b_m increases sharply toward unrealistic durations of 300 h and longer, causing the long-term predictions of wave extremes to be underestimated significantly, as we will show later.

Given $P(H_s > h)$ and $\overline{b}(a)$, we can now compute $p_A(a)$ by using (14) and make predictions for both the return period $R(H_s > h)$ and the mean persistence $\overline{D}(h)$ from (18) and (19), respectively. Figure 9 shows the latter predictions based on the data of buoy 46006 for various values of λ. The results in Fig. 9a suggest that parabolic-type storms yield predictions consistent with those of the ETS model ($\lambda = 1$). However, for values of λ below the threshold $\lambda_c = 0.7$, the EPS model drastically underestimates the extreme significant wave heights in comparison to the ETS predictions. Figure 8 suggests that these are related to equivalent storms with durations longer than 400–500 h, which are unrealistic. A similar trend is also observed in the predictions of the mean persistence $\overline{D}(h)$ of buoy 46006 plotted in Fig. 9b.

Figure 10 illustrates the predictions for $R(H_s > h)$ and $\overline{D}(h)$ computed from the data of the buoy 41008 for the triangular ($\lambda = 1$), parabolic ($\lambda = 2$), and cusp ($\lambda = 0.75$) models. For the same values of λ, the predictions of the return period $R(H_{\max} > H)$ are computed using (24) for buoy 46006 and shown in Fig. 11.

Our analysis reveals that if a cusp model is adopted ($\lambda < 1$), then λ should be greater than $\lambda_c = 0.7$ to avoid unrealistic predictions. Moreover, when $\lambda > \lambda_c$, the predictions based on the EPS model are somewhat insensitive to changes in the shape parameter λ, and they tend to be robust. Particularly for the optimal model that best fits the distribution's tail of the maximum wave height observed in a given storm ($\lambda = 0.75$; see Fig. 3), the EPS predictions are slightly more conservative than those from the ETS model ($\lambda = 1$), but the predicted levels differ by less than 1.5%.

Figure 12 shows the conditional pdf computed from (27) for a cusp model ($\lambda = 0.75$) for various values of $H_{\max}$ for buoy 46006. As $H_{\max}$ increases, the conditional pdf (27) tends to become symmetric relative to its mean $\mu_{A|F}$. Lastly, Fig. 13a shows the coefficient of variation $\gamma = \sigma_{A|F}/\mu_{A|F}$ as function of $H_{\max}$. Note that γ is slightly sensitive to λ, and it tends to decrease for larger wave heights. For $H_{\max} > 25$ m, γ is very small and nearly

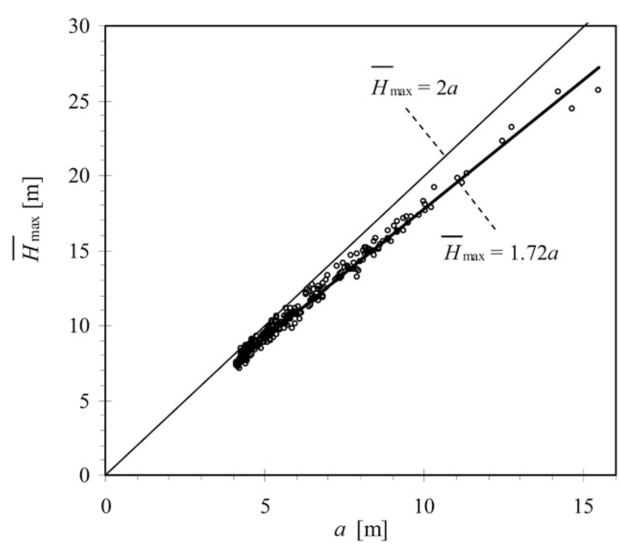

FIG. 6. NOAA 46006 buoy in the Pacific Ocean: regression of $\overline{H}_{\max}$ and the peak intensity a of the actual storms recorded during 2000–07.

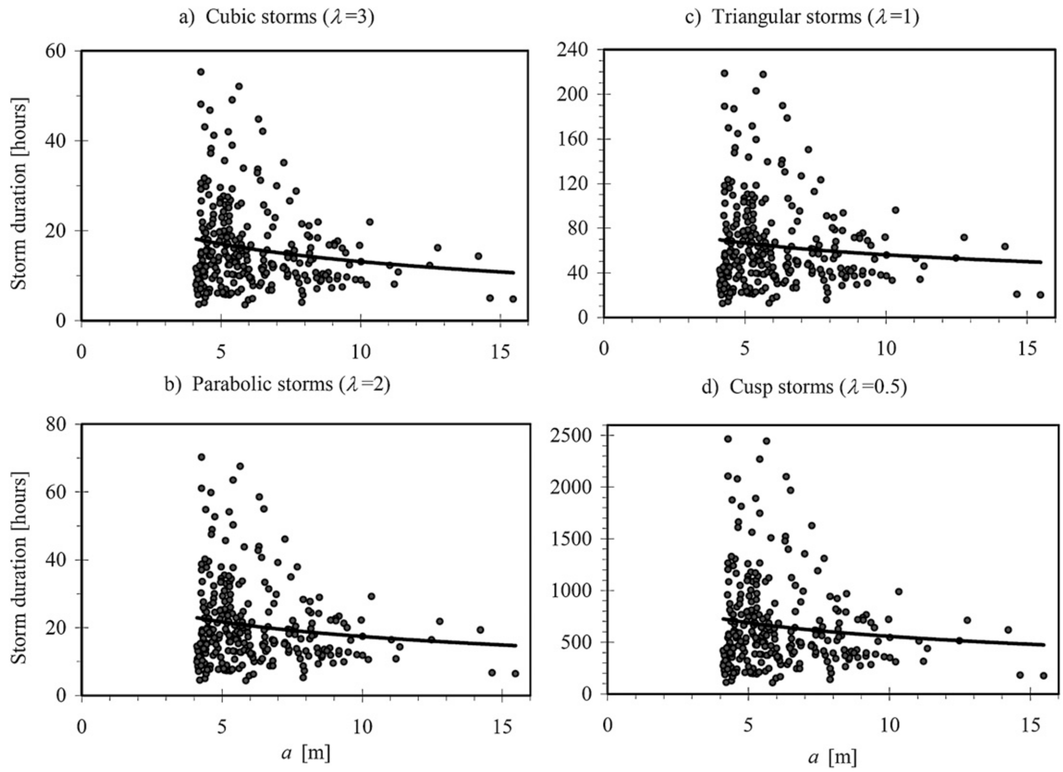

FIG. 7. NOAA buoy 46006 in the Pacific Ocean: regression (28) of the $\overline{b}(a)$ as function of the storm peak a for different values of λ from data recorded during 2000–07: (a) cubic, (b) triangular, (c) parabolic, and (d) cusp storms.

0.11, irrespective of λ. This implies that an exceptional wave event most probably occurs during a storm whose maximum significant wave height A is very close to $\mu_{A|F}$. Further, the ratio $H_{\max}/\mu_{A|F}$ is also slightly sensitive to λ, and the maximum wave height does not exceed twice the value of its expected storm peak intensity, as clearly seen in Fig. 13b.

5. Conclusions

We have presented a generalization of the ETS model of Boccotti (2000). To introduce flexibility in modeling the significant wave height history locally at storm peaks, we define a sequence of random equivalent storms with parabolic ($\lambda > 1$) or cusped ($\lambda < 1$) shapes, with λ as a free positive parameter. This approximation relies on the measured data, specifically, on the exceedance distribution $P(h)$ of the significant wave heights observed and the conditional base $\overline{b}(a)$, both of which can be estimated via regression. The storm-peak density p_A follows from the solution of a Volterra integral equation of the first kind by requiring that the average time interval during which the significant wave height lingers above a given threshold is identical in both the equivalent storm sequence and actual storms. No data fitting is then required in describing wave extremes and associated statistics. Hence, we were able to derive the return periods $R(H_s > h)$ and $R(H_{\max} > H)$ as explicit functions of p_A and to describe the statistical properties of the largest waves in storms based solely on probabilistic principles.

As applications, we have examined the statistics of extreme waves in numerous storms recorded by two NOAA buoys located in the Atlantic and Pacific Oceans. Our analysis reveals that λ should be greater than the critical value $\lambda_c = 0.7$ to avoid unrealistic predictions. For $\lambda > \lambda_c$, the EPS model yields robust predictions, being slightly sensitive to changes in the shape parameter. We also observed that for a given storm, the

TABLE 2. Parameters K_1 and K_2 (h) of the base height regression $\overline{b}(a)$ for NOAA NODC buoys for different values of λ.

λ	0.50	0.75	1	2	3
			Buoy 46006		
K_1	−189.28	−34.277	−14.892	−6.197	−5.588
K_2	993.15	204.14	90.351	31.694	25.978
			Buoy 41008		
K_1	−0.1200	−0.1847	−3.7648	−5.6872	−6.4029
K_2	680.35	145.47	70.112	29.503	26.064

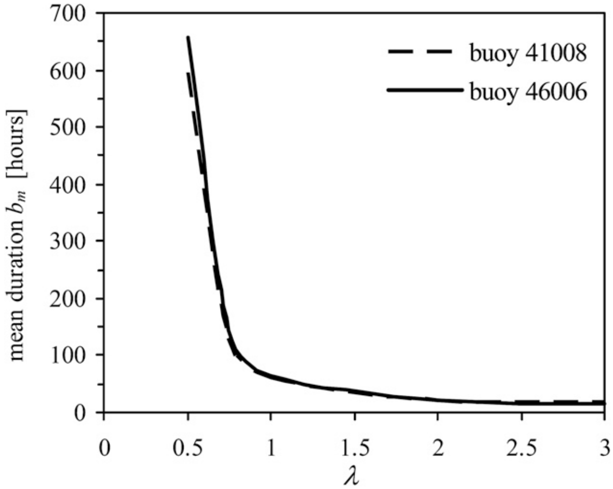

FIG. 8. Mean duration $b_m(\lambda)$ of storms recorded by NOAA buoys 46006 and 41008 in the Pacific and Atlantic Oceans, respectively, during the period 2000–07.

ETS model slightly overestimates the exceedance probability $P(H_{\max} > H)$ of the maximum wave height observed in the storms. The EPS model is then exploited to improve the ETS predictions by choosing the optimal value of the shape parameter λ that best fits the tail of the maximum wave height distribution observed in a storm. For the storms analyzed in this study, we find that the optimal λ roughly equals 0.75. For these optimal models, the prediction of extremes is about 1.5%

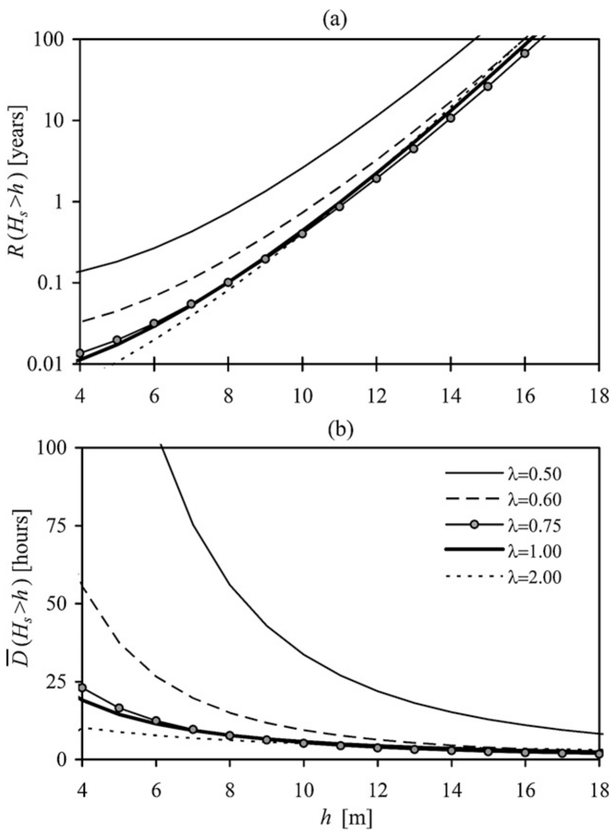

FIG. 9. NOOA 46006 buoy in the Pacific Ocean: (a) $R(H_s > h)$ and (b) $\overline{D}(h)$ computed for different values of λ.

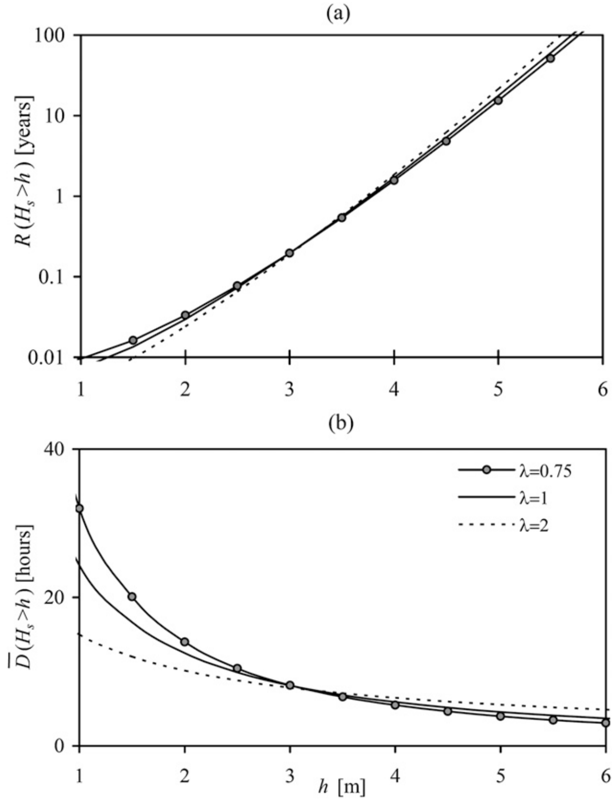

FIG. 10. Same as Fig. 9, except for NOOA buoy 41008 in the Atlantic Ocean.

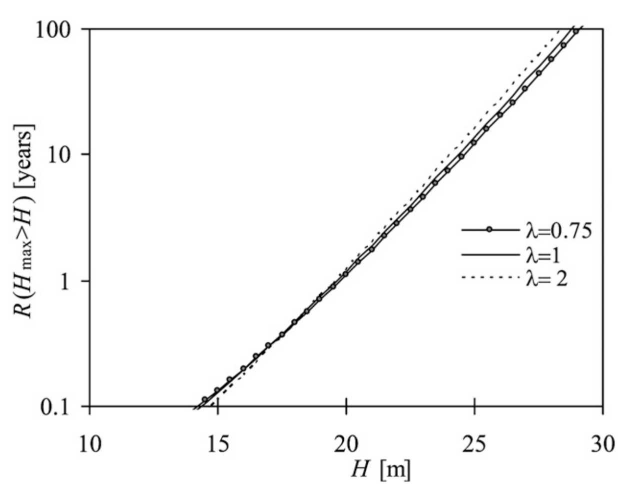

FIG. 11. NOOA 46006 buoy in the Pacific Ocean: $R(H_{\max} > H)$ computed for different values of λ.

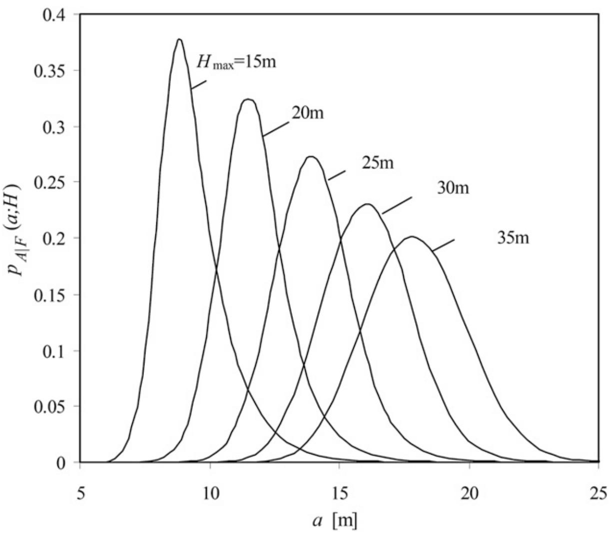

FIG. 12. NOAA 46006 buoy in the Pacific Ocean: conditional probability $p_{A|F}(a;\, H)$ of the intensity A of the equivalent cusp storm ($\lambda = 0.75$) given the event $F = \{H_{max} > H\}$, where H_{max} is the crest-to-trough height of the largest wave of the storm.

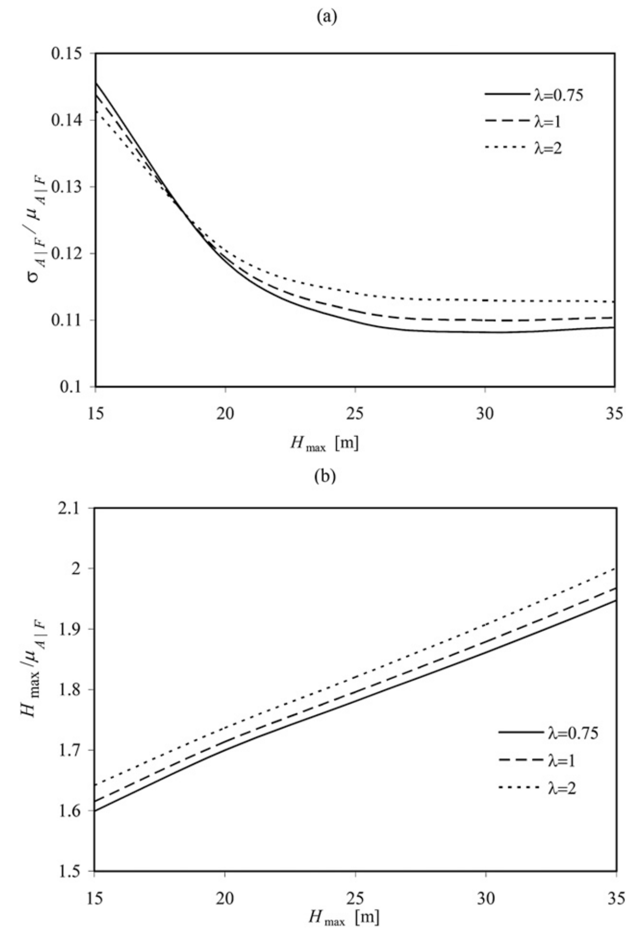

FIG. 13. NOAA 46006 buoy in the Pacific Ocean: (a) the variation coefficient $\gamma = \sigma_{A|F}/\mu_{A|F}$ of the storm peak intensity A given $F = (H_{max} > H)$ and (b) the associated ratio between H_{max} and the conditional mean $\mu_{A|F}$ computed for different values of λ, where H_{max} is the crest-to-trough height of the largest wave of the storm.

larger and thus more conservative than those of the ETS model.

APPENDIX

Solutions for Probability Density Function $p_A(a)$

In (13), define

$$G(a) = \frac{N(\tau)\overline{b}(a)}{\tau a^{1/\lambda}} p_A(a) \tag{A1}$$

to obtain an integral Volterra equation of first kind for G, namely,

$$P(h) = \int_h^\infty G(a)(a-h)^{1/\lambda}\, da. \tag{A2}$$

The solution of this equation varies depending on if $\lambda > 1$, $\lambda = 1$, or $0 < \lambda < 1$, namely,

$$G(\lambda, a) = \begin{cases} \dfrac{\sin(\pi/\lambda)}{\pi/\lambda} \displaystyle\int_1^\infty \left.\frac{d^2P}{dz^2}\right|_{a,x} (x-1)^{-1/\lambda}\, dx, & \text{if} \quad \lambda > 1 \\ \dfrac{d^2P}{da^2}, & \text{if} \quad \lambda = 1 \\ \dfrac{(-1)^n a^n}{n!} \dfrac{\sin(\pi\mu)}{\pi\mu} \displaystyle\int_1^\infty \left.\frac{d^{n+2}P}{dz^{n+2}}\right|_{a,x} (x-1)^{-\mu}\, dx, & \text{if} \quad \lambda = (n+\mu)^{-1} < 1\,, \end{cases} \tag{A3}$$

with (integer) $n > 1$ and $0 < \mu < 1$. If $\lambda = 1/n$ is rational, that is, $\mu = 0$, then from (A3)

$$G(\lambda, a) = -\frac{(-1)^n\, a^n}{n!} \frac{d^{n+1}P}{da^{n+1}}. \tag{A4}$$

In applications, $G(\lambda, a)$ is computed via numerical integration if $\lambda \neq 1$ or $\mu \neq 0$. In the following we will present the formal derivation of the Volterra integral Eq. (A2) that leads to (A3).

a. Solution for $\lambda = 1$

In this case, (A2) reduces to

$$P(h) = \int_h^\infty G(a)(a-h)\,da. \qquad \text{(A5)}$$

The solution for G proceeds by differentiating both members of (A5) twice with respect to h and setting $h = a$. This yields

$$G(a) = \frac{d^2P}{da^2}. \qquad \text{(A6)}$$

b. Solution for $\lambda > 1$

Consider a solution of (A2) of the form

$$G(a) = \int_a^\infty g(z)H(z-a)\,dz, \qquad \text{(A7)}$$

where $g(z \ge 0)$ and $H(z \ge a)$ are arbitrary functions. Substituting (A7) into (A2) yields

$$P(h) = \int_{a=h}^\infty \int_{z=a}^\infty g(z)H(z-a)(a-h)^{1/\lambda}\,da\,dz. \qquad \text{(A8)}$$

For z given, we first integrate with respect to a and then over z. On this basis, (A8) can be rewritten as

$$P(h) = \int_h^\infty g(z)K(h,z)\,dz, \qquad \text{(A9)}$$

where

$$K(h,z) = \int_h^z H(z-a)(a-h)^{1/\lambda}\,da. \qquad \text{(A10)}$$

Now, (A9) can be easily solved for g if the arbitrary function H in (A10) is chosen to yield $K \sim (z-h)$ from (A10). To do so, we first make a change of variables

$$a = \frac{z+h}{2} + \frac{z-h}{2}\cos\theta \qquad \text{(A11)}$$

to rewrite (A10) as

$$K(h,z) = -\int_\pi^0 H\left[\frac{z-h}{2}(1-\cos\theta)\right]\left(\frac{z-h}{2}\right)^{1/\lambda} \times (1+\cos\theta)^{1/\lambda}\frac{z-h}{2}\sin\theta\,d\theta. \qquad \text{(A12)}$$

On the assumption that

$$H(z-a) = \frac{1}{(z-a)^{1/\lambda}}, \qquad \text{(A13)}$$

it will follow that for $\lambda > 1$,

$$K(h,z) = \frac{z-h}{2}\int_0^\pi \left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{1/\lambda}\sin\theta\,d\theta = \frac{\pi(z-h)}{\lambda\sin(\pi/\lambda)}. \qquad \text{(A14)}$$

Thus, (A9) is simplified to

$$P(h) = \frac{\pi/\lambda}{\sin(\pi/\lambda)}\int_h^\infty g(z)(z-h)\,dz. \qquad \text{(A15)}$$

This is the same type of Volterra equation as in case (*a*). We thus solve for g by differentiating both sides of (A15) twice with respect to h and then set $h = z$. This yields

$$g(z) = \frac{\sin(\pi/\lambda)}{\pi/\lambda}\frac{d^2P}{dz^2}. \qquad \text{(A16)}$$

From (A13) and (A16), (A7) leads to the solution

$$G(a) = \frac{\sin(\pi/\lambda)}{\pi/\lambda}\int_a^\infty \frac{d^2P}{dz^2}\frac{1}{(z-a)^{1/\lambda}}\,dz, \quad \lambda > 1. \qquad \text{(A17)}$$

c. Solution for $0 < \lambda < 1$

In this case, we seek an integer $n > 1$ and a real number $0 < \mu < 1$ such that

$$\frac{1}{\lambda} = n + \mu. \qquad \text{(A18)}$$

On this basis, (A2) becomes

$$P(h) = \int_h^\infty G(a)(a-h)^{n+\mu}\,da. \qquad \text{(A19)}$$

Differentiate both sides of the preceding expression n times to get

$$\frac{(-1)^n}{n!}\frac{d^nP}{dh^n} = \int_h^\infty G(a)(a-h)^\mu\,da, \quad 0 < \mu < 1. \qquad \text{(A20)}$$

This is the same type of integral equation as in case (*b*). Thus,

$$G(a) = \frac{(-1)^n}{n!}\frac{\sin(\pi\mu)}{\pi\mu}\int_a^\infty \frac{d^{n+2}P}{dz^{n+2}}\frac{1}{(z-a)^\mu}\,dz. \qquad \text{(A21)}$$

REFERENCES

Arena, F., 2004: On the prediction of extreme sea waves. *Environmental Sciences and Environmental Computing,* Vol. 2, P. Zannetti, Ed., EnviroComp Institute, 1–50. [Available online at http://www.envirocomp.org/esecII/esecII_flyer.pdf.]

——, and D. Pavone, 2006: Return period of nonlinear high wave crests. *J. Geophys. Res.,* **111,** C08004, doi:10.1029/2005JC003407.

——, and ——, 2009: A generalized approach for the long-term modelling of extreme sea waves. *Ocean Modell.,* **26,** 217–225.

Battjes, J. A., 1972: Long-term wave height distributions at seven stations around the British Isles. *Deutsch. Hydrogr. Z.,* **25,** 179–189.

Boccotti, P., 1981: On the highest waves in a stationary Gaussian process. *Atti Accad. Ligure Sci. Lett., Genoa,* **38,** 271–302.

——, 1986: On coastal and offshore structure risk analysis. *Excerpta Ital. Contrib. Field Hydraul. Eng.,* **1,** 19–36.

——, 1997: A general theory of three-dimensional wave groups. *Ocean Eng.,* **24,** 265–300.

——, 2000: *Wave Mechanics for Ocean Engineering*. Oceanography Series, Vol. 64, Elsevier, 496 pp.

Borgman, L. E., 1970: Maximum wave height probabilities for a random number of random intensity storms. *Proc. 12th Conf. Coastal Engineering,* Washington, DC, ASCE, 53–64.

——, 1973: Probabilities for highest wave in hurricane. *J. Waterw., Harbors Coastal Eng. Div., Amer. Soc. Civ. Eng.,* **99,** 185–207.

Cardone, V. J., R. E. Jensen, D. T. Resio, V. R. Swail, and A. T. Cox, 1996: Evaluation of contemporary ocean wave models in rare extreme events: The ''Halloween Storm'' of October 1991 and the ''Storm of the Century'' of March 1993 *J. Atmos. Oceanic Technol.,* **13,** 198–230.

Ferreira, J. A., and C. Guedes Soares, 2000: Modelling distributions of significant wave height. *Coastal Eng.,* **40,** 361–374.

Goda, Y., 1999: *Random Seas and Design of Maritime Structures*. World Scientific, 443 pp.

Guedes Soares, C., 1989: Bayesian prediction of design wave height. *Reliability and Optimization of Structural System '88,* P. Thoft-Christensen, Ed., Springer-Verlag, 311–323.

Haver, S., 1985: Wave climate off northern Norway. *Appl. Ocean Res.,* **7,** 85–92.

Isaacson, M., and N. G. Mackenzie, 1981: Long-term distributions of ocean waves: A review. *J. Waterw., Port Coastal Ocean Div., Amer. Soc. Civ. Eng.,* **107,** 93–109.

Krogstad, H. E., 1985: Height and period distributions of extreme waves. *Appl. Ocean Res.,* **7,** 158–165.

Longuet-Higgins, M. S., 1952: On the statistical distribution of the heights of sea waves. *J. Mar. Res.,* **11,** 245–266.

Ochi, M. K., 1998: *Ocean Waves: The Stochastic Approach*. Ocean Technology Series, Vol. 6, Cambridge University Press, 319 pp.

Prevosto, M., H. E. Krogstad, and A. Robin, 2000: Probability distributions for maximum wave and crest heights. *Coastal Eng.,* **40,** 329–360.

Rice, S. O., 1944: Mathematical analysis of random noise. *Bell Syst. Tech. J.,* **23,** 282–332.

——, 1945: Mathematical analysis of random noise. *Bell Syst. Tech. J.,* **24,** 46–156.