TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 352, Number 9, Pages 4271–4293
S 0002-9947(00)02460-0
Article electronically published on April 14, 2000

ESTIMATES FOR FUNCTIONS OF THE LAPLACE OPERATOR ON HOMOGENEOUS TREES

MICHAEL COWLING, STEFANO MEDA, AND ALBERTO G. SETTI

Abstract. In this paper, we study the heat equation on a homogeneous graph, relative to the natural (nearest–neighbour) Laplacian. We find pointwise estimates for the heat and resolvent kernels, and the L^p-L^q mapping properties of the corresponding operators.

In recent years, random walks on graphs have been studied as analogues of diffusions on manifolds (see, e.g., T. Coulhon [Cn], M. Kanai [K1], [K2], and N. Th. Varopoulos [V1], [V2]). More recently, I. Chavel and E. A. Feldman [CF] and M. M. H. Pang [P] considered the semigroup $(e^{-t\Delta})_{t>0}$. In this paper, we consider the heat semigroup associated to the canonical Laplace operator on a homogeneous tree, which is perhaps the basic example of a graph where the cardinality of the "ball of radius r" grows exponentially in r. We study the properties of the heat and resolvent operators, and discover very close analogies with diffusions on hyperbolic spaces, in line with the philosophy of the above-mentioned authors.

A homogeneous tree $\mathfrak{X}$ of degree $q+1$ is defined to be a connected graph with no loops, in which every vertex is adjacent to $q+1$ other vertices. We denote by d the natural distance on $\mathfrak{X}$, $d(x,y)$ being the number of edges between the vertices x and y, and by $L^p(\mathfrak{X})$ the Lebesgue space with respect to counting measure on $\mathfrak{X}$. On $\mathfrak{X}$ the canonical "Laplace operator" $\mathcal{L}$ is defined thus:

$$\mathcal{L}f(x) = f(x) - \frac{1}{q+1} \sum_{\substack{y\in\mathfrak{X} \\ d(x,y)=1}} f(y) \quad \forall x \in \mathfrak{X};$$

it is easily seen to be bounded on $L^p(\mathfrak{X})$ for every p in $[1,\infty]$, and self–adjoint on $L^2(\mathfrak{X})$. Denote by $\sigma_2(\mathcal{L})$ the $L^2(\mathfrak{X})$ spectrum of $\mathcal{L}$, and by P_λ the associated spectral resolution of the identity, such that

$$\mathcal{L}f = \int_{\sigma_2(\mathcal{L})} \lambda \, dP_\lambda f \quad \forall f \in L^2(\mathfrak{X}),$$

and denote $\inf \sigma_2(\mathcal{L})$ by b_2. For θ in $[0,1]$ and α with $\operatorname{Re}(\alpha) > 0$, the heat operator and the (θ,α)-resolvent of $\mathcal{L}$ are the operators defined by the formulae

$$\mathcal{H}_t f = \int_{\sigma_2(\mathcal{L})} e^{-t\lambda} \, dP_\lambda f \quad \forall t \in \mathbb{R}^+,$$

Received by the editors October 4, 1996.
1991 *Mathematics Subject Classification.* Primary 43A85; Secondary 20E08, 43A90, 22E35.
Key words and phrases. Homogeneous trees, Laplace–Beltrami operator, spherical functions, harmonic analysis.
Work partially supported by the Australian Research Council and the Italian M.U.R.S.T.

and

$$\mathcal{R}_\theta^\alpha f = \int_{\sigma_2(\mathcal{L})} (\lambda - \theta b_2)^{-\alpha/2}\, dP_\lambda f,$$

for all $L^2(\mathfrak{X})$ functions f for which the integrals define L^2 functions. Since the operators $\mathcal{H}_t$ and $\mathcal{R}_\theta^\alpha$ are invariant under the action of the group of automorphisms of $\mathfrak{X}$, they are represented by convolution kernels denoted by h_t (the heat kernel) and r_θ^α respectively.

The first part of this paper is devoted to a study of the behaviour of the heat kernel, and of the L^p–L^r operator norms of the heat operator. The second part of the paper is concerned with the description of the region where the (θ, α) resolvent is L^p–L^r bounded. In the last part of the paper we establish some results about variants of the Littlewood–Paley–Stein g–function associated to the heat operator.

1. Notation and preliminaries on harmonic analysis on $\mathfrak{X}$

Let o be a fixed reference point on $\mathfrak{X}$, and write $|x|$ for $d(x, o)$. We say that a function f on $\mathfrak{X}$ is radial if $f(x)$ depends only on $|x|$. If $E(\mathfrak{X})$ is a function space on $\mathfrak{X}$, we will denote by $E(\mathfrak{X})^\sharp$ the subspace of radial elements in $E(\mathfrak{X})$.

Let G be the group of automorphisms of the tree, i.e., of isometries of $(\mathfrak{X}, d)$, and let G_o be the isotropy subgroup of o. Then $\mathfrak{X}$ may be canonically identified with the coset space G/G_o, and functions and radial functions on $\mathfrak{X}$ may be identified with G_o–right-invariant and G_o–bi-invariant functions on G respectively. By using this identification, we may define the convolution of two functions f_1 and f_2 on $\mathfrak{X}$ by the formula

$$f_1 * f_2(g \cdot o) = \int_G f_1(h \cdot o)\, f_2(h^{-1} g \cdot o)\, dh \quad \forall g \in G,$$

whenever the integral makes sense. When f_2 is radial,

$$f_1 * f_2(x) = \sum_{n=0}^{\infty} f_2(x_n) \sum_{d(x,y)=n} f_1(y) \quad \forall x \in \mathfrak{X},$$

where x_n is chosen such that $|x_n| = n$ for every n in $\mathbb{N}$.

Let δ_o denote the Dirac measure at o and ν the uniformly distributed probability measure supported on $\{x \in \mathfrak{X} : |x| = 1\}$. The Laplace operator is given by right convolution with the function $\delta_o - \nu$. Every G–invariant (in the sense that $\mathcal{K}(f \circ g) = (\mathcal{K} f) \circ g$ for every f in $L^p(\mathfrak{X})$ and g in G) continuous operator $\mathcal{K}$ from $L^p(\mathfrak{X})$ to $L^r(\mathfrak{X})$ (weak-star continuous if $p = \infty$) is given by right convolution with a G_o–bi-invariant kernel k:

$$\mathcal{K} f(x) = f * k(x) \quad \forall x \in \mathfrak{X}.$$

We shall denote by $Cv_p^r(\mathfrak{X})$ the space of such convolution kernels. For an operator $\mathcal{K}$, we denote its usual L^p–L^r operator norm by $\|\mathcal{K}\|_{p;r}$; by abuse of notation, if k is in $Cv_p^r(\mathfrak{X})$, we also denote by $\|k\|_{p;r}$ the L^p–L^r operator norm of the corresponding operator. A simple argument shows that operators in $Cv_p^r(\mathfrak{X})$ are equal to their transposes, so that, if $k \in Cv_p^r(\mathfrak{X})$, then $k \in Cv_{r'}^{p'}(\mathfrak{X})$, and the norms in the two spaces coincide.

We will use the "variable constant convention", and denote by C, possibly with sub- or superscripts, a constant that may vary from place to place and may depend on any factor quantified (implicitly or explicitly) before its occurrence, but not on factors quantified afterwards. Given functions A and B, both defined on a set $\mathbb{D}$, we write $A(t) \sim B(t)$ for all t in $\mathbb{D}$ if there exist positive constants C and C' such that

$$C\,A(t) \leq B(t) \leq C'\,A(t) \quad \forall t \in \mathbb{D}.$$

The expression $A(t) \asymp B(t)$ as t tends to t_0 means that $A(t)/B(t)$ tends to 1 when t tends to t_0.

We let τ denote $2\pi/\log q$, and for every p in $[1,\infty]$, we write $\delta(p)$ for $1/p - 1/2$ and p' for the conjugate index $p/(p-1)$. We also let $\mathbb{S}_p$ and $\overline{\mathbb{S}}_p$ denote the strip $\{z \in \mathbb{C} : |\mathrm{Im}\,(z)| < |\delta(p)|\}$ and its closure.

We now summarise the main features of spherical harmonic analysis on $\mathfrak{X}$. The spherical functions are the radial eigenfunctions of the Laplace operator $\mathcal{L}$ satisfying the normalisation condition $\phi(o) = 1$, and are given by the formula

$$\phi_z(x) = \begin{cases} \left(1 + \frac{q-1}{q+1}\,|x|\right) q^{-|x|/2} & \forall z \in \tau\mathbb{Z}, \\ \left(1 + \frac{q-1}{q+1}\,|x|\right) q^{-|x|/2}(-1)^{|x|} & \forall z \in \tau/2 + \tau\mathbb{Z}, \\ \mathbf{c}(z)\,q^{(iz-1/2)|x|} + \mathbf{c}(-z)\,q^{(-iz-1/2)|x|} & \forall z \in \mathbb{C} \setminus (\tau/2)\mathbb{Z}, \end{cases} \tag{1.1}$$

where $\mathbf{c}$ is the meromorphic function defined by the rule

$$\mathbf{c}(z) = \frac{q^{1/2}}{q+1}\,\frac{q^{1/2+iz} - q^{-1/2-iz}}{q^{iz} - q^{-iz}} \quad \forall z \in \mathbb{C} \setminus (\tau/2)\mathbb{Z}. \tag{1.2}$$

It can be shown (see, e.g., [CMS]) that $z \mapsto \phi_z(x)$ is an entire function for each x in $\mathfrak{X}$, and

$$|\phi_z(x)| \leq 1 \quad \forall x \in \mathfrak{X} \quad \forall z \in \overline{\mathbb{S}}_1. \tag{1.3}$$

It should perhaps be noted that we use a different parametrisation of the spherical functions from Figà-Talamanca and his collaborators (see, e.g., [FTP] and [FTN]); our ϕ_z corresponds to their $\phi_{1/2+iz}$, and $\mathbf{c}$ is reparametrised similarly.

The spherical Fourier transform $\widetilde{f}$ of a function f in $L^1(\mathfrak{X})^\sharp$ is given by the formula

$$\widetilde{f}(z) = \sum_{x \in \mathfrak{X}} f(x)\,\phi_z(x)\,dx \quad \forall z \in \overline{\mathbb{S}}_1.$$

The symmetry properties of the spherical functions imply that $\widetilde{f}$ is even and τ-periodic in the strip $\mathbb{S}_1$. We denote the torus $\mathbb{R}/\tau\mathbb{Z}$ by $\mathbb{T}$, and usually identify it with $[-\tau/2, \tau/2)$.

Let μ denote the Plancherel measure on $\mathbb{T}$, given by the formula

$$d\mu(s) = c_G\,|\mathbf{c}(s)|^{-2}\,ds, \qquad \text{where} \qquad c_G = \frac{q\log q}{4\pi(q+1)}. \tag{1.4}$$

The following theorems are well known.

Theorem 1.1. *The spherical Fourier transformation extends to an isometry of $L^2(\mathfrak{X})^\sharp$ onto $L^2(\mathbb{T},\mu)$, and corresponding Plancherel and inversion formulae hold:*

$$\|f\|_2 = \left[\int_{\tau/2}^{\tau/2} \left|\widetilde{f}(s)\right|^2 d\mu(s)\right]^{1/2} \qquad \forall f \in L^2(\mathfrak{X})^\sharp,$$

and

$$f(x) = \int_{-\tau/2}^{\tau/2} \widetilde{f}(s)\,\phi_s(x)\,d\mu(s) \quad \forall x \in \mathfrak{X}$$

for "nice" radial functions f on $\mathfrak{X}$.

Proof. See, for instance, Chapter 2 of [FTN]. □

The next result is the version of the Hausdorff–Young inequality valid in our context. Note that there is a partial converse, which is not classical.

Theorem 1.2. *Suppose that $1 \le p < 2$ and f is in $L^p(\mathfrak{X})^\sharp$. Then $\widetilde{f}$ extends to a holomorphic function in the strip $\mathbb{S}_p$, with boundary values $\widetilde{f}(\cdot \pm i\delta(p))$ belonging to the conjugate space $L^{p'}(\mathbb{T})$, and*

$$\left[\int_{-\tau/2}^{\tau/2} \left|\widetilde{f}(s \pm i\delta(p))\right|^{p'} ds\right]^{1/p'} \le C\|f\|_p.$$

Further, if $1 \le p < 2$ and f is a radial function on $\mathfrak{X}$ such that $f\phi_{i\delta(p)}$ is in $L^1(\mathfrak{X})^\sharp$, then f is in $L^p(\mathfrak{X})$ and

$$\|f\|_p \le C\big\|f\phi_{i\delta(p)}\big\|_1^{2\delta(p)} \left[\int_{-\tau/2}^{\tau/2} |f(s + i\delta(p))|^2\,ds\right]^{1/2-\delta(p)}.$$

Proof. See [CMS], Theorems 1.1 and 1.2. □

Theorem 1.3. *Suppose that $1 \le p, r \le \infty$, and that $s = \min(r, p')$. Suppose also that k is an element of $Cv_p^r(\mathfrak{X})$. Then the following hold:*

(i) *if $p \le r$, then k is in $L^s(\mathfrak{X})^\sharp$, and $\|k\|_s \le |\!|\!| k |\!|\!|_{p;r}$;*

(ii) *if $s < 2$, then $\widetilde{k}$ extends to a holomorphic function in $\mathbb{S}_s$;*

(iii) *if $p = r \ne 2$, then $\widetilde{k}$ extends to a bounded holomorphic function in $\mathbb{S}_s$.*

Proof. First, $p \le r$ by the trivial generalisation to $\mathfrak{X}$ of a well known theorem of Hörmander [Hr]. Further, k is in $L^r(\mathfrak{X})^\sharp$ because δ_o is in $L^p(\mathfrak{X})$ and $k = \delta_o * k$; moreover, $\|k\|_r \le |\!|\!| k |\!|\!|_{p;r}$ because $\|\delta_o\|_p = 1$. By duality, the same holds when r is replaced by p', and (i) is proved. Now (ii) is a consequence of (i) and Theorem 1.2. Finally, (iii) is a straightforward generalisation of the Clerc–Stein condition [CS] to this situation. See [CMS] for more details. □

Theorem 1.4. *Suppose that $1 \le p < r \le 2$. The following convolution inclusions hold:*

(i) $L^p(\mathfrak{X}) * L^r(\mathfrak{X})^\sharp \subset L^r(\mathfrak{X})$, *i. e.,* $L^r(\mathfrak{X})^\sharp \subseteq Cv_p^r(\mathfrak{X})$ *and* $|\!|\!| k |\!|\!|_{p;r} \le C\,\|f\|_r$;

(ii) $L^r(\mathfrak{X}) * L^p(\mathfrak{X})^\sharp \subset L^r(\mathfrak{X})$, *i. e.,* $L^p(\mathfrak{X})^\sharp \subseteq Cv_r^r(\mathfrak{X})$ *and* $|\!|\!| k |\!|\!|_{r;r} \le C\,\|f\|_p$;

(iii) $L^p(\mathfrak{X}) * L^{r'}(\mathfrak{X})^\sharp \subset L^{r'}(\mathfrak{X})$, *i. e.,* $L^{r'}(\mathfrak{X})^\sharp \subseteq Cv_p^{r'}(\mathfrak{X})$ *and* $|\!|\!| k |\!|\!|_{p;r'} \le C\,\|f\|_{r'}$;

(iv) $L^{r'}(\mathfrak{X}) * L^p(\mathfrak{X})^\sharp \subset L^{r'}(\mathfrak{X})$, *i. e.,* $L^p(\mathfrak{X})^\sharp \subseteq Cv_{r'}^{r'}(\mathfrak{X})$ *and* $|\!|\!| k |\!|\!|_{r';r'} \le C\,\|f\|_p$.

Proof. These are all simple consequences of the Kunze–Stein phenomenon, proved for the group G by C. Nebbia [N]. See [CMS] for further information. □

Corollary 1.5. *Suppose that $1 \le p < r \le 2$, and that k is a radial function on $\mathfrak{X}$. Then $k \in L^r(\mathfrak{X})$ if and only if $k \in Cv_p^r(\mathfrak{X})$, and $\|k\|_r \sim |\!|\!| k |\!|\!|_{p;r}$ for all k in $L^r(\mathfrak{X})^\sharp$.*

Proof. This is an immediate consequence of Theorems 1.3 and 1.4. □

Let γ be the entire function defined by the formula

$$\gamma(z) = \frac{q^{1/2}}{q+1}\left(q^{iz} + q^{-iz}\right).$$

Then

$$\gamma(z) = \frac{2q^{1/2}}{q+1}\cos(z\log q) = \gamma(0)\cos(z\log q). \tag{1.5}$$

The spherical Fourier transform of $\delta_o - \nu$ is $1-\gamma$, and using this one may show that the L^p spectrum $\sigma_p(\mathcal{L})$ of $\mathcal{L}$ is the image of $\overline{\mathbb{S}}_p$ under the map $1-\gamma$ (see Chapter 2 of [FTN]). A simple computation shows that $\sigma_p(\mathcal{L})$ is the region of all w in $\mathbb{C}$ such that

$$\left[\frac{1-\mathrm{Re}\,(w)}{\gamma(0)\cosh(\delta(p)\log q)}\right]^2 + \left[\frac{\mathrm{Im}\,(w)}{\gamma(0)\sinh(\delta(p)\log q)}\right]^2 \leq 1.$$

In particular $\sigma_2(\mathcal{L})$ degenerates to the real segment $[1-\gamma(0), 1+\gamma(0)]$. We denote by b_p the infimum of $\mathrm{Re}\,(\sigma_p(\mathcal{L}))$. From the expression above we deduce that

$$b_p = \gamma(0)\cosh(\delta(p)\log q) = 1 - \gamma(i\delta(p)). \tag{1.6}$$

2. Estimates for the heat operator

The heat semigroup generated by the Laplacian $\mathcal{L}$ is denoted $(\mathcal{H}_t)_{t>0}$. Since $\mathcal{L}$ is bounded on $L^p(\mathfrak{X})$ whenever $p \geq 1$ and $t > 0$, $\mathcal{H}_t$ is given by the formula

$$\mathcal{H}_t = e^{-t\mathcal{L}} = \sum_{n=0}^{\infty} \frac{(-t\mathcal{L})^n}{n!};$$

the series converges in the uniform operator topology, and $(\mathcal{H}_t)_{t>0}$ is a uniformly continuous semigroup on $L^p(\mathfrak{X})$. At the kernel level we have

$$\mathcal{H}_t f = f * h_t,$$

where

$$h_t(x) = \sum_{n=0}^{\infty} \frac{t^n(\nu - \delta_o)^{(*n)}}{n!} = e^{-t}\sum_{n=0}^{\infty} \frac{t^n \nu^{(*n)}}{n!},$$

$\nu^{(*k)}$ denoting the k-th convolution power of ν. Since ν is a probability measure, the second series converges in $L^1(\mathfrak{X})$, hence uniformly, and

$$\|h_t\|_1 = e^{-t}\sum_{n=0}^{\infty} \frac{t^n \left\|\nu^{(*n)}\right\|_1}{n!} = 1.$$

Thus $\mathcal{H}_t$ is contractive on $L^p(\mathfrak{X})$ for every p in $[1,\infty]$. It is immediate to check that $\mathcal{H}_t$ is symmetric, so that $(\mathcal{H}_t)_{t>0}$ is a symmetric contraction semigroup. This section is devoted to the study of the heat semigroup $(\mathcal{H}_t)_{t>0}$ and the heat kernels h_t. We investigate the dependence of $|\!|\!|\mathcal{H}_t|\!|\!|_{p;r}$ on t as t tends to 0 or ∞, and we estimate h_t pointwise and in the $L^p(\mathfrak{X})$ norm.

The heat kernel h_t is associated to the Fourier multiplier $\exp[-t(1-\gamma)]$, and, by spherical Fourier inversion,

$$h_t(x) = \int_{-\tau/2}^{\tau/2} \exp\left[-t(1-\gamma(s))t\right]\,\phi_s(x)\,d\mu(s) \quad \forall x \in \mathfrak{X} \quad \forall t \in \mathbb{R}^+.$$

We will also need to consider the case where $q = 1$, when the tree $\mathfrak{X}$ may be identified with the integers $\mathbb{Z}$. To avoid ambiguity, we will use a sub- or superscript $\mathbb{Z}$ to denote objects defined on $\mathbb{Z}$. So $\mathcal{L}^{\mathbb{Z}}$ denotes the Laplacian on $\mathbb{Z}$, given by

$$\mathcal{L}^{\mathbb{Z}} F(j) = F(j) - \frac{F(j+1) + F(j-1)}{2} = F *_{\mathbb{Z}} \left[\delta_0 - \frac{\delta_1 + \delta_{-1}}{2}\right],$$

$h_t^{\mathbb{Z}}$ denotes the heat kernel associated to $\mathcal{L}^{\mathbb{Z}}$, and so forth.

Since the Fourier transform of the convolution kernel of $\mathcal{L}^{\mathbb{Z}}$ is $1 - \cos(\cdot \log q)$, that of $h_t^{\mathbb{Z}}$ is $\exp\left[-t(1 - \cos(\cdot \log q))\right]$, and, by Fourier inversion,

$$\begin{aligned} h_t^{\mathbb{Z}}(j) &= \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} \exp\left[t(\cos(s \log q) - 1)\right] q^{-isq}\, ds \\ &= \frac{e^{-t}}{2\pi} \int_{-\pi}^{\pi} \exp\left[t \cos s\right] \cos(sj)\, ds \quad \forall j \in \mathbb{Z} \quad \forall t \in \mathbb{R}^+. \end{aligned}$$

Clearly the semigroup $(\mathcal{H}_t^{\mathbb{Z}})_{t>0}$ shares with $(\mathcal{H}_t)_{t>0}$ all the properties described above.

The following lemma describes the asymptotic behaviour of the L^p-norms of the heat kernel h_t for large values of t.

Lemma 2.1. *Let h_t be the heat kernel, described above. Then the following hold:*

(i) *if $p = 2$, then*

$$\|h_t\|_p \sim t^{-3/4} \exp(-b_2 t) \quad \forall t \in [1, \infty);$$

(ii) *if $p = \infty$, then*

$$\|h_t\|_p \sim t^{-3/2} \exp(-b_2 t) \quad \forall t \in [1, \infty);$$

(iii) *if p is in $[1, 2)$, then*

$$\|h_t\|_p \sim t^{-1/2p'} \exp(-b_p t) \quad \forall t \in [1, \infty).$$

Proof. The proof is a modification of that of Lemma 3.1 of [CGM]. To prove (i), observe first that

$$|\mathbf{c}(s)|^{-2} = \frac{4\,(q+1)^2 \sin^2(s \log q)}{(q+1)^2 \sin^2(s \log q) + (q-1)^2 \cos^2(s \log q)} \quad \forall s \in \mathbb{T}.$$

Since

$$(q-1)^2 \le (q+1)^2 \sin^2(s \log q) + (q-1)^2 \cos^2(s \log q) \le (q+1)^2 \quad \forall s \in \mathbb{T},$$

the Plancherel formula (Theorem 1.1) and formula (1.5) for γ imply that

$$\begin{aligned} \|h_t\|_2^2 &= \int_{-\tau/2}^{\tau/2} \left|\widetilde{h}_t(s)\right|^2 \, d\mu(s) \\ &\sim \int_{-\tau/2}^{\tau/2} \exp\left[-2t(1 - \gamma(0) \cos(s \log q))\right] \sin^2(s \log q)\, ds \quad \forall t \in \mathbb{R}^+. \end{aligned}$$

After the change of variables $v = s\sqrt{t} \log q$, the preceding integral becomes

$$\frac{t^{-3/2} e^{-2b_2 t}}{\log q} \int_{\infty}^{\infty} \chi_{I(t)}(v)\, t\, \sin^2(v/\sqrt{t})\, \exp\left[-2t\gamma(0)(1 - \cos(v/\sqrt{t}))\right] dv,$$

where $I(t)$ denotes the interval $[-\pi\sqrt{t}, \pi\sqrt{t}]$ and $\chi_{I(t)}$ denotes its characteristic function. The integrand converges locally uniformly to $v^2 \exp\left[-\gamma(0)\, v^2\right]$ as t tends to ∞, and it is easy to verify that, for some positive c, it is bounded above by $v^2 \exp\left[-cv^2\right]$, first on the interval $I(t)$, and hence on all of $\mathbb{R}$. By a variant of the Lebesgue dominated convergence theorem, it follows that as t tends to infinity the integral tends to

$$\int_{-\infty}^{\infty} v^2 \exp\left[-\gamma(0)\, v^2\right]\, dv$$

(which is equal to $\frac{1}{2}\pi^{1/2}\gamma(0)^{-3/2}$), and (i) is proved.

To prove (ii), recall from (1.3) that $|\phi_z(x)| \leq 1$ for all z in $\overline{\mathbb{S}}_1$ and x in $\mathfrak{X}$. By combining this fact with the inversion formula (Theorem 1.1), it may be seen that

$$|h_t(x)| \leq h_t(o) \quad \forall x \in \mathfrak{X},$$

so that $\|h_t\|_\infty = h_t(o)$. Proceeding as in the proof of (i), it is easy to see that

$$\|h_t\|_\infty \sim \int_{-\tau/2}^{\tau/2} \exp\left[-t(1-\gamma(s))\right] \sin^2(s\log q)\, ds \sim t^{-3/2}\, e^{-b_2 t} \quad \forall t \in [1,\infty).$$

The hardest part of this lemma is (iii). If $1 < p < 2$, then, by Theorem 1.2,

$$\begin{aligned} \|h_t\|_p^{p'} &\geq C \int_{-\tau/2}^{\tau/2} \left|\widetilde{h}_t\left(s + i\delta(p)\right)\right|^{p'} ds \\ &= C\, e^{-p' b_p t} \int_{-\tau/2}^{\tau/2} \exp\left[\, -tp' \operatorname{Re}\left[\gamma(i\delta(p)) - \gamma(s+i\delta(p))\right]\,\right] ds \end{aligned}$$

for all t in $\mathbb{R}^+$. Noting that

$$\operatorname{Re}\left[\gamma(i\delta(p)) - \gamma(s+i\delta(p))\right] = \gamma(i\delta(p))\left[1-\cos(s\log q)\right],$$

and arguing as before to estimate the last integral, we obtain the lower bound

$$\|h_t\|_p \geq C\, t^{-1/2p'} e^{-b_p t} \quad \forall t \in [1,\infty).$$

To prove a comparable upper bound, we apply Theorem 1.2, and estimate:

$$\begin{aligned} \|h_t\|_p &\leq C\left[\widetilde{h}_t\left(i\delta(p)\right)\right]^{2\delta(p)} \left[\int_{-\tau/2}^{\tau/2} \left|\widetilde{h}_t\left(s+i\delta(p)\right)\right|^2 ds\right]^{1/2-\delta(p)} \\ &\leq C\, \left[\exp(-b_p t)\right]^{2\delta(p)} \left[t^{-1/4}\, \exp(-b_p t)\right]^{1-2\delta(p)} \\ &= C\, t^{-1/2p'}\, \exp(-b_p t). \end{aligned}$$

This concludes the proof of the second half of (iii), and of Lemma 2.1. □

The results of Lemma 2.1 leave open the question of how $\|h_t\|_p$ behaves when t is small or when $p > 2$. Since $\|h_t\|_p = \|\!|\mathcal{H}_t|\!\|_{1;p}$ for any p in $[1,\infty]$, this question, and more, is addressed in Theorem 2.2.

Theorem 2.2. *Let $(\mathcal{H}_t)_{t>0}$ be the heat semigroup described above. Then the following hold:*

(i) *for all p in $[1,\infty]$,*

$$\|\!|\mathcal{H}_t|\!\|_p = \exp(-b_p t) \quad \forall t \in \mathbb{R}^+;$$

(ii) *for all* p *and* r *such that* $1 \leq p < r \leq \infty$,

$$|||\mathcal{H}_t|||_{p;r} \sim 1 \quad \forall t \in (0,1];$$

(iii) *for all* p *and* r *such that* $1 \leq p < r < 2$,

$$|||\mathcal{H}_t|||_{p;r} \sim t^{-1/2r'} \exp(-b_r t) \quad \forall t \in [1,\infty);$$

(iv) *for all* p *and* r *such that* $2 < p < r \leq \infty$,

$$|||\mathcal{H}_t|||_{p;r} \sim t^{-1/2p} \exp(-b_p t) \quad \forall t \in [1,\infty).$$

(v) *for all* p *and* r *such that either* $1 \leq p < r = 2$ *or* $2 = p < r \leq \infty$,

$$|||\mathcal{H}_t|||_{p;r} \sim t^{-3/4} \exp(-b_2 t) \quad \forall t \in [1,\infty);$$

(vi) *for all* p *and* r *such that* $1 \leq p < 2 < r \leq \infty$,

$$|||\mathcal{H}_t|||_{p;r} \sim t^{-3/2} \exp(-b_2 t) \quad \forall t \in [1,\infty).$$

Proof. The proof follows that of Theorem 3.2 in [CGM]. If p is in $[1,2]$, then

$$|||\mathcal{H}_t|||_p = \widetilde{h}_t(1/p) = \exp(-b_p t) \quad \forall t \in \mathbb{R}^+,$$

by Herz's *principe de majoration* (see Proposition 2.3 of [CMS]). The case where p is in $[2,\infty)$ follows by duality, and (i) is proved.

Whenever $1 \leq q \leq s \leq \infty$, $L^q(\mathfrak{X})$ embeds into $L^s(\mathfrak{X})$ without increasing norms, and it follows that

$$h_t(o) = \|\mathcal{H}_t\|_{1;\infty} \leq \|\mathcal{H}_t\|_{p;r} \leq \|\mathcal{H}_t\|_{p;p} \leq 1 \quad \forall t \in \mathbb{R}^+.$$

Further, $h_t(o)$ is a monotone decreasing function of t, by the semigroup property. Thus

$$h_1(o) \leq \|\mathcal{H}_t\|_{p;r} \leq 1 \quad \forall t \in [0,1]$$

and (ii) is proved.

By Corollary 1.5, if $1 \leq p < r \leq 2$, then $|||h_t|||_{p;r} \sim \|h_t\|_r$, and then (iii) and the first case of (v) follow from this fact and Lemma 2.1. By duality, (iv) follows from (iii), and the second case of (v) follows from the first.

Finally, (vi) follows from the inequality

$$\|h_t\|_\infty = |||\mathcal{H}_t|||_{1;\infty} \leq |||\mathcal{H}_t|||_{p;r} \leq |||\mathcal{H}_{t/2}|||_{p;2} \, |||\mathcal{H}_{t/2}|||_{2;r}$$

for all t in $\mathbb{R}^+$, and the estimates of Lemma 2.1 and (ii) and (v). ∎

Our next goal is to obtain pointwise estimates for the heat kernel $h_t(x)$ on $\mathfrak{X}$, uniform for x in $\mathfrak{X}$ and t in $\mathbb{R}^+$. To achieve this we first consider the heat kernel $h_t^{\mathbb{Z}}$ on $\mathbb{Z}$. We are interested in the behaviour of $h_t^{\mathbb{Z}}(j)$ when j^2+t^2 tends to infinity. We may restrict our investigation to positive j since $h_t^{\mathbb{Z}}$ is an even function. Exercise 10.1 of [Ol] (p. 60) expresses $h_t^{\mathbb{Z}}$ in terms of the modified Bessel function of imaginary argument, namely

$$h_t^{\mathbb{Z}}(j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left[t(\cos s - 1)\right] \cos(sj)\, ds = e^{-t}\, I_{|j|}(t) \quad \forall j \in \mathbb{Z} \quad \forall t \in \mathbb{R}^+.$$

From the uniform asymptotic expansion of modified Bessel functions of large order, we obtain the following theorem.

Theorem 2.3. *Let $\xi : \mathbb{R}^+ \to \mathbb{R}$ and $F : \mathbb{Z} \times \mathbb{R}^+ \to \mathbb{R}$ be the functions defined by the rules*

$$\xi(z) = (1+z^2)^{1/2} + \log\left(\frac{z}{1+(1+z^2)^{1/2}}\right)$$

and

$$F(j,t) = \begin{cases} (2\pi)^{-1/2} \frac{\exp[-t+|j|\,\xi(t/|j|)]}{(1+j^2+t^2)^{1/4}} & \text{if } j \neq 0, \\ \\ (2\pi)^{-1/2}\,(1+t^2)^{-1/4} & \text{if } j = 0. \end{cases}$$

Then $h_t^{\mathbb{Z}}(j) \asymp F(j,t)$ as j^2+t^2 tends to ∞, and $h_t^{\mathbb{Z}}(j) \sim F(j,t)$ for all j in $\mathbb{Z}$ and t in $\mathbb{R}^+$.

Proof. To prove the first assertion, it suffices to show that, if $\epsilon > 0$, there exists an integer M_ϵ such that

$$(1-\epsilon)\,F(j,t) \leq h_t^{\mathbb{Z}}(j) \leq (1+\epsilon)\,F(j,t),$$

or equivalently

$$(1-\epsilon)\,e^t\,F(j,t) \leq I_{|j|}(t) \leq (1+\epsilon)\,e^t\,F(j,t),$$

when $(j^2+t^2) \geq M_\epsilon$. Let ϵ be given.

Let $G : \mathbb{Z} \times \mathbb{R}^+ \to \mathbb{R}$ be the auxiliary function defined by the rule

$$G(j,t) = \frac{(1+j^2+t^2)^{1/4}}{(j^2+t^2)^{1/4}}\,F(j,t) \quad \forall j \in \mathbb{Z} \quad \forall t \in \mathbb{R}^+.$$

Formula 7.16 of [Ol] (p. 377) shows that there exists η_ϵ in $\mathbb{R}^+$ such that, if $\eta \geq \eta_\epsilon$ and $z > 0$, then

$$(1-\epsilon)\,\frac{e^{\eta\xi(z)}}{(2\pi\eta)^{1/2}(1+z^2)^{1/4}} \leq I_\eta(\eta z) \leq (1+\epsilon)^{1/2}\,\frac{e^{\eta\xi(z)}}{(2\pi\eta)^{1/2}(1+z^2)^{1/4}}.$$

We rewrite these inequalities with $t/|j|$ in place of z and $|j|$ in place of η, and deduce that

$$(1-\epsilon)\,e^t\,G(j,t) \leq I_{|j|}(t) \leq (1+\epsilon)^{1/2}\,e^t\,G(j,t)$$

when $|j| \geq \eta_\epsilon$ and $t \in \mathbb{R}^+$. Provided that η_ϵ is taken so large that $(1+\eta_\epsilon^2)^{1/2} \leq (1+\epsilon)\eta_\epsilon$, it follows that

$$(1-\epsilon)\,e^t\,F(j,t) \leq I_{|j|}(t) \leq (1+\epsilon)\,e^t\,F(j,t) \tag{2.1}$$

whenever $|j| \geq \eta_\epsilon$ and $t \in \mathbb{R}^+$. Next, for every fixed $|j|$,

$$I_{|j|}(t) \asymp (2\pi t)^{-1/2}\,e^t \quad \text{as} \quad t \to \infty$$

(see, e.g., formula 5.11.8 of [L] (p. 123) or section 3.7.4 of [Ol] (p. 83)), and therefore there exists $T_{1,\epsilon}$ such that

$$(1-\epsilon)^{1/2}(2\pi t)^{-1/2}\,e^t \leq I_{|j|}(t) \leq (1+\epsilon)^{1/2}(2\pi t)^{-1/2}\,e^t$$

when $|j| \leq \eta_\epsilon$ and $t \geq T_{1,\epsilon}$. Moreover, it is easy to verify that

$$F(j,t) \asymp (2\pi t)^{-1/2} \quad \text{as } t \to \infty$$

if $|j| \leq \eta_\epsilon$. Therefore there exists $T_{2,\epsilon}$ such that

$$(1-\epsilon)^{1/2}\, e^t\, F(j,t) \leq (2\pi t)^{-1/2}\, e^t \leq (1+\epsilon)^{1/2}\, e^t\, F(j,t) \tag{2.2}$$

when $|j| \leq \eta_\epsilon$ and $t \geq T_{2,\epsilon}$. It follows that

$$(1-\epsilon)\, e^t\, F(j,t) \leq I_{|j|}(t) \leq (1+\epsilon)\, e^t\, F(j,t),$$

if $|j| \leq \eta_\epsilon$ and $t \geq \max(T_{1,\epsilon}, T_{2,\epsilon})$. By combining this with the uniform estimate (2.1), we may conclude that

$$(1-\epsilon)\, e^t\, F(j,t) \leq I_{|j|}(t) \leq (1+\epsilon)\, e^t\, F(j,t)$$

if $j^2 + t^2 \geq 2\max(\eta_\epsilon^2, T_{1,\epsilon}^2, T_{2,\epsilon}^2)$, as required.

To show that $h_t^{\mathbb{Z}}(j) \sim F(j,t)$ for all j in $\mathbb{Z}$ and t in $\mathbb{R}^+$, on the one hand we observe that, from the series expansion of $I_{|j|}(t)$ (see, e.g., formula 5.7.1 of [L] (p. 108)), namely

$$h_t^{\mathbb{Z}}(j) = e^{-t} \sum_{n=0}^{\infty} \frac{(t/2)^{|j|+2n}}{n!\,(n+|j|)!} \quad \forall j \in \mathbb{Z} \quad \forall t \in \mathbb{R}^+,$$

it follows that

$$h_t^{\mathbb{Z}}(j) \asymp \frac{(t/2)^{|j|}}{|j|!} \qquad \text{as } t \to 0^+,$$

uniformly when $|j| \leq \eta_\epsilon$. On the other hand, we observe that, when $1 \leq |j| \leq \eta_\epsilon$,

$$-t + |j|\; \xi(t/\,|j|) = |j|\,(\log t + 1 - \log(2\,|j|)) + o(1) \qquad \text{as } t \to 0^+,$$

so that, if $0 \leq |j| \leq \eta_\epsilon$,

$$F(j,t) \asymp (2\pi)^{-1/2}(1+j^2)^{-1/4}\, \frac{t^{|j|} e^{|j|}}{(2\,|j|)^{|j|}} \qquad \text{as } t \to 0^+$$

(where $(2\,|j|)^{|j|}$ is taken to be 1 if $j=0$). Consequently,

$$h_t^{\mathbb{Z}}(j) \sim F(j,t)$$

when $|j| \leq \eta_\epsilon$ and $t \leq T_{2,\epsilon}$. Combining this with (2.1) and (2.2), it follows that

$$h_t^{\mathbb{Z}}(j) \sim F(j,t) \quad \forall j \in \mathbb{Z} \quad \forall t \in \mathbb{R}^+,$$

as required. This concludes the proof of the theorem. □

It is easy to verify that the bounds for $h_t^{\mathbb{Z}}$ that we obtain here improve somewhat those of M. M. Pang [P].

We next derive a formula that expresses the heat kernel h_t of $\mathfrak{X}$ in terms of $h_t^{\mathbb{Z}}$. This formula is reminiscent of the relationship between the heat kernel of a symmetric space and that of a maximal flat submanifold. Combining this with the estimates of Theorem 2.3 above will yield bounds for h_t.

We start with a lemma that relates the derivative of $h_t^{\mathbb{Z}}$ with respect to t with $h_t^{\mathbb{Z}}$ itself.

Lemma 2.4. *The following hold for every j in $\mathbb{N}$ and t in $\mathbb{R}^+$:*

(i) $$\frac{d}{dt} h_t^{\mathbb{Z}}(j) = h_t^{\mathbb{Z}}(j+1) - h_t^{\mathbb{Z}}(j) + \frac{j}{t} h_t^{\mathbb{Z}}(j);$$

(ii) $$h_t^{\mathbb{Z}}(j) - h_t^{\mathbb{Z}}(j+2) = \frac{2(j+1)}{t}\, h_t^{\mathbb{Z}}(j+1).$$

Proof. We first prove (i). Substituting the integral representation for $I_j(t)$ (see, e.g., formula 5.22 in [L]) in the expression for $h_t^{\mathbb{Z}}$ yields

$$h_t^{\mathbb{Z}}(j) = \frac{2\,e^{-t}\,(t/2)^j}{\sqrt{\pi}\,\Gamma(j+1/2)} \int_0^1 (1-u^2)^{j-1/2}\,\cosh(tu)\,du.$$

Therefore the derivative of $h_t^{\mathbb{Z}}$ with respect to t may be written as the sum of three terms, obtained by differentiating the factors e^{-t}, $(t/2)^j$, and $\cosh(tu)$ respectively. The first two summands contribute $-h_t^{\mathbb{Z}}(j)$ and $\dfrac{j}{t}h_t^{\mathbb{Z}}(j)$ respectively. By differentiating inside the integral and integrating by parts, we obtain that

$$\frac{d}{dt}\int_0^1 (1-u^2)^{j-1/2}\,\cosh(tu)\,du = \frac{t/2}{j+1/2}\int_0^1 (1-u^2)^{j+1/2}\,\cosh(tu)\,du.$$

Thus the third summand is equal to

$$\frac{2\,e^{-t}\,(t/2)^{j+1}}{\sqrt{\pi}\,(j+1/2)\,\Gamma(j+1/2)}\int_0^1 (1-u^2)^{j+1/2}\,\cosh(tu)\,du = h_t^{\mathbb{Z}}(j+1),$$

as required.

We now turn to (ii). Recall that $h_t^{\mathbb{Z}}(j)$ is a solution to the initial value problem

$$\begin{cases} \dfrac{d}{dt}u(j,t) + \mathcal{L}^{\mathbb{Z}}u(j,t) = 0 \quad \forall t\in\mathbb{R}^+ \quad \forall j\in\mathbb{Z}, \\[2ex] u(j,0) = \delta_0(j) \quad \forall j\in\mathbb{Z}. \end{cases}$$

Thus $h_t^{\mathbb{Z}}$ satisfies the relation

$$h_t^{\mathbb{Z}}(j+1) - h_t^{\mathbb{Z}}(j) + \frac{j}{t}h_t^{\mathbb{Z}}(j) = \frac{d}{dt}h_t^{\mathbb{Z}}(j) = \frac{h_t^{\mathbb{Z}}(j+1)+h_t^{\mathbb{Z}}(j-1)}{2} - h_t^{\mathbb{Z}}(j),$$

whence

$$h_t^{\mathbb{Z}}(j-1) - h_t^{\mathbb{Z}}(j+1) = \frac{2j}{t}h_t^{\mathbb{Z}}(j).$$

Substituting $j+1$ in place of j completes the proof of the lemma. □

We are now ready to state and prove our main result, in which we establish formulae relating h_t with $h_t^{\mathbb{Z}}$, and derive upper and lower bounds for h_t.

Proposition 2.5. *The following hold for all t in $\mathbb{R}^+$ and x in $\mathfrak{X}$:*

(i) $$h_t(x) = e^{-b_2 t}\,q^{-|x|/2}\sum_{k=0}^{\infty} q^{-k}\Big[h_{t\gamma(0)}^{\mathbb{Z}}(|x|+2k) - h_{t\gamma(0)}^{\mathbb{Z}}(|x|+2k+2)\Big];$$

(ii) $$h_t(x) = \frac{2\,e^{-b_2 t}}{\gamma(0)\,t}\,q^{-|x|/2}\sum_{k=0}^{\infty} q^{-k}\,(|x|+2k+1)\,h_{t\gamma(0)}^{\mathbb{Z}}(|x|+2k+1);$$

(iii) $$\frac{(q-1)^3}{q^{1/2}(q+1)^3}\,h_t(x) \le \frac{e^{-b_2 t}}{t}\,\phi_0(x)\,h_{t\gamma(0)}^{\mathbb{Z}}(|x|+1) \le \frac{q^{1/2}}{(q+1)}\,h_t(x).$$

Proof. We first prove (i). By spherical Fourier inversion (see Theorem 1.1),

$$h_t(x) = 2\,c_G e^{-t}\int_{-\tau/2}^{\tau/2} e^{t\gamma(s)}\,\mathbf{c}(-s)^{-1}q^{-(1/2+is)|x|}\,ds.$$

By using formulae (1.4), (1.5), and (1.2), for c_G, γ, and $\mathbf{c}$, we may deduce that

$$\begin{aligned} h_t(x) &= \frac{q\log q}{2\,\pi} e^{-t} q^{-|x|/2} \int_{-\tau/2}^{\tau/2} e^{t\gamma(0)\cos(s\log q)}\, \frac{q^{-1/2-is}-q^{-1/2+is}}{q^{1/2-is}-q^{-1/2+is}}\, q^{is|x|}\, ds \\ &= \frac{\log q}{2\pi} e^{-t} q^{-|x|/2} \int_{-\tau/2}^{\tau/2} e^{t\gamma(0)\cos(s\log q)}\, \frac{1-q^{2is}}{1-q^{2is-1}}\, q^{is|x|}\, ds. \end{aligned}$$

Now we expand the function $s \mapsto \dfrac{1-q^{2is}}{1-q^{2is-1}}$, change variables, and use formula (1.6), relating b_2 and $\gamma(0)$, to obtain the formula

$$h_t(x) = \frac{1}{2\pi} e^{-b_2 t}\, q^{-|x|/2} \int_{-\pi}^{\pi} e^{-t\gamma(0)(1-\cos u)} \sum_{k=0}^{\infty} q^{-k} \left(e^{2iuk} - e^{2iu(k+1)}\right) e^{iu|x|}\, du.$$

Since the series converges uniformly, we may interchange the sum and the integral, and since

$$h^{\mathbb{Z}}_{t\gamma(0)}(j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-t\gamma(0)(1-\cos u)} e^{iuj}\, du \quad \forall j \in \mathbb{N},$$

we may conclude that

$$h_t(x) = e^{-b_2 t}\, q^{-|x|/2} \sum_{k=0}^{\infty} q^{-k} \left(h^{\mathbb{Z}}_{t\gamma(0)}(|x|+2k) - h^{\mathbb{Z}}_{t\gamma(0)}(|x|+2k+2)\right),$$

and (i) is proved.

Formula (ii) is an immediate consequence of (i) and Lemma 2.4 (ii).

We now turn to (iii). On the one hand, since all the summands in the series (ii) are positive, we may estimate the series by its first term, deducing that

$$h_t(x) \geq \frac{2\, e^{-b_2 t}}{\gamma(0)\, t}\, q^{-|x|/2} (|x|+1)\, h^{\mathbb{Z}}_{t\gamma(0)}(|x|+1).$$

On the other hand, $|x| + 2k + 1 \leq (2k+1)(|x|+1)$, and by Lemma 2.4 (ii), $h^{\mathbb{Z}}_{t\gamma(0)}(n+2j) \leq h^{\mathbb{Z}}_{t\gamma(0)}(n)$. Thus, from (ii),

$$\begin{aligned} h_t(x) &= \frac{2\, e^{-b_2 t}}{\gamma(0)\, t}\, q^{-|x|/2} \sum_{k=0}^{\infty} q^{-k}\, (|x|+2k+1)\, h^{\mathbb{Z}}_{t\gamma(0)}(|x|+2k+1) \\ &\leq \frac{2\, e^{-b_2 t}}{\gamma(0)\, t}\, q^{-|x|/2} \sum_{k=0}^{\infty} q^{-k}\, (2k+1)\, (|x|+1)\, h^{\mathbb{Z}}_{t\gamma(0)}(|x|+1) \\ &= \frac{2\, e^{-b_2 t}}{\gamma(0)\, t}\, q^{-|x|/2}\, \frac{q(q+1)}{(q-1)^2}\, (|x|+1)\, h^{\mathbb{Z}}_{t\gamma(0)}(|x|+1). \end{aligned}$$

The definition of the spherical functions (1.1) shows that

$$\frac{q-1}{q+1}\phi_0(x) \leq (|x|+1)\, q^{-|x|/2} \leq \phi_0(x) \quad \forall x \in \mathfrak{X},$$

and the proposition follows by combining all these inequalities and formula (1.5) for $\gamma(0)$. ∎

It is perhaps worth noting that the ratio of the left hand side to the right hand side of inequality (iii) of the preceding proposition tends to one as q becomes large.

The last result of this section concerns the weak $(1,1)$ boundedness of the maximal heat operators on $\mathbb{Z}$ and on $\mathfrak{X}$; these are defined by the formulae

$$\mathcal{H}_*^{\mathbb{Z}}F(j) = \sup_{t\in\mathbb{R}^+} \left|F *_{\mathbb{Z}} h_t^{\mathbb{Z}}(j)\right| \quad \forall j \in \mathbb{Z}$$

and

$$\mathcal{H}_* f(x) = \sup_{t\in\mathbb{R}^+} |f * h_t(x)| \quad \forall x \in \mathfrak{X}.$$

Theorem 2.6. *The maximal heat operators $\mathcal{H}_*^{\mathbb{Z}}$ and $\mathcal{H}_*$ are of weak type $(1,1)$ and of strong type (p,p) for every p in $(1,\infty)$.*

Proof. Let $\mathfrak{M}$ denote the class of all radial probability measures on $\mathfrak{X}$. According to Corollary 3.2 of [CMS], the operator $\mathcal{M}_s$, defined by the rule

$$\mathcal{M}_s f(x) = \sup_{\mu\in\mathfrak{M}} (|f| * \mu) \quad \forall x \in \mathfrak{X},$$

is of weak type $(1,1)$ and of strong type (p,p) for every p in $(1,\infty)$.

Since $\sum_{x\in\mathfrak{X}} h_t(x) = 1$,

$$\mathcal{H}_* f(x) \leq \mathcal{M}_s f(x) \quad \forall x \in \mathfrak{X}$$

for every bounded function f on $\mathfrak{X}$, and the boundedness of $\mathcal{H}_*$ follows.

We turn now to the boundedness of $\mathcal{H}_t^{\mathbb{Z}}$. The case where p is in $(1,\infty)$ is classical (see [S] and [Co1]), so we only need to prove the weak $(1,1)$ boundedness of $\mathcal{H}_t^{\mathbb{Z}}$. Since the analogue of Corollary 3.2 of [CMS] does not hold on $\mathbb{Z}$, we use a different argument, which is an adaptation of an idea in [CGGM], and relies on the sharp estimates for the heat kernel $h_t^{\mathbb{Z}}$ obtained in Theorem 2.3.

Let $\mathcal{E}_t^{\mathbb{Z}}$ denote the ergodic operator associated to $\mathcal{H}_t^{\mathbb{Z}}$, namely

$$\mathcal{E}_t^{\mathbb{Z}}F(j) = \frac{1}{t}\int_t^{2t} \mathcal{H}_s^{\mathbb{Z}}F(j)\, ds \quad \forall j \in \mathbb{Z} \quad \forall t \in \mathbb{R}^+,$$

and let $\mathcal{E}_*^{\mathbb{Z}}$ be the the corresponding maximal operator:

$$\mathcal{E}_*^{\mathbb{Z}}F(j) = \sup_{t\in\mathbb{R}^+} \left|\mathcal{E}_t^{\mathbb{Z}}F(j)\right| \quad \forall x \in \mathbb{Z}.$$

The Hopf–Dunford–Schwartz theorem implies that $\mathcal{E}_*^{\mathbb{Z}}$ is of weak type $(1,1)$, so, as in [CGGM], it suffices to prove that the ergodic kernel, defined by the rule

$$\varepsilon_t^{\mathbb{Z}}(j) = \frac{1}{t}\int_t^{2t} h_s^{\mathbb{Z}}(j)\, ds \quad \forall j \in \mathbb{Z} \quad \forall t \in \mathbb{R}^+,$$

dominates the heat kernel. From Theorem 2.3,

$$h_s^{\mathbb{Z}}(j) \sim (1 + j^2 + s^2)^{-1/4} \exp\left[|j|\ (-s/\,|j| + \xi(s/\,|j|)\right]$$

for all j in $\mathbb{Z}\setminus\{0\}$ and t in $\mathbb{R}^+$, and since the function $z \mapsto -z + \xi(z)$ is increasing, while $s \mapsto (1 + j^2 + s^2)^{-1/4}$ is decreasing,

$$\begin{aligned} \varepsilon_t^{\mathbb{Z}}(j) &\geq C\,(1 + j^2 + 4t^2)^{-1/4} \exp\left[|j|\ (-t/\,|j| + \xi(t/\,|j|))\right] \\ &\geq 2^{-1/2}C\,(1 + j^2 + t^2)^{-1/4} \exp\left[|j|\ (-t/\,|j| + \xi(t/\,|j|))\right] \\ &\geq C'\, h_t^{\mathbb{Z}}(j) \quad \forall j \in \mathbb{Z}\setminus\{0\} \quad \forall t \in \mathbb{R}^+. \end{aligned}$$

Similar estimates (but without the exponential term) hold when $j = 0$, so

$$\varepsilon_t^{\mathbb{Z}}(j) \geq C' \, h_t^{\mathbb{Z}}(j) \quad \forall j \in \mathbb{Z} \quad \forall t \in \mathbb{R}^+,$$

and the weak type $(1,1)$ boundedness for $\mathcal{H}_*^{\mathbb{Z}}$ follows. □

3. Estimates for the resolvent

Throughout this section, α denotes a complex parameter and a its real part. For θ in $[0,1)$, let $\mathcal{L}_\theta$ and $\mathcal{H}_{t,\theta}$ denote the operators $\mathcal{L} - \theta b_2$ and $e^{-t\mathcal{L}_\theta}$. By Theorem 2.2 (i),

$$|||\mathcal{H}_{t,\theta}|||_p = e^{-[1-b_2\theta-\gamma(i\delta(p))]\,t} \quad \forall t \in \mathbb{R}^+.$$

Let p_θ in $[1,2]$ be the threshold index for which $\mathcal{H}_{t,\theta}$ is contractive on $L^p(\mathfrak{X})$, or, equivalently, for which the L^p spectrum of $\mathcal{L}_\theta$ is contained in the right half plane. Thus p_θ is the unique solution in the interval $[1,2]$ of the equation

$$\gamma\left(i\delta(p_\theta)\right) = 1 - b_2\theta, \tag{3.1}$$

i.e.,

$$\frac{1}{p_\theta} = \frac{1}{2} + \frac{1}{\log q} \cosh^{-1}\left(\frac{1-\theta b_2}{1-b_2}\right).$$

We denote by $\mathcal{R}_\theta^\alpha$ the operator $\mathcal{L}_\theta^{-\alpha/2}$ and by r_θ^α its integral kernel. This section is devoted to studying the behaviour of r_θ^α at infinity and the L^p–L^q boundedness properties of $\mathcal{R}_\theta^\alpha$.

We need a technical lemma, which may be extracted from p. 48 of [E].

Lemma 3.1. *Suppose that $-\infty < c < d < \infty$, $0 < \mathrm{Re}(\lambda) \leq 1$, and $\eta \in C^2([c,d])$. The following hold:*
(i) *if $\eta^{(k)}(d) = 0$ when $k = 0$ and 1, then*

$$\int_c^d e^{ixt}\,(t-c)^{\lambda-1}\,\eta(t)\,dt = \Gamma(\lambda)\,e^{i\lambda\pi/2}\,\eta(c)\,x^{-\lambda}\,e^{icx} + E(x),$$

where the error term E is $O(x^{-\lambda-1})$ as x tends to ∞;
(ii) *if $\eta^{(k)}(c) = 0$ when $k = 0$ and 1, then*

$$\int_c^d e^{ixt}\,(d-t)^{\lambda-1}\,\eta(t)\,dt = \Gamma(\lambda)\,e^{-i\lambda\pi/2}\,\eta(d)\,x^{-\lambda}\,e^{idx} + E(x),$$

where the error term E is $O(x^{-\lambda-1})$ as x tends to ∞.

Proof. Notice that the proof in [E] extends to the case when $0 < \mathrm{Re}\,(\lambda) \leq 1$, without change. The required conclusion is obtained by taking the first two terms in the resulting asymptotic expansion. □

The next proposition describes the asymptotic behaviour of r_θ^α at infinity.

Proposition 3.2. *Suppose that $0 < \theta < 1$, $\alpha \neq 0$, and $a \geq 0$. Then*

$$r_\theta^\alpha(x) \asymp c_{\alpha,\theta}\,|x|^{\alpha/2-1}\,q^{-|x|/p_\theta} \qquad \textit{as } |x| \to \infty,$$

where

$$c_{\alpha,\theta} = \frac{q}{q+1}\,\frac{(\gamma(0)\,\sinh(\delta(p_\theta)\log q))^{-\alpha/2}}{\mathbf{c}(-i\delta(p_\theta))\,\Gamma(\alpha/2)}.$$

Proof. Since r_θ^α is associated to the multiplier $(1-\theta b_2-\gamma)^{-\alpha/2}$, the spherical Fourier inversion formula (Theorem 1.1) and formula (3.1) relating θb_2 and p_θ imply that

$$\begin{aligned} r_\theta^\alpha(x) &= 2\, c_G \int_{-\tau/2}^{\tau/2} (1-\theta b_2-\gamma(s))^{-\alpha/2}\, \mathbf{c}(-s)^{-1}\, q^{(-1/2+is)|x|}\, ds \\ &= 2\, c_G \int_{-\tau/2}^{\tau/2} (\gamma(i\delta(p_\theta))-\gamma(s))^{-\alpha/2}\, \mathbf{c}(-s)^{-1}\, q^{(-1/2+is)|x|}\, ds \quad \forall x \in \mathfrak{X}. \end{aligned}$$

From (1.5), for all u and v in $\mathbb{R}$, $\gamma(i\delta(p_\theta)) - \gamma(u+iv)$ is equal to

$$\gamma(0)\left[\cosh(\delta(p_\theta)\log q) - \cosh(v)\cos(u) + i\sinh(v)\sin(u)\right],$$

so the set of zeroes of $\gamma(i\delta(p_\theta)) - \gamma(\cdot)$ is $\tau\mathbb{Z} \pm i\delta(p_\theta)$, and $\mathrm{Re}\,(\gamma(i\delta(p_\theta)) - \gamma(\cdot)) > 0$ in $\overline{\mathbb{S}}_{p_\theta} \setminus (\tau\mathbb{Z} \pm i\delta(p_\theta))$. Observe also that $\mathbf{c}^{-1}$ is meromorphic in $\mathbb{C}$, with simple poles on the line $\mathrm{Im}\,(z) = 1/2$, and no other singularities, because

$$\mathbf{c}(z) = \frac{q^{1/2}}{q+1}\, \frac{\sin[(z-i/2)\log q]}{\sin[z\log q]} \quad \forall z \in \mathbb{C} \setminus (\tau/2)\mathbb{Z},$$

from formula (1.2). Thus, for d in $\mathbb{N}$ and α such that $a \geq 0$, we may define an analytic function $f_\alpha : \mathbb{S}_{p_\theta} \to \mathbb{C}$ by the rule

$$f_\alpha(z) = [\gamma(i\delta(p_\theta)) - \gamma(z)]^{-\alpha/2}\, \mathbf{c}(-z)^{-1}, \tag{3.2}$$

where we take the principal branch of the logarithm to compute the power, and f_α has (almost everywhere) boundary values on the lines $\mathrm{Im}\,(z) = \pm\delta(p_\theta)$. Further,

$$r_\theta^\alpha(x) = 2\, c_G \int_{-\tau/2}^{\tau/2} f_\alpha(s)\, q^{(is-1/2)|x|}\, ds.$$

The key idea of the proof is to vary the path of integration. Suppose that $1 < p < 2$, and consider the rectangle with corners $\pm\tau/2$ and $\pm\tau/2 + i\delta(p)$. If f is analytic inside the rectangle, then Cauchy's theorem implies that

$$\begin{aligned} \int_{-\tau/2}^{\tau/2} f_\alpha(s)\, q^{(is-1/2)|x|}\, ds &= \int_{-\tau/2}^{\tau/2} f_\alpha(s+i\delta(p))\, q^{(is-1/p)|x|}\, ds \\ &= q^{-|x|/p} \int_{-\tau/2}^{\tau/2} f_\alpha(s+i\delta(p))\, q^{is|x|}\, ds, \end{aligned}$$

because the two integrals along the vertical sides of the rectangle cancel, by periodicity. Consequently, if $|f_\alpha(\cdot + i\delta(p))|$ is integrable, then the integral is bounded independently of $|x|$, and

$$r_\theta^\alpha(x) = O(q^{-|x|/p}) \quad \forall x \in \mathfrak{X}.$$

The proof will involve variants and refinements of this idea.

First, we consider the case in which $\alpha = 2k$, where $k \in \mathbb{Z}^+$. In this case, f_α is meromorphic. By applying the above technique for some p in $(1, p_\theta)$, and taking into account the pole of f inside the rectangle, we see that

$$r_\theta^\alpha(x) = 4\pi i c_G\, \mathrm{Res}\left(f_\alpha(z)\, q^{(iz-1/2)|x|}; z = i\delta(p_\theta) \right) + E_1(x) \quad \forall x \in \mathfrak{X}, \tag{3.3}$$

where the error term $E_1(x)$ is $O\left(q^{-|x|/p}\right)$ for all x in $\mathfrak{X}$. Now

$$\begin{aligned}&\operatorname{Res}\Big(f_\alpha(z)\,q^{(iz-1/2)|x|}; z=i\delta(p_\theta)\Big)\\&=\frac{1}{(k-1)!}\left(\frac{d}{dz}\right)^{k-1}\left[(z-i\delta(p_\theta))^k\,f_\alpha(z)\,q^{(iz-1/2)|x|}\right]\Big|_{z=i\delta(p_\theta)}.\end{aligned}$$

The expression in square brackets is equal to $g(z)\,q^{(iz-1/2)|x|}$, where $g(z)=(z-i\delta(p_\theta))^k f_\alpha(z)$; thus g is independent of x, holomorphic near $i\delta(p_\theta)$, and $g(i\delta(p_\theta))\neq 0$. By Leibniz's rule,

$$\begin{aligned}&\frac{1}{(k-1)!}\left(\frac{d}{dz}\right)^{k-1}\left[g(z)\,q^{(iz-1/2)|x|}\right]\Big|_{z=i\delta(p_\theta)}\\&=\sum_{j=0}^{k-1}\frac{1}{j!\,(k-1-j)!}\,g^{(k-1-j)}(i\delta(p_\theta))\,(i\,|x|\log q)^j\,q^{-|x|/p_\theta}\\&=g(i\delta(p_\theta))\,\frac{(i\,|x|\log q)^{k-1}}{(k-1)!}\,q^{-|x|/p_\theta}+E_2(x),\end{aligned}$$

where the error term $E_2(x)$ is $O((1+|x|)^{k-2}q^{-|x|/p_\theta})$ for all x in $\mathfrak{X}$. From formula (3.2) for f_α,

$$g(i\delta(p_\theta))=\lim_{z\to i\delta(p_\theta)}\left[\frac{z-i\delta(p_\theta)}{\gamma(i\delta(p_\theta))-\gamma(z)}\right]^k\frac{1}{\mathbf{c}(-z)}=\left[\frac{-1}{\gamma'(i\delta(p_\theta))}\right]^k\frac{1}{\mathbf{c}(-i\delta(p_\theta))}.$$

Thus

$$\begin{aligned}&\operatorname{Res}\Big(f_\alpha(z)\,q^{(iz-1/2)|x|}; z=i\delta(p_\theta)\Big)\\&=\left[\frac{-1}{\gamma'(i\delta(p_\theta))}\right]^k\frac{(i\,|x|\log q)^{k-1}}{\mathbf{c}(-i\delta(p_\theta))\,(k-1)!}\,q^{-|x|/p_\theta}+E_2(x),\end{aligned}$$

and by using formulae (3.3) for r_θ^α, (1.4) for c_G, and (1.5) for γ, we see that

$$\begin{aligned}r_\theta^\alpha(x)&=4\pi i\,c_G\,\operatorname{Res}\Big(f_\alpha(z)\,q^{(iz-1/2)|x|}; z=i\delta(p_\theta)\Big)+E_1(x)\\&=\frac{q}{q+1}\left(\frac{-i\log q}{\gamma'(i\delta(p_\theta))}\right)^k\frac{1}{\mathbf{c}(-i\delta(p_\theta))\,\Gamma(\alpha/2)}\,|x|^{k-1}\,q^{-|x|/p_\theta}+E_3(x),\\&=\frac{q}{q+1}\,\frac{[\gamma(0)\,\sinh(\delta(p_\theta)\log q)]^{-\alpha/2}}{\mathbf{c}(-i\delta(p_\theta))\,\Gamma(\alpha/2)}\,|x|^{\alpha/2-1}\,q^{-|x|/p_\theta}+E_3(x),\end{aligned}$$

where the error term $E_3(x)$ is $O((1+|x|)^{k-2}\,q^{-|x|/p_\theta})$, for all x in $\mathfrak{X}$.

We now consider the case where $\alpha\notin 2\mathbb{Z}$. First of all, we shift the interval of integration using periodicity:

$$r_\theta^\alpha(x)=2c_G\int_0^\tau f_\alpha(s)\,q^{(-1/2+is)|x|}\,ds.$$

We first assume that $a<2$. In this case f_α has integrable singularities at $i\delta(p_\theta)$ and $\tau+i\delta(p_\theta)$, so that we may consider the rectangle with corners 0, τ, $i\delta(p_\theta)$, and $\tau+i\delta(p_\theta)$, and shift the contour of integration as before, to obtain

$$r_\theta^\alpha(x)=2c_G\,q^{-|x|/p_\theta}\int_0^\tau f_\alpha(s+i\delta(p_\theta))\,q^{is|x|}\,ds.$$

Let $I_\alpha(x)$ denote the integral above, and take functions ψ_1 and ψ_2 in $C^\infty([0,\tau])$ such that $\psi_1+\psi_2=1$, ψ_1 takes the value 0 near τ, and ψ_2 takes the value 0 near 0. Then

$$\begin{aligned} I_\alpha(x) &= \int_0^\tau \left[f_\alpha(s+i\delta(p_\theta))\, s^{\alpha/2}\, \psi_1(s)\right] s^{-\alpha/2}\, e^{is|x|\log q}\, ds \\ &\qquad + \int_0^\tau \left[f_\alpha(s+i\delta(p_\theta))\, (\tau-s)^{\alpha/2}\, \psi_2(s)\right] (\tau-s)^{-\alpha/2}\, e^{is|x|\log q}\, ds \\ &= I_{\alpha,1}(x)+I_{\alpha,2}(x) \quad \forall x\in\mathfrak{X}, \end{aligned}$$

say. By applying Lemma 3.1 to $I_{\alpha,1}$ and $I_{\alpha,2}$, it follows that

$$\begin{aligned} I_\alpha(x) &= \Gamma(1-\alpha/2)\,(|x|\,\log q)^{\alpha/2-1}\,\left[e^{i(1-\alpha/2)\pi/2}\, v_1 + e^{-i(1-\alpha/2)\pi/2}\, v_2\right] + E_1(x) \\ &= i\,\Gamma(1-\alpha/2)\,(|x|\,\log q)^{\alpha/2-1}\,\left[e^{-i\alpha\pi/4}\, v_1 - e^{i\alpha\pi/4}\, v_2\right] + E_1(x) \end{aligned}$$

for all x in $\mathfrak{X}$, where the error term $E_1(x)$ is $O(x^{\alpha/2-2})$ as $|x|$ tends to ∞, and where

$$v_1 = \lim_{s\to 0^+} f_\alpha(s+i\delta(p_\theta))\, s^{\alpha/2} \qquad \text{and} \qquad v_2 = \lim_{s\to\tau^-} f_\alpha(s+i\delta(p_\theta))\,(\tau-s)^{\alpha/2}.$$

Now

$$\begin{aligned} v_1 &= \lim_{s\to 0^+}\left[\frac{s}{\gamma(i\delta(p_\theta))-\gamma(s+i\delta(p_\theta))}\right]^{\alpha/2} \frac{1}{\mathbf{c}(-s-i\delta(p_\theta))} \\ &= \left[\frac{-1}{\gamma'(i\delta(p_\theta))}\right]^{\alpha/2} \frac{1}{\mathbf{c}(-i\delta(p_\theta))} \\ &= e^{-i\alpha\pi/4}\,\frac{(\gamma(0)\,\log q\,\sinh(\delta(p_\theta)\log q))^{-\alpha/2}}{\mathbf{c}(-i\delta(p_\theta))}, \end{aligned}$$

and similarly

$$v_2 = e^{i\alpha\pi/4}\,\frac{(\gamma(0)\,\log q\,\sinh(\delta(p_\theta)\log q))^{-\alpha/2}}{\mathbf{c}(-i\delta(p_\theta))}.$$

Consequently, using the formulae $\Gamma(z)\,\Gamma(1-z)\,\sin(\pi z)=\pi$ (see, e.g., formula 1.2.2 of [L] (p. 3)) and (1.4) for c_G, we see that

$$\begin{aligned} I_\alpha(x) &= \frac{i\,\Gamma(1-\alpha/2)\,(\gamma(0)\,\sinh(\delta(p_\theta)\log q))^{-\frac{\alpha}{2}}}{\mathbf{c}(-i\delta(p_\theta))\,\log q}\left[e^{-i\alpha\frac{\pi}{2}} - e^{i\alpha\frac{\pi}{2}}\right]|x|^{\frac{\alpha}{2}-1} + E_1(x), \\ &= \frac{2\pi\,(\gamma(0)\,\sinh(\delta(p_\theta)\log q))^{-\alpha/2}}{\mathbf{c}(-i\delta(p_\theta))\,\log q\,\Gamma(\alpha/2)}\,|x|^{\alpha/2-1} + E_1(x) \quad \forall x\in\mathfrak{X}, \end{aligned}$$

where $E_1(x)$ is $O(x^{\alpha/2-2})$ as $|x|$ tends to ∞, and

$$r_\theta^\alpha(x) = \frac{q}{q+1}\,\frac{(\gamma(0)\,\sinh(\delta(p_\theta)\log q))^{-\alpha/2}}{\mathbf{c}(-i\delta(p_\theta))\,\Gamma(\alpha/2)} + E_2(x) \quad \forall x\in\mathfrak{X},$$

where $E_2(x) = 2c_G\, q^{-|x|/p_\theta}\, E_1(x)$.

We now prove the estimate when $a\geq 2$, and α is not an even integer. Performing a k–fold integration by parts in the formulae that define $I_{\alpha,1}(x)$ and $I_{\alpha,2}(x)$, we deduce that

$$
\begin{aligned}
r_\theta^\alpha(x) &= 2c_G\, q^{-|x|/p_\theta}\,(I_{\alpha,1}(x)+I_{\alpha,2}(x)) \\
&= 2c_G\, \frac{q^{-|x|/p_\theta}}{(1-\alpha/2)\cdots(k-\alpha/2)}\,(I_{\alpha,1,k}(x)+I_{\alpha,2,k}(x)) \quad \forall x\in\mathfrak{X},
\end{aligned} \tag{3.4}
$$

where

$$I_{\alpha,1,k}(x) = (-1)^k \int_0^\tau s^{k-\alpha/2}\,\left(\frac{d}{ds}\right)^k \left[f_\alpha(s+i\delta(p_\theta))\,s^{\alpha/2}\,\psi_1(s)\,e^{is|x|\log q}\right] ds$$

and

$$I_{\alpha,2,k}(x) = \int_0^\tau (\tau-s)^{k-\alpha/2}\,\left(\frac{d}{ds}\right)^k \left[f_\alpha(s+i\delta(p_\theta))\,(\tau-s)^{\alpha/2}\,\psi_2(s)\,e^{is|x|\log q}\right] ds,$$

when $a<2$. By analytic continuation, equality (3.4) continues to hold when $a<2(k+1)$ and $\alpha\neq 2,4,\dots,2k$. The required estimate when $2k\leq a<2(k+1)$ is then obtained by noting that the main contribution comes from the term where all the derivatives are applied to the factor $q^{is|x|}$, and arguing as before to estimate the resulting integrals. □

Corollary 3.3. *If a is positive, then the resolvent kernel r_θ^α is in the Lorentz space $L^{p_\theta,r}(\mathfrak{X})$ if and only if $a<2/r'$. If u is in $\mathbb{R}\setminus\{0\}$, then r_θ^{iu} is in $L^{p_\theta,r}(\mathfrak{X})$ when $r>1$, but is not in $L^{p_\theta,1}(\mathfrak{X})$.*

Proof. Let $\{x_0,x_1,x_2,\dots\}$ be a half geodesic emanating from o, so that $|x_d|=d$ for every d in $\mathbb{N}$. Then a radial function f on $\mathfrak{X}$ is in $L^{p,r}(\mathfrak{X})^\sharp$ if and only if the function defined on $\mathbb{N}$ by $d\mapsto f(x_d)\,q^{d/p}$ is in $L^r(\mathbb{N})$, and the expression

$$\left[\sum_{d=0}^\infty |f(x_d)|^r\,q^{rd/p}\right]^{1/r}$$

defines an equivalent norm on $L^{p,r}(\mathfrak{X})^\sharp$. This fact and Proposition 3.2 imply that if $a>0$, then r_θ^α is in $L^{p_\theta,r}(\mathfrak{X})^\sharp$ if and only if

$$\sum_{d=1}^\infty d^{r(a/2-1)}<\infty.$$

This happens if and only if $r(a/2-1)<-1$, i.e., if and only if $a<2/r'$, as required. The second statement is proved analogously. □

Theorem 3.4. *Suppose that $0\leq\theta<1$, $1\leq p\leq r\leq\infty$, $s=\min(r,p')$, and $a\geq 0$. Then the operator $\mathcal{R}_\theta^\alpha$ is bounded from $L^p(\mathfrak{X})$ to $L^r(\mathfrak{X})$ if and only if one of the following conditions holds:*

(i) $\alpha=0$;
(ii) $\alpha\neq 0$, $s>p_\theta$;
(iii) $\alpha\neq 0$, $a=0$, $1\neq s=p_\theta$, $p=r$;
(iv) $\alpha\neq 0$, $0\leq a<2/p_\theta$, $s=p_\theta$, $p<r$.

Proof. If $\alpha=0$, then $\mathcal{R}_\theta^\alpha$ is just the identity operator, which is trivially bounded from $L^p(\mathfrak{X})$ to $L^r(\mathfrak{X})$. We may therefore assume that $\alpha\neq 0$ for the rest of this proof.

Assume that $r_\theta^\alpha \in Cv_p^r(\mathfrak{X})$. We shall show that one of (ii), (iii), and (iv) must hold. According to Theorem 1.3, if $s < 2$, then $\widetilde{r_\theta^\alpha}$ must be analytic in the strip $\mathbb{S}_s$; since

$$\widetilde{r_\theta^\alpha}(z) = [\gamma(i\delta(p_\theta)) - \gamma(z)]^{-\alpha/2},$$

it is necessary that $s \geq p_\theta$. Further, if $s \geq 2$, then $s \geq p_\theta$ trivially. When $s = p_\theta$, Theorem 1.3 gives further information. On the one hand, if $p = r$, then $\widetilde{r_\theta^\alpha}$ must be bounded, whence $a = 0$; if, in addition, $s = 1$, then r_θ^α must be in $L^1(\mathfrak{X})$, which is excluded by Corollary 3.3. On the other hand, if $p < r$, then r_θ^α must lie in $L^s(\mathfrak{X})$, which implies that $0 \leq a < 2/p_\theta$, again by Corollary 3.3.

Conversely, we shall show that $r_\theta^\alpha \in Cv_p^r(\mathfrak{X})$ if one of (ii), (iii) and (iv) holds. Suppose first that (ii) holds, and take t in $(p_\theta, 2)$ such that $t < s$. Since $\widetilde{r_\theta^\alpha}$ is smooth on $\overline{\mathbb{S}}_t$, $r_\theta^\alpha \in L^{t,1}(\mathfrak{X})$; by a theorem of Pytlik [Py], $r_\theta^\alpha \in Cv_t^t(\mathfrak{X})$. By duality and interpolation, $r_\theta^\alpha \in Cv_t^t(\mathfrak{X})$ for all t in (p_θ, p_θ'), and for a suitable choice of t, $p < t < r$, so $Cv_t^t(\mathfrak{X}) \subseteq Cv_p^r(\mathfrak{X})$, as required. Next, if (iii) holds, the argument of Theorem 4.1 of [CGM] is applicable. Finally, if (iv) holds, then $r_\theta^\alpha \in L^{p_\theta}(\mathfrak{X})$, and then $r_\theta^\alpha \in Cv_p^r(\mathfrak{X})$ by Theorem 1.4. ◻

Corollary 3.5. *Let $1 \leq p < 2$. Then the inclusions*

$$L^{p,1}(\mathfrak{X})^\sharp \subset Cv_p^p(\mathfrak{X}) \subset L^p(\mathfrak{X})^\sharp,$$

and

$$Cv_p^{p'}(\mathfrak{X}) \subset L^{p'}(\mathfrak{X})^\sharp$$

are strict.

Proof. We may choose θ such that $p = p_\theta$. To prove the first part of the statement, note that by Theorem 3.4 r_θ^{iu} is in $Cv_{p_\theta}^{p_\theta}(\mathfrak{X})$ for every real u. By Corollary 3.3, r_θ^{iu} is in the Lorentz space $L^{p_\theta,r}(\mathfrak{X})$ when $r > 1$, but it is not in $L^{p_\theta,1}(\mathfrak{X})$, and so $L^{p_\theta,1}(\mathfrak{X})$ is strictly included in $Cv_{p_\theta}^{p_\theta}(\mathfrak{X})$.

Again by Corollary 3.3, if a is in $(0, 2/r')$, then the resolvent kernel r_θ^α is in the Lorentz space $L^{p_\theta,r}(\mathfrak{X})$, but it is not in $Cv_{p_\theta}^{p_\theta}(\mathfrak{X})$, because its spherical Fourier transform is not bounded on the strip $\mathbb{S}_{p_\theta}$. This shows that $Cv_{p_\theta}^{p_\theta}(\mathfrak{X})$ does not contain the Lorentz space $L^{p_\theta,r}(\mathfrak{X})^\sharp$ for any r in $(1, \infty)$; in particular it is properly contained in $L^{p_\theta}(\mathfrak{X})^\sharp$.

We prove the second part of the statement by contradiction. If it were true that $L^{p_\theta'}(\mathfrak{X})^\sharp \subseteq Cv_{p_\theta}^{p_\theta'}(\mathfrak{X})$, then it would follow that

$$\begin{aligned} |\langle f * k, g\rangle| &\leq \|f * k\|_{p_\theta'} \|g\|_{p_\theta} \leq \|f\|_{p_\theta} \|g\|_{p_\theta} \|k\|_{Cv_{p_\theta}^{p_\theta'}(\mathfrak{X})^\sharp} \\ &\leq C \|f\|_{p_\theta} \|g\|_{p_\theta} \|k\|_{p_\theta'} \end{aligned}$$

for every f and g in $L^{p_\theta}(\mathfrak{X})^\sharp$ and k in $L^{p_\theta'}(\mathfrak{X})^\sharp$. Since f is radial, $\langle f*k, g\rangle = \langle k, f*g\rangle$, so we would have

$$\|f * g\|_{p_\theta} = \sup_{\|k\|_{p_\theta'}=1} |\langle k, f * g\rangle| \leq C \|f\|_{p_\theta} \|g\|_{p_\theta}.$$

Thus the bilinear map $(f, g) \mapsto f * g$ would be continuous from $L^{p_\theta}(\mathfrak{X})^\sharp \times L^{p_\theta}(\mathfrak{X})^\sharp$ to $L^{p_\theta}(\mathfrak{X})^\sharp$. We show that this is false. If a is in the interval $(1/p_\theta', 2/p_\theta')$, then the resolvent kernel r_θ^α is in $L^{p_\theta}(\mathfrak{X})^\sharp$ by Corollary 3.3, while $r_\theta^\alpha * r_\theta^\alpha$ is not, for otherwise its spherical Fourier transform would satisfy the estimate

$$\left[\int_0^{\tau_r} |(r_\theta^\alpha * r_\theta^\alpha)^\sim(s+i\delta(p_\theta))|^{p_\theta'}\, ds\right]^{1/p_\theta'} \leq C\, \|r_\theta^\alpha * r_\theta^\alpha\|_{p_\theta}$$

by Theorem 1.2. A simple computation shows that the integral on the left is divergent, and the proof is complete. □

4. On the Littlewood–Paley–Stein g–function

In this section we present some results on the L^p boundedness of the Littlewood–Paley–Stein g–function associated to the semigroup $(\mathcal{H}_{t,\theta})_{t>0}$. Again, α denotes a complex parameter and a its real part.

Suppose that M is a σ–finite measure space, and that $\mathcal{A}$ is a positive self-adjoint operator on $L^2(M)$; then $\mathcal{A}$ generates a symmetric contraction semigroup $(\mathcal{T}_t)$ on $L^2(M)$. Assume that $\mathcal{T}_t$ is subpositive, and that, for some r in $[1,2)$,

$$\|\mathcal{T}_t f\|_p \leq \|f\|_p \quad \forall f \in L^p(M)\cap L^2(M),$$

whenever p is in $[r,r']$. For α in $\mathbb{Z}^+$, we consider the nonlinear functional g_α, defined by

$$g_\alpha(f) = (2\pi)^{-1/2}\left[\int_0^\infty \left| t^\alpha \left(\frac{\partial}{\partial t}\right)^\alpha e^{-t\mathcal{A}} f\right|^2 dt\right]^{1/2} \qquad \forall f \in L^2(M),$$

which was introduced by Stein [S]. It may be shown (see [Co2]) that

$$g_\alpha(f) = (2\pi)^{-1/2}\left[\int_{\mathbb{R}} \left|\Gamma(\alpha - iu)\,\mathcal{A}^{iu} f\right|^2 du\right]^{1/2} \qquad \forall f \in L^2(M). \tag{4.1}$$

Since Γ is a meromorphic function with simple poles in $-\mathbb{N}$, (4.1) makes sense for all complex α in the "admissible set" $\{\alpha\in\mathbb{C} : -a \notin \mathbb{N}\}$. Thus we may define g_α by (4.1) for all α in the admissible set. It is known that if $r=1$, i.e., if the semigroup is contractive on $L^p(M)$ for all p in $[1,\infty]$, then g_α is bounded on $L^p(M)$ for all p in $(1,\infty)$ and α in $\mathbb{R}^+$.

We summarise some of the properties of the functional g_α in the next theorem.

Theorem 4.1. *Suppose that α is admissible. Then the functional g_α is bounded on $L^p(M)$ if one of the following holds:*
(i) $p = r'$ *and* $a < -1/2$;
(ii) $p = r$ *and* $a < -1$;
(iii) $r < p < r'$.

Proof. Part (iii) was proved in [Co2] and [CDMY]; (i) and (ii) are from [M]. □

Suppose that α is admissible and $a > 0$. Theorem 4.1 does not give any information about the boundedness of the functional g_α on $L^r(M)$ or $L^{r'}(M)$. It was shown in [M] that on noncompact symmetric spaces the g_α–functional associated to the θ–heat semigroup, which is positivity preserving and contractive on L^p for all p in the interval $[p_\theta, p_\theta']$, is unbounded on L^{p_θ} and on $L^{p_\theta'}$ when $a > 0$. We prove here a similar result for the g_α–functional associated to the θ–heat operator on $\mathfrak{X}$.

For every θ in $(0,1)$, we consider the modified semigroup $\mathcal{H}_{t,\theta}$, and for any admissible α, we define $g_{\alpha,\theta}$ by the rule

$$g_{\alpha,\theta}(f) = (2\pi)^{-1/2}\left[\int_{-\infty}^{\infty}\left|\Gamma(\alpha-iu)\,\mathcal{L}_\theta^{\,iu} f\right|^2 du\right]^{1/2} \qquad \forall f\in L^2(\mathfrak{X}).$$

Our aim is to prove some endpoint results for $g_{\alpha,\theta}$.

Theorem 4.2. *Suppose that θ is in $(0,1)$ and α is admissible. Then the following hold:*
(i) *$g_{\alpha,\theta}$ is bounded on $L^{p_\theta}(\mathfrak{X})$ if $a<-1$ and on $L^{p_\theta{}'}(\mathfrak{X})$ if $a<-1/2$;*
(ii) *$g_{\alpha,\theta}$ is bounded on $L^p(\mathfrak{X})$ if $p\in(p_\theta, p_\theta{}')$;*
(iii) *$g_{\alpha,\theta}$ is not bounded on $L^{p_\theta}(\mathfrak{X})$ or on $L^{p_\theta{}'}(\mathfrak{X})$ if $a>0$.*

Proof. The statements (i) and (ii) are immediate consequences of Theorem 4.1.

We now suppose that $a>0$, and we show that $g_{\alpha,\theta}$ is not bounded on $L^{p_\theta{}'}(\mathfrak{X})$. Observe that $\mathcal{L}_\theta^{\,iu}\phi_z = (\gamma(i\delta(p_\theta))-\gamma(z))^{-iu}\phi_z$, so

$$\begin{aligned}
|||g_{\alpha,\theta}|||_{p_\theta'}^2 &\geq \|g_{\alpha,\theta}(\phi_z)\|_{p_\theta'}^2/\|\phi_z\|_{p_\theta}^2\\
&= (2\pi)^{-1}\int_{-\infty}^{\infty}|\Gamma(\alpha-iu)|^2\left|(\gamma(i\delta(p_\theta))-\gamma(z))^{-iu}\right|^2 du,
\end{aligned}$$

for all z in $\mathbb{S}_{p_\theta}$. Denoting the integrand by $A(u,z)$, it follows that

$$|||g_{\alpha,\theta}|||_{p_\theta'}^2 \geq \sup_{z\in\mathbb{S}_{p_\theta}}(2\pi)^{-1}\int_{-\infty}^{\infty}A(u,z)\,du.$$

By using the asymptotics of the Γ–function, viz,

$$|\Gamma(\alpha-iu)|\asymp\sqrt{2\pi}\,e^{-\pi|u|/2}|u|^{\alpha-1/2}\quad \text{as } u\to\pm\infty \text{ in } \mathbb{R},$$

it is immediate to check that

$$\begin{aligned}
A(u,z) &= |\Gamma(\alpha-iu)|^2\exp\big[2u\arg[\gamma(i\delta(p_\theta))-\gamma(z)]\big]\\
&\sim \exp(-\pi|u|)\,|u|^{2a-1}\exp\big[2u\arg[\gamma(i\delta(p_\theta))-\gamma(z)]\big]\\
&= |u|^{2a-1}\exp\big[-\pi|u|+2u\arg[\gamma(i\delta(p_\theta))-\gamma(z)]\big]
\end{aligned}$$

for all u in $\mathbb{R}$ and z in $\mathbb{S}_{p_\theta}$. Denote the last term on the right hand side by $B(u,z)$.

We claim that, if z approaches the point $i\delta(p_\theta)$ from inside the strip $\mathbb{S}_{p_\theta}$ along a suitable path, then $\arg[\gamma(ip_\theta)-\gamma(z)]$ converges to $\pi/2$. Indeed, for small real y,

$$\begin{aligned}
&\gamma(i\delta(p_\theta))-\gamma(y-iy^2+i\delta(p_\theta))\\
&\quad= 2\gamma(0)\big(\cos(i\delta(p_\theta))-\cos(y-iy^2)\cos(i\delta(p_\theta))+\sin(y-iy^2)\sin(i\delta(p_\theta))\big)\\
&\quad= 2\gamma(0)\big(\cosh(\delta(p_\theta))\,y^2/2+i\sinh(\delta(p_\theta))\,(y-iy^2)\big)+O(y^3).
\end{aligned}$$

Clearly, as $y\to\pm0$, $\arg\big[\gamma(i\delta(p_\theta))-\gamma(y-iy^2+i\delta(p_\theta))\big]\to\pm\pi/2$.

From our claim it follows that

$$\begin{aligned}
\|g_{\alpha,\theta}\|^2_{p_\theta'} &\geq \sup_{z\in\mathbb{S}_{p_\theta}} (2\pi)^{-1} \int_{-\infty}^{\infty} A(u,z)\,du \\
&\geq C \sup_{z\in\mathbb{S}_{p_\theta}} \int_0^\infty B(u,z)\,du \\
&= C \sup_{z\in\mathbb{S}_{p_\theta}} \int_0^\infty u^{2a-1} \exp\left[-\pi u + 2u \arg\left[\gamma(i\delta(p_\theta)) - \gamma(z)\right]\right]\,du \\
&\geq C \sup_{\epsilon>0} \int_0^\infty u^{2a-1} \exp\left[-\pi u + 2u(\pi/2-\epsilon)\right]\,du \\
&= \infty,
\end{aligned}$$

as required.

The proof of the rest of (iii) is that of Theorem 2.2.(ii) of [M], *mutatis mutandis*. The proof of the theorem is now complete. □

References

[CF] I. Chavel and E. A. Feldman, *Modified isoperimetric constants and large time heat diffusion in Riemannian manifolds*, Duke Math. J. **64** (1991), 473–499. MR **92m:**58132

[CS] J. L. Clerc and E. M. Stein, *L^p multipliers for noncompact symmetric spaces*, Proc. Nat. Acad. Sci. U. S. A. **71** (1974), 3911–3912. MR **51:**3803

[Cn] T. Coulhon, *Noyau de la chaleur et discrétisation d'une variété riemannienne*, Israel J. Math. **80** (1992), 289–300. MR **93k:**58213

[Co1] M. G. Cowling, *Harmonic analysis on semigroups*, Ann. Math. **117** (1983), 267–283. MR **84h:**43004

[Co2] ———, *Square functions in Banach spaces*, Miniconference in Linear Analysis and Function Spaces (A. McIntosh and A. Pryde, eds.), Proc. Centre Math. Anal. Austral. Nat. Univ., vol. 9, Austral. Nat. Univ., Canberra, 1985, pp. 177–184. MR **87i:**43003

[CDMY] M. G. Cowling, I. Doust, A. McIntosh and A. Yagi, *Banach space operators with a bounded H_∞ functional calculus*, J. Austral. Math. Soc. Ser. A **60** (1996), 51–89. MR **97d:**47023

[CGGM] M. G. Cowling, G. I. Gaudry, S. Giulini and G. Mauceri, *Weak type (1,1) estimates for heat kernel maximal functions on Lie groups*, Trans. Amer. Math. Soc. **323** (1991), 637–649. MR **91e:**43011

[CGM] M. G. Cowling, S. Giulini, and S. Meda, *$L^p - L^q$ estimates for functions of the Laplace operator on noncompact symmetric spaces. I*, Duke Math. J. **72** (1993), 109–150. MR **95b:**22031

[CMS] M. G. Cowling, S. Meda and A. G. Setti, *On spherical analysis on groups of isometries of homogeneous trees*, preprint.

[E] A. Erdélyi, *Asymptotic Expansions*, Dover, New York, 1956. MR **17:**1202c

[FTN] A. Figà-Talamanca and C. Nebbia, *Harmonic Analysis and Representation Theory for Groups Acting on Homogeneous Trees*, London Math. Soc. Lecture Notes Series, 162, Cambridge Univ. Press, Cambridge, 1991. MR **93f:**22004

[FTP] A. Figà-Talamanca and M. Picardello, *Harmonic Analysis on Free Groups*, Dekker, New York, 1983. MR **85j:**43001

[H] C. S. Herz, *Sur le phénomène de Kunze–Stein*, C. R. Acad. Sci. Paris (Série A) **271** (1970), 491–493. MR **43:**6741

[Hr] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. **104** (1960), 93–140. MR **22:**12389

[K1] M. Kanai, *Rough isometries and combinatorial approximations of geometries of non-compact Riemannian manifolds*, J. Math. Soc. Japan **37** (1985), 391–413. MR **87d:**53082

[K2] ———, *Rough isometries and the parabolicity of Riemannian manifolds*, J. Math. Soc. Japan **38** (1986), 227–238. MR **87e:**53066

[L] N. N. Lebedev, *Special Functions and Their Applications*, Dover, New York, 1972. MR **50:**2568

[M] S. Meda, *On the Littlewood–Paley–Stein g–function*, Trans. Amer. Math. Soc. **347** (1995), 2201–2212. MR **95m:**47072

[N] C. Nebbia, *Groups of isometries of a tree and the Kunze–Stein phenomenon*, Pacific J. Math. **133** (1988), 141–149. MR **89h:**43005

[Ol] F. W. Olver, *Asymptotics and Special Functions*, Academic Press, New York, 1974. MR **55:**8655

[P] M. M. Pang, *Heat kernels of graphs*, J. London Math. Soc. **47** (1993), 50–64. MR **93k:**60177

[Py] T. Pytlik, *Radial convolutors on free groups*, Studia Math. **78** (1984), 178–183. MR **86j:**43001

[S] E. M. Stein, *Topics in Harmonic Analysis Related to the Littlewood–Paley Theory*, Ann. of Math. Studies, vol. 63, Princeton, N. J, 1970. MR **40:**6176

[V1] N. Th. Varopoulos, *Brownian motion and random walks on manifolds*, Ann. Inst. Fourier (Grenoble) **34** (1984), fasc. 2, 243–269. MR **85m:**58186

[V2] N. Th. Varopoulos, *Long range estimates for Markov chains*, Bull. Sci. Math. (2) **109** (1985), 225–252. MR **87j:**60100

School of Mathematics, University of New South Wales, Sydney NSW 2052, Australia

E-mail address: `m.cowling@unsw.edu.au`

Dipartimento di Matematica, Politecnico di Milano, via Bonardi 9, 20133 Milano, Italy

Current address: Department of Statistics, University of Milan-Bicocca, Edificio U7 II piano, v. Le Sarca 202, I-20100 Milan, Italy

E-mail address: `stemed@ipmma1.mate.polimi.it`

Dipartimento di Matematica, Università di Milano, via Saldini 50, 20133 Milano, Italy

Current address: Faculty of Science, Universitá dell'Insubria-1 Como via Lucini 3, I-22100 Como, Italy

E-mail address: `setti@fis.unico.it`