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THE SINGLE-VALUED EXTENSION PROPERTY FOR BILATERAL OPERATOR WEIGHTED SHIFTS

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ABSTRACT. In this paper, we give necessary and sufficient conditions for a bilateral operator weighted shift to enjoy the single-valued extension property.

1. INTRODUCTION

Let $\mathcal{X}$ be a complex Banach space, and let $\mathcal{L}(\mathcal{X})$ be the algebra of bounded linear operators on $\mathcal{X}$. For an operator $T \in \mathcal{L}(\mathcal{X})$, we denote, as usual, by $\sigma(T)$ and $\sigma_p(T)$ the spectrum and the point spectrum of T, respectively. An operator $T \in \mathcal{L}(\mathcal{X})$ is said to have the *single-valued extension property* provided that for every open subset U of $\mathbb{C}$ the only analytic solution $\phi : U \to \mathcal{X}$ of the equation

$$(T - \lambda)\phi(\lambda) = 0 \quad (\lambda \in U)$$

is the identically zero function.

Throughout this paper, $\mathcal{H}$ will denote a complex Hilbert space and $(A_n)_{n\in\mathbb{Z}}$ is a two–sided sequence of uniformly bounded invertible operators of $\mathcal{L}(\mathcal{H})$. For $1 \leq p < \infty$, let

$$l^p(\mathcal{H}, \mathbb{Z}) := \left\{ x = (x_n)_{n\in\mathbb{Z}} \subset \mathcal{H} : \|x\|_p = \left(\sum_{n\in\mathbb{Z}} \|x_n\|^p \right)^{\frac{1}{p}} < +\infty \right\}.$$

It is a Banach space under the norm $\|.\|_p$. For $p = +\infty$, let

$$l^\infty(\mathcal{H}, \mathbb{Z}) := \left\{ x = (x_n)_{n\in\mathbb{Z}} \subset \mathcal{H} : \|x\|_\infty = \sup_{n\in\mathbb{Z}} \|x_n\| < +\infty \right\}.$$

This space is also a Banach space under the norm $\|.\|_\infty$. A linear operator S on $l^p(\mathcal{H}, \mathbb{Z})$, $(1 \leq p \leq \infty)$, is called a *bilateral operator weighted shift with the weight sequence* $(A_n)_{n\in\mathbb{Z}}$ if

$$S(..., x_{-2}, x_{-1}, [x_0], x_1, x_2, ...) = (..., A_{-2}x_{-2}, [A_{-1}x_{-1}], A_0x_0, A_1x_1, ...),$$

where for an element $x = (..., x_{-2}, x_{-1}, [x_0], x_1, x_2, ...) \in l^p(\mathcal{H}, \mathbb{Z})$, $[x_0]$ denotes the central ($0th$) term of x.

In [8], Li Jue Xian proved that if $\dim \mathcal{H} = m < +\infty$, then a bilateral operator weighted shift S has the single-valued extension property if and only if the cardinal number of $\sigma_p(S) \cap \mathbb{R}^+$ is not greater than m, where $\mathbb{R}^+$ is the set of positive

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real numbers. Here, we completely settle the question of which bilateral operator weighted shift S on $l^p(\mathcal{H},\mathbb{Z})$ has the single-valued extension property even when $\mathcal{H}$ is an infinite-dimensional Hilbert space. Our proof is simple and is based on a recent result of [3] on a local version of the single-valued extension property.

In the sequel, let $(B_n)_{n\in\mathbb{Z}}$ be the two–sided sequence given by

$$B_n := \begin{cases} A_{n-1}A_{n-2}...A_1A_0 & \text{if } n>0, \\ 1 & \text{if } n=0, \\ A_n^{-1}A_{n+1}^{-1}...A_{-2}^{-1}A_{-1}^{-1} & \text{if } n<0. \end{cases}$$

For a nonzero $x\in\mathcal{H}$, we set

$$n(S,x) := \liminf_{n\to+\infty} \|B_{-n}x\|^{-\frac{1}{n}},\ p(S,x) := \limsup_{n\to+\infty} \|B_n x\|^{\frac{1}{n}},$$

$$n^*(S,x) := \liminf_{n\to+\infty} \|(B_n^{-1})^*x\|^{-\frac{1}{n}}, \quad \text{and } p^*(S,x) := \limsup_{n\to+\infty} \|(B_{-n}^{-1})^*x\|^{\frac{1}{n}}.$$

Moreover, we also introduce the following notation:

(*i*) $x^{(n)} = (\delta_{n,k}x)_{k\in\mathbb{Z}},\ (n\in\mathbb{Z})$, where $\delta_{n,k}$ is the usual Kronecker-delta symbol.
(*ii*) $E^p(x)$ denotes the closed linear span of $\{(B_nx)^{(n)} : n\in\mathbb{Z}\}$ in $l^p(\mathcal{H},\mathbb{Z})$.
(*iii*) S_x denotes the restriction of S to $E^p(x)$.

Finally, wherever it is more convenient, we will write $y = \sum_{n\in\mathbb{Z}} \oplus y_n$ instead of $y = (y_n)_{n\in\mathbb{Z}} \in l^p(\mathcal{H},\mathbb{Z})$.

2. Main results

The single-valued extension property plays an important and crucial role in local spectral theory. A local version of this property which dates back to Finch [7] has been recently investigated in the local spectral theory and Fredholm theory by many authors (see [1], [2], [3], and the references contained therein). Recall that a bounded linear operator T on a complex Banach space $\mathcal{X}$ is said to have the single-valued extension property at a point $\lambda_0\in\mathbb{C}$ if for every open disc U centered at λ_0 the only analytic function $\phi : U\to\mathcal{X}$ that satisfies the equation

$$(T-\lambda)\phi(\lambda) = 0\ (\lambda\in U)$$

is the identically zero function $\phi\equiv 0$. The set of all points on which T fails to have the single-valued extension property will be denoted by $\Re(T)$. It is an open subset of $\mathbb{C}$ contained in $\sigma_p(T)$, and it is empty precisely when T has the single-valued extension property. The local resolvent set, $\rho_T(x)$, of an operator $T\in\mathcal{L}(\mathcal{X})$ at a point $x\in\mathcal{X}$ is the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $\phi : U\to\mathcal{X}$ that satisfies

$$(T-\lambda)\phi(\lambda) = x\ (\lambda\in U).$$

The local spectrum of T at x is defined by

$$\sigma_T(x) := \mathbb{C}\backslash\rho_T(x).$$

It is clearly a closed subset of $\sigma(T)$. In [3], P. Aiena and O. Monsalve established a useful characterization of the operators that do not have the single-valued extension property at a given point $\lambda_0\in\mathbb{C}$. They showed that an operator $T\in\mathcal{L}(\mathcal{X})$ does not have the single-valued extension property at a point $\lambda_0\in\mathbb{C}$ precisely when there exists a nonzero $x \in \ker(T-\lambda_0)$ for which $\sigma_T(x)=\emptyset$. For more on local spectral theory, the reader may consult [6] and [9].

We are now able to state and prove the main result of this paper.

Theorem 2.1. *The following properties hold.*

(*i*) $\sigma_p(S)=\bigcup_{x\neq 0}\sigma_p(S_x)$.

(*ii*) $\Re(S)=\bigcup_{x\neq 0}\Re(S_x)=\bigcup_{x\neq 0}\{\lambda\in\mathbb{C}: p(S,x)<|\lambda|<n(S,x)\}$.

Moreover, the following statements are equivalent.

(*a*) *S has the single-valued extension property.*

(*b*) *Each S_x has the single-valued extension property.*

(*c*) *$n(S,x)\le p(S,x)$ for all nonzero x in $\mathcal{H}$.*

Proof. Let x be a nonzero element of $\mathcal{H}$. It is clear that S_x is similar to the bilateral scalar weighted shift on $l^p(\mathbb{Z})$ with the weight sequence $\big(\frac{\|B_{n+1}x\|}{\|B_n x\|}\big)_{n\in\mathbb{Z}}$. Therefore,

$$\{\lambda\in\mathbb{C}: p(S,x)<|\lambda|<n(S,x)\}\subset\sigma_p(S_x)\subset\{\lambda\in\mathbb{C}: p(S,x)\le|\lambda|\le n(S,x)\}$$

(see [9] and [11]).

(*i*) Now, suppose that $\lambda\in\mathbb{C}$ is an eigenvalue for S and $y:=\sum_{n\in\mathbb{Z}}\oplus y_n\in l^p(\mathcal{H},\mathbb{Z})$ is a corresponding eigenvector. We obviously have $\lambda\neq 0$ and

$$A_n y_n=\lambda y_{n+1} \text{ for all } n\in\mathbb{Z}.$$

Therefore,

$$y_n=\frac{1}{\lambda^n}B_n y_0 \text{ for all } n\in\mathbb{Z};$$

this shows that $y\in E^p(y_0)$. Hence, $\lambda\in\sigma_p(S_{y_0})$ and therefore,

$$\sigma_p(S)\subset\bigcup_{x\neq 0}\sigma_p(S_x).$$

The reverse inclusion is trivial since S coincides with S_x when it is restricted to each $E^p(x)$.

(*ii*) First, let us prove that for every nonzero $x\in\mathcal{H}$, we have

$$\sigma_S(y)=\sigma_{S_x}(y) \quad \text{for all } y\in E^p(x). \tag{2.1}$$

Let x be a nonzero element of $\mathcal{H}$, and let $y=\sum_{n\in\mathbb{Z}}\oplus y_n\in E^p(x)$. Since S coincides with S_x when restricted to $E^p(x)$,

$$\sigma_S(y)\subset\sigma_{S_x}(y).$$

Conversely, let $\phi=\sum_{n\in\mathbb{Z}}\oplus\phi_n$ be a $l^p(\mathcal{H},\mathbb{Z})$-valued analytic function on some open set $U\subset\rho_S(y)$ such that

$$(S-\lambda)\phi(\lambda)=y \ (\lambda\in U).$$

For every $n\in\mathbb{Z}$, we have

$$A_{n-1}\phi_{n-1}(\lambda)-\lambda\phi_n(\lambda)=y_n \ (\lambda\in U). \tag{2.2}$$

For every $n\in\mathbb{Z}$, let

$$F_n(\lambda):=P_n\phi_n(\lambda) \ (\lambda\in U),$$

where P_n is the canonical projection from $\mathcal{H}$ onto $M_n := \operatorname{span}\{B_n x\}$. We clearly have $A_n M_n = M_{n+1}$ for all $n \in \mathbb{Z}$; therefore, it follows from (2.2) that, for every $n \in \mathbb{Z}$,

$$A_{n-1}F_{n-1}(\lambda) - \lambda F_n(\lambda) = y_n \ (\lambda \in U). \tag{2.3}$$

Since $\|F_n(.)\| \leq \|\phi_n(.)\|$ for all $n \in \mathbb{Z}$, the function

$$F(\lambda) := \sum_{n\in\mathbb{Z}} \oplus F_n(\lambda) \ (\lambda \in U)$$

is well defined and is, in fact, an $E^p(x)$-valued analytic function on U. Moreover, in view of (2.3), this function satisfies the equation

$$(S-\lambda)F(\lambda) = (S_x - \lambda)F(\lambda) = y \ (\lambda \in U).$$

This shows that $U \subset \rho_{S_x}(y)$; therefore, $\sigma_{S_x}(y) \subset \sigma_S(y)$. Thus (2.1) is established.

Next, we let x be a nonzero element of $\mathcal{H}$ and note that if $\sigma_p(S_x) \neq \emptyset$, we have

$$(S-\lambda)k_x(\lambda) = 0 \ (\lambda \in \sigma_p(S_x)),$$

where

$$k_x(\lambda) = \sum_{n\in\mathbb{Z}} \oplus \frac{1}{\lambda^n} B_n x.$$

Moreover, we have

$$\begin{aligned} \Re(S_x) &= \{\lambda \in \mathbb{C} : r_3^+(S,x) < |\lambda| < r_2^-(S,x)\} \\ &= \{\lambda \in \sigma_p(S_x) : \sigma_{S_x}(k_x(\lambda)) = \emptyset\} \\ &= \{\lambda \in \sigma_p(S_x) : \sigma_S(k_x(\lambda)) = \emptyset\}. \end{aligned}$$

Indeed, since all eigenvalues of S_x are simple (see [9, theorem 9]), the equalities

$$\begin{aligned} \Re(S_x) &= \{\lambda \in \sigma_p(S_x) : \sigma_{S_x}(k_x(\lambda)) = \emptyset\} \\ &= \{\lambda \in \sigma_p(S_x) : \sigma_S(k_x(\lambda)) = \emptyset\} \end{aligned}$$

hold by applying [3, theorem 1.9] and (2.1). Now, let us show that

$$\Re(S_x) = \{\lambda \in \mathbb{C} : p(S,x) < |\lambda| < n(S,x)\}.$$

Since $\sigma_p(S_x) \subset \{\lambda \in \mathbb{C} : p(S,x) \leq |\lambda| \leq n(S,x)\}$, we have

$$\Re(S_x) \subset \{\lambda \in \mathbb{C} : p(S,x) < |\lambda| < n(S,x)\}. \tag{2.4}$$

Conversely, suppose that $p(S,x) < n(S,x)$ and let

$$O := \{\lambda \in \mathbb{C} : p(S,x) < |\lambda| < n(S,x)\}.$$

We have $O \subset \sigma_p(S_x)$, and k_x is clearly a nonzero identically analytic function on O and satisfies the equation

$$(S_x - \lambda)k_x(\lambda) = 0 \ (\lambda \in O).$$

This establishes the reverse inclusion of (2.4), as desired.

Finally, we shall deduce that

$$\Re(S) = \bigcup_{x\neq 0} \Re(S_x) = \bigcup_{x\neq 0} \{\lambda \in \mathbb{C} : p(S,x) < |\lambda| < n(S,x)\}.$$

Indeed, we trivially see that $\bigcup_{x\neq 0} \Re(S_x) \subset \Re(S)$. Conversely, let $\lambda_0 \in \Re(S)$. By [3, theorem 1.9] there exists a nonzero $y = \sum_{n\in\mathbb{Z}} \oplus y_n \in \ker(S - \lambda_0)$ such that

$$\sigma_S(y) = \emptyset.$$

As in the proof of (i), we see that $y_0 \neq 0$ and $y = k_{y_0}(\lambda_0) \in \ker(S_{y_0} - \lambda_0)$. By (2.1), we have

$$\sigma_S(y) = \sigma_S(k_{y_0}(\lambda_0)) = \sigma_{S_{y_0}}(k_{y_0}(\lambda_0)) = \emptyset.$$

This shows that $\lambda_0 \in \Re(S_{y_0})$ and therefore

$$\Re(S) \subset \bigcup_{x\neq 0} \Re(S_x).$$

This completes the proof. $\square$

Assume that $1 \leq p < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. The dual of $l^p(\mathcal{H}, \mathbb{Z})$ can be identified with $l^q(\mathcal{H}, \mathbb{Z})$, with the duality being implemented by the formula

$$\langle y, z\rangle = \sum_{n\in\mathbb{Z}} \langle y_n, z_n\rangle \ (y = \sum_{n\in\mathbb{Z}} \oplus y_n \in l^q(\mathcal{H}, \mathbb{Z}), \text{ and } z = \sum_{n\in\mathbb{Z}} \oplus z_n \in l^p(\mathcal{H}, \mathbb{Z})).$$

The adjoint, S^*, of S is given by

$$S^* y = (..., A^*_{-2} y_{-1}, A^*_{-1} y_0, [A^*_0 y_1], A^*_1 y_2, A^*_2 y_3, ...) \ (y = \sum_{n\in\mathbb{Z}} \oplus y_n \in l^q(\mathcal{H}, \mathbb{Z})).$$

It is similar to the bilateral operator weighted shift, $\tilde{S}$, on $l^q(\mathcal{H}, \mathbb{Z})$ with the weight sequence $(\tilde{A}_n)_{n\in\mathbb{Z}}$, where $\tilde{A}_n = A^*_{-n-1}$ for all $n \in \mathbb{Z}$.

Corollary 2.2. *The following properties hold.*

(i) $\sigma_p(S^*) = \bigcup_{x\neq 0} \sigma_p(\tilde{S}_x)$.

(ii) $\Re(S^*) = \bigcup_{x\neq 0} \Re(\tilde{S}_x) = \bigcup_{x\neq 0} \{\lambda \in \mathbb{C} : p^*(S, x) < |\lambda| < n^*(S, x)\}$.

Moreover, the following statements are equivalent.

(a) *S^* has the single-valued extension property.*

(b) *Each $\tilde{S}_x$ has the single-valued extension property.*

(c) *$n^*(S, x) \leq p^*(S, x)$ for all nonzero x in $\mathcal{H}$.*

Unlike in the scalar case, both S and S^* need not have the single-valued extension property.

Example 2.3. Assume that $(e_n)_{n\geq 0}$ is an orthonormal basis of $\mathcal{H}$, and let T and R be the diagonal operators with the diagonal sequences $(2, 4, 1, 1, 1, ...)$ and $(4, 2, 1, 1, 1, ...)$, respectively. We set

$$A_n := \begin{cases} T & \text{if } n \geq 0, \\ R & \text{if } n < 0. \end{cases}$$

We have

$$p(S, e_0) = p^*(S, e_1) = 2 \text{ and } n(S, e_0) = n^*(S, e_1) = 4.$$

In view of theorem 2.1 and corollary 2.2, we see that

$$\{\lambda \in \mathbb{C} : 2 < |\lambda| < 4\} \subset \sigma_p(S) \cap \sigma_p(S^*),$$

and neither S nor S^* has the single-valued extension property.

In the sequel, for a nonzero $y = \sum_{n\in\mathbb{Z}} \oplus y_n \in l^p(\mathcal{H},\mathbb{Z})$, let

$$R^-(S,y) := \limsup_{n\to+\infty} \|B_{-n}^{-1}y_{-n}\|^{\frac{1}{n}} \text{ and } R^+(S,y) := 1/\limsup_{n\to+\infty} \|B_n^{-1}y_n\|^{\frac{1}{n}}.$$

Remark 2.4. Let x be a nonzero element of $\mathcal{H}$. In view of (2.1) and [5], we have the following results.

(i) For every nonzero finitely supported element y of $E^p(x)$,

$$\sigma_S(y) = \{\lambda \in \mathbb{C} : n(S,x) \le |\lambda| \le p(S,x)\}.$$

(ii) Assume that $n(S,x) \le p(S,x)$, and let y be a nonzero element of $E^p(x)$. If $R^-(S,y) < n(S,x)$ and $p(S,x) < R^+(S,y)$, then

$$\sigma_S(y) = \{\lambda \in \mathbb{C} : n(S,x) \le |\lambda| \le p(S,x)\}.$$

Otherwise,

$$\{\lambda \in \mathbb{C} : \max\big(R^-(S,y), n(S,x)\big) < |\lambda| < \min\big(R^+(S,y), p(S,x)\big)\} \subset \sigma_S(y).$$

(iii) Assume that $n(S,x) = 0$, and let y be a nonzero negatively finitely supported element of $E^p(x)$. If $p(S,x) < R^+(S,y)$, then

$$\sigma_S(y) = \{\lambda \in \mathbb{C} : |\lambda| \le p(S,x)\}.$$

Otherwise,

$$\{\lambda \in \mathbb{C} : |\lambda| \le R^+(S,y)\} \subset \sigma_S(y).$$

In [4], Ben-Artzi and Gohberg introduced the concepts of Bohl exponent and canonical splitting projection to describe the spectrum and the essential spectrum of operator weighted shifts of *finite multiplicity* (see also [10]). However, in the general setting of bilateral operator weighted shifts of *infinite multiplicity*, the complete description of the spectrum and its parts is not yet settled.

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