

Convective-Dynamical Instability in Radiation-Supported Accretion Disks

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ABSTRACT

We study radiation-hydrodynamical normal modes of radiation-supported accretion disks in the WKB limit. It has long been known that in the large optical depth limit the standard equilibrium is unstable to convection. We study how the growth rate depends on location within the disk, optical depth, disk rotation, and the way in which the local dissipation rate depends on density and pressure. The greatest growth rates are found near the disk surface. Rotation stabilizes vertical wavevectors, so that growing modes tend to have nearly-horizontal wavevectors. Over the likely range of optical depths, the linear growth rate for convective instability has only a weak dependence on disk opacity. Perturbations to the dissipation have little effect on convective mode growth rates, but can cause growth of radiation sound waves.

1. Introduction

Shakura & Sunyaev (1973) predicted that the inner portions of accretion disks that extend into relativistically-deep gravitational potentials should be radiation pressure-dominated when the accretion rate is greater than a modest fraction of the Eddington rate. In that regime, they found that disks could achieve hydrostatic balance in the vertical direction if the local dissipation rate were proportional to the local mass density. Given that assumption, upward radiation flux could support the disk matter against gravity if the density were essentially constant as a function of height (falling sharply to zero at the top surface) and the radiation pressure fell gradually from the disk midplane to the surface.

Soon after this equilibrium was discovered, it was found to suffer from several sorts of instabilities. Lightman & Eardley (1974) pointed out that if the viscous stress is proportional to the total pressure (in this case, dominated by radiation), perturbations with radial wavelengths long compared to the vertical thickness h , but short compared to a radius r , grow on the (comparatively long) viscous inflow timescale. Shakura & Sunyaev (1976) then observed that in these conditions perturbations in the same range of wavelengths would also grow on the (shorter) thermal timescale. Bisnovatyi-Kogan & Blinnikov (1977) noticed that if the radiation is locked to the gas even on short lengthscales (i.e., if, for the purpose of dynamics, the optical depth is treated as effectively infinite), such disks should be convectively unstable, for the specific entropy decreases outward; the linear growth rate for convective “bubbles” was worked out by Lominadze & Chagelishvili (1984). More recently, Gammie (1998) has demonstrated that a magnetic field in radiation-supported disks can catalyze a short-wavelength ($kh \gg 1$) overstable wave mode. In view of these instabilities, it has long been a puzzle just what sort of equilibrium would actually be found in Nature when the accretion rate is high enough that radiation pressure-domination might be expected (see, e.g. Shapiro, Lightman & Eardley 1976; Liang 1977; Coroniti 1981;

Svensson & Zdziarski 1994; Szuszkiewicz & Miller 1997; Krolik 1998).

In this paper, we take a closer look at the nature of the short wavelength modes in radiation-supported disks without magnetic fields. Our goal (motivated by a companion work on radiation-hydrodynamics simulations of such disks: Agol & Krolik 2000b) is to examine more closely which modes can be expected to grow most quickly, what happens when finite optical depth permits some photon diffusion, and what role, if any, is played by associated perturbations in the local dissipation rate.

2. Problem Definition

We begin by writing down the equations of non-relativistic radiation hydrodynamics so that we may first describe the equilibrium in this language, and then discuss linear perturbations to this equilibrium. Because we are interested in accretion disks, it is convenient to write them in a rotating frame. The first is the usual equation of mass conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0. \quad (1)$$

Our notation is the usual one, in which ρ is the mass density and \vec{v} is the fluid velocity.

Next is the the fluid force equation:

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla p_g + \rho \vec{g} + (\kappa \rho / c) \vec{\mathcal{F}} + 2\rho \vec{v} \times \vec{\Omega} - \rho v_r (\partial \Omega / \partial \ln r) \hat{\phi} \quad (2)$$

where p_g is the gas pressure, \vec{g} is the local gravity, κ is the opacity per unit mass, $\vec{\mathcal{F}}$ is the radiation flux, and Ω is the rotation rate of the fluid. Although all the fluid quantities are defined in the rotating frame, the radiation quantities (e.g., $\vec{\mathcal{F}}$) are defined in the frame of the local fluid motion, i.e. including any departures from corotation (Mihalas & Mihalas 1984). Note that we have omitted magnetic forces.

For the two equations describing radiation energy density and momentum density,

we follow Buchler (1979), but write the Lagrangian time derivatives explicitly, *i.e.*, $D/Dt = (\partial/\partial t + \vec{v} \cdot \nabla)$. The evolution of radiation energy density E is described by:

$$\frac{\partial E}{\partial t} + \vec{v} \cdot \nabla E + \nabla \cdot \vec{\mathcal{F}} + \mathbf{p}_r : \nabla \vec{v} + \frac{E}{c^2} \nabla \cdot \vec{v} + \frac{2}{c^2} \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) \cdot \vec{\mathcal{F}} = Q. \quad (3)$$

Here \mathbf{p}_r is the radiation pressure tensor and Q is the net local emissivity. Finally, there is the equation describing the time-dependence of the radiation momentum density $(1/c^2)\mathcal{F}$:

$$\frac{1}{c} \frac{\partial \vec{\mathcal{F}}}{\partial t} + \frac{\vec{v}}{c} \cdot \nabla \vec{\mathcal{F}} + c \nabla \cdot \mathbf{p}_r + \frac{1}{c} \left(\vec{\mathcal{F}} \cdot \nabla \vec{v} + \vec{\mathcal{F}} \nabla \cdot \vec{v} \right) + \frac{1}{c} (E \mathbf{I} + \mathbf{p}_r) \cdot \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) = \vec{q}. \quad (4)$$

In this equation, \mathbf{I} is the identity matrix and \vec{q} is the net rate per unit volume at which photon momentum is created by radiation (usually negative in the fluid frame because photon momentum is lost due to opacity, while newly-created photons are usually isotropic in the fluid frame).

In the equilibrium, $\partial/\partial t = \vec{v} = 0$. To isolate the effect of radiation support, we also take the extreme limit of $p_g \ll p_r$. If we regard the rotation of the disk matter as cancelling the radial component of gravity, the only non-trivial remark to make about the equilibrium is that $\mathcal{F}_z = cg/\kappa$, where $g = g_z(z, r)$ is the local vertical component of gravitational acceleration [in a thin disk, $g_z(z, r) \simeq GMz/r^3$ for central mass M].

Now consider perturbed versions of equations (1) through (4). In order to write these perturbations in Fourier-transform form (*i.e.*, for any quantity X , the perturbation is $\delta \tilde{X} = \delta X \exp\{i(k_r r + k_z z) - i\omega t\}$), we will suppose that the wavevectors obey three conditions: that $k = (k_z^2 + k_r^2)^{1/2} \ll \kappa\rho$; that $k_z \gg 1/h$; and that $k_r \gg 1/r$. The first limit means that the diffusion approximation applies, *i.e.* $\mathbf{p}_r = p_r \mathbf{I}$, so that $E = 3p_r$. The second is the WKB approximation, as applied to variations in both the radial and vertical directions. Note that we further restrict our attention to axisymmetric perturbations. The condition $k_z h \gg 1$ also means that we can ignore any gradients in the gravity or equilibrium radiation flux. In addition, assuming $k_r \gg 1/r$ allows us to neglect the terms in vector

divergences arising from cylindrical geometry. For example, after Fourier-transforming, $\nabla \cdot \delta \vec{v}$ becomes $ik_r \delta v_r + \delta v_r / r + ik_z \delta v_z \simeq ik_r \delta v_r + ik_z \delta v_z$. We then find:

$$-i\omega \delta \rho + \rho \vec{k} \cdot \delta \vec{v} = 0 \quad (5)$$

$$-i\omega \rho \delta v_r = (\kappa \rho / c) \delta \mathcal{F}_r + 2\rho \Omega \delta v_\phi \quad (6)$$

$$-i\omega \rho \delta v_z = (\kappa \rho / c) \delta \mathcal{F}_z \quad (7)$$

$$-i\omega \rho \delta v_\phi = (\kappa \rho / c) \delta \mathcal{F}_\phi - (1/2)\rho \Omega \delta v_r \quad (8)$$

$$-3i\omega \delta p_r - 3\rho g \delta v_z + ik_z \delta \mathcal{F}_z + ik_r \delta \mathcal{F}_r + 4p_r (ik_z \delta v_z + ik_r \delta v_r) - i\frac{2\omega}{c^2} \delta v_z F_z = \delta Q \quad (9)$$

$$-i(\omega/c) \delta \mathcal{F}_r + ik_r c \delta p_r + ik_z (g/\kappa) \delta v_r - 4i(\omega/c) p_r \delta v_r = -\kappa \rho \delta \mathcal{F}_r \quad (10)$$

$$-i\left(\frac{\omega}{c}\right) \delta \mathcal{F}_z + ik_z c \delta p_r + \frac{g}{\kappa} (2ik_z \delta v_z + ik_r \delta v_r) - 4i\frac{\omega}{c} p_r \delta v_z = -cg \delta \rho - \kappa \rho \delta \mathcal{F}_z \quad (11)$$

$$-i(\omega/c) \delta \mathcal{F}_\phi + i(k_z/c) \mathcal{F}_z \delta v_\phi - 4i(\omega/c) p_r \delta v_\phi = -\kappa \rho \delta \mathcal{F}_\phi \quad (12)$$

The second and third equations in this set are the two components of the fluid force equation, with the vertical component reduced by the fact that the radiation flux exactly balances gravity in the equilibrium. The last three equations are the three components of the radiation flux equation. These equations may be further simplified by the assumption (verifiable *post hoc*) that $\omega \ll \kappa \rho c$; that is, the wave frequency is very small compared to the mean time between photon scattering events.

With the further approximation that $p_r \ll \rho c^2$, equations (5) – (12) may be manipulated to yield a dispersion relation. This relation is most conveniently displayed in terms of dimensionless quantities, so that $\omega = \omega_* \sqrt{g/h}$ and $\vec{k} = \vec{k}_*/h$. To satisfy the WKB approximation, $k_{*z} \gg 1$ and $k_{*r} \gg h/r$. The dispersion relation is then seen to depend on six dimensionless combinations of parameters:

$$P = \frac{p_r}{\rho g h} \quad (13)$$

$$G = \frac{\sqrt{g h}}{c} \quad (14)$$

$$\Omega_*^2 = h\Omega^2/g \quad (15)$$

$$\tau = \kappa\rho h \quad (16)$$

$$\mathcal{Q}_p = (\partial Q/\partial p_r)(h/g)^{1/2} \quad (17)$$

$$\mathcal{Q}_\rho = (\partial Q/\partial \rho)(hg^3)^{-1/2} \quad (18)$$

Note that here g is $g = g_z(z)$, the *local* value of the vertical gravity. In any thin disk, the quantity $G \ll 1$, for it is of order the free-fall speed from the top of the disk to the midplane in units of c . The relative importance of rotational effects is given by Ω_*^2 , which is simply h/z for Keplerian rotation. The parameter P is also a function of height above the midplane. In an optically thick disk, $P = \frac{1}{2}[1 + 1/\tau - (z/h)^2]/(z/h)$. Thus, the two parameters P and Ω_*^2 could be replaced by the single parameter z/h . The last two dimensionless parameters describe the sensitivity of the dissipation to the radiation pressure and the density, respectively. If one could extrapolate the “ α ” prescription to local fluctuations, one might expect that $\mathcal{Q}_p \simeq \alpha$.

3. The Dispersion Relation

3.1. General considerations

In terms of these dimensionless quantities, the dispersion relation is:

$$\begin{aligned} & 3\omega_*^5 + \left[ik_*^2/(G\tau) - 3k_{*z}G/\tau - i\mathcal{Q}_p \right] \omega_*^4 \\ & - \left(4k_*^2P + 3\Omega_*^2 + ik_{*z}^3/\tau^2 - ik_{*z}G\mathcal{Q}_p/\tau \right) \omega_*^3 \\ & + \left[4k_{*z}^3PG/\tau - ik_*^2\Omega_*^2/(G\tau) + i\mathcal{Q}_p(-ik_{*z} + \Omega_*^2) - ik_*^2\mathcal{Q}_\rho \right] \omega_*^2 \\ & + \left[4k_{*z}^2P\Omega_*^2 - 3k_{*r}^2 + k_{*z}^2G(ik_{*z}\mathcal{Q}_\rho - \mathcal{Q}_p)/\tau \right] \omega_* \\ & + k_{*z}k_{*r}^2\Omega_*^2/(G\tau) + k_{*z}\Omega_*^2(ik_{*z}\mathcal{Q}_\rho - \mathcal{Q}_p) = 0. \end{aligned}$$

In the limit of $\tau \rightarrow \infty$ and $\mathcal{Q}_p = \mathcal{Q}_\rho = 0$, this dispersion relation simplifies to

$$\omega_*^5 - [(4/3)k_*^2 P + \Omega_*^2] \omega_*^3 + [(4/3)k_{*z}^2 P \Omega_*^2 - k_{*r}^2] \omega_* = 0. \quad (19)$$

One root is clearly $\omega_* = 0$. The other four are given by:

$$\omega_*^2 = \begin{cases} (4/3)k_*^2 P \\ [2k_{*z}^2 \Omega_*^2 - (3/2)k_{*r}^2 / P] / k_*^2 \end{cases} \quad (20)$$

The first pair of roots are the familiar radiation-supported sound waves. The second pair describe buoyancy behavior (cf. Balbus 1999). If $k_{*z}^2 \Omega_*^2 \geq (3/4)k_{*r}^2 / P$, there are two neutrally stable gravity (epicyclic) waves; on the other hand, if $k_{*z}^2 \Omega_*^2 < (3/4)k_{*r}^2 / P$, one mode is damped, but the other grows exponentially with essentially no oscillation. It is this last mode that corresponds to convection. Although rotation tends to have a stabilizing effect, when $\Omega_* > 0$ one can always find a mode with k_r/k_z large enough to satisfy this criterion. As a consequence, when radiation pressure dominates gas pressure in a very optically thick disk, the standard equilibrium is always unstable to convection. However, relatively strong rotational effects (i.e., $\Omega_*^2 P \sim (h/z)^2$ relatively large, or location near the midplane) do diminish the range of wavevector directions that is unstable.

3.2. Dependence on parameters

As already remarked, both P and Ω_* are determined by z/h , and both should be, except very near the midplane, ~ 1 . Because the growth rate of the fastest growing mode in the optically thick limit is $\simeq P^{-1/2}$, and P decreases upward, convective instability should develop most rapidly near the top of the disk. This expectation is borne out when the full equations are solved: as predicted by equation 20, when $k_r \gg k_z$, the growth rate is nearly independent of $|\vec{k}|$ and is $\simeq 7\times$ greater at $z/h = 0.9$ than at $z/h = 0.1$.

The optical depth τ in a radiation pressure-dominated Shakura-Sunyaev disk is

$$\tau = \frac{4/3}{\alpha} \dot{m}^{-1} x^{3/2} \left(\frac{\kappa}{\kappa_T} \right)^{-1} \frac{R_z R_T}{R_R^2}, \quad (21)$$

where α is the usual dimensionless stress parameter, \dot{m} is the accretion rate in Eddington units for unit efficiency, x is the radius in units of GM/c^2 , κ/κ_T is the opacity relative to pure Thomson opacity, and $R_{R,T,z}$ are the relativistic corrections to the dissipation rate, torque, and vertical gravity (Page & Thorne 1974; Abramowicz, Lanza & Percival 1997; see also Agol & Krolik 2000a). Because we expect $\alpha, \dot{m} < 1$, but x must be > 1 , these disks should be optically thick. However, in the inner part of the disk, τ might be as little as $\sim 10^2$.

Fig. 1 shows how the growth rate and phase velocity of the unstable convective modes depend on wavenumber and k_z/k_r when there is no perturbation to the dissipation rate and the optical depth is very large ($\tau = 10^6$). To illustrate the dependence on optical depth, the same curves are shown in Fig. 2 for a case with the smallest optical depth one might expect, $\tau = 10^2$. In both cases, the range of meaningful wavenumbers (and hence the range of wavenumbers displayed in the figures) is limited by two criteria. On the one hand, the WKB approximation demands that $k_z h \gg 1$; on the other, the diffusion approximation is only consistent with $kh \ll \tau$. We present results extending out to $kh \sim \tau$; at the highest wavenumbers shown, the diffusion approximation has only marginal validity.

For fixed optical depth, the growth rate depends on k_z/k_r ; when $k_z/k_r \gtrsim 1.2$, the unstable convective mode disappears. For smaller k_z/k_r (< 0.1), the growth rate is independent of k_z/k_r . Decreasing the optical depth reduces the extent of the “flat” portion of the growth rate curve for a given k_z/k_r . However, the largest value of the dimensionless growth rate, $(\omega_{*i})_{max}$, in the wavenumber range of physical interest is almost independent of τ over this range.

Figs. 1 and 2 also show the phase velocity of the unstable convective mode, normalized

to the free fall speed in the disk. For large optical depths, the real frequency of convective modes is so close to zero that they can be considered as non-oscillatory; decreasing the optical depth towards the smallest value ($\tau = 10^2$) expected for radiation pressure dominated disks leads to an increase in the real frequency of the mode, endowing it with a non-negligible phase velocity and, as a consequence, it becomes oscillatory.

In the absence of perturbations to the dissipation rate, finite (although large) optical depth primarily affects the radiation sound waves (cf. Agol & Krolik 1998). Not surprisingly, as the opacity falls, short wavelength sound waves become damped. Less intuitively, convective modes remain unstable even when the diffusion rate ($\sim k_*^2 c / (h\tau)$) is larger than the characteristic frequency $\sqrt{g/h}$, i.e. when $k_* > \sqrt{G\tau}$. In this regime, epicyclic motions cause convective modes to acquire an oscillatory character. Convection remains unstable even in the face of strong radiative diffusion because the equilibrium gas density is constant. A parcel that begins to fall because its specific entropy is too small is squeezed as it encounters higher pressure. Its radiation pressure rises, but because diffusion is so effective, only up to the ambient level. Meanwhile, however, the increased gas density means that the parcel's density exceeds that of its neighbors, and it continues to fall.

Such impunity to the effects of radiation diffusion does not persist all the way to zero opacity, however. When $k_* > \tau$, the diffusion approximation is no longer a valid description of radiation transfer. For wavelengths that short (or opacity that small), the radiation streams freely, entirely independent of gas motions. In that limit, if there is nothing to alter the distribution of dissipation, there is no perturbation to the flux as a function of position, and therefore, no perturbation to the force. The result, of course, is that convection disappears.

Again applying the Shakura-Sunyaev disk solution, we expect G to be small, for it is

given by

$$G = \frac{3}{2} \dot{m} \left(\frac{\kappa}{\kappa_T} \right) \left(\frac{z}{h} \right)^{1/2} x^{-3/2} \frac{R_R}{R_z^{1/2}}. \quad (22)$$

G becomes irrelevant over most of the wavenumber interval of interest when $\tau \rightarrow \infty$, as every factor in which it appears in the dispersion relation is divided by τ ; the value of G can still have some influence even at large τ only for very large wavenumbers, $k_* \gtrsim 0.01\tau$, since some of those same terms of the dispersion relation actually are $\propto k_*^2/\tau$ or even $\propto k_*^3/\tau$. However, even when τ is as large as 10^5 , some dependence on G begins to appear across the range of interesting wavenumbers. When the optical depth is no larger than this, diminishing G tends to enhance instability, particularly at short wavelengths. When $\tau \sim 10^5$, the growth rate of the very short wavelength convective modes increases by factors of several when G falls from 10^{-2} to 10^{-4} . When $\tau \lesssim 10^4$, and $G \sim 10^{-4}$, a second mode, in addition to the convective mode, becomes unstable at short wavelengths. This mode is weakly oscillatory, and its very low phase velocity ($\sim 2.5 \times 10^{-2} - 1.6 \times 10^{-3}$ the free fall velocity) behaves similarly to that of damped radiation sound waves at the same wavelengths. Analysis of the corresponding amplitudes of the various physical quantities' perturbations shows that this mode has magnitude and especially phase relationships between perturbations in pressure, density and velocity components that are intermediate between those of convective modes and those characteristic of radiation sound waves.

It is important to note, however, that if the canonical disk solution applies, the product

$$G\tau = 2 \left(\frac{z}{h} \right)^{1/2} \alpha^{-1} \frac{R_T R_z^{1/2}}{R_R}. \quad (23)$$

In other words, the product $G\tau$ should be very nearly independent of radius (if α is). Even where the relativistic corrections are substantial, $G\tau$ hardly changes because R_R and R_T are very nearly proportional, and R_z has a total range of at most a factor of two (Krolik 1999). Thus, the criterion for which scaled wavelengths should be substantially affected by photon diffusion (i.e., $k_* > \sqrt{G\tau}$) is almost independent of radius, but does depend somewhat on

z/h .

Given our estimate of G (equation 22), we can also verify our expectation that $\omega/c\kappa\rho \ll 1$. Rewritten in terms of our dimensionless parameters, this ratio is ω_*G/τ . As we have just estimated, $G \ll 1$ and $\tau \gg 1$ in radiation pressure-supported α -disks. Consequently, unless ω_* is extremely large, $\omega/c\kappa\rho \ll 1$ is a very safe assumption.

As already pointed out, \mathcal{Q}_p is much like a local version of the vertically-averaged quantity α , the ratio of the r - ϕ stress to the total pressure. On this basis we expect \mathcal{Q}_p to be somewhat less than unity. The (dimensional) dissipation perturbation coefficient $\mathcal{Q}_\rho \equiv (\partial Q/\partial\rho)$, on the other hand, is most closely related to the ratio of the flux to the surface density. In a Shakura-Sunyaev disk, this quantity (reduced to a dimensionless number according to equation 18), is

$$\left(\frac{\tilde{F}}{\tilde{\Sigma}}\right) \equiv \frac{\mathcal{F}}{\Sigma} \frac{1}{(hg^3)^{1/2}} = \frac{1}{2} \left(\frac{3}{2}\right)^{3/2} \alpha \dot{m}^{3/2} \left(\frac{\kappa}{\kappa_T}\right)^{3/2} x^{-3/2} \frac{R_R^{5/2}}{R_z^2 R_T}. \quad (24)$$

Although we don't know for certain that $\mathcal{Q}_\rho \sim \tilde{F}/\tilde{\Sigma}$, it is a likely estimator. On this basis, we expect $\mathcal{Q}_\rho \ll 1$, and should be particularly small at larger radius or in disks with relatively low accretion rate.

Like τ , varying \mathcal{Q}_p or \mathcal{Q}_ρ affects the radiation sound waves much more than the convective modes. The growth rate of the near-zero real frequency convective mode is almost independent of \mathcal{Q}_p and \mathcal{Q}_ρ over a large range in these parameters, but the longer-wavelength radiation sound waves become unstable when these parameters are positive. For example, when $z/h = 0.5$, and $\tau \gg 1$, the growth rates for positive \mathcal{Q}_p and null \mathcal{Q}_ρ are $\simeq (0.1 - 0.16)\mathcal{Q}_p$, and, similarly, those for positive \mathcal{Q}_ρ and null \mathcal{Q}_p turn out to be $\simeq (0.1 - 0.16)\mathcal{Q}_\rho$.

Unlike the convective mode, for radiation sound waves, growth persists to $k_r/k_z < 1$. Although the maximum growth rate does not depend on the value of this ratio, the range

of wavelengths for which growth occurs shrinks and moves towards larger wavelengths as the ratio k_r/k_z decreases.

Smaller optical depth can also counteract the instability created by positive \mathcal{Q}_p or \mathcal{Q}_ρ (see also Agol & Krolik 1998 for a discussion of radiation diffusion damping of MHD waves). In the very large τ limit, the maximum growth rate for radiation sound waves is almost independent of the value of the optical depth, but for $\tau \lesssim 10^4$ it decreases with decreasing τ . The radiation sound waves are all damped for $\tau < 10^3$ or so; at larger optical depth, the shorter wavelength modes remain damped (because, of course, diffusion is most effective acting on them), but growth appears for longer wavelengths, with the range of growing wavelengths stretching as τ increases. Fig. 3 shows the behaviour of the imaginary part of the frequency and the normalised phase velocity for radiation sound waves with $\mathcal{Q}_p = 1$ and $\mathcal{Q}_\rho = 0$ as functions of wavenumber for a variety of optical depths.

Summarizing this discussion, we conclude that there are two sorts of instabilities of importance in radiation pressure-supported disks: convective instabilities, which grow on roughly the dynamical timescale, should be very nearly ubiquitous unless the disk becomes almost optically thin; and radiation sound waves, which can be driven unstable (at a slower rate) if \mathcal{Q}_p or \mathcal{Q}_ρ is positive.

3.3. Comparison with the Classical Convection Problem

In standard treatments of convection in rotating fluids (e.g., Chandrasekhar 1961), it is shown that convection begins when the “Rayleigh number”

$$\mathcal{R} = \frac{g\alpha_V\beta h^4}{\kappa_{th}\nu} \quad (25)$$

exceeds a critical value (generally $\gg 1$) which depends on the “Taylor number”

$$\mathcal{T} = 4\frac{\Omega^2 h^4}{\nu^2}. \quad (26)$$

Writing T for the temperature, we define $\alpha_V \equiv |\partial \ln \rho / \partial T|$, $\beta \equiv |dT/dz|$, κ_{th} is the thermal conductivity, and ν is the kinematic viscosity.

This analysis applies partially, but not entirely, to the circumstances of radiation-dominated accretion disks. The fact that, rather than entering the fluid solely at the bottom, heat is actually generated throughout the fluid probably alters the quantitative criteria for convective instability but should not change anything at a qualitative level. On the other hand, certain contrasts are of greater importance.

One that requires mention is that the pressure in these circumstances, unlike in classical fluids, depends only on the radiation, and is therefore independent of density. Consequently, thermal diffusion can alter the pressure with no change in density. In the classical analog, thermal diffusion can quench convective instability by adding heat to falling low-entropy fluid elements, thereby forcing them to expand until their density no longer exceeds the surrounding density. This occurs when κ_{th} is large enough that \mathcal{R} falls below the critical value. By contrast, in the radiation-dominated case, the density remains unchanged in the face of thermal diffusion. This is why convection persists even when $k_* > \sqrt{G\tau}$; i.e., the radiation analog of the Rayleigh number does not accurately predict the effect of thermal diffusion on the instability.

Perhaps the most important contrast with classical fluids is that there is no easy definition of the kinematic viscosity. Bulk orbital shear leads to angular momentum transfer via magnetic forces (Balbus & Hawley 1998); these are, of course, entirely absent in our treatment. However, stresses arising from other shears may be quite different, whether they are caused by magnetic forces or other mechanisms (for example, photon diffusion: Agol & Krolik 1998). For this reason, it is unclear whether the conventional equation of the $r - \phi$ shear stress with a nominal viscosity (as in the Shakura-Sunyaev α formalism) is appropriate here. In any event, none of these possible viscosity mechanisms is present in

our equations; not even radiation viscosity is considered, for no radiation shear stress exists in the pure diffusion approximation. Thus, in a formal sense, neither \mathcal{R} nor \mathcal{T} exists in our treatment of this problem.

Nonetheless, having expressed these *caveats*, it is still of some interest to estimate what \mathcal{T} and \mathcal{R} may be in radiation-dominated accretion disks in order to make some contact with the classical theory. To do so we make the suspect identification discussed in the preceding paragraph: we suppose that the shear viscosity in all directions is given by the Shakura-Sunayev α . If so, $\mathcal{T} \simeq 10\alpha^{-2}$, independent of radius (or anything else). Within the radiation-dominated regime, the Rayleigh number is similarly constant: $\mathcal{R} \sim O(10)(\kappa/\kappa_T)^2$. Interestingly, the critical \mathcal{R} for $10 < \mathcal{T} < 10^5$ is $\sim 10^3 - 10^4$ (Chandrasekhar 1961). Thus, if the most appropriate analog to ordinary fluid viscosity is correctly estimated by this means, the strong convective instability we find might be quenched; however, as we have stressed, it is by no means clear that this analogy applies.

4. Implications

We have shown that if an accretion disk is placed in the Shakura-Sunayev radiation-dominated equilibrium, the growth rate of the convective instability is hardly affected by large changes in parameters such as the opacity or the way in which the local dissipation might be perturbed. This behavior is, perhaps, not surprising, when one considers that the convective instability is essentially dynamical, whereas the opacity and dissipation rates pertain to the (rather more slowly accomplished) thermal balance.

Given that the Shakura-Sunayev equilibrium is unstable to convection for all reasonable parameters, we can hardly expect that it describes the actual state of radiation pressure-dominated disks. One's first guess might be that what happens instead is that

entropy is redistributed so as to make the disks at most marginally unstable to convection (Bisnovatyi-Kogan & Blinnikov 1977). If the entropy distribution is at most weakly unstable to convection, then departures from hydrostatic equilibrium should be small, and the equilibrium could be determined by applying these two conditions (i.e. isentropy and hydrostatic balance). However, marginal stability to convection also suggests that the amount of heat carried by convection is small, so that radiative diffusion dominates energy exchange. Unfortunately, this condition is inconsistent with the first two. Any two of these three conditions suffice to determine the equilibrium; the solution is over-determined if all three are applied. The result must be, then, that in real disks none of them is satisfied exactly, but all three are approximately correct. The quantitative character of this balancing act can be determined only by detailed calculations that carry convection into the nonlinear regime (e.g. Agol & Krolik 2000b).

Deviation from constant specific entropy can certainly be expected near the disk surface. There, radiation diffusion should be able to consistently dominate convection as a heat-loss mechanism because the diffusion time is much shorter than in the body of the disk. Consequently, the specific entropy distribution near the surface must be clearly stable, i.e. rise with increasing altitude.

The sense of heat redistribution by convection is to raise the specific entropy near the top and diminish it below. For the same reason that radiative diffusion must dominate near the top, this redistribution can have the side effect of changing the overall rate at which heat is able to leave the disk. The mean specific entropy in equilibrium is therefore not necessarily the same as the value predicted by the original Shakura-Sunyaev equilibrium.

These remarks do not bear on whether the thermal and viscous instabilities continue to plague radiation-dominated disks. It is possible that convection might be able to remove extra heat generated in the course of the thermal instability, and might consequently be

able to quench it. Whether that is so can only be determined by detailed calculation of heat transport by convection in the nonlinear regime (Agol & Krolik 2000b). Whether the viscous instability acts under these conditions depends on the character of dynamical coupling (see, e.g., Agol & Krolik 1998) between the radiation and the MHD turbulence that most likely accounts for angular momentum transport in disks (Balbus & Hawley 1998).

We have also shown that radiation sound waves may be driven unstable if the emission of radiation is proportional to either the gas density or the local radiation pressure (the latter possibility is suggested by a straight-forward extrapolation of the “ α -model”). Although phenomenological guesses of this variety are plausible, we do not yet know enough about the physical mechanisms of dissipation in accretion disks to say whether this is what should actually happen. However, if it does occur, growth of radiation sound waves (and, presumably, magnetosonic waves) could provide an additional source of turbulence in disks, supplementing magneto-rotational instability and convection. Indeed, if dissipation leads to renewed stirring, there might be interesting twists to the character of the MHD turbulent cascade.

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Fig. 1.— Dispersion curves of unstable convection modes for very large optical depth ($\tau = 10^6$). Non-dimensional growth rate, $\omega_{*i} \equiv \omega_i/\sqrt{g/h}$, and logarithm of absolute value of non-dimensional phase velocity, $u_{ph} \equiv v_{ph}/\sqrt{gh} = \omega_{*r}/k_*$, are shown as functions of the non-dimensional wavenumber $k_* = kh$. In both plots the curves are labelled with the corresponding value of the ratio of vertical to radial wavenumber k_z/k_r ; curves for $k_z/k_r < 0.1$ are not shown, since, in the physically meaningful wavenumber range, the growth rates are nearly identical to those found for $k_z/k_r = 0.1$.

Fig. 2.— Same as in Fig. 1, but for the smallest value of the optical depth expectable in the disk regime under examination, that is $\tau = 10^2$. The range of non-dimensional wavenumbers shown is much smaller than for Fig. 1 because of the restriction imposed by using the diffusion approximation.

Fig. 3.— Dispersion curves for radiation sound waves in a case including perturbation of dissipation rate. The curves shown all refer to $k_z/k_r = 0.7$ as a reference value; see the text for discussion of the dependence of the dispersion curves on the value of the ratio k_z/k_r . Different curves are labelled with the corresponding value of the optical depth τ . The wavenumber interval over which the mode is unstable depends on τ , growing in extent towards larger wavenumbers with increasing τ .