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On nonatomicity for non-additive functions

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ABSTRACT

The notion of nonatomicity for set functions plays a key role in classical measure theory and its applications. For classical measures taking values in finite dimensional Banach spaces, it guarantees the connectedness of range. Even just replacing σ-additivity with finite additivity for measures requires some stronger nonatomicity property for the same conclusion to hold. In the present paper, we deal with non-additive functions – called 's-outer' and 'quasi-triangular' – defined on rings and taking values in Hausdorff topological spaces. No algebraic structure is required on their target spaces. In this context, we make use of a notion of strong nonatomicity involving just the behavior of functions on ultrafilters of their underlying Boolean domains. This notion is proved to be equivalent to that proposed in earlier contributions concerning Lyapunov-types theorems in additive and non-additive frameworks. Thus, in particular, our analysis allows to generalize, improve and unify several known results on this topic.

1. Introduction

In classical measure theory, measures are required to satisfy an additive property. Additivity is effective and convenient in some applications, but turns out to be inadequate for many others. In economy, for instance, the efficiency of a set of collaborating workers is not realistically represented as the addition of the efficiency of each individual working on his own.

Nowadays, there is a considerable literature devoted to different types of non-additive functions; see, e.g., [12,15,29,35] and the references therein. Customary examples are sub-measures, k-triangular functions, decomposable measures, fuzzy measures, distorted probabilities, and multisubmeasures. All of them share however the following property: their target spaces possess an algebraic structure. Either derived from a partial order, or from a pseudoaddition, this allows, in a sense, to replace additivity with some monotonicity or weakened additivity.

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In this note, the (non-additive) functions coming into play – called 's-outer' and 'quasi-triangular' – take instead values in a Hausdorff topological space, where no algebraic structure is required. Neither monotonicity, nor pseudoadditivity is therefore a meaningful notion in our setting. A definition of these non-additive (classes of) functions involves the topological structure of their target spaces. Loosely speaking, a function φ, acting on a Boolean ring $\mathcal{R}$, is quasi-triangular whenever it enjoys the following property: for any disjoint $a, b \in \mathcal{R}$, if any two of the quantities $\varphi(a)$, $\varphi(b)$, $\varphi(a \vee b)$ are 'close' enough to $\varphi(0)$, then the remaining one is 'close' to $\varphi(0)$. A function φ is instead s-outer provided that $\varphi(a \vee b)$ and $\varphi(b)$ are 'close' whenever $\varphi(a)$ and $\varphi(0)$ are 'close' enough. Precise definitions and examples are given in Section 2. Classical measures, including finitely additive ones, as well as the other examples mentioned above, turn out to be special instances of quasi-triangular functions. Moreover, most of them are also s-outer. Therefore, heuristically speaking, our approach allows to provide a unified framework for different types of non-additive functions as well.

It is our aim here to investigate the notion of nonatomicity in such a general framework.

Nonatomic set functions frequently appear in measure theory and its applications. Typical examples are provided by Lyapunov-type theorems, and by the Aumann–Shapley approach to games [2]. Nonatomic games actually represent a good model for those games which are determined only by the behavior of infinite coalitions of players, while each single player is immaterial. Namely, a single player's choice effect is 'negligible'. This is the case, for instance, in decisions taken by voting, and in the determination of an equilibrium market price. On the other hand, in the Lyapunov theorem nonatomicity is a sufficient condition for the convexity (hence, for the connectedness) of the range of classical measures, defined on σ-algebras and taking values in finite dimensional Banach spaces. However, just replacing such measures with (nonnegative) bounded finitely additive functions does require a 'stronger nonatomicity' of functions in order to ensure the connectedness of their ranges (see e.g. [25], [26, Theorem 1], [1] and [24]). Contributions to this and closely related issues include [8,9,1,27,36,4,33,23,11] (in the finitely additive setting), and [19,28,21,16].

In the present paper, the notion of 'strong nonatomicity' for a function is formulated in terms of its behavior on the ultrafilters of its underlying Boolean ring (see $(\mathcal{C}_2)$ in Section 3). We prove that for any quasi-triangular function satisfying customary conditions, such notion is equivalent to that of 'strong continuity' in the sense of [3]. This is the content of Theorem 5.1, Section 5. On focusing on s-outer functions, further characterizations are exhibited (Theorem 5.2), which enable us to achieve Lyapunov-types results (Theorem 6.1, Section 6). We conclude this note by exhibiting that 'nonatomicity' and 'strong nonatomicity' are equivalent notions as long as s-outer functions φ acting on δ-rings $\mathcal{R}$ and taking values in metrizable spaces (fulfilling mild continuity properties such as exhaustivity and order-continuity) are considered (Theorem 7.4, Section 7). For these functions both notions are thus equivalent to the fact that for each $a \in \mathcal{R}$ the set $\{\varphi(x)\colon \mathcal{R} \ni x \leqslant a\}$ is arcwise-connected, and nonatomicity is still a sufficient condition for the connectedness of their ranges (Corollary 7.5, Section 7).

It may be worth pointing out that our results are obtained by exploiting a connection between quasi-triangular functions and finitely additive functions taking values in topological groups, established in [6], and stated below in Theorem 3.2. This result amounts to the fact that any quasi-triangular function is equivalent, in a sense, to a finitely additive function taking values in a topological group. Here, we further investigate on such equivalence (Section 3), and illustrate how it allows to translate verbatim classical results on strong nonatomicity from the finitely additive case to our non-additive setting.

2. Quasi-triangular and s-outer functions: definitions and examples

We hereafter assume that $\mathcal{R}$ is a Boolean ring, and $\mathcal{S} = (S, \tau)$ is a Hausdorff topological space. A function $\varphi : \mathcal{R} \to \mathcal{S}$ is said to be *quasi-triangular* whenever for any neighborhood U of $\varphi(0)$ in $\mathcal{S}$ ($U \in \tau[\varphi(0)]$, for short) there exists some $V \in \tau[\varphi(0)]$ such that, for all disjoint $d_1, d_2 \in \mathcal{R}$,

(I) if $\varphi(d_1), \varphi(d_2) \in V$, then $\varphi(d_1 \vee d_2) \in U$;
(II) if $\varphi(d_1), \varphi(d_1 \vee d_2) \in V$, then $\varphi(d_2) \in U$.

In the case when $\mathcal{S} = (S, \mathcal{U})$ is a Hausdorff uniform space, φ is called *s-outer* whenever for any entourage $U \in \mathcal{U}$ there exists some $V \in \mathcal{U}$ such that, for all disjoint $d_1, d_2 \in \mathcal{R}$,

(III) if $(\varphi(d_1), \varphi(0)) \in V$, then $(\varphi(d_1 \vee d_2), \varphi(d_2)) \in U$.

Let us emphasize that the notion of s-outer function does require some regularity on the target space of the function φ, namely, $\mathcal{S}$ must be at least a Tychonoff space. Notice that any Hausdorff topological group G is a Tychonoff space (see e.g. [17, Theorem 5, Chap. III]), and it is easily seen that every G-valued finitely additive function on $\mathcal{R}$ is an s-outer function. Moreover, each s-outer function is quasi-triangular [6, Proposition 2.1], but the converse fails (see Remarks 4.3).

We now illustrate by means of simple examples that the class of quasi-triangular functions includes various collections of non-additive functions extensively studied in the literature. Most of them are also s-outer.

Examples 2.1. Let $\varphi : \mathcal{R} \to [0, +\infty]$ with $\varphi(0) = 0$. Adopt the convention $\infty - \infty = 0$, and consider the usual uniformity on $[0, +\infty]$.

(a) If φ is a *submeasure* in the sense of Drewnowski (see e.g. [13,14]), i.e.

$$\max\{\varphi(d_1), \varphi(d_2)\} \leqslant \varphi(d_1 \vee d_2) \leqslant \varphi(d_1) + \varphi(d_2) \tag{1}$$

for all disjoint $d_1, d_2 \in \mathcal{R}$, then φ is s-outer.
In particular, outer measures (and hence classical measures) are s-outer functions.

(b) If φ is *k-triangular*, with $k \geqslant 1$ (see e.g. [29,32]) i.e. for some $k \geqslant 1$

$$\left|\varphi(d_1 \vee d_2) - \varphi(d_2)\right| \leqslant k\varphi(d_1)$$

for all disjoint $d_1, d_2 \in \mathcal{R}$, then φ is s-outer.
In particular, *measuroids* (see e.g. [31,37]), namely 1-triangular functions, are s-outer.

(c) Let $\oplus$ be a triangular-conorm (t-conorm) on $[0, +\infty]$, i.e. a commutative, associative and monotonic binary operation on $[0, +\infty]$ with 0 as the neutral element.
If φ is an *$\oplus$-decomposable function* (see e.g. [15,38,39,29,30]) i.e. $\oplus$ is continuous at 0 and

$$\varphi(d_1 \vee d_2) = \varphi(d_1) \oplus \varphi(d_2)$$

for all disjoint $d_1, d_2 \in \mathcal{R}$, then φ is quasi-triangular.
Standard instances of t-conorms on $[0, +\infty]$ continuous at 0 are provided by $\bigoplus_p(x, y) = \{x^p + y^p\}^{1/p}$ for $p > 0$, $\bigoplus_\infty(x, y) = \max\{x, y\}$, the Sugeno t-conorm $\bigoplus_\lambda(x, y) = x + y + \lambda xy$ for $\lambda \geqslant -1$, and the Łukasiewicz t-conorm $\bigoplus_L(x, y) = \min\{x, y\}$. For more sophisticated examples of t-conorms we refer the reader to [20]. As far as the above-mentioned t-conorms are considered, $\bigoplus_p$-decomposable functions are also s-outer, for all $p \in]0, \infty]$.

(d) If φ is a *quasi-submeasure*, i.e. some constants $C_1, C_2 \geqslant 1$ exist such that

$$\frac{1}{C_1}\max\{\varphi(d_1), \varphi(d_2)\} \leqslant \varphi(d_1 \vee d_2) \leqslant C_2 \max\{\varphi(d_1), \varphi(d_2)\} \tag{2}$$

for all disjoint $d_1, d_2 \in \mathcal{R}$, then φ is quasi-triangular.

Remarks 2.2. **(i)** The denomination here adopted of 'quasi-submeasures' for functions of item (d) is justified by the fact that condition (2) weakens (1). Let us notice that quasi-submeasures can be neither monotone, nor sub-additive, and even not s-outer. To see this, consider the σ-algebra Σ of Lebesgue measurable subset of $\mathbb{R}$, and let λ be the Lebesgue measure on Σ. Define $\varphi : \Sigma \to [0, +\infty]$ by $\varphi(E) := \lambda(E)$ if $\lambda(E) \leqslant 1$, and $\lambda(E) - 1/2$ elsewhere. It can be easily checked that φ is a quasi-submeasure – with $C_1 = C_2 = 2$ in (2) – but φ fails to be either monotone, or sub-additive or even s-outer. **(ii)** Several instances of quasi-submeasures (hence, of quasi-triangular functions) are provided by the composite function $f \circ \varphi$ of a submeasure $\varphi : \mathcal{R} \to [0, +\infty[$ with a function $f : [0, +\infty[\to [0, +\infty[$ fulfilling $f(0) = 0$ and one of the following conditions

(a) f is increasing and quasi-concave,
(b) f is convex (namely, a *Young function*) and satisfies the Δ_2-condition – i.e. a positive constant $c \geqslant 1$ exists such that $f(2t) \leqslant cf(t)$ for all t.

When (a) is in force, $f \circ \varphi$ is actually a submeasure (hence, an s-outer function). This follows from the fact that quasi-concave functions are always sub-additive. Computation are left to the reader. In the specific situation of $\varphi = \mathcal{P}$, where $\mathcal{P}$ is any finitely additive probability on a Boolean algebra $\mathcal{A}$, and $f : [0, 1] \to [0, 1]$, with $f(\mathcal{P}(0)) = 0$ and $f(\mathcal{P}(1)) = 1$, satisfies either condition (a) or (b) one thus recovers *distorted probabilities* (see e.g. [12,29]) inside the class of quasi-triangular functions. For *distortions* f of type (a), they are also s-outer.

Thus, as an example, for any $p > 0$, $\alpha \geqslant 0$, and non-negative measure φ, then φ^p and $(1+\varphi)^p(\log(1+\varphi))^\alpha$ are quasi-triangular functions, and also s-outer provided $p \leqslant 1$.

Example 2.3. Let $\mathcal{P}_f(X)$ be the collection of non-empty closed sets of a Banach space X, endowed with the extended metric defined by

$$\rho(F, H) := \max\Big\{ \sup_{x\in F} \inf_{y\in H} \|x - y\|, \sup_{y\in H} \inf_{x\in F} \|y - x\| \Big\} \quad \text{for } F, H \in \mathcal{P}_f(X),$$

where $\|\cdot\|$ is the norm of X. Consider the Minkowski addition $\dot{+}$ in $\mathcal{P}_f(X)$ defined by $F \dot{+} H = \overline{F + H}$, where $F + H := \{x + y\colon\ x \in F,\ y \in H\}$ and $\overline{F + H}$ is the closure of $F + H$ with respect to the topology induced by the norm of X.

If $\varphi : \mathcal{R} \to \mathcal{P}_f(X)$ is a *multisubmeasure* (see e.g. [16]), i.e. $\varphi(0) = \{0\}$ and

$$\varphi(d_1) \cup \varphi(d_2) \subseteq \varphi(d_1 \vee d_2) \subseteq \varphi(d_1) \dot{+} \varphi(d_2)$$

for all disjoint $d_1, d_2 \in \mathcal{R}$, then φ is s-outer.

3. Equivalence among functions valued in topological spaces

Let $\varphi : \mathcal{R} \to \mathcal{S}$ and $\nu : \mathcal{R} \to \mathcal{S}'$ be any two functions, both defined on $\mathcal{R}$, with values in Hausdorff topological spaces which are not-necessarily related. Then ν is said to be *continuous with respect to* φ – writing $\nu \ll \varphi$ – whenever for any $U \in \tau'[\nu(0)]$ there exists some $V \in \tau[\varphi(0)]$ such that if $\varphi([0, a]) \subseteq V$ for $a \in \mathcal{R}$, then $\nu([0, a]) \subseteq U$. Here, and in the sequel, $[0, a]$ stands for the set $\{b \in \mathcal{R}\colon\ b \leqslant a\}$.

Henceforth φ and ν are said to be *equivalent* – writing $\varphi \approx \nu$ – if they are continuous with respect to each other, namely,

$$\varphi \approx \nu \quad \overset{def}{\Longleftrightarrow} \quad \nu \ll \varphi \text{ and } \varphi \ll \nu. \tag{3}$$

Once defined the *kernel of* φ as

$$N(\varphi) := \{a \in \mathcal{R}\colon\ \varphi(x) = \varphi(0) \text{ for all } x \in [0, a]\}, \tag{4}$$

one clearly has that

$$\varphi \approx \nu \quad \Longrightarrow \quad N(\varphi) = N(\nu). \tag{5}$$

Our first result describes the behavior of equivalent functions with respect to the following continuity properties, which will be taken into account hereafter. A function $\varphi : \mathcal{R} \to (S, \tau)$ is said to be

($\mathcal{C}_0$): *locally exhaustive* whenever $\lim_k \varphi(d_k) = \varphi(0)$ for all pairwise disjoint sequence $(d_k)_{k\in\mathbb{N}}$ in $\mathcal{R}_a := \{a \wedge x\colon\ x \in \mathcal{R}\}$ and $a \in \mathcal{R}$;

($\mathcal{C}_1$): *locally strongly continuous* whenever for any $a \in \mathcal{R}$ and $U \in \tau[\varphi(0)]$ there exists a finite partition $\mathcal{D} \subset \mathcal{R}_a$ of a such that $\varphi(d \wedge x) \in U$ for every $d \in \mathcal{D}$ and $x \in \mathcal{R}$;

($\mathcal{C}_2$): *strongly nonatomic* whenever $\lim_{b\in\mathcal{F}} \varphi(b) = \varphi(0)$ for all ultrafilters $\mathcal{F}$ in $\mathcal{R} \setminus N(\varphi)$.

Incidentally, let us mention that no general agreement appears in the literature – from finitely additive investigations on – about the definition of strongly nonatomic function. Several contributions adopt a definition of strong nonatomicity as a global ($\mathcal{C}_1$) property, namely, of a strong continuity in the sense of [3]. Property ($\mathcal{C}_2$) – firstly considered in [8] for σ-additive functions valued in topological groups – is naturally patterned on the fact that, roughly speaking, an atom for a classical measure φ determines at least an ultrafilter $\mathcal{F}$ such that $\lim_{b\in\mathcal{F}} \varphi(b) \neq \varphi(0)$. We will discuss this and related topics in Section 7. Besides, in our generalized non-additive setting, properties ($\mathcal{C}_1$) and ($\mathcal{C}_2$) are demonstrated to be equivalent by Theorem 5.1.

Lemma 3.1. *Let $\varphi : \mathcal{R} \to \mathcal{S}$. For any function ν on $\mathcal{R}$ such that $\nu \approx \varphi$*

(i) *ν enjoys property ($\mathcal{C}_i$) for $i = 0, 1$ if (and only if) φ does;*
(ii) *ν enjoys both properties ($\mathcal{C}_0$) and ($\mathcal{C}_2$) if (and only if) φ does.*

Proof. (i) Let φ enjoy property ($\mathcal{C}_0$), namely, φ is locally exhaustive. Then, $\lim_k \varphi(d_k \wedge x) = \varphi(0)$ for all disjoint sequence $(d_k)_{k\in\mathbb{N}}$ in $\mathcal{R}_a$ and $a \in \mathcal{R}$, uniformly with respect to $x \in \mathcal{R}$. This is easily verified on arguing by contradiction. Thus, via (3), the claim follows. The case ($\mathcal{C}_1$) can be straightforwardly deduced from (3).

(ii) Let φ enjoy both properties ($\mathcal{C}_0$) and ($\mathcal{C}_2$). On arguing by contradiction, notice that $\lim_{b\in\mathcal{F}} \varphi(b \wedge x) = \varphi(0)$ for all ultrafilters $\mathcal{F}$ in $\mathcal{R} \setminus N(\varphi)$, uniformly with respect to $x \in \mathcal{R}$. Hence, an application of (3) ends the proof. □

Heuristically speaking, Lemma 3.1 tells us that properties ($\mathcal{C}_0$)–($\mathcal{C}_1$) are actually invariant under replacements of the reference function φ with one equivalent in the sense of (3). The same holds for property ($\mathcal{C}_2$) provided ($\mathcal{C}_0$) is in force for φ.

As noted at the end of Section 1, the equivalence $\approx$ described in (3) is significant for us because of results in [6] – whose proofs rely on the use of Fréchet–Nikodým topologies – that we here rewrite collectively as

Theorem 3.2. *(See [6].) Let $\mathcal{R}$ be a Boolean ring, and $\mathcal{S}$ be a Hausdorff topological space. If $\varphi : \mathcal{R} \to \mathcal{S}$ is quasi-triangular, then there exists a finitely additive function $\mu : \mathcal{R} \to G$, where G is a topological commutative group, such that $\mu \approx \varphi$, i.e. $\varphi \ll \mu$ and $\mu \ll \varphi$.*

In particular, G is the algebraic group $(\mathcal{R}, \Delta)$ *with* Δ *denoting the symmetric difference, endowed with the* φ*-topology induced on* $\mathcal{R}$ *by the neighborhood base at each* $a \in \mathcal{R}$

$$\Gamma_\varphi[a] := \left(\left\{x \in \mathcal{R}:\ \varphi\left([0, x\Delta a]\right) \subseteq U\right\}\right)_{U \in \mathcal{B}}, \tag{6}$$

where $\mathcal{B}$ *is a neighborhood base at the point* $\varphi(0)$.

Our next result reveals that the regularity of the topological group G described in Theorem 3.2 only depends on the kernel of the quasi-triangular function taken into account.

Lemma 3.3. *Let* $\varphi : \mathcal{R} \to \mathcal{S}$ *be quasi-triangular. Then* $N(\varphi)$ *is an ideal of* $\mathcal{R}$*. Moreover, the* φ*-topology – determined on* $\mathcal{R}$ *by* (6) *– is Hausdorff if, and only if, the kernel of* φ *is trivial, i.e.* $N(\varphi) = \{0\}$.

Proof. By (4), to verify that $N(\varphi)$ is an ideal of $\mathcal{R}$ it suffices to show that $a \vee b \in N(\varphi)$ for $a, b \in N(\varphi)$. For this, take any $x \in [0, a \vee b]$. Of course, $x = (a \wedge x) \vee ((b \setminus a) \wedge x)$. The conclusion $\varphi(x) = \varphi(0)$ thus follows by applying the quasi-triangularity of φ and the Hausdorff property of $\mathcal{S}$. The arbitrariness of x implies that $a \vee b \in N(\varphi)$. The rest of the statement can be easily deduced from (6) and (4). $\square$

4. Null-additive functions

In this section we focus on functions $\varphi : \mathcal{R} \to \mathcal{S}$ with non-trivial kernels, i.e. $N(\varphi) \neq \{0\}$. In the case when $\mathcal{S} = \mathbb{R}$, such functions are usually called 'not faithful'. The motivation comes from Lemma 3.3 above and the following observation. For a quasi-triangular $\varphi : \mathcal{R} \to \mathcal{S}$ with $N(\varphi) \neq \{0\}$, the quotient ring $\hat{\mathcal{R}} = \mathcal{R}/N(\varphi)$ and the canonical map

$$\pi : a \in \mathcal{R} \longmapsto \hat{a} := a\Delta N(\varphi) \in \hat{\mathcal{R}}$$

are actually well-defined. Nevertheless, setting

$$\hat{\varphi}(\hat{a}) := \varphi(a) \quad \text{for } a \in \mathcal{R} \tag{7}$$

does not define a function on $\hat{\mathcal{R}}$ in general. As an example, consider the function φ defined on $\mathcal{R} = \{0, a, 1\setminus a, 1\}$ by $\varphi(0) = \varphi(a) = 0$, $\varphi(1\setminus a) = 1$ and $\varphi(1) = 2$. Then φ is quasi-triangular with $N(\varphi) = \{0, a\}$, but $\hat{\varphi}$ can be not defined, inasmuch $1 \setminus a \in \hat{1}$ and $\varphi(1 \setminus a) \neq \varphi(1)$.

The reason for this lack is displayed by our next result in full generality.

Lemma 4.1. *For any function* $\varphi : \mathcal{R} \to \mathcal{S}$*, the following are equivalent:*

(i) *The kernel* $N(\varphi)$ *of* φ *is an ideal of* $\mathcal{R}$*, and the function*

$$\hat{\varphi} : \hat{a} \in \hat{\mathcal{R}} = \frac{\mathcal{R}}{N(\varphi)} \longmapsto \varphi(a) \in \mathcal{S} \tag{8}$$

is well-defined;

(ii) φ *is null-additive, that is,*

$$\varphi(a) = \varphi(a \vee b) \quad \text{for all } a \in \mathcal{R} \text{ and } b \in N(\varphi), \text{ with } a \wedge b = 0. \tag{9}$$

Proof. We can assume that $N(\varphi) \neq \{0\}$, being the statement trivial otherwise. Suppose (i) be in force. If (ii) failed, then there would exist some $a \in \mathcal{R} \setminus \{0\}$ and $b \in N(\varphi)$, with $a \wedge b = 0$, such that $\varphi(a) \neq \varphi(a \vee b)$.

But this contradicts (i), because $a \vee b \in \hat{a}$. Thus (ii) holds true. The converse implication can be easily checked. $\square$

Remark 4.2. Let $\varphi : \mathcal{R} \to \mathcal{S}$. It can be easily checked that the set

$$\mathcal{I} := \{a \in \mathcal{R}:\ \varphi(a \vee b) = \varphi(b) \text{ for all } b \in \mathcal{R}\}$$

is an ideal of $\mathcal{R}$; moreover, with no additional assumptions on φ, the function

$$\tilde{\varphi} : \tilde{a} = a\Delta\mathcal{I} \in \frac{\mathcal{R}}{\mathcal{I}} \longmapsto \varphi(a) \in \mathcal{S}$$

is well-defined. Notice, however, that for the function φ of the example before Lemma 4.1 $\mathcal{I} = \{0\}$, whereas $N(\varphi) = \{0, a\}$. The null-additivity requirement on a function φ comes thus into play because we are interested in the quotient ring $\frac{\mathcal{R}}{N(\varphi)}$, and consequently in the function $\hat{\varphi}$ described by (7), on account of Lemmas 3.3 and 4.1.

Remarks 4.3. **(i)** Quasi-triangular functions having non-trivial kernels may fail to be null-additive, as shown at the beginning of this section. Equality (9) is instead always in force for s-outer functions φ valued in Hausdorff spaces. Let us stress that in the present paper (see Section 2) target spaces of functions are always assumed to be Hausdorff. Notice also that Lemma 4.1 in particular tells us that the class of s-outer functions is strictly included in that of quasi-triangular functions. Moreover, in the former class – unlike in the latter – the correspondence $\varphi \mapsto \hat{\varphi}$ is granted. **(ii)** It is noteworthy that coupling the requirement of quasi-triangularity with that of null-additivity for a function taking values in a uniform space do not gain an s-outer function. As an example, consider the σ-algebra Σ of the Lebesgue measurable subsets of $\mathbb{R}$, and denote by λ the Lebesgue measure on Σ. The function φ, defined by $\varphi(E) = \lambda(E)$ for $E \in \Sigma$ with $\lambda(E) \leqslant 1$, and $\varphi(E) = 2$ elsewhere, is both quasi-triangular and null-additive, but φ is not s-outer. **(iii)** The concept of null-additivity formulated in (9) extends that introduced in [34] and deeply investigated by E. Pap (see e.g. [28–30]).

In an analogy with the situation described in Lemma 3.1, we now exhibit that properties for functions we are concerned with are actually invariant under the correspondence $\varphi \mapsto \hat{\varphi}$, defined in the class of null-additive functions, according to Lemma 4.1. Each s-outer function does belong to such a class, as just observed in **(i)**.

Theorem 4.4. *Let $\varphi : \mathcal{R} \to S$, with $N(\varphi) \neq \{0\}$, be null-additive. Then*

(i) *$\hat{\varphi}$ is quasi-triangular if, and only if, φ is quasi-triangular*;
(ii) *$\hat{\varphi}$ is s-outer if, and only if, φ is s-outer*;
(iii) *$\hat{\varphi}$ enjoys property $(\mathcal{C}_i)$ for $i = 0, 1, 2$ if, and only if, φ does.*

Proof. (i) Let φ be quasi-triangular. This means that for any $U \in \tau[\varphi(0)]$ some $V \in \tau[\varphi(0)]$ exists such that, for all disjoint $d_1, d_2 \in \mathcal{R}$,

(I) if $\varphi(d_1), \varphi(d_2) \in V$, then $\varphi(d_1 \vee d_2) \in U$;
(II) if $\varphi(d_1), \varphi(d_1 \vee d_2) \in V$, then $\varphi(d_2) \in U$.

Note that for any disjoint $\hat{d}_1, \hat{d}_2$ in $\hat{\mathcal{R}}$, one can always determine some disjoint $d_1^*, d_2^* \in \mathcal{R}$ such that $d_1^* \in \hat{d}_1$ and $d_2^* \in \hat{d}_2$. So Lemma 4.1 tells us that if $\hat{\varphi}(\hat{d}_1), \hat{\varphi}(\hat{d}_2) \in V$, then $\varphi(d_1^*), \varphi(d_2^*) \in V$. Hence applying (I) entails that $\varphi(d_1^* \vee d_2^*) \in U$. Thus, being the canonical map a Boolean homomorphism, $\hat{\varphi}(\widehat{d_1^* \vee d_2^*}) = \hat{\varphi}(\hat{d}_1 \vee \hat{d}_2) \in U$. Furthermore, if $\hat{\varphi}(\hat{d}_1), \hat{\varphi}(\hat{d}_1 \vee \hat{d}_2) \in V$, then $\varphi(d_1^*), \varphi(d_1^* \vee d_2^*) \in V$. By (II), $\varphi(d_2^*) = \hat{\varphi}(\hat{d}_2) \in U$. Therefore, $\hat{\varphi}$ is quasi-triangular. The reverse implication is straightforward.

(ii) The statement follows by arguing as in the previous case. We leave it to the reader.

(iii) First, assume that φ enjoys property $(\mathcal{C}_0)$, namely, φ is locally exhaustive. Let $\hat{a} \in \hat{\mathcal{R}}$ be arbitrarily given, and consider any pairwise disjoint sequence $(\hat{d}_k)_{k\in\mathbb{N}}$ in $\hat{\mathcal{R}}_{\hat{a}}$. Arguing by induction enables to provide the existence of a pairwise disjoint sequence $(d_k^*)_{k\in\mathbb{N}}$ in $\mathcal{R}_a$ such $d_k^* \in \hat{d}_k$ for all k. Then, an application of Lemma 4.1 entails that

$$\lim_k \hat{\varphi}(\hat{d}_k) = \lim_k \varphi\big(d_k^*\big) = \varphi(0) = \hat{\varphi}(\hat{0}).$$

Namely, $\hat{\varphi}$ is locally exhaustive. The converse is trivial.

Second, focus on property $(\mathcal{C}_1)$. We limit ourselves to proving that when $\hat{\varphi}$ enjoys property $(\mathcal{C}_1)$ then φ does as well, since the other implication straightforwardly follows from the fact that the canonical map is an order-preserving Boolean homomorphism. Let $a \in \mathcal{R}$ be arbitrarily given, and consider $\hat{a} \in \hat{\mathcal{R}}$. The local strong continuity of $\hat{\varphi}$ assures the existence of a finite partition $\hat{\mathcal{D}} \subset \hat{\mathcal{R}}_{\hat{a}}$ of $\hat{a}$ such that $\hat{\varphi}(\hat{d} \wedge \hat{x}) \in U$ for every $\hat{d} \in \hat{\mathcal{D}}$ and $\hat{x} \in \hat{\mathcal{R}}$. Hence, some easy computations allow to determine a finite partition $\mathcal{D} \subset \mathcal{R}_a$ of a fulfilling the property that each element $d^* \in \mathcal{D}$ does belong exactly to one $\hat{d} \in \hat{\mathcal{D}}$. Therefore, Lemma 4.1 together with the fact that the canonical map is a Boolean homomorphism imply that $\varphi(d^* \wedge x) = \hat{\varphi}(\widehat{d^* \wedge x}) = \hat{\varphi}(\hat{d} \wedge \hat{x}) \in U$ for every $d^* \in \mathcal{D}$ and $x \in \mathcal{R}$. Namely, φ is locally strongly continuous, as desired.

Finally, let φ enjoy property $(\mathcal{C}_2)$, namely, φ is strongly nonatomic. Let $\hat{\mathcal{F}}$ be any ultrafilter of $\hat{\mathcal{R}}$, and define $\mathcal{F} := \{b \in \mathcal{R}\colon\ \hat{b} \in \hat{\mathcal{F}}\}$. It is not difficult to check that $\mathcal{F}$ is actually an ultrafilter in $\mathcal{R} \setminus \mathcal{N}(\varphi)$. Then, coupling the strongly nonatomicity of φ with Lemma 4.1 gives

$$\lim_{\hat{b}\in\hat{\mathcal{F}}} \hat{\varphi}(\hat{b}) = \lim_{b\in\mathcal{F}} \varphi(b) = \varphi(0) = \hat{\varphi}(\hat{0}).$$

Namely, $\hat{\varphi}$ is strongly nonatomic. In order to prove the reverse implication, it suffices to notice that for any ultrafilter $\mathcal{F}$ in $\mathcal{R} \setminus \mathcal{N}(\varphi)$ the set $\hat{\mathcal{F}} := \{\hat{b} \in \hat{\mathcal{R}}\colon\ b \in \mathcal{F}\}$ is an ultrafilter of $\hat{\mathcal{R}}$. This follows from the maximality of $\mathcal{F}$ and the fact that the canonical map is an order-preserving Boolean homomorphism. Hence, if $\hat{\varphi}$ is strongly nonatomic, an application of Lemma 4.1 allows to conclude that

$$\lim_{b\in\mathcal{F}} \varphi(b) = \lim_{\hat{b}\in\hat{\mathcal{F}}} \hat{\varphi}(\hat{b}) = \hat{\varphi}(\hat{0}) = \varphi(0),$$

i.e. φ is strongly nonatomic. □

5. On strong nonatomicity for quasi-triangular and s-outer functions

Here, we firstly provide a characterization of strong nonatomicity in the (non-additive) class of quasi-triangular functions. Namely, our notion of strong nonatomicity involving just the behavior of functions on ultrafilters of their underlying Boolean domains is proved to be therein equivalent to that of strong continuity in the sense of [3], which is proposed in earlier contributions concerning Lyapunov-types theorems in additive and non-additive frameworks.

Theorem 5.1. *Let $\varphi : \mathcal{R} \to S$ be a locally exhaustive quasi-triangular function. Assume, in addition, that φ is null-additive whenever $N(\varphi) \neq \{0\}$. Then the following are equivalent*:

$(\mathcal{C}_1)$ *φ is locally strongly continuous*;
$(\mathcal{C}_2)$ *φ is strongly nonatomic.*

Proof. Let $\varphi : \mathcal{R} \to S$ be quasi-triangular and locally exhaustive. In the case when $N(\varphi) = \{\mathit{0}\}$, coupling Theorem 3.2 and Lemma 3.3 provides the existence of a finitely additive function $\mu : \mathcal{R} \to G$, where G is actually a Hausdorff topological group, such that $\mu \approx \varphi$. By (i) in Lemma 3.1, μ is locally exhaustive. This assures that local strong continuity and strong nonatomicity for μ are equivalent concepts (see e.g. [10, (2.7), Chap. VI]). Such an equivalence pertains to φ, because of Lemma 3.1. This ends the proof for $N(\varphi) = \{\mathit{0}\}$.

In the case when $N(\varphi) \neq \{\mathit{0}\}$, φ is also null-additive. Thus Lemma 4.1 enables to consider the function $\hat{\varphi}$ defined on $\mathcal{R}/N(\varphi)$ by (8). Such a $\hat{\varphi}$ is quasi-triangular and locally exhaustive, owing to (i) and (iii) of Theorem 4.4, and $N(\hat{\varphi}) = \{\hat{\mathit{0}}\}$. An application of the previous case to $\hat{\varphi}$ in place of φ yields the equivalence for $\hat{\varphi}$ of properties $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$. Such an equivalence pertains to φ, by applying (iii) of Theorem 4.4. □

Focusing on s-outer functions, strong nonatomicity can be further characterized. For this, recall that a Boolean ring $\mathcal{R}$ is said to be an **F**-*ring* or, equivalently, to enjoy the *interpolation property* (see e.g. [1]) if for any pair of sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ in $\mathcal{R}$ such that $a_n \leqslant a_{n+1} \leqslant b_{n+1} \leqslant b_n$ for each n, there exists some $c \in \mathcal{R}$ fulfilling the property that $a_n \leqslant c \leqslant b_n$ for all n. Notice that δ-rings are clearly **F**-rings.

Theorem 5.2. *Let $\varphi : \mathcal{R} \to (S,\mathcal{U})$ be a locally exhaustive s-outer function. Then the following are equivalent*:

- $(\mathcal{C}_2)$ *φ is strongly nonatomic*;
- (Π) *for each $a \in \mathcal{R}$ and pseudometric p belonging to the gauge $\mathcal{G}$ of $\mathcal{U}$, there exists an increasing function $f_a : D \to \mathcal{R}_a$, where D is a dense subset of $[0,1]$ with $0,1 \in D$, such that*:
  - (α) *$f_a(0) = \mathit{0}$ and $f_a(1) = a$*;
  - (β) *the functions $\gamma_x(t) := \varphi(f_a(t) \wedge x)$, $t \in D$, are $\mathcal{U}_p$-uniformly continuous on D, uniformly with respect to $x \in \mathcal{R}$.*

*In addition, if $\mathcal{R}$ is an **F**-ring, then $(\mathcal{C}_2)$ is equivalent to (Π) with $[0,1]$ instead of D.*

Proof. Let $\varphi : \mathcal{R} \to \mathcal{S}$ be s-outer and locally exhaustive. In the case when $N(\varphi) = \{\mathit{0}\}$, Theorem 3.2 and Lemma 3.3 entail the existence of a finitely additive function $\mu : \mathcal{R} \to G$, where G is actually a Hausdorff topological group, such that $\mu \approx \varphi$. Moreover, by Lemma 3.1, μ is strongly nonatomic if, and only if, φ enjoys property $(\mathcal{C}_2)$.

Now, for any pseudometric $p \in \mathcal{G}$, define $U_n := \{(x,y) \in S \times S\colon\ p(x,y) < \frac{1}{n}\}$ for $n \in \mathbb{N}$, and set $U_o := \{(x,y) \in S \times S\colon\ p(x,y) < 2\}$. Since φ is s-outer, it is easily seen that a decreasing sequence $(V_n)_{n\in\mathbb{N}}$ in $\mathcal{U}$ exists obeying the following property

$$\big(\varphi(d_1), \varphi(\mathit{0})\big) \in V_n \quad \Longrightarrow \quad \big(\varphi(d_1 \vee d_2), \varphi(d_2)\big) \in U_{n-1}$$

for all n and disjoint $d_1, d_2 \in \mathcal{R}$.

For each n the set $V'_n := \{x \in S\colon\ (x, \varphi(\mathit{0})) \in V_n\}$ is a neighborhood of $\varphi(\mathit{0})$. Thus Theorem 3.2 tells us that each $W_n := \{x \in \mathcal{R}\colon\ \varphi([\mathit{0}, x\Delta a]) \subseteq V'_n\}$ is actually a neighborhood of $\mathit{0}$ in G. By the made construction, $(W_n)_{n\in\mathbb{N}}$ is therefore a decreasing sequence of neighborhoods of $\mathit{0}$ in G. The existence of such a sequence allows us to argue as in the proof of [11, Teorema I] and entails the equivalence for μ of its strong nonatomicity and the existence for each $a \in \mathcal{R}$ of an increasing function $f_a : D \to \mathcal{R}_a$, where D is dense subset of $[0,1]$ containing $\{0,1\}$, such that

- (α) $f_a(0) = \mathit{0}$ and $f_a(1) = a$;
- $(*)$ the functions $\psi_x : t \in D \mapsto \mu(f_a(t) \wedge x)$ are $\mathcal{U}_p$-uniformly continuous on D, uniformly with respect to $x \in \mathcal{R}$.

This together with the fact that φ is s-outer and $\varphi \approx \mu$ as well yield the first part of the statement.

When $\mathcal{R}$ is an **F**-ring, for each $a \in \mathcal{R}$ the function $f_a : D \to \mathcal{R}_a$ can be extended on $[0,1]$ preserving monotonicity. In fact, for any $t \in [0,1] \setminus D$ there are sequences $(t'_n)_{n\in\mathbb{N}}$, $(t''_n)_{n\in\mathbb{N}}$ in D such that $t'_n \uparrow t$ and $t''_n \downarrow t$. Then being f_a increasing implies that the sequences $(f_a(t'_n))_{n\in\mathbb{N}}$, $(f_a(t''_n))_{n\in\mathbb{N}}$ in $\mathcal{R}_a$ satisfy the estimate

$$f_a(t'_n) \leqslant f_a(t'_{n+1}) \leqslant f_a(t''_{n+1}) \leqslant f_a(t''_n)$$

for each n. Since $\mathcal{R}_a$ is actually an **F**-ring, some $c \in \mathcal{R}_a$ exists such that

$$f_a(t'_n) \leqslant c \leqslant f_a(t''_n) \quad \text{for all } n.$$

It is easily verified that setting $f_a(t) := c$ provides an extension of f_a to $[0,1]$ fulfilling the required properties. This ends the proof of the theorem in the case when $N(\varphi) = \{0\}$.

For $N(\varphi) \neq \{0\}$, according to **(i)** in Remarks 4.3 and Lemma 4.1, consider the function $\hat{\varphi}$ defined on $\hat{\mathcal{R}} = \mathcal{R}/N(\varphi)$ by (8). Such a $\hat{\varphi}$ is s-outer and locally exhaustive, because of (ii)–(iii) of Theorem 4.4, and $N(\hat{\varphi}) = \{\hat{0}\}$. Moreover, $\hat{\varphi}$ is strongly nonatomic if (and only if) φ enjoys $(\mathcal{C}_2)$, by (iii) of Theorem 4.4. Besides, $\hat{\mathcal{R}}$ is an **F**-ring if $\mathcal{R}$ does, since the canonical map $a \in \mathcal{R} \mapsto \hat{a} \in \hat{\mathcal{R}}$ is an order-preserving Boolean homomorphism. Thus the result follows from the previous case just replacing φ with $\hat{\varphi}$ and using (8) to conclude. □

Incidentally, let us notice that proofs of Theorems 5.1–5.2 display the role of null-additivity for the reference function φ. That is, null-additivity for φ suffices to reduce matters to the consideration of a 'faithful' φ. The replacement of φ with $\hat{\varphi}$ is in fact allowed via Lemma 4.1, and Theorem 4.4 guarantees the invariance under such a replacement of properties taken into account. Null-additivity does not appear explicitly in Theorem 5.2, unlike Theorem 5.1, just because s-outer functions are always null-additive.

6. Lyapunov-type results for s-outer functions

We are now ready to establish connectedness for ranges of strongly nonatomic s-outer functions $\varphi : \mathcal{R} \to \mathcal{S}$. Henceforth, in this section $\mathcal{S}$ will always stand for a Hausdorff uniform space $(S, \mathcal{U})$.

The investigation of the non-metrizable case for $\mathcal{S}$ employs the notion of U-chain, allowed by the uniform structure of $\mathcal{S}$. Recall that for any $U \in \mathcal{U}$ and points $y, z \in S$, a U-chain from y to z is a finite ordered set of points of $\mathcal{S}$, say $s_o = y, s_1, \ldots, s_n = z$, such that $(s_{i-1}, s_i) \in U$ for each $i \in \{1, \ldots, n\}$.

Theorem 6.1. *Let $\varphi : \mathcal{R} \to \mathcal{S}$ be a locally exhaustive s-outer function, where $\mathcal{R}$ is an **F**-ring.*

(i) *If φ is strongly nonatomic and each of the sets $\varphi(\mathcal{R}_a)$ is compact for $a \in \mathcal{R}$, then the range $\varphi(\mathcal{R})$ is connected.*

(ii) *When $\mathcal{S}$ is metrizable, then φ is strongly nonatomic if, and only if, $\varphi(\mathcal{R}_a)$ is arcwise connected for every $a \in \mathcal{R}$.*
*Thus, in particular, strongly nonatomicity for φ implies $\varphi(\mathcal{R})$ is arcwise connectedness. The converse holds for $\mathcal{R}$ being an **F**-algebra.*

Proof. (i) Under the made assumption, it suffices to show for each $a \in \mathcal{R}$ the connectedness of $\varphi(\mathcal{R}_a)$ with respect to its relative uniformity. Indeed, $\varphi(\mathcal{R}) = \bigcup_{a\in\mathcal{R}} \varphi(\mathcal{R}_a)$ and each $\varphi(\mathcal{R}_a)$ includes $\varphi(0)$.

Let $a \in \mathcal{R}$ be arbitrarily fixed. Since $\varphi(\mathcal{R}_a)$ is assumed to be compact, its connectedness is actually equivalent to its chained connectedness (see e.g. [5, Proposition 6, p. 204]), namely, to the validity of the equality

$$\varphi(\mathcal{R}_a) = \bigcap_{U\in\mathcal{U}} C_U\big[\varphi(0)\big], \tag{10}$$

where $U \in \mathcal{U}$ is symmetric and

$$C_U\big[\varphi(0)\big] := \big\{z \in \varphi(\mathcal{R}_a)\colon \text{ there is a } U\text{-chain from } \varphi(0) \text{ to } z\big\}.$$

Note that for any symmetric $U \in \mathcal{U}$ there exist at least a pseudometric p belonging to the gauge of $\mathcal{U}$ and some $\epsilon > 0$ such that

$$U_o := \big\{(x, y) \in S \times S\colon\ p(x, y) < \epsilon\big\} \subseteq U.$$

So Theorem 5.2 – in the case when $\mathcal{R}$ is an **F**-ring – provides for any $x \in \mathcal{R}_a$ the existence of a $\mathcal{U}_p$-path $\gamma_x : [0, 1] \to \varphi(\mathcal{R}_a)$ joining $\varphi(x)$ to $\varphi(0)$. The $\mathcal{U}_p$-uniform continuities of such paths are uniform with respect to $x \in \mathcal{R}_a$. This enables us to get the existence of some $\delta > 0$, depending only on U_o, such that for any partition $\{t_0, \ldots, t_n\}$ of $[0, 1]$ whose mesh is less than δ one has $(\gamma_x(t_{i-1}), \gamma_x(t_i)) \in U_o \subseteq U$ for all i and $x \in \mathcal{R}_a$. Thus, $\varphi(\mathcal{R}_a) = C_U[\varphi(0)]$. Hence (10) holds and the proof ends.

(ii) Let ρ be a metric on S such that $\mathcal{U} = \mathcal{U}_\rho$. By Theorem 5.2, the strong nonatomicity for φ is actually equivalent to the fact that for every $a \in \mathcal{R}$ and $x \in \mathcal{R}_a$ there is a path $\gamma_x : [0, 1] \to \varphi(\mathcal{R}_a)$ joining $\varphi(x)$ to $\varphi(0)$. Since $\mathcal{S}$ is Hausdorff, this in turn is equivalent to the fact that each of the sets $\varphi(\mathcal{R}_a)$ is arcwise connected for $a \in \mathcal{R}$. Hence, in particular, strong nonatomicity for φ implies that $\varphi(\mathcal{R}) = \bigcup_{a\in\mathcal{R}} \varphi(\mathcal{R}_a)$ is arcwise connected, since $\varphi(\mathcal{R}_a) \ni \varphi(0)$ for all $a \in \mathcal{R}$. □

Remarks 6.2. **(i)** The assumption of $\mathcal{R}$ being an **F**-ring in Theorem 6.1 cannot be disregarded. Indeed, without such an assumption, Theorem 6.1 fails. As an example, consider the algebra $\mathcal{R}$ generated by the collection of closed intervals with rational end-points of $[0, 1]$. Such an algebra does not clearly satisfy the interpolation property. Then let φ be the restriction of the Lebesgue measure to $\mathcal{R}$. Of course, φ is s-outer and locally exhaustive. Moreover, φ is (locally) strongly continuous, as one can easily check. Thus, by Theorem 5.1, φ is strongly nonatomic. Finally, notice that the range of φ coincides with $\mathbb{Q} \cap [0, 1]$, which is not connected. **(ii)** On replacing in (i) of Theorem 6.1 the assumption of compactness of each $\varphi(\mathcal{R}_a)$ by that of their relative compactness, the connectedness of the closure of $\varphi(\mathcal{R})$ shall be stated. **(iii)** Part (ii) of Theorem 6.1 holds even when the assumption of metrizability of $\mathcal{S}$ is replaced by that of metrizability of each of the sets $\varphi(\mathcal{R}_a)$ for $a \in \mathcal{R}$. The latter is satisfied by σ-additive functions defined on δ-rings and with values in Hausdorff topological groups. These functions are indeed locally exhaustive, and enjoy the countable chain condition, namely, any family of pairwise disjoint non-negligible elements is (at most) countable. Hence, the metrizability of each $\varphi(\mathcal{R}_a)$ follows from [22, Lemma].

7. Nonatomicity and strong nonatomicity for s-outer functions

In our general non-additive context, notions of atom and nonatomicity for functions can be formulated as follows. Given any $\varphi : \mathcal{R} \to \mathcal{S}$, an element $a \in \mathcal{R} \setminus N(\varphi)$ is an *atom* for φ (a φ-atom, for short) whenever $\varphi(a) \neq \varphi(0)$, and either x or $a \setminus x$ belongs to $N(\varphi)$ for each $x \in]0, a[$. Then φ is called *nonatomic* if there are no φ-atoms.

Remarks 7.1. **(i)** For a nonatomic function φ, the set $At(\mathcal{R})$ of atoms of $\mathcal{R}$ is contained into the kernel of φ, i.e. $At(\mathcal{R}) \subseteq N(\varphi)$. **(ii)** For a null-additive function φ, if a is a φ-atom, then $\varphi = \varphi(a)$ on the set $[a \setminus y, a] := \{z \in \mathcal{R}\colon\ a \setminus y \leqslant z \leqslant a\}$ for all $y \in \mathcal{R}_a \cap N(\varphi)$. Hence, $\varphi(\mathcal{R}_a) = \{\varphi(0), \varphi(a)\}$. Thus, as far as null-additive functions are taken into account, nonatomicity is a necessary condition for the connectedness of any set $\varphi(\mathcal{R}_a)$ for $a \in \mathcal{R}$.

Our goal in this final section is to compare the notion of nonatomicity with that of strong nonatomicity considered throughout this note. As mentioned in Section 3, the latter is naturally patterned on the fact that, roughly speaking, an atom for a classical measure φ determines at least an ultrafilter $\mathcal{F}$ such that $\lim_{b\in\mathcal{F}}\varphi(b)\neq\varphi(0)$. We focus on s-outer functions, in view of Theorem 6.1 and its applications. These functions are null-additive (see **(i)** in Remarks 4.3) and thereby item **(ii)** in Remarks 7.1 does hold for them. Moreover a standard argument enables us to state

Lemma 7.2. *Let $\varphi:\mathcal{R}\to\mathcal{S}$ be a locally exhaustive s-outer function. For any decreasing net $(b_\lambda)_{\lambda\in\Lambda}$ in $\mathcal{R}$, the net $(\varphi(b_\lambda\wedge x))_{\lambda\in\Lambda}$ is Cauchy, uniformly with respect to $x\in\mathcal{R}$.*

In particular, for any ultrafilter $\mathcal{F}$ of $\mathcal{R}$ the net $(\varphi(b\wedge x))_{b\in\mathcal{F}}$ converges in the completion of $\mathcal{S}$, uniformly with respect to $x\in\mathcal{R}$.

Our next result shows that [27, Theorem 1] – which concerns the case of bounded non-negative finitely additive functions – can be extended verbatim to the (non-additive) context of locally exhaustive s-outer functions.

Lemma 7.3. *Let $\varphi:\mathcal{R}\to\mathcal{S}$ be a locally exhaustive s-outer function.*

(i) *If a is a φ-atom, then there exists a unique ultrafilter $\mathcal{F}_a$ of $\mathcal{R}$ such that $a\in\mathcal{F}_a$, and $\varphi(a)$ is the limit of the net $(\varphi(b))_{b\in\mathcal{F}_a}$.*
(ii) *For any φ-atoms a and b, $\mathcal{F}_a=\mathcal{F}_b$ if, and only if, $a\Delta b\in\mathcal{N}(\varphi)$.*
(iii) *If φ is strongly nonatomic, then φ is nonatomic.*

Proof. (i) Let $a\in\mathcal{R}$ be a φ-atom, and define $\mathcal{F}_a:=\{x\in\mathcal{R}\colon\ a\setminus x\in N(\varphi)\}$. It is not difficult to check that $\mathcal{F}_a$ is an ultrafilter in $\mathcal{R}\setminus N(\varphi)$. Computation are left to the reader. Clearly, $a\in\mathcal{F}_a$. Moreover, by Lemma 7.2 together with **(ii)** in Remarks 7.1 for $a\notin At(\mathcal{R})$, the net $(\varphi(b))_{b\in\mathcal{F}_a}$ converges to $\varphi(a)$. The uniqueness part follows directly from properties of φ-atoms and ultrafilters.

(ii) The statement trivially follows by applying the above-definition of $\mathcal{F}_a$ and the fact that $N(\varphi)$ is an ideal of $\mathcal{R}$ via Lemma 3.3.

(iii) Let φ be strongly nonatomic. If a φ-atom existed, then (i) would imply the existence of an ultrafilter $\mathcal{F}\subseteq\mathcal{R}$ such that $\lim_{b\in\mathcal{F}}\varphi(b)\neq\varphi(0)$, which is a contradiction. Thus, φ is nonatomic. □

It is well-known that the converse of part (iii) fails since bounded non-negative finitely additive functions defined on rings are considered (see e.g. [10, p. 152], [26]).

Conditions on the underlying Boolean domain $\mathcal{R}$ and on the locally exhaustive s-outer function φ ensuring the converse are now exhibited.

For this, let us recall that a function $\varphi:\mathcal{R}\to\mathcal{S}$ is said to be *order continuous* whenever $\lim_k\varphi(b_k)=\varphi(0)$ for all decreasing sequence $(b_k)_{k\in\mathbb{N}}$ in $\mathcal{R}$ whose infimum is 0. Let us notice that Maharam submeasures as well as uniform vector-valued Dobrakov submeasures (see e.g. [18]) are special instances of order-continuous s-outer functions. The order-continuity requirement is in fact integrated in their own definitions.

Theorem 7.4. *Let $\varphi:\mathcal{R}\to\mathcal{S}$ be a locally exhaustive s-outer function, where $\mathcal{R}$ is a δ-ring and $\mathcal{S}$ metrizable. If φ is order-continuous, then the following are equivalent*:

($\mathcal{A}$-f) *φ is nonatomic*;
($\mathcal{C}_2$) *φ is strongly nonatomic.*

Proof. The implication $(\mathcal{C}_2)\Rightarrow(\mathcal{A}\text{-f})$ is the content of (iii) in Lemma 7.3, and holds without the additional assumptions that $\mathcal{R}$ is a δ-ring and $\mathcal{S}$ is metrizable. To prove the converse, assume, by way of contradiction, that φ is not strongly nonatomic. According to Lemma 7.2, this means that at least an ultrafilter $\mathcal{F}$ of $\mathcal{R} \setminus N(\varphi)$ exists such that

$$\lim_{b\in\mathcal{F}} \varphi(b \wedge x) = \xi \neq \varphi(0),$$

uniformly with respect to $x \in \mathcal{R}$, where ξ belongs to the completion of $\mathcal{S}$. Hence, for any decreasing sequence $(b_k)_{k\in\mathbb{N}}$ in $\mathcal{F}$,

$$\lim_{k\in\mathbb{N}} \varphi(b_k \wedge x) = \xi \neq \varphi(0) \tag{11}$$

uniformly with respect to $x \in \mathcal{R}$.

Notice that, since $\mathcal{R}$ is a δ-ring, $b_o := \bigwedge_{k\in\mathbb{N}} b_k$ now exists and belongs to $\mathcal{R}$.

First, we claim that $b_o \notin \mathcal{N}(\varphi)$ and, in particular, $\varphi(b_o) = \xi \neq \varphi(0)$. In fact, $(b_k \setminus b_o)_{k\in\mathbb{N}}$ is a decreasing sequence in $\mathcal{R}$ whose infimum is 0. Therefore, coupling the order continuity for φ with its local exhaustivity provides that

$$\lim_{k\in\mathbb{N}} \varphi\big((b_k \setminus b_o) \wedge x\big) = \varphi(0),$$

uniformly with respect to $x \in \mathcal{R}$. From this, using the fact that φ is s-outer provides that

$$\lim_{k\in\mathbb{N}} \varphi(b_k \wedge x) = \varphi(b_o) \tag{12}$$

uniformly with respect to $x \in \mathcal{R}$. The claim thus just follows from coupling (11) and (12).

Second, we observe that $b_o \in \mathcal{F}$. Indeed, if $b_o \notin \mathcal{F}$, then $(b_k \setminus b_o)_{k\in\mathbb{N}}$ would be a decreasing sequence in $\mathcal{F}$ with infimum 0. An application of the order continuity of φ would thus yield that $\lim_{k\in\mathbb{N}} \varphi(b_k \wedge x) = \varphi(0)$, namely, a contradiction to (11).

In the case when b_o is an atom of $\mathcal{R}$, the proof ends, since b_o is actually a φ-atom and this contradicts the nonatomicity of φ.

Assume henceforth that $b_o \notin At(\mathcal{R})$, and consider a finitely additive function $\mu : \mathcal{R} \to G$, where G is a topological group, such that $\mu \approx \varphi$. Its existence is established by Theorem 3.2. Notice that μ is locally exhaustive, via Lemma 3.1. Moreover, μ is σ-additive and, in particular, order-continuous by [7, Lemma 3.1.3], being φ order continuous. Finally, μ fails to be strongly nonatomic according to Lemma 3.1 and the fact that – by way of contradiction – φ is not strongly nonatomic.

It is easily verified – on applying the fact that μ is φ-continuous – that, with respect to the above-described $\mathcal{F}$, $(b_k)_{k\in\mathbb{N}}$ and b_o, the behavior of μ is as follows:

$$\lim_{b\in\mathcal{F}} \mu(b \wedge x) = y \neq 0 \tag{13}$$

uniformly with respect to $x \in \mathcal{R}$, where y belongs to the completion of G;

$$\lim_{k\in\mathbb{N}} \mu(b_k \wedge x) = \mu(b_o) \neq 0 \tag{14}$$

uniformly with respect to $x \in \mathcal{R}$; $b_o \notin \mathcal{N}(\mu)$ according to (5) and the first claim above.

Because $b_o \notin At(\mathcal{R})$, there is some $x \in \mathcal{R}$ such that $0 < x < b_o$. Being $\mathcal{F}$ an ultrafilter of $\mathcal{R}$, either x or $b_o \setminus x$ does belong to $\mathcal{F}$. Assume the latter. By (13) and (14),

$$\mu(b_o \setminus x) = \lim_{k\in\mathbb{N}} \mu(b_k \setminus x) = y = \mu(b_o).$$

Hence, since μ is additive, $\mu(x) = 0$. As $[b_o \setminus x, b_o] \subset \mathcal{F}$ for $b_o \setminus x \in \mathcal{F}$, one thus concludes that $x \in \mathcal{N}(\mu)$. The arbitrariness of $x \in \mathcal{R}$ fulfilling the condition $0 < x < b_o$ tells us that b_o is a μ-atom. Therefore, via (5), b_o is a φ-atom as well. This contradicts the nonatomicity of φ, and proves that ($\mathcal{A}$-f) ⇒ ($\mathcal{C}_2$). □

Theorem 7.4 in particular generalizes [27, Theorem 2], which concerns the case of finite (non-negative) classical measures on σ-algebras.

By combining Theorems 5.1, 6.1 and 7.4, one thus infers

Corollary 7.5. *For any order-continuous, locally exhaustive, s-outer function $\varphi : \mathcal{R} \to \mathcal{S}$, where $\mathcal{R}$ is a δ-ring and $\mathcal{S}$ metrizable, the following are equivalent:*

($\mathcal{A}$-f) *φ is nonatomic;*
($\mathcal{C}_1$) *φ is locally strongly continuous;*
($\mathcal{L}$) *$\varphi(\mathcal{R}_a)$ is arcwise connected for every $a \in \mathcal{R}$.*

Proof. The implication ($\mathcal{A}$-f) ⇒ ($\mathcal{C}_1$) follows directly from Theorem 7.4 and Theorem 5.1, whereas ($\mathcal{C}_1$) ⇔ ($\mathcal{L}$) by coupling Theorem 5.1 and Theorem 6.1. For ($\mathcal{L}$) ⇒ ($\mathcal{A}$-f) it suffices to note that if a φ-atom a existed then $\varphi(\mathcal{R}_a) = \{\varphi(0), \varphi(a)\}$ by item **(ii)** in Remarks 7.1. This would contradict ($\mathcal{L}$). □

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