COADJOINT ORBIT OF THE VIRASORO GROUP AND THE INDEX THEOREM ON DIFF S^1/S^1

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We use the methods of coadjoint orbits and the corresponding representations of the Virasoro group to formulate an index theorem for the BRST operator. A few concrete examples are set forth, one of them implies the vanishing theorem of Frenkel, Garland and Zukerman.

The philosophy underlying the recent upsurge of interest in loop space formulation [1–3] of string theories is that one can gain more intricate structures of string field theory. The beautiful result of Bowick and Rajeev [1] suggests that non-perturbative string equations of motion do indeed emerge through manifestations of geometric quantization of an infinite-dimensional system. However, as Witten had suggested several years ago [4], some refined global quantities on loop space can be used to probe field theory information of a perturbative nature, such as supersymmetry breaking patterns. Bearing this in mind, some intriguing attempts are made in refs. [5,6] to set up an index theorem of the Dirac–Ramond operator in loop space. These studies borrowed the fixed point formulas of Atiyah and coworkers [7], so as to reduce the formula into a finite-dimensional manifold instead of loop space itself. On the other hand, as the Dirac–Ramond operator can be identified with supercharge of a certain supersymmetric $(1+1)$-dimensional non-linear sigma model, it is not clear what is the string nature of the index theorems achieved so far. To explore this question a step further, one may generalize these studies to get another version of the index theorem which is able to incorporate string field theory more closely. An immediate response seems to be an index theorem for the BRST operator. BRST formalisms have been developed mainly to study the constrained systems, they prove especially useful in quantizing string theory which contains infinite degrees of freedom and its constraint algebra is also infinite dimensional. A series of recent mathematical literature [8] pointed out a deep relation between BRST cohomology and some basic results in string theory, e.g. the no-ghost theorem. This hinted that one may arrive at unexpected results along the way towards a better understanding of the BRST formalism. In this paper, we try to set up a more stringy version of fixed point formulas which can be interpreted as an index theorem for the BRST operator. In doing so, we discover the close relation between Virasoro representations, character formulas and the coadjoint orbits of the Virasoro group.

The major difference between the index of a Dirac–Ramond operator in loop space and that of the BRST operator defined as a certain exterior derivative on an elliptic complex arises from the following two facts: (1) the BRST operator, in view of the string field theory proposed by Witten, is dynamical so that unlike the Dirac–Ramond operator, which can be identified with the supercharge, there is no consistent definition of its index in terms of a classical complex such as the de Rham complex over a spacetime manifold; (2) classically, a BRST operator is nilpotent, $Q^2=0$, this means that when we calculate its trace on the space of Q eigenvalues we get exactly zero due to the triviality of cohomology. Thus the only non-trivial way to establish an index theorem for the BRST operator is to consider its quantized version, i.e. take

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the trace over the entire eigenspace consisting of first quantized string states in contrast to the Dirac–Ramond case where loop space is viewed as a configuration space of a closed string. These two facts suggest that a new version of an index theorem should emerge only after manifestations of the algebraic structures of string field theory within the framework of the BRST formalism. We will find in what follows coadjoint orbits associated to the symplectic reduction of the infinite constrained system especially useful in this vein of ideas.

But let us first start with little ambition. The Nambu–Goto action which is an expression of the worldsheet area

$$S=-\int \mathrm{d}\sigma\,\mathrm{d}\tau\sqrt{-\det\partial_\alpha x^\mu\partial_\beta x_\mu}\,, \tag{1}$$

is invariant under reparametrizations generated by

$$L_n=\int_0^{2\pi}\mathrm{d}\sigma\left(\frac{\delta S}{\delta\partial_\tau X_\mu}-\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}\sigma}X^\mu\right)^2\exp(-in\sigma)\,. \tag{2}$$

The quantized version of L_n is

$$L_n=\frac{1}{2}\sum_{-\infty}^{\infty}:\alpha_{n-m}\alpha_m:\,,$$

$$L_0=\tfrac{1}{2}\alpha_0^2+\sum_{n=1}^{\infty}:\alpha_{-n}\alpha_n:\,. \tag{3}$$

What the BRST procedure amounts to is that one rewrites the N–G action in such a way that the two-dimensional metric has a gauge degree of freedom and introduces the ghosts to compensate the BRST transformations, in order to render the action BRST invariant. This indicates that the total action is

$$\mathscr{L}=\mathscr{L}_{\text{inv}}+\mathscr{L}_{\text{gauge-fixing}}+\mathscr{L}_{\text{F-P}}\,. \tag{4}$$

Now the BRST charge takes the form

$$\begin{aligned}Q&=\tfrac{1}{2}i\pi\int_{-\pi}^{\pi}\mathrm{d}\sigma\, j_{\text{BRST}}(\sigma)\\&=\sum_{-\infty}^{\infty}:(L_{-m}^{(\alpha)}+\tfrac{1}{2}L_{-m}^{(c)}-a\delta_m)c_m:\\&=\sum_{-\infty}^{\infty}L_{-m}^{(\alpha)}c_m-\tfrac{1}{2}\sum(m-n):c_{-m}c_{-n}b_{m+n}:-ac_0\,,\end{aligned} \tag{5}$$

which is the first order differential operator of the Virasoro algebra acting on the first quantized string states. Its nilpotency ($Q^2=0$) is equivalent to the positiveness of the Hilbert space norm and thus the no-ghost theorem.

A more formal view of this procedure is in terms of the semi-infinite complex with its exterior derivative precisely the BRST charge. This was shown in ref. [8] to be equivalent to the symplectic reduction of an infinite-dimensional hamiltonian system with infinite-dimensional constraint algebra. Because its geometric feature is more fascinating, one expects this to give rise to much deep information about string field theory.

Given a classical system of infinite degrees of freedom, the symplectic reduction is a method to simplify the equations of motion by going to a certain subquotient. Thus let $\mathrm{N}=\mathscr{L}\mathrm{M}$ be the loop space consisting of all closed loops in M, it has the natural interpretation of being the phase space of open bosonic strings. The cotangent manifold T*N is a manifold on which the classical action (1) is defined. This action is highly singular since it contains first class constraints; the typical form of its hamiltonian is

$$H=\int(\pi_\alpha\dot\varphi^\alpha-\mathscr{L}_0+\lambda^n L_n)\,,\quad \pi_\alpha=\frac{\partial\mathscr{L}}{\partial\dot\varphi^\alpha}\,, \tag{6}$$

where λ^n are Lagrange multipliers, and L_n serve as constraints of the form

$$(L_0-1)|\,\rangle=0\,,\quad L_n|\,\rangle=0\,,\quad n>0\,, \tag{7}$$

which constitutes an infinite-dimensional algebra.

It is useful to denote the inverse image of L_n under a certain Hamilton map by C which is a closed subspace in T*N. The basic ingredient of symplectic reduction of a triple (M, C, Φ) together with a Poisson algebra, $\mathscr{G}$. Where M is a canonical phase space in general. For example in the finite-dimensional case, M is spanned by $2k$ canonical coordinates q^i, p_i ($i=1, ..., k$), the Poisson algebra $\mathscr{G}$ is generated by functions on M which obey $f_\mu(q,p)=0$, as

$$\{f_\mu,f_\nu\}=\sum c^\sigma_{\mu\nu}f_\sigma\,. \tag{8}$$

The LHS is defined barring the symplectic structure on M, $\omega=\Sigma\mathrm{d}p^i\wedge\mathrm{d}q^i$. And C is simply the solution space of $f_\mu=0$ which is coisotropic in M in many cases. In (8) the "structure constant" may render $\mathscr{G}$ to be semi-simple, hence there exists a hamiltonian action of G, the corresponding group of $\mathscr{G}$, such that [9] G acts as a symmetry of the symplectic structure ω. The set of all smooth functions on M are naturally endowed with the structure of the Poisson algebra, it is not difficult to show that the Poisson algebra of functions on N (where N is the underlying configuration space) reduces to a subalgebra generated by those functions whose restrictions on C vanish.

Given a hamiltonian action of G, there is associated a moment map defined by

$$\Phi: \mathrm{M} \to \mathscr{G}^* , \tag{9}$$

where $\mathscr{G}^*$ is the space dual to $\mathscr{G}$, and $\mathrm{M}=\mathrm{T}^*\mathrm{N}$. The image $\Phi(m)$ (valued at $\xi\in\mathscr{G}$) consists of the coadjoint orbits Θ of the G action in $\mathscr{G}^*$. It turns out that [9] the coadjoint orbits constitute a symplectic G space. On the other hand, the inverse image $\Phi^{-1}(\Theta)=\mathrm{C}$ is the submanifold of M which can be identified as the coisotropic submanifold described above, modulo a certain technical hypothesis.

The symplectic reduction is in general a process of going from functions on M, $F(\mathrm{M})$ to that on C, $F(\mathrm{C})$ in a G equivariant sense, i.e. there exists the following sequence:

$$0 \to \mathrm{G} \to F(\mathrm{C}) \xrightarrow{\mathrm{d}^*} F(\mathrm{M}) \xrightarrow{\delta} \mathrm{I} \to 0 , \tag{10}$$

where $\mathrm{Ker}\,\mathrm{d}^*=\Phi^{-1}\cdot g$, $g\in\mathscr{G}$; δ is the derivative of the hamiltonian action, and I consists of functions vanishing on C. The corresponding cohomology description via Koszul resolution of the $\mathscr{G}$-module is the statement that

$$H^0_{\mathrm{D}}(\wedge\mathscr{G}^*\otimes\wedge\mathscr{G}\otimes F(\mathrm{M}))=F(\mathrm{C})^{\mathrm{G}} , \tag{11}$$

where D is the derivative of a double complex $\wedge\mathscr{G}^*\otimes\wedge\mathscr{G}\otimes F(\mathrm{M})$, which is similar to the classical BRST complex with D identified as the BRST charge. We see the close relation between the cohomology of the BRST complex and symplectic reduction.

Although the method of symplectic reduction seems trivially when applied to a finite-dimensional system, it is non-trivial when applied to an infinite-dimensional system such as string theory where both the phase space and the constraint group (Virasoro group) are infinite-dimensional. The non-triviality lies on the second cohomology group of the Virasoro algebra.

Let the triple be ($\mathrm{T}^*\mathrm{N}$, C, Φ), where $\mathrm{N}=\mathscr{L}\mathrm{M}$ is the loop space, note that in fact $\mathrm{C}=\Phi^{-1}(0)$, Φ being the moment map for a hamiltonian action of the Virasoro group $\mathrm{Diff}\,\mathrm{S}^1$ (we momentarily ignore the central extension $\widehat{\mathrm{Diff}\,\mathrm{S}^1}$). We wish to study functions on C which are invariant with respect to $\mathrm{G}=\mathrm{Diff}\,\mathrm{S}^1$. To find an example of this function, we note that the moment map

$$\Phi: \mathrm{T}^*\mathrm{N} \to \mathscr{G}^* \tag{12}$$

is a function $\Phi(m)[\xi]$ which is the same as the function $\delta(\xi)(m)$, where for $\xi\in\mathscr{G}$, $\delta(\xi)$ is a linear function valued in M. Thus one can take $\Phi(\mathrm{C})$ to be the specific function which, by definition, vanishes when it is restricted to a coadjoint orbit consisting of a single point, the origin. The vanishing function is of course invariant under $\mathrm{Diff}\,\mathrm{S}^1$; to find non-vanishing functions which are invariant under $\mathrm{Diff}\,\mathrm{S}^1$, one simply lets Φ (as a function) run over all coadjoint orbits, and the requirement that they are $\mathrm{Diff}\,\mathrm{S}^1$ invariant means that $F(\mathrm{C})$ has a quotient structure isomorphic to $\mathrm{Diff}\,\mathrm{S}^1/\mathrm{G}_0$, where G_0 is the isotropic subgroup associated to the null foliation of C. This is because for a hamiltonian G-space M and a moment map Φ:$\mathrm{M}\to\mathscr{G}^*$, there exists [9] a locally constant map

$$\Phi\circ \mathrm{S}_{\mathrm{M}} - \mathrm{Ad}^*(s)\circ\Phi: \mathrm{M}\to\mathscr{G}^* , \tag{13}$$

where $s\in\mathrm{G}$, and Ad^* denotes the coadjoint representation of G in $\mathscr{G}^*$. Among other things, this simply means that one can find a G-equivariant group isomorphism between $F(\mathrm{C})$ and the coadjoint orbits $\mathrm{Diff}\,\mathrm{S}^1/\mathrm{G}_0$ for every element s of G that is left invariant by G_0. Thus there is a correspondence between the coadjoint orbits and the $\mathrm{Diff}\,\mathrm{S}^1$ invariant $F(\mathrm{C})$ (Virasoro $\mathscr{G}$-module). A theorem of Borel, Weil and Kostant asserts that there is a one–one correspondence between the coadjoint orbits and the irreducible representations of the Virasoro group. One is led to interpret $F(\mathrm{C})$ as space of irreducible representations of the Virasoro group. This is natural in view of a recent work of Witten [10] on the classification of possible coadjoint orbits of the Virasoro group where he argued that the possible orbits compatible with geometric quantization are $\mathrm{Diff}\,\mathrm{S}^1/\mathrm{N}$, where $\mathrm{N}=\{\mathrm{I}\}$, S^1, $\mathrm{SL}^{(1)}(2,\mathbb{R})$.

Only given a structure on $F(\mathrm{C})$, this is not enough because it corresponds to the light-cone formalism of a bosonic string. The full covariant version should include ghosts amounting to considering the whole $F(\mathrm{M})$, i.e. an arbitrary $\mathscr{G}$-module not necessarily being irreducible. In view of (11), one wishes to study the BRST cohomology in an infinite-dimensional setting. The analogue of eq. (11) is a beautiful result of Frenkel, Garland and Zukerman [8] about the vanishing theorem of semi-infinite cohomology:

$$H^n_\infty(\mathscr{G}, \mathscr{G}_0, \mathrm{V}) = 0 , \quad n > 0 , \tag{14}$$

$$\dim H^0_\infty(\mathscr{G}, \mathscr{G}_0, \mathrm{V}) = \dim F(C)^{\mathrm{G}} , \tag{15}$$

where V is an arbitrary Fock module. In terms of Virasoro representation theory, this result can be interpreted as singling out only unitary, irreducible Verma modules consisting of highest weight representations of the Virasoro algebra. Unitarity means that the norm of any states of the form $\prod_{n=1}^{\infty} L_{-n}^{a_n}|0\rangle$ should be positive definite:

$$|L_{-n}|0\rangle|^2 = \langle 0|[L_1, L_{-1}]|0\rangle$$
$$= 2\langle 0|L_0|0\rangle = 2h . \tag{16}$$

Thus any states with $h<0$ cannot be unitary.

The analogy of the elliptic complex of (11) is achieved by making the following identification of the BRST operator acting on the complex $(\wedge \mathscr{G}^* \otimes \wedge \mathscr{G} \otimes F(\mathrm{M}), Q)$:

$$Q \to \mathrm{D} = \mathrm{d} + (-1)^p 2\delta ,$$
$$\mathrm{D}: \wedge^p \mathscr{G}^* \otimes \wedge^q \mathscr{G} \otimes F(\mathrm{M})$$
$$\to \wedge^{p+1} \mathscr{G}^* \otimes \wedge^q \mathscr{G} \otimes F(\mathrm{M})$$
$$+ \wedge^p \mathscr{G}^* \otimes \wedge^{q-1} \mathscr{G} F(\mathrm{M}), \quad \mathrm{d}^2 = 0 , \quad \delta^2 = 0$$
$$\Rightarrow \mathrm{D}^2 = 0 . \tag{17}$$

The sole importance of quantization is that

$$\mathrm{D}^2_{\text{quantum}} = H = \mathrm{d} * \delta + * \delta \mathrm{d} , \tag{18}$$

which serves as the Hodge theory laplacian, its algebraic property is mainly embodied in the following projective representation:

$$[\rho(x), [\rho(y), \mathrm{D}^2]] = \{\alpha([x, y]) - [\alpha(x), \alpha(y)]\} . \tag{19}$$

Kostant and Sternberg [11] have shown that this projective representation, corresponding to the 2-cocycle in the second cohomology group, can be paired with another representation with opposite class so as to trivialize cohomology.

From what we gained so far, we know that the BRST complex is no other than the elliptic complex modelled on the space of representations of the Virasoro group, or, what is the same, a Diff S^1-invariant $\mathscr{G}$-module which is, à la Borel, Weil and Kostant, in one-to-one correspondence with the coadjoint orbits Diff S^1/N, with N normal subgroups of Diff S^1, e.g. S^1. Thus to try to get a better understanding of a deep result like (14)–(15) is the same thing as to obtain an index theorem on Diff $\mathrm{S}^1/\mathrm{S}^1$.

However, as we noted from the beginning, there will be difficulties in order to correctly set up, and then prove of course, an index theorem for the BRST operator. What we have in mind is a less ambitious aim, i.e. to use the theory of the representation of an infinite-dimensional Lie group together with the coadjoint orbit method we discussed earlier, to prove certain fixed point formula for the BRST operator within simplified circumstances.

Even for this purpose, we need more structure on the BRST complex. Happily this is provided by the existence of a sort of Poincaré duality on the cohomology level [12]:

$$H^q_{\mathrm{D}} \approx H^{3-q}_{\mathrm{D}} , \quad H^0_{\mathrm{D}} \approx H^3_{\mathrm{D}} = \mathrm{C} . \tag{20}$$

One then has a pairing

$$H^1_{\mathrm{D}} \times H^2_{\mathrm{D}} \to H^3_{\mathrm{D}} = C , \tag{21}$$

under this daulity, the BRST operator is skew-adjoint:

$$Q^* \to -Q .$$

One needs to clarify the eigenvalue space of Q. Recall that Q acts on spaces of highest weight string states by creating ghost number one. Denote the eigenspace of Q with odd (even) ghost number states by $\Gamma_+ (\Gamma_-)$. It is natural to define the index of Q as

$$I(Q) = \dim(\Gamma_+) - \dim(\Gamma_-) ,$$

but now we have to deal with infinite kernels. The more convenient definition is the G-character valued one:

$$F_Q(q) = \mathrm{Tr}_{\Gamma_+} g - \mathrm{Tr}_{\Gamma_-} g , \quad g \in \mathrm{G} , \tag{22}$$

This can be shown to be equal to the Lefshetz number on the BRST complex:

$$L_Q(q) = \mathrm{Tr}_\Gamma(* \cdot g - *^{-1} \cdot g) = \mathrm{Tr}_\Gamma \, \alpha , \tag{23}$$

where $\Gamma = \Gamma_+ \oplus \Gamma_-$, and α is an automorphism of the $\mathscr{G}$-module Γ. The traces in (23) are distributional traces [13,14], so that one can average a smooth operator over G with suitable group measure. This enables us to define

$$\overset{0}{\mathrm{Tr}_\Gamma}\, \alpha = \lim_{Q\to 1} \mathrm{Tr}_\Gamma(\alpha\cdot Q) \tag{24}$$

(where Q should not be confused with the BRST charge). In the most simplified cases, the traces in (23), (24) can be calculated as fixed point formulas on Γ. (As a side remark, we point out that it is possible to calculate the index as defined in (22) directly by mimicking the quantum field theory method, just as what one is doing on the Dirac–Ramond index [5]. For example, the following quantity seems calculable by field theory methods:

$$\mathrm{Tr}_\Gamma(*\cdot \exp(\mathrm{i}Q^2))\ , \tag{25}$$

where Q is the quantum BRST charge of the Virasoro algebra. We have not yet developed a viable method to calculate (25) [15].)

The simplest such fixed point formula seems to be the Lefshetz number of a representation, which is given as follows [13]. Let G be a group and H its normal subgroup, the quotient has a bundle structure. The left action of G on G/H now lifts naturally to an action on the bundle endomorphism T_g:

$$T_g : \mathrm{G}\to \mathrm{Aut}\,\Gamma(l*F)\ , \tag{26}$$

which is called the induced representation. Denote the map in the base space by f, and its bundle lifting by φ, then

$$L(T_g) = \sum_p \frac{\sum_k (-1)^k \,\mathrm{Tr}\,\varphi_p^k}{|\det(1-\mathrm{d}f_p)|} \tag{27}$$

gives the Lefshetz number of the induced representation. Recall that $L(T)=\sum_k(-1)^k\,\mathrm{Tr}\,H^k(T)$, if the higher dimensional cohomology groups vanish, this L automatically reduces to the character formula of the chosen representation:

$$\mathrm{ch}_\lambda(\mathrm{q}) = \sum_n m_\lambda(\lambda - n\delta)\exp(\mathrm{i}\pi n\delta)$$

$$= \sum_n \dim(V^{-n})q^n \tag{28}$$

where $\dim V = m_\lambda$ is defined [16] as the multiplicity of an irreducible representation occurring in a particular $\mathcal{G}$-module.

It is quite natural to compare (27) with the G-character valued index defined in (22), (23). Indeed, by our results in the first half of this paper, the effective way to calculate (23) is just to calculate the Lefshetz number for the representation. This justifies our use of the representation theory of infinite-dimensional Lie algebras in the following two examples to get concrete results out from the proposed Q-index theorem. Actually, one of them will give rise to the famous F–G–Z results, eq. (15).

(1) Let us focus on the fixed point version of (23). Here the fixed point set in Γ can be identified with the Weyl group consisting of reflections of Lie algebra bases. It has a well defined partition function [17] since the Weyl group is a subgroup of GL(F) belonging to the type $\mathrm{A}_l^{(1)}$:

$$\mathrm{ch}_{L(\Lambda)}(q) = \sum_{n\geq 0} m_{L(\Lambda)}(\Lambda - n\delta)q^n$$

$$= \prod_{n\geq 1}(1-q^n)^{-l}\ . \tag{29}$$

Now define the generating function

$$c_\lambda^\Lambda = \exp(-s_{\Lambda,\lambda}\delta)\sum_{n\geq 0} m_{L(\Lambda)}(\Lambda - n\delta)\exp(-n\delta)\ , \tag{30}$$

which is related to $\mathrm{ch}_{L(\Lambda)}$ by

$$\exp(-s_\Lambda\delta)\,\mathrm{ch}_{L(\Lambda)}(q) = \sum_{\lambda\in \mathrm{P}} c_\lambda^\Lambda \Theta_\lambda\ , \tag{31}$$

where Θ_λ is the classical theta function.

The function c_λ^Λ so defined is called a string function, it is a modular form of weight $-l/2$ with respect to the modular group $\Gamma(rm)\cap\Gamma(r(m+g))$ [17]. It is easily shown that under the modular transformation $\tau\to\tau+1$, it picks up a phase corresponding to the conformal anomaly:

$$c_\lambda^\Lambda(\tau+1) = \exp(2\pi\mathrm{i}s_{\Lambda,\lambda})c_\lambda^\Lambda(\tau)\ . \tag{32}$$

$s_{\Lambda,\lambda}$ is defined by

$$s_{\Lambda,\lambda} = \frac{|\Lambda+\rho|^2}{2(m+g)} - \frac{|\rho|^2}{2g} - \frac{\pm|\lambda|^2}{m}\ , \tag{33}$$

When the Lie algebra is chosen as one of the types $\mathrm{A}_l^{(1)}$, $\mathrm{D}_l^{(1)}$, $\mathrm{E}_l^{(1)}$ and $\mathrm{A}_{2l}^{(2)}$ (just as the one chosen in (29)), the quantity $w(s_{\Lambda,\lambda}) = s_{\Lambda_0,\Lambda_0} = -1/24$. Now apply the anomaly cancelation arguments of Schellekens and Warner [18], than one concludes that $s_{\Lambda_0,\Lambda_0}=$ integers n. Picking $n=-1$, one determines the power in (29) to be $l=24$, thus it follows that

$$m_{L(\Lambda)}(\Lambda - n\delta) = p_{(24)}(n) = \dim F(c)^{\mathrm{G}} . \qquad (34)$$

where $p_{(l)}(n)$ is the coefficient of q^n in the expansion of $\prod_{n\geq 1}(1-q^n)^{-l}$, this is the result of ref. [8].

(2) Let V be the $\mathcal{G}$-module identified with the homogeneous G-space, i.e. quotient by its maximal "torus". The Lie algebra decomposes into direct sum of 2-planes E_k, $k=1, ..., m$:

$$\mathcal{G}/\eta = \sum_{k=1}^{m} \mathrm{E}_k . \qquad (35)$$

Pick an automorphism of the Lie algebra (35) corresponding to rotations through angles θ_k in E_k. The group is necessarily abelian. Now evaluating the following trace [17]:

$$\mathrm{tr}_{L(\Lambda)} \exp(2\pi \mathrm{i} \rho^{\vee}/r) , \qquad (36)$$

one gets

$$\prod_{\alpha\in\Delta_+^{\vee}} \sin\frac{\pi}{r}\langle \Lambda+\rho, \alpha\rangle \Big/ \sin\frac{\pi}{r}\langle \rho, \alpha\rangle$$

$$= \prod_{k=1}^{m} \cot(\theta_k/2) , \qquad (37)$$

where we have defined $\theta_k = (2\pi/r)\langle \rho, \alpha\rangle$. In writing (37), we assumed that the highest weight $\Lambda\in\mathrm{P}_+$ is integral divisible by the root vector $\alpha\in\Delta_+^{\vee}$. The eq. (37) coincides with a special fixed point formula obtained in ref. [13].

We conclude by pointing out that the profound structures of the coadjoint orbits of the Virasoro group may provide a clue to understand what is the true gauge invariance of string field theory [12]. It seems that there is a large variety of alternatives to the gauge algebra of the BRST system. The two more promising are the proposal of Witten [12] of the outer derivative algebra and the one advocated by the authors of ref. [19] as the algebra generated by BRST and anti-BRST operators together with the transformation sp(2) relating them. These are better frameworks into which our formalism fits. Work along this direction and the details of the materials included here are in progress [15].

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