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Regularity of solutions of elliptic equations in divergence form in modified local generalized Morrey spaces

V. S. Guliyev[1,2,3] · M. N. Omarova[4,5] · M. A. Ragusa[3,6] · A. Scapellato[6]

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Abstract

Aim of this paper is to prove regularity results, in some Modified Local Generalized Morrey Spaces, for the first derivatives of the solutions of a divergence elliptic second order equation of the form

$$\mathscr{L}u := \sum_{i,j=1}^{n} \left(a_{ij}(x)u_{x_i}\right)_{x_j} = \nabla \cdot f, \qquad \text{for almost all } x \in \Omega$$

where the coefficients a_{ij} belong to the Central (that is, Local) Sarason class CVMO and f is assumed to be in some Modified Local Generalized Morrey Spaces $\widetilde{LM}^{p,\varphi}_{\{x_0\}}$. Heart of the paper is to use an explicit representation formula for the first derivatives of the solutions of the elliptic equation in divergence form, in terms of singular integral operators and commutators with Calderón–Zygmund kernels. Combining the representation formula with some Morrey-type estimates for each operator that appears in it, we derive several regularity results.

Keywords Morrey-type spaces · Integral operators · VMO · Elliptic equations

Mathematics Subject Classification 35B45 · 42B20 · 42B35 · 42B37

1 Introduction and mathematical background

In this note we consider the following divergence form elliptic equation

$$\mathscr{L}u := \sum_{i,j=1}^{n} \left(a_{ij}(x)u_{x_i}\right)_{x_j} = \nabla \cdot f, \qquad \text{for almost all } x \in \Omega \tag{1.1}$$

in a bounded set $\Omega \subset \mathbb{R}^n$, $n \geq 3$.

Extended author information available on the last page of the article

We assume that $\mathscr{L}$ is a linear elliptic operator and its coefficients belong to the space VMO and the vectorial field $f = (f_1, f_2, \ldots, f_n)$ is such that $f_i \in LM^{p,\varphi}$ for $i = 1, \ldots, n$, with $1 < p < \infty$ and φ positive and measurable function. The space VMO was introduced by Sarason and it is the proper subspace of the John-Nirenberg space BMO whose BMO norm over a ball vanishes as the radius of the ball tends to zero.

In the last few years have been studied several differential problems on nonstandard function spaces (see for instance [21–23]) and, in particular, several results have been obtained on Generalized Morrey Spaces (see, for instance, [12]).

Recently, in [5,27,28] the authors studied some regularity results for solutions of linear partial differential equations with discontinuous coefficients in nondivergence form.

Our main result in this paper is the study of local regularity in the Generalized Morrey Spaces $LM^{p,\varphi}$ of the first derivatives of the solutions of the equation under consideration as in the past has been done in L^p−spaces and in $L^{p,\lambda}$−spaces.

See, for instance, [2] where the author obtains local regularity in the classical Lebesgue spaces L^p for the first derivatives of the solutions of the equation with discontinuous coefficients. See, also, [24] in which has been done the same in the Morrey spaces $L^{p,\lambda}$. Hearth of the technique is the use of an integral representation formula for the first derivatives of the solutions of Equation (1.1) and the boundedness in $L^{p,\varphi}$ of some integral operators and commutators appearing in this formula.

Precisely, in this work we apply the boundedness on Generalized *local* Morrey Spaces of singular integral operators and its commutators obtained in [13]. We would like to point out that in the last decades a lot of authors studied the boundedness of such operators in several functional spaces (see e.g. [1,4,14]).

Throughout the paper, we set $d = \sup_{x,y\in\Omega} |x-y|$, $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$ and $\Omega(x,r) = \Omega \cap B(x,r)$. Furthermore, by $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

Let Ω be an open bounded subset of $\mathbb{R}^n$, with $n \ge 3$, and f be a locally integrable function on Ω. We say that f belongs to the John-Nirenberg space BMO of the functions with bounded mean oscillation if

$$\|f\|_* := \sup_B \frac{1}{|B|} \int_B |f(x) - f_B|\,\mathrm{d}x < \infty$$

where B ranges in the set of the balls contained in Ω and f_B is the integral average of f over B, namely

$$f_B := \frac{1}{B} \int_B f(x)\,\mathrm{d}x.$$

We say that the number $\|f\|_*$ is the BMO-norm of f.

If $f \in BMO$ and r is a positive number, we set

$$\eta(r) := \sup_{\substack{x\in\mathbb{R}^n \\ \rho\le r}} \frac{1}{|B_\rho|}\int_{B_\rho} |f(x) - f_{B_\rho}|\,\mathrm{d}x,$$

where B_ρ stands for a ball with radius ρ less than or equal to r. The function $\eta(r)$ is called VMO-modulus of f. We say that $f \in BMO$ is in the space VMO of functions with vanishing mean oscillation if

$$\lim_{r\to 0^+} \eta(r) = 0.$$

In the sequel we denote η_{ij} the VMO-modulus of the coefficient a_{ij} and

$$\eta(r) = \left(\sum_{i,j=1}^{n} \eta_{ij}^2(r)\right)^{\frac{1}{2}}.$$

For further details on the VMO space, we refer the reader to [25] and to the references therein.

The definition of local BMO space is as follows.

Definition 1.1 Let $1 \le q < \infty$. A function $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ is said to belong to the $CBMO^q_{\{x_0\}}(\mathbb{R}^n)$ (central BMO space), if

$$\|f\|_{CBMO^q_{\{x_0\}}} = \sup_{r>0}\left(\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} |f(y) - f_{B(x_0,r)}|^q dy\right)^{1/q} < \infty.$$

We set

$$CBMO^q_{\{x_0\}}(\mathbb{R}^n) = \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n) \;:\; \|f\|_{CBMO^q_{\{x_0\}}} < \infty\}.$$

In [16], Lu and Yang introduced the central BMO space $CBMO^q(\mathbb{R}^n) = CBMO^q_{\{0\}}(\mathbb{R}^n)$. Note that, $BMO(\mathbb{R}^n) \subset CBMO^q_{\{x_0\}}(\mathbb{R}^n)$, $1 \le q < \infty$. The space $CBMO^q_{\{x_0\}}(\mathbb{R}^n)$ can be regarded as a local version of $BMO(\mathbb{R}^n)$, the space of bounded mean oscillation, at the origin. But, they have quite different properties. The classical John-Nirenberg inequality shows that functions in $BMO(\mathbb{R}^n)$ are locally exponentially integrable. This implies that, for any $1 \le q < \infty$, the functions in $BMO(\mathbb{R}^n)$ can be described by means of the condition:

$$\sup_{r>0}\left(\frac{1}{|B|}\int_B |f(y) - f_B|^q dy\right)^{1/q} < \infty,$$

where B denotes an arbitrary ball in $\mathbb{R}^n$. However, the space $CBMO^q_{\{x_0\}}(\mathbb{R}^n)$ depends on q. If $q_1 < q_2$, then $CBMO^{q_2}_{\{x_0\}}(\mathbb{R}^n) \subsetneqq CBMO^{q_1}_{\{x_0\}}(\mathbb{R}^n)$. Therefore, there is no analogy of the famous John-Nirenberg inequality of $BMO(\mathbb{R}^n)$ for the space $CBMO^q_{\{x_0\}}(\mathbb{R}^n)$. One can imagine that the behavior of $CBMO^q_{\{x_0\}}(\mathbb{R}^n)$ may be quite different from that of $BMO(\mathbb{R}^n)$.

Lemma 1.2 ([17]) *Let b be a function in $CBMO^p_{\{x_0\}}(\mathbb{R}^n)$, $1 \le p < \infty$ and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{|B(x_0,r_1)|}\int_{B(x_0,r_1)}|b(y)-b_{B(x_0,r_2)}|^p dy\right)^{\frac{1}{p}} \le C\left(1+\left|\ln\frac{r_1}{r_2}\right|\right)\|b\|_{CBMO^p_{\{x_0\}}},$$

where $C > 0$ is independent of b, r_1 and r_2.

We say that $f \in CBMO^p_{\{x_0\}}$ is in the space $CVMO^p_{\{x_0\}}$ of functions with vanishing mean oscillation if

$$\lim_{r\to 0^+} \eta(r) = 0.$$

The following condition is essential to the proof of the main result of the paper: A function b is said to satisfy the well known mean value inequality if there exists a constant $C > 0$ such that for any ball $B \subset \mathbb{R}^n$

$$\|b(\cdot)-b_B\|_{L^\infty(\mathbb{R}^n)} \lesssim \frac{1}{|B|}\int_B |b(x)-b_B|dx. \tag{1.2}$$

Also, we recall the definition of the classical Morrey Spaces, formulated by Morrey in 1938 in [19].

For $1 < p < \infty$, $0 < \lambda < n$, we say that a measurable function f belong to the Morrey space $L^{p,\lambda}(\Omega)$ if its norm, defined by

$$\|f\|^p_{L^{p,\lambda}(\Omega)} = \sup_{\substack{x\in\Omega\\ \rho>0}} \frac{1}{\rho^\lambda}\int_{B(x,\rho)\cap\Omega}|f(y)|^p \mathrm{d}y$$

is finite.

The first author, Mizuhara and Nakai [6,18,20] extended the previous definition of Morrey Space, introducing the Generalized Morrey Spaces (see, also [7,8,26]).

Definition 1.3 Let $\varphi(x,r)$ be a positive measurable function on $\Omega \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M^{p,\varphi}(\Omega)$ ($WM^{p,\varphi}(\Omega)$) the Generalized Morrey space (the weak Generalized Morrey space), the space of all functions $f \in L^p_{\mathrm{loc}}(\Omega)$ with finite quasinorm

$$\|f\|_{M^{p,\varphi}(\Omega)} = \sup_{\substack{x\in\Omega\\ 0<r<d}} \frac{1}{\varphi(x,r)}\frac{1}{|B(x,r)|^{\frac{1}{p}}}\|f\|_{L^p(\Omega(x,r))}$$

$$\left(\|f\|_{WM^{p,\varphi}(\Omega)} = \sup_{\substack{x\in\Omega\\ 0<r<d}} \frac{1}{\varphi(x,r)}\frac{1}{|B(x,r)|^{\frac{1}{p}}}\|f\|_{WL^p(\Omega(x,r))}\right).$$

According to this definition we obtain, for $0 \le \lambda < n$, the Morrey space $L^{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$L^{p,\lambda} = M^{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

In this note we are interested in the study of regularity properties of solutions to elliptic equations in the local version of Generalized Morrey Spaces. In order to achieve this purpose we need the following definitions.

Definition 1.4 Let $\varphi(x,r)$ be a positive measurable function on $\Omega\times(0,d)$ and $1 \le p < \infty$. Fixed $x_0 \in \Omega$, we denote by by $LM^{p,\varphi}_{\{x_0\}}(\Omega)$ $(WLM^{p,\varphi}_{\{x_0\}}(\Omega))$ the local Generalized Morrey space (the weak local Generalized Morrey space), the space of all functions $f \in L^p_{\text{loc}}(\Omega)$ with finite quasinorm

$$\|f\|_{LM^{p,\varphi}_{\{x_0\}}(\Omega)} = \sup_{0<r<d} \frac{1}{\varphi(x_0,r)} \frac{1}{|B(x_0,r)|^{\frac{1}{p}}} \|f\|_{L^p(\Omega(x_0,r))}$$
$$\left(\|f\|_{WLM^{p,\varphi}_{\{x_0\}}(\Omega)} = \sup_{0<r<d} \frac{1}{\varphi(x_0,r)} \frac{1}{|B(x_0,r)|^{\frac{1}{p}}} \|f\|_{WL^p(\Omega(x_0,r))}\right).$$

Definition 1.5 Let $\varphi(x,r)$ be a positive measurable function on $\Omega\times(0,d)$ and $1 \le p < \infty$. We denote by $\widetilde{M}^{p,\varphi}(\Omega)$ $\left(W\widetilde{M}^{p,\varphi}(\Omega)\right)$ the modified Generalized Morrey space (the modified weak Generalized Morrey space), the space of all functions $f \in L^p(\Omega)$ with finite norm

$$\|f\|_{\widetilde{M}^{p,\varphi}(\Omega)} = \|f\|_{M^{p,\varphi}(\Omega)} + \|f\|_{L^p(\Omega)}$$
$$\left(\|f\|_{W\widetilde{M}^{p,\varphi}(\Omega)} = \|f\|_{WM^{p,\varphi}(\Omega)} + \|f\|_{WL^p(\Omega)}\right).$$

According to this definition we obtain, for $\lambda \ge 0$, the local Morrey Space $LM^{p,\lambda}_{\{x_0\}}$ under the choice $\varphi(x_0,r) = r^{\frac{\lambda-n}{p}}$:

$$LM^{p,\lambda}_{\{x_0\}}(\Omega) = LM^{p,\varphi}_{\{x_0\}}(\Omega)\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}.$$

Definition 1.6 Let $\varphi(x,r)$ be a positive measurable function on $\Omega\times(0,\infty)$ and $1 \le p < \infty$. Fixed $x_0 \in \Omega$, we denote by $\widetilde{LM}^{p,\varphi}_{\{x_0\}}(\Omega)$ $\left(\widetilde{LM}^{p,\varphi}_{\{x_0\}}(\Omega)\right)$ the modified local Generalized Morrey space (the modified weak local Generalized Morrey space), the space of all functions $f \in L^p(\Omega)$ with finite norm

$$\|f\|_{\widetilde{LM}^{p,\varphi}_{\{x_0\}}(\Omega)} = \|f\|_{LM^{p,\varphi}_{\{x_0\}}(\Omega)} + \|f\|_{L^p(\Omega)}$$
$$\left(\|f\|_{W\widetilde{LM}^{p,\varphi}_{\{x_0\}}(\Omega)} = \|f\|_{WLM^{p,\varphi}_{\{x_0\}}(\Omega)} + \|f\|_{WL^p(\Omega)}\right).$$

Remark 1.7 For further details on Local Generalized Morrey Spaces, see for instance [10,11,15].

Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 3$, let us consider

$$\mathscr{L}u \equiv -\sum_{i,j=1}^{n} \left(a_{ij}(x)u_{x_i}\right)_{x_j} = \nabla \cdot f, \qquad \text{a.e. } x \in \Omega, \tag{1.3}$$

and, fixed $x_0 \in \mathbb{R}^n$, we suppose that there exists $p \in]1, +\infty[$ and a positive measurable function φ defined on $\mathbb{R}^n \times (0, \infty)$ such that:

$$f = (f_1, \ldots, f_n) \in \left[LM^{p,\varphi}_{\{x_0\}}(\Omega)\right]^n; \tag{1.4}$$

$$a_{ij}(x) \in L^\infty \cap CVMO^{\max\{p,p'\}}_{\{x_0\}}, \forall i, j = 1, \ldots, n; \tag{1.5}$$

$$a_{ij}(x) = a_{ji}(x), \quad \forall i, j = 1, \ldots, n, \text{ a.a. } x \in \Omega; \tag{1.6}$$

$$\exists \kappa > 0 : \kappa^{-1}|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \kappa|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \text{ a.a. } x \in \Omega. \tag{1.7}$$

We say that a function u is a solution of (1.3) if $u, \partial_{x_i} u \in L^p(\Omega), \forall i = 1, \ldots, n$ and for some $1 < p < \infty$ and

$$\int_\Omega a_{ij}u_{x_i}\varphi_{x_j}\,\mathrm{d}x = -\int_\Omega f_i\varphi_{x_i}\,\mathrm{d}x, \qquad \forall \varphi \in C_0^\infty(\Omega).$$

2 Calderón–Zygmund kernel and preliminary results

In order to present the representation formula for the first derivatives of a solution of 1.3, we find it convenient to present the definition of Calderón–Zygmund kernel:

Definition 2.1 Let $k : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$. We say that $k(x)$ is a Calderón–Zygmund kernel (C-Z kernel) if: .

(1) $k \in C^\infty(\mathbb{R}^n \backslash \{0\})$;
(2) $k(x)$ is homogeneous of degree $-n$;
(3) $\int_\Sigma k(x)\,\mathrm{d}x = 0$, where $\Sigma = \{x \in \mathbb{R}^n : |x| = 1\}$.

Many authors obtained several boundedness results for integral operators involving Calderón–Zygmund kernels. For instance, in [3] the authors studied the boundedness of Calderón–Zygmund singular integral operators and commutators on Morrey Spaces. Recently, in [13] the authors extended the previous results in Generalized Local Morrey Spaces.

The previous theorem was proved using the following important result contained in [10].

Theorem 2.2 *Let $x_0 \in \mathbb{R}^n$, $1 \le q < \infty$, K be a Calderón–Zygmund singular integral operator and the functions φ_1, φ_2 satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x_0,\tau)\, \tau^{\frac{n}{q}}}{t^{\frac{n}{q}+1}}\, dt \le C\, \varphi_2(x_0,r), \tag{2.1}$$

where C does not depend on r. Then for $1 < q < \infty$ the operator K is bounded from $LM^{q,\varphi_1}_{\{x_0\}}(\mathbb{R}^n)$ to $LM^{q,\varphi_2}_{\{x_0\}}(\mathbb{R}^n)$ and for $1 \le q < \infty$ the operator K is bounded from $LM^{q,\varphi_1}_{\{x_0\}}(\mathbb{R}^n)$ to $WLM^{q,\varphi_2}_{\{x_0\}}(\mathbb{R}^n)$. Moreover, for $1 < q < \infty$

$$\|Kf\|_{LM^{q,\varphi_2}_{\{x_0\}}} \le c\, \|f\|_{LM^{q,\varphi_1}_{\{x_0\}}},$$

where c does not depend on x_0 and f and for $q = 1$

$$\|Kf\|_{WLM^{1,\varphi_2}_{\{x_0\}}} \le c\, \|f\|_{LM^{1,\varphi_1}_{\{x_0\}}},$$

where c does not depend on x_0 and f.

Precisely, using the boundedness of the Calderón–Zygmund singular integral operators from $LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ in itself (see [10]), the following theorem is valid that will be crucial in the sequel.

Theorem 2.3 *Let $x_0 \in \mathbb{R}^n$, $1 < p < +\infty$, K be a Calderón–Zygmund singular integral operator and the measurable function $\varphi : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}^+$ satisfy the conditions*

$$\int_r^\infty \left(1 + \ln\frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\, \varphi_2(x_0,r), \tag{2.2}$$

where C does not depend on r and x_0.

If $a \in CBMO^{\max\{p,p'\}}_{\{x_0\}}(\mathbb{R}^n)$, the commutator

$$[a,K](f) = aKf - K(af)$$

is a bounded operator from $LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$ in itself.

Precisely, for every $f \in LM^{p,\varphi}_{\{x_0\}}(\mathbb{R}^n)$, we have

$$\|[a,K](f)\|_{LM^{p,\varphi}_{\{x_0\}}} \le c\|a\|_{CBMO^{\max\{p,p'\}}_{\{x_0\}}} \|f\|_{LM^{p,\varphi}_{\{x_0\}}}.$$

To prove Theorem 2.3, we first give some auxiliary lemmas.

In this section we are going to use the following statement on the boundedness of the weighted Hardy operator

$$H^*_w g(t) := \int_t^d g(s)w(s)ds,\ 0 < t < d < \infty,$$

where w is a fixed function non-negative and measurable on $(0, d)$.

The following lemma was proved in [10], see also [9].

Lemma 2.4 *Let v_1, v_2 and w be positive almost everywhere and measurable functions on $(0, d)$. The inequality*

$$\operatorname*{ess\,sup}_{0<t<d} v_2(t) H_w^* g(t) \le C \operatorname*{ess\,sup}_{0<t<d} v_1(t) g(t) \tag{2.3}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0, d)$ if and only if

$$B := \operatorname*{ess\,sup}_{0<t<d} v_2(t) \int_t^d \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<d} v_1(\tau)} < \infty. \tag{2.4}$$

Moreover, if C^* is the minimal value of C in (2.3), then $C^* = B$.

Remark 2.5 In (2.3) and (2.4) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.

Lemma 2.6 *Let $x_0 \in \mathbb{R}^n$, $1 < p < \infty$, $b \in CBMO_{\{x_0\}}^{\max\{p,p'\}}(\mathbb{R}^n)$ and K be a Calderón–Zygmund singular integral operator. Then the inequality*

$$\|[b, K](f)\|_{L^p(B)} \lesssim \|b\|_{CBMO_{\{x_0\}}^{\max\{p,p'\}}} r^{\frac{n}{p}} \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big) t^{-\frac{n}{p}-1} \|f\|_{L^p(B(x_0,t))} dt$$

holds for any ball $B = B(x_0, r)$ and for all $f \in L^p_{\rm loc}(\mathbb{R}^n)$.

Proof Let $1 < p < \infty$, $b \in BMO(\mathbb{R}^n)$, and K be a Calderón–Zygmund singular integral operator. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\complement_{(2B)}}$. Hence

$$[b, K](f)(x) \equiv J_1 + J_2 + J_3 + J_4 = \big(b(x) - b_B\big)K(f_1)(x)$$
$$- K\Big(\big(b(\cdot) - b_B\big) f_1\Big)(x) + \big(b(x) - b_B\big)K(f_2)(x) - K\Big(\big(b(\cdot) - b_B\big) f_2\Big)(x).$$

We get

$$\|[b, K](f)\|_{L^p(B)} \le \|J_1\|_{L^p(B)} + \|J_2\|_{L^p(B)} + \|J_3\|_{L^p(B)} + \|J_4\|_{L^p(B)}.$$

From the boundedness of K on $L^p(\mathbb{R}^n)$, (1.2) and Lemma 1.2 (see [29] [inequality (1.3)]) it follows that:

$$\begin{aligned}
\|J_1\|_{L^p(B)} &\le \|\big(b(\cdot)-b_B\big)K(f_1)(\cdot)\|_{L^p(B)}\\
&\le \|b(\cdot)-b_B\|_{L^\infty(B)}\|K(f_1)\|_{L^p(B)}\\
&\lesssim |B|^{-1}\|b(\cdot)-b_B\|_{L^1(B)}\,\|f_1\|_{L^p(\mathbb{R}^n)}\\
&\approx |B|^{-1+\frac{1}{p'}}\,\|b(\cdot)-b_B\|_{L^p(B)}\,\|f\|_{L^p(2B)}\,r^{\frac{n}{p}}\int_{2r}^\infty t^{-1-\frac{n}{p}}dt\\
&\lesssim \|b\|_{CBMO^p_{\{x_0\}}}\,r^{\frac{n}{p}}\int_{2r}^\infty t^{-\frac{n}{p}-1}\|f\|_{L^p(B(x_0,t))}dt.
\end{aligned}$$

From (1.2) and Lemma 1.2 (see [29] [inequality (1.3)]) for J_2 we have

$$\begin{aligned}
\|J_2\|_{L^p(B)} &\le \|K\big(b(\cdot)-b_B\big)f_1\|_{L^p(B)}\\
&\lesssim \|b(\cdot)-b_B\|_{L^\infty(B)}\|K(f_1)\|_{L^p(B)}\\
&\lesssim |B|^{-1}\|b(\cdot)-b_B\|_{L^1(B)}\,\|f\|_{L^p(2B)}\\
&\approx |B|^{-1+\frac{1}{p'}}\,\|b(\cdot)-b_B\|_{L^p(B)}\,\|f\|_{L^p(2B)}\,r^{\frac{n}{p}}\int_{2r}^\infty t^{-1-\frac{n}{p}}dt\\
&\lesssim \|b\|_{CBMO^p_{\{x_0\}}}\,r^{\frac{n}{p}}\int_{2r}^\infty t^{-\frac{n}{p}-1}\|f\|_{L^p(B(x_0,t))}dt.
\end{aligned}$$

For J_3, it is known that $x\in B$, $y\in {}^{\complement}(2B)$, which implies $\frac{1}{2}|x_0-y|\le|x-y|\le\frac{3}{2}|x_0-y|$.

By Fubini's theorem and applying Hölder inequality we have

$$\begin{aligned}
|K(f_2)(x)| &\lesssim \int_{^{\complement}(2B)}\frac{|f(y)|}{|x_0-y|^n}dy\\
&\approx \int_{2r}^\infty\int_{2r<|x_0-y|<t}|f(y)|dy\,t^{-1-n}dt\\
&\lesssim \int_{2r}^\infty\int_{B(x_0,t)}|f(y)|dy\,t^{-1-n}dt\\
&\lesssim \int_{2r}^\infty \|f\|_{L^p(B(x_0,t))}\,|B(x_0,t)|^{1-\frac{1}{p}}\,\frac{dt}{t^{n+1}}\\
&\lesssim \int_{2r}^\infty t^{-\frac{n}{p}-1}\,\|f\|_{L^p(B(x_0,t))}\,dt.
\end{aligned}$$

Hence, from Lemma 1.2 we get

$$\begin{aligned}\|J_3\|_{L^p(B)} &= \|\big(b(\cdot)-b_B\big)K(f_2)(\cdot)\|_{L^p(B)}\\ &\lesssim \|b(\cdot)-b_B\|_{L^p(B)} \int_{2r}^{\infty} t^{-\frac{n}{p}-1}\,\|f\|_{L^p(B(x_0,t))}\,dt\\ &\lesssim \|b\|_{CBMO^p_{\{x_0\}}}\, r^{\frac{n}{p}} \int_{2r}^{\infty} t^{-\frac{n}{p}-1}\,\|f\|_{L^p(B(x_0,t))}\,dt.\end{aligned}$$

For $x\in B$ by Fubini's theorem applying Hölder inequality and from Lemma 1.2 we have

$$\begin{aligned}&|K\Big(\big(b(\cdot)-b_B\big)f_2\Big)(x)| \lesssim \int_{\complement_{(2B)}} |b(y)-b_B|\,\frac{|f(y)|}{|x-y|^n}dy\\ &\lesssim \int_{\complement_{(2B)}} |b(y)-b_B|\,\frac{|f(y)|}{|x_0-y|^n}dy\\ &\approx \int_{2r}^{\infty}\int_{2r<|x_0-y|<t} |b(y)-b_B|\,|f(y)|dy\,\frac{dt}{t^{n+1}}\\ &\lesssim \int_{2r}^{\infty}\int_{B(x_0,t)} |b(y)-b_{B(x_0,t)}|\,|f(y)|dy\frac{dt}{t^{n+1}}\\ &\quad+\int_{2r}^{\infty} |b_{B(x_0,r)}-b_{B(x_0,t)}|\int_{B(x_0,t)} |f(y)|dy\frac{dt}{t^{n+1}}\\ &\lesssim \int_{2r}^{\infty} \|(b(\cdot)-b_{B(x_0,t)})\|_{L^{p'}(B(x_0,t))}\,\|f\|_{L^p(B(x_0,t))}\,\frac{dt}{t^{n+1}}\\ &\quad+\int_{2r}^{\infty} |b_{B(x_0,r)}-b_{B(x_0,t)}|\|f\|_{L^p(B(x_0,t))}\,|B(x_0,t)|^{1-\frac{1}{p}}\,t^{-n-1}dt\\ &\lesssim \|b\|_{CBMO^{p'}_{\{x_0\}}} \int_{2r}^{\infty} |B(x_0,t)|^{\frac{1}{p'}}\,\|f\|_{L^p(B(x_0,t))}t^{-n-1}\,dt\\ &\quad+\|b\|_{CBMO^{p'}_{\{x_0\}}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\,t^{-\frac{n}{p}-1}\,\|f\|_{L^p(B(x_0,t))}dt\\ &\lesssim \|b\|_{CBMO^{p'}_{\{x_0\}}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\,t^{-\frac{n}{p}-1}\,\|f\|_{L^p(B(x_0,t))}dt.\end{aligned}$$

□

Remark 2.7 The statement of Theorem 2.3 follows by Lemmas 2.4 and 2.6.

In order to achieve the regularity results, we must prove the following theorem.

Theorem 2.8 *Let Ω be an open bounded subset of $\mathbb{R}^n$, $d=\sup_{x,y\in\Omega}|x-y|<\infty$, $\Omega(x_0,r)=\Omega\cap B(x_0,r)$, $x_0\in\Omega$, $0<r\le d$, $1\le q<p<\infty$, $\frac{1}{q}=\frac{1}{p}+\frac{1}{n}$ and*

$$Tg(x)=\int_{\Omega}\frac{g(y)}{|x-y|^{n-1}}dy.$$

(i) Let $1 < q < \infty$. *If* $g \in L^q(\Omega)$ *such that*

$$\int_r^d t^{-\frac{n}{p}-1} \|g\|_{L^q(\Omega(x_0,t))}\, dt < \infty \quad \text{for all} \quad r \in (0,d), \tag{2.5}$$

then for any $r \in (0,d)$ *the inequality*

$$\|Tg\|_{L^p(\Omega(x_0,r))} \le c r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \|g\|_{L^q(\Omega(x_0,t))}\, dt + c r^{\frac{n}{p}} \|g\|_{L^q(\Omega)} \tag{2.6}$$

holds with constant $c > 0$ *independent of* g, x_0 *and* r.

(ii) Let $q = 1$. *If* $g \in L^1(\Omega)$ *satisfies condition* (2.5), *then for any* $r \in (0,d)$ *the inequality*

$$\|Tg\|_{WL^p(\Omega(x_0,r))} \le c r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \|g\|_{L^1(\Omega(x_0,t))}\, dt + c r^{\frac{n}{p}} \|g\|_{L^1(\Omega)} \tag{2.7}$$

holds with constant $c > 0$ *independent of* g, x_0 *and* r.

Proof Let $1 \le q < p < \infty$. Since

$$r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \|g\|_{L^q(\Omega(x_0,t))}\, dt \ge r^{\frac{n}{p}} \|g\|_{L^q(\Omega(x_0,r))} \int_r^d t^{-\frac{n}{p}-1}\, dt$$
$$\approx \|g\|_{L^q(\Omega(x_0,r))} (d^{\frac{n}{p}} - r^{\frac{n}{p}}), \quad r \in (0,d),$$

we get that

$$\|g\|_{L^q(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \|g\|_{L^q(\Omega(x_0,t))}\, dt + r^{\frac{n}{p}} \|g\|_{L^q(\Omega)}, \quad r \in (0,d). \tag{2.8}$$

(i). Assume that $1 < q < \infty$. Let $r \in (0, d/2)$. We write $g = g_1 + g_2$ with $g_1 = g\chi_{\Omega(x_0,2r)}$ and $g_2 = g\chi_{\Omega\setminus\Omega(x_0,2r)}$. Taking into account the linearity of T, we have

$$\|Tg\|_{L^p(\Omega(x_0,r))} \le \|Tg_1\|_{L^p(\Omega(x_0,r))} + \|Tg_2\|_{L^p(\Omega(x_0,r))}. \tag{2.9}$$

Since $g_1 \in L^q(\Omega)$, in view of (2.8), the boundedness of T from $L^q(\Omega)$ to $L^p(\Omega)$ implies that

$$\|Tg_1\|_{L^p(\Omega(x_0,r))} \le \|Tg_1\|_{L^p(\Omega)} \lesssim \|g_1\|_{L^q(\Omega)} \approx \|g\|_{L^q(\Omega(x_0,2r))}$$
$$\lesssim r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \|g\|_{L^q(\Omega(x_0,t))}\, dt + r^{\frac{n}{p}} \|g\|_{L^q(\Omega)}, \tag{2.10}$$

where the constant is independent of g, x_0 and r.

We have

$$|Tg_2(x)| \lesssim \int_{\Omega\setminus\Omega(x_0,2r)} \frac{|g(y)|}{|x-y|^{n-1}}\, dy, \qquad x \in \Omega(x_0,r).$$

It is clear that $x \in \Omega(x_0,r)$, $y \in \Omega\backslash(\Omega(x_0,2r))$ implies $\frac{1}{2}|x_0-y| \le |x-y| < \frac{3}{2}|x_0-y|$. Therefore we obtain that

$$\|Tg_2\|_{L^p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{\Omega\backslash(\Omega(x_0,2r))} \frac{|g(y)|}{|x_0-y|^{n-1}}\,dy.$$

By Fubini's theorem, we get that

$$\begin{aligned}
&\int_{\Omega\backslash\Omega(x_0,2r)} \frac{|g(y)|}{|x_0-y|^{n-1}}\,dy \\
&\approx \int_{\Omega\backslash\Omega(x_0,2r)} |g(y)| \left(1+\int_{|x_0-y|}^{d} \frac{ds}{s^n}\right) dy \\
&= \int_{\Omega\backslash\Omega(x_0,2r)} |g(y)|\,dy + \int_{\Omega\backslash\Omega(x_0,2r)} |g(y)| \left(\int_{|x_0-y|}^{d} \frac{ds}{s^n}\right) dy \\
&= \int_{\Omega\backslash\Omega(x_0,2r)} |g(y)|\,dy + \int_{2r}^{d} \left(\int_{2r\le|x_0-y|\le s} |g(y)|\,dy\right) \frac{ds}{s^n} \\
&\le \int_{\Omega} |g(y)|\,dy + \int_{2r}^{d} \left(\int_{\Omega(x_0,s)} |g(y)|\,dy\right) \frac{ds}{s^n}.
\end{aligned}$$

Applying Hölder's inequality, we obtain

$$\int_{\Omega\backslash\Omega(x_0,2r)} \frac{|g(y)|}{|x_0-y|^{n}}\,dy \lesssim \|g\|_{L^q(\Omega)} + \int_{2r}^{d} s^{-\frac{n}{p}-1} \|g\|_{L^q(\Omega(x_0,s))}\,ds.$$

Thus the inequality

$$\|Tg_2\|_{L^p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{r}^{d} s^{-\frac{n}{p}-1} \|g\|_{L^q(\Omega(x_0,s))}\,ds + r^{\frac{n}{p}} \|g\|_{L^q(\Omega)} \tag{2.11}$$

holds for all $r \in (0, d/2)$ for $q \ge 1$.

Finally, combining (2.10) and (2.11), we obtain that

$$\|Tg\|_{L^p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{r}^{d} s^{-\frac{n}{p}-1} \|g\|_{L^q(\Omega(x_0,s))}\,ds + r^{\frac{n}{p}} \|g\|_{L^q(\Omega)}$$

holds for all $r \in (0, d/2)$ with a constant independent of f, x_0 and r.

Let now $r \in [d/2, d)$. Then, using $(L^q(\Omega), L^p(\Omega))$-boundedness of T, we obtain

$$\|Tg\|_{L^p(\Omega(x_0,r))} \le \|Tg\|_{L^p(\Omega)} \lesssim \|g\|_{L^q(\Omega)} \approx r^{\frac{n}{p}} \|g\|_{L^q(\Omega)},$$

and inequality (2.6) holds.

(ii). Assume that $q = 1$. Let again $r \in (0, d/2)$. We write $g = g_1 + g_2$ with $g_1 = g\chi_{\Omega(x_0,2r)}$ and $g_2 = g\chi_{\Omega\backslash\Omega(x_0,2r)}$. Taking into account the linearity of T, we have

$$\|Tg\|_{L^p(\Omega(x_0,r))} \le \|Tg_1\|_{L^p(\Omega(x_0,r))} + \|Tf_2\|_{L^p(\Omega(x_0,r))}. \tag{2.12}$$

Since $g_1 \in L^q(\Omega)$, in view of (2.8), the boundedness of T from $L^1(\Omega)$ to $WL^p(\Omega)$ implies that

$$\begin{aligned}\|Tg_1\|_{WL^p(\Omega(x_0,r))} &\le \|Tg_1\|_{WL^p(\Omega)} \lesssim \|g_1\|_{L^1(\Omega)} \approx \|g\|_{L^1(\Omega(x_0,2r))}\\ &\lesssim r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \|g\|_{L^1(\Omega(x_0,t))}\,dt + r^{\frac{n}{p}} \|g\|_{L^1(\Omega)},\end{aligned} \tag{2.13}$$

where the constant is independent of f, x_0 and r.

On the other hand, since

$$\|Tg_2\|_{WL^p(\Omega(x_0,r))} \le \|Tg_2\|_{L^p(\Omega(x_0,r))}$$

using (2.11), we get that

$$\|Tg_2\|_{WL^p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_r^d s^{-\frac{n}{p}-1} \|g\|_{L^1(\Omega(x_0,s))}\,ds + r^{\frac{n}{p}} \|g\|_{L^1(\Omega)} \tag{2.14}$$

holds true for all $r \in (0, d/2)$.

Combining (2.12), (2.13) and (2.14), we see that inequality (2.7) holds true for all $r \in (0, d/2)$ with a constant independent of g, x_0 and r.

If $r \in [d/2, d)$, then, using the boundedness of T from $L^1(\Omega)$ to $WL^p(\Omega)$, we obtain that

$$\|Tg\|_{WL^p(\Omega(x_0,r))} \le \|Tg\|_{WL^p(\Omega)} \lesssim \|g\|_{L^1(\Omega)} \approx r^{\frac{n}{p}} \|g\|_{L^1(\Omega)},$$

and, inequality (2.7) holds. □

In order to achieve the regularity results, we must prove the following theorem.

Theorem 2.9 *Let Ω be an open bounded subset of $\mathbb{R}^n$, $x_0 \in \Omega$, $1 \le q < p < \infty$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{n}$. Let also $\varphi_1(x,r)$ and $\varphi_2(x,r)$ two positive measurable functions defined on $\Omega \times (0,d)$ such that the following condition is fulfilled:*

$$\int_r^d \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_2(x_0,\tau)\,\tau^{\frac{n}{q}}}{t^{\frac{n}{p}+1}}\,dt \le C\,\varphi_1(x_0,r), \tag{2.15}$$

where C does not depend on r. Then, in the case $q > 1$ for every $g \in \widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(\Omega)$, the function $Tg(x)$ is a.e. defined, Tg belongs to the space $\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(\Omega)$ and there exists $c = c(q, \varphi_1, \varphi_2, n) > 0$ such that

$$\|Tg\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(\Omega)} \le c\|g\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(\Omega)}.$$

In the case $q = 1$ the function Tg belongs to the space $\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(\Omega)$ and there exists $c = c(\varphi_1, \varphi_2, n) > 0$ such that

$$\|Tg\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(\Omega)} \le c\|g\|_{\widetilde{LM}^{1,\varphi_2}_{\{x_0\}}(\Omega)}.$$

Proof By Theorem 2.8 and Theorem 2.4 with $v_2(r) = \varphi_1(x_0, r)^{-1}$, $v_1(r) = \varphi_2(x_0, r)^{-1} r^{-\frac{n}{q}}$ and $w(r) = r^{-\frac{n}{p}}$ for $q > 1$ we have

$$\begin{aligned}
\|Tg\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(\Omega)} &\lesssim \sup_{0<r<d} \varphi_1(x_0, r)^{-1} \int_r^d \|f\|_{L^q(\Omega(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}} + \|Tg\|_{L^p(\Omega)} \\
&\lesssim \sup_{0<r<d} \varphi_2(x_0, r)^{-1} r^{-\frac{n}{q}} \|g\|_{L^q(\Omega(x_0,r))} + \|g\|_{L^q(\Omega)} \\
&= \|g\|_{LM^{q,\varphi_2}_{\{x_0\}}(\Omega)} + \|g\|_{L^q(\Omega)} \\
&= \|g\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(\Omega)}
\end{aligned}$$

and for $q = 1$

$$\begin{aligned}
\|Tg\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(\Omega)} &\lesssim \sup_{0<r<d} \varphi_1(x_0, r)^{-1} \int_r^d \|f\|_{L^1(\Omega(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}} + \|Tg\|_{L^p(\Omega)} \\
&\lesssim \sup_{0<r<d} \varphi_2(x_0, r)^{-1} r^{-n} \|g\|_{L^1(\Omega(x_0,r))} + \|g\|_{L^1(\Omega)} \\
&= \|g\|_{LM^{1,\varphi_2}_{\{x_0\}}(\Omega)} + \|g\|_{L^1(\Omega)} \\
&= \|g\|_{\widetilde{LM}^{1,\varphi_2}_{\{x_0\}}(\Omega)}.
\end{aligned}$$

□

From Theorem 2.9 we get the following corollary.

Corollary 2.10 *Let Ω be an open bounded subset of $\mathbb{R}^n$, $1 \le q < p < \infty$, $\frac{1}{q} = \frac{1}{p} + \frac{1}{n}$. Let also $\varphi_1(x, r)$ and $\varphi_2(x, r)$ two positive measurable functions defined on $\Omega \times (0, d)$ such that the following condition is fulfilled:*

$$\int_r^d \frac{\operatorname*{ess\,inf}_{t<\tau<d} \varphi_2(x, \tau)\, \tau^{\frac{n}{q}}}{t^{\frac{n}{p}+1}}\, dt \le C\, \varphi_1(x, r), \tag{2.16}$$

where C does not depend on x and r. Then, in the case $q > 1$ for every $g \in \widetilde{M}^{q,\varphi_2}(\Omega)$, the function $Tg(x)$ is a.e. defined, Tg belongs to the space $\widetilde{M}^{p,\varphi_1}(\Omega)$ and there exists $c = c(q, \varphi_1, \varphi_2, n) > 0$ such that

$$\|Tg\|_{\widetilde{M}^{p,\varphi_1}(\Omega)} \le c\|g\|_{\widetilde{M}^{q,\varphi_2}(\Omega)}.$$

In the case $q = 1$ *the function* Tg *belongs to the space* $W\widetilde{M}^{p,\varphi_1}(\Omega)$ *and there exists* $c = c(\varphi_1, \varphi_2, n) > 0$ *such that*

$$\|Tg\|_{W\widetilde{M}^{p,\varphi_1}(\Omega)} \leq c\|g\|_{\widetilde{M}^{1,\varphi_2}(\Omega)}.$$

3 Application to partial differential equations

Let us consider the divergence form elliptic equation (1.3), in a bounded set $\Omega \subset \mathbb{R}^n$, $n \geq 3$. We set

$$\Gamma(x,t) = \frac{1}{n(2-n)\omega_n\sqrt{\det\{a_{ij}(x)\}}}\left(\sum_{i,j=1}^{n} A_{ij}(x)t_i t_j\right)^{\frac{2-n}{2}},$$
$$\Gamma_i(x,t) = \frac{\partial}{\partial t_i}\Gamma(x,t), \qquad \Gamma_{ij}(x,t) = \frac{\partial}{\partial t_i \partial t_j}\Gamma(x,t),$$
$$M = \max_{i,j=1,\dots,n}\max_{|\alpha|\leq 2n}\left\|\frac{\partial^\alpha \Gamma_{ij}(x,t)}{\partial t^\alpha}\right\|_{L^\infty(\Omega\times\Sigma)},$$

for a.a. $x \in B$ and $\forall t \in \mathbb{R}^n \backslash \{0\}$, where A_{ij} denote the entries of the inverse matrix of the matrix $\{a_{ij}(x)\}_{i,j=1,\dots,n}$, and ω_n is the measure of the unit ball in $\mathbb{R}^n$.

It is well known that $\Gamma_{ij}(x,t)$ are Calderón–Zygmund kernels in the t variable.

Let $r, R \in \mathbb{R}^+, r < R$ and $\varphi \in C^\infty(\Omega)$ be a standard cut-off function such that for every ball $B_R \subset \Omega$,

$$\varphi(x) = 1 \quad \text{in } B_r, \qquad \varphi(x) = 0, \quad \text{in } \Omega\backslash B_R.$$

Then if u is a solution of (1.3) and $v = \varphi u$ we have

$$L(v) = \nabla \cdot G + g,$$

where

$$G = \varphi f + uA\nabla\varphi,$$
$$g = \langle A\nabla u, \nabla\varphi\rangle - \langle f, \nabla\varphi\rangle.$$

Using the notations above, we are able to recall an integral representation formula for the first derivatives of a solution u of (1.3).

Lemma 3.1 *For every* $i = 1, \dots, n$, *let* $a_{ij} \in L^\infty(\mathbb{R}^n) \cap CBMO^{\max\{p,p'\}}_{\{x_0\}}$ *satisfy* (1.6) *and* (1.7), *let* u *be a solution of* (1.3) *and let* φ, g *and* G *defined as above. Then, for every* $i = 1, \dots, n$ *we have*

$$\partial_{x_i}(\varphi u) = \sum_{h,j=1}^{n} P.V. \int_{B_R} \Gamma_{ij}(x, x-y)\{(a_{jh}(x) - a_{jh}(y))\partial x_h(\varphi u)(y) - G_j(y)\}\, \mathrm{d}y$$
$$- \int_{B_R} \Gamma_i(x, x-y) g(y)\, \mathrm{d}y + \sum_{h=1}^{n} c_{ih}(x) G_h(x), \qquad \forall x \in B_R,$$

setting $c_{ih} = \int_{|t|=1} \Gamma_i(x,t) t_h \, \mathrm{d}\sigma_t$.

Using the representation formula stated in Lemma 3.1, we can obtain a regularity result for the solutions to (1.3).

Theorem 3.2 *Let a_{ij} be such that (1.5), (1.6), (1.7) are true, we assume that the condition (2.15) is fulfilled and that $\varphi_2 \gtrsim \varphi_1$. Let also suppose that u is a solution of (1.3) such that $\partial_{x_i} u \in \widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(\Omega)$, for all $i = 1, \ldots, n$, $f \in [\widetilde{LM}^{q,\varphi_1}_{\{x_0\}}(\Omega)]^n$, $x_0 \in \Omega$. Let $\varphi \in C^\infty(\Omega)$ a standard cut-off function. Then, for any $K \subset \Omega$ compact there exists a constant $c(n, p, \varphi_1, \varphi_2, dist(K, \partial\Omega))$ such that*

$$(i) \qquad \partial_{x_i} u \in \widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K), \qquad \forall i = 1, \ldots, n,$$
$$(ii) \qquad \|\partial_{x_i} u\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} \lesssim \|u\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(\Omega)} + \|\partial_{x_i} u\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(\Omega)} + \|f\|_{\widetilde{LM}^{q,\varphi_1}_{\{x_0\}}(\Omega)},$$
$$\forall i = 1, \ldots, n,$$

where $\frac{1}{p} = \frac{1}{q} + \frac{1}{n}$.

Proof Let $K \subset \Omega$ be a compact set. Using Lemma and the boundedness of the commutator proved in [13], we obtain the following estimate:

$$\begin{aligned} \|\partial_{x_i}(\varphi u)\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} &\le \|C[a_{ij}, \varphi]\partial_{x_h}(u\varphi)\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} + \|KG\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} \\ &\quad + \|Tg\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} + \|G\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} \\ &\le c\|a\|_{CVMO^{\max\{p,p'\}}_{\{x_0\}}} \|\partial_{x_h}(u\varphi)\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} + \|G\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)} \\ &\quad + \|g\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(K)} \\ &\quad + \|G\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)}, \end{aligned}$$

where the norm $\|a\|_{CVMO^{\max\{p,p'\}}_{\{x_0\}}}$ is taken in the set B_R.

Taking into account that $a \in CVMO^{\max\{p,p'\}}_{\{x_0\}}$, we can choose the radius R of the ball B_R such that $c\|a\|_{CVMO^{\max\{p,p'\}}_{\{x_0\}}} < \frac{1}{2}$. This remark allow us to write

$$
\begin{aligned}
&\|\partial_{x_i}(\varphi u)\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)}\\
&\quad\le \|G\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)}+\|g\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(K)}+\|G\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)}\\
&\quad\approx \|G\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)}+\|g\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(K)}\\
&\quad= \|\varphi f+uA\nabla\varphi\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)}+\|\langle A\nabla u,\nabla\varphi\rangle-\langle f,\nabla\varphi\rangle\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(K)}\\
&\quad\le \|f\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)}+\|u\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)}+\|\partial_{x_i}u\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(K)}+\|f\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(K)}.
\end{aligned}
$$

Now we apply the hypothesis $\varphi_2 \gtrsim \varphi_1$, obtaining the following estimate for the norm $\|f\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}}$:

$$
\begin{aligned}
\|f\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(K)} &\le \sup_{0<r<d}\frac{1}{\varphi_2(x_0,r)}\frac{1}{|B(x_0,r)|^{\frac{1}{q}}}\|f\|_{L^q(|B(x_0,r)\cap K)}+\|f\|_{L^q(K)}\\
&\lesssim \sup_{0<r<d}\frac{1}{\varphi_1(x_0,r)}\frac{1}{|B(x_0,r)|^{\frac{1}{q}}}\|f\|_{L^q(|B(x_0,r)\cap K)}+\|f\|_{L^q(K)}\\
&= \|f\|_{LM^{q,\varphi_1}_{\{x_0\}}(K)}+\|f\|_{L^q(K)}=\|f\|_{\widetilde{LM}^{q,\varphi_1}_{\{x_0\}}(K)}.
\end{aligned}
$$

Using the previous estimate we finally obtain that

$$
\|\partial_{x_i}u\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(K)}\le C\left(\|u\|_{\widetilde{LM}^{p,\varphi_1}_{\{x_0\}}(\Omega)}+\|\partial_{x_i}u\|_{\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(\Omega)}+\|f\|_{\widetilde{LM}^{q,\varphi_1}_{\{x_0\}}(\Omega)}\right),\quad \forall i=1,\dots,n,
$$

□

From Theorem 3.2 we get the following corollary.

Corollary 3.3 *Let $a_{ij}\in L^\infty(\mathbb{R}^n)\cap VMO$ such that (1.6), (1.7) are true, we assume that the condition (2.16) is fulfilled and that $\varphi_2\gtrsim\varphi_1$. Let also suppose that u is a solution of (1.3) such that $\partial_{x_i}u\in\widetilde{LM}^{q,\varphi_2}_{\{x_0\}}(\Omega)$, for all $i=1,\dots,n$, $f\in[\widetilde{M}^{p,\varphi_1}(\Omega)]^n$. Let $\varphi\in C^\infty(\Omega)$ a standard cut-off function. Then, for any $K\subset\Omega$ compact there exists a constant $c(n,p,\varphi_1,\varphi_2,dist(K,\partial\Omega))$ such that*

$$
\begin{aligned}
&(i)\quad \partial_{x_i}u\in\widetilde{M}^{p,\varphi_1}(K),\qquad \forall i=1,\dots,n,\\
&(ii)\quad \|\partial_{x_i}u\|_{\widetilde{M}^{p,\varphi_1}(K)}\lesssim\|u\|_{\widetilde{M}^{p,\varphi_1}(\Omega)}+\|\partial_{x_i}u\|_{\widetilde{M}^{q,\varphi_2}(\Omega)}+\|f\|_{\widetilde{M}^{q,\varphi_1}(\Omega)},\\
&\qquad \forall i=1,\dots,n,
\end{aligned}
$$

where $\frac{1}{p}=\frac{1}{q}+\frac{1}{n}$.

In the case $\varphi_1(x,r)=\varphi_2(x,r)$ we get the following corollaries.

Corollary 3.4 *Let a_{ij} be such that (1.5), (1.6), (1.7) are true, we assume that $\varphi(x,r)$ positive measurable function defined on $\Omega\times(0,d)$ and the following condition is* *fulfilled:*

$$\int_r^d \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x_0,\tau)\,\tau^{\frac{n}{q}}}{t^{\frac{n}{p}+1}}\,dt \le C\,\varphi(x_0,r),$$

where C does not depend on r.

Let also suppose that u is a solution of (1.3) *such that $\partial_{x_i}u \in \widetilde{LM}^{q,\varphi}_{\{x_0\}}(\Omega)$, for all $i=1,\dots,n$, $f \in [\widetilde{LM}^{q,\varphi}_{\{x_0\}}(\Omega)]^n$, $x_0 \in \Omega$. Let $\varphi \in C^\infty(\Omega)$ a standard cut-off function. Then, for any $K \subset \Omega$ compact there exists a constant $c(n,p,\varphi,dist(K,\partial\Omega))$ such that*

$$(i)\quad \partial_{x_i}u \in \widetilde{LM}^{p,\varphi}_{\{x_0\}}(K), \qquad \forall i=1,\dots,n,$$

$$(ii)\quad \|\partial_{x_i}u\|_{\widetilde{LM}^{p,\varphi}_{\{x_0\}}(K)} \lesssim \|u\|_{\widetilde{LM}^{p,\varphi}_{\{x_0\}}(\Omega)} + \|\partial_{x_i}u\|_{\widetilde{LM}^{q,\varphi}_{\{x_0\}}(\Omega)} + \|f\|_{\widetilde{LM}^{q,\varphi}_{\{x_0\}}(\Omega)}, \quad \forall i=1,\dots,n,$$

where $\frac{1}{p}=\frac{1}{q}+\frac{1}{n}$.

Corollary 3.5 *Let $a_{ij} \in L^\infty(\mathbb{R}^n) \cap VMO$ satisfy* (1.6), (1.7) *are true, we assume that $\varphi(x,r)$ positive measurable function defined on $\Omega \times (0,d)$ and the following condition is fulfilled:*

$$\int_r^d \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi(x,\tau)\,\tau^{\frac{n}{q}}}{t^{\frac{n}{p}+1}}\,dt \le C\,\varphi(x,r),$$

where C does not depend on x, r.

Let also suppose that u is a solution of (1.3) such that $\partial_{x_i}u \in \widetilde{M}^{q,\varphi}(\Omega)$, for all $i=1,\dots,n$, $f \in [\widetilde{M}^{q,\varphi}(\Omega)]^n$. Let $\varphi \in C^\infty(\Omega)$ a standard cut-off function. Then, for any $K \subset \Omega$ compact there exists a constant $c(n,p,\varphi,dist(K,\partial\Omega))$ such that

$$(i)\quad \partial_{x_i}u \in \widetilde{M}^{p,\varphi}(K), \qquad \forall i=1,\dots,n,$$

$$(ii)\quad \|\partial_{x_i}u\|_{\widetilde{M}^{p,\varphi}(K)} \lesssim \|u\|_{\widetilde{M}^{p,\varphi}(\Omega)} + \|\partial_{x_i}u\|_{\widetilde{M}^{q,\varphi}(\Omega)} + \|f\|_{\widetilde{M}^{q,\varphi}(\Omega)}, \quad \forall i=1,\dots,n,$$

where $\frac{1}{p}=\frac{1}{q}+\frac{1}{n}$.

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Affiliations

V. S. Guliyev[1,2,3] · M. N. Omarova[4,5] · M. A. Ragusa[3,6] · A. Scapellato[6]

✉ A. Scapellato
scapellato@dmi.unict.it

V. S. Guliyev
vagif@guliyev.com

M. N. Omarova
mehriban_omarova@yahoo.com

M. A. Ragusa
maragusa@dmi.unict.it

1 Institute of Applied Mathematics, Baku State University, AZ1148 Baku, Azerbaijan

2 Department of Mathematics, Dumlupinar University, Kutahya 43100, Turkey

3 S.M. Nikolskii Institute of Mathematics at RUDN University, Moscow, Russia 117198

4 Baku State University, AZ1148 Baku, Azerbaijan

5 Institute of Mathematics and Mechanics of NAS of Azerbaijan, AZ1141 Baku, Azerbaijan

6 Dipartimento di Matematica e Informatica, Università di Catania, Catania, Italy