Chapter 5
Experimental Approaches to Theoretical Thinking: Artefacts and Proofs

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1 Introduction

From straight-edge and compass to a variety of computational and drawing tools, throughout history instruments have been deeply intertwined with the genesis and development of abstract concepts and ideas in mathematics. Their use introduces an "experimental" dimension into mathematics, as well as a dynamic tension between the *empirical nature* of activities with them, which encompasses perceptual and operational components– and the *deductive nature* of the discipline, which entails rigorous and sophisticated formalisation. As Pierce writes of this peculiarity:

> (It) has long been a puzzle how it could be that, on the one hand, mathematics is purely deductive in its nature, and draws its conclusions apodictically, while on the other hand, it presents as rich and apparently unending a series of surprising discoveries as any observational science.
>
> (Peirce, C.P., 3.363: quoted in Dörfler 2005, p. 57)

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The main goal of our chapter centres on the dynamic tension between the empirical and the theoretical nature of mathematics. Our purpose is to underline the elements of historical continuity in the stream of thought today called experimental mathematics, and show the concrete possibilities it offers to today's teachers for pursuing the learning of proof in the classroom, especially through the use of their computer tools.

Specifically, we examine how this dynamic tension regulates the actions of students who are asked to solve mathematical problems by first making explorations with technological tools, then formulating suitable conjectures and finally proving them.

The latest developments in computer and video technology have provided a multiplicity of computational and symbolic tools that have rejuvenated mathematics and mathematics education. Two important examples of this revitalisation are *experimental mathematics* and *visual theorems*:

> Experimental mathematics is the use of a computer to run computations – sometimes no more than trial-and-error tests – to look for patterns, to identify particular numbers and sequences, to gather evidence in support of specific mathematical assertions that may themselves arise by computational means, including search. Like contemporary chemists – and before them the alchemists of old – who mix various substances together in a crucible and heat them to a high temperature to see what happens, today's experimental mathematicians put a hopefully potent mix of numbers, formulas, and algorithms into a computer in the hope that something of interest emerges.
>
> (Borwein and Devlin 2009, p. 1)

> Briefly, a visual theorem is the graphical or visual output from a computer program – usually one of a family of such outputs – which the eye organizes into a coherent, identifiable whole and which is able to inspire mathematical questions of a traditional nature or which contributes in some way to our understanding or enrichment of some mathematical or real world situation.
>
> (Davis 1993, p. 333)

Such developments throw a fresh light on mathematical epistemology and on the processes of mathematical discovery; consequently, we must also rethink the nature of mathematical learning processes. In particular, the new epistemological and cognitive viewpoints have challenged and reconsidered the phenomenology of learning proof (cf. Balacheff 1988, 1999; Boero 2007; de Villiers 2010; Chap. 3). These recent writers have scrutinised and revealed not only *deductive* but also *abductive* and *inductive* processes crucial in all mathematical activities, emphasising the importance of experimental components in teaching proofs. The related didactical phenomena become particularly interesting when instructors plan proving activities in a technological environment (Arzarello and Paola 2007; Jones et al. 2000), where they can carefully design their interventions. By "technological environment", we do not mean just digital technologies but any environment where instruments are used to learn mathematics (for a non-computer technology, see Bartolini Bussi 2010). We discuss this issue from different linked perspectives: historical, epistemological, didactical and pedagogical.

In Part 1, we consider some emblematic events from the history of Western mathematics where instruments have played a crucial role in generating mathematical concepts.

Next, Part 2 analyses some didactical episodes from classroom life, where the use of instruments in proving activities makes the dynamic tension palpable.

We carefully analyse students' procedures whilst using tools and derive some theoretical frameworks that explain how that tension can be used to design suitable didactical situations. Within these, students can learn practices with the tools that help them pass from the empirical to the theoretical side of mathematics. In particular, we discuss the complex interactions between inductive, abductive and deductive modalities in that transition. By analysing the roles for technologies within our framework, we show that instructors can and should make the history and cultural aspects of experimental mathematics visible to students.

Last, in Part 3 we show how a general pedagogical framework (Activity Theory) makes sense of the previous microanalyses within a general, unitary educational standpoint.

2 Part 1: From Straight-Edge and Compass to Dynamic Geometry Software

2.1 Classical European Geometry

Since antiquity, geometrical constructions have had a fundamental theoretical importance in the Greek and later Western traditions (Heath 1956, p. 124); indeed, construction problems were central to Euclid's work. This centrality is clearly illustrated by the later history of the classic 'impossible' problems, which so puzzled Euclid and other Greek geometers (Henry 1993). Despite their apparent practical objective, geometrical constructions (like drawings produced on papyrus or parchment) do have a theoretical meaning. In Euclid's masterpiece, the *Elements*, no real, material tools are envisaged; rather their use is objectified into the geometrical objects defined by definitions and axioms. However, Arsac (1987) shows that the observational, empirical component was also present in the *Elements*. Euclid was aware of the dialectic between the decontextualised aspects of pure geometry and the phenomenology of our perception of objects in space and our representations of them in the plane. In his *Optics* (Euclide 1996) masterpiece, he gives a rationale for this tension. Giusti writes: "the mathematical objects are not generated through abstraction from real objects […] but they formalize human operations"[1](Giusti 1999). In addition, they are shaped by the tools with which people perform such operations.

Consequently, the tools and the rules for their use have a counterpart in the axioms and theorems of a theoretical system, so that we may conceive of any construction as a theoretical problem stated inside a specific theoretical system. The solution of a problem is correct; therefore, insofar as we can validate it within such a

[1] 'gli oggetti matematici provengono non dall'astrazione da oggetti reali […] ma formalizzano l'operare umano'.

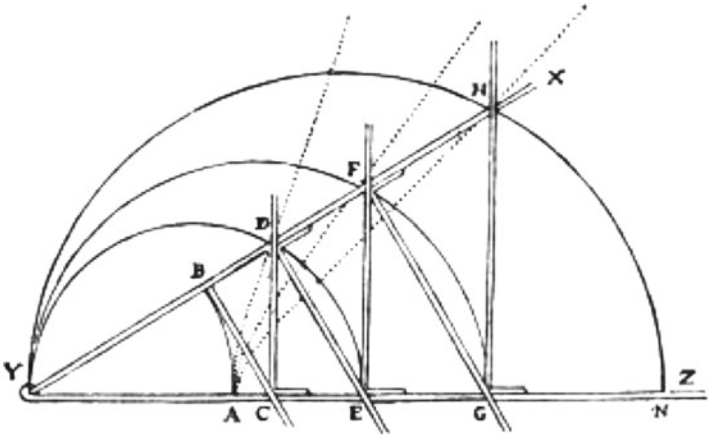


Fig. 5.1 Descartes compass

theoretical system, any successful construction corresponds to a specific theorem, and *validates* the specific relationships between the properties of the geometrical figure represented by the drawing, obtained after the construction.

From the perspective of classical geometry, drawing tools, despite their empirical manifestation, may also be conceived as theoretical tools defining a particular geometry. Hence, the tradition of calling classic Euclidean geometry "straight-edge and compass geometry" refers to both the origins and limitations of its objects.

2.2 The Modern Age in Europe

In the Western Euclidean tradition, interesting European developments about the theoretical status of drawing tools and geometric objects have accumulated since the seventeenth century. For example, in the '*Géométrie*', Descartes clearly states two methods of representing curves: (a) by a continuous motion and (b) by an equation (Bos 1981). Descartes invented a tool, his compass (Fig. 5.1), to make evident what he meant: moving the different t-squares YBC, DCE, EDF, … moves a point like H and generates a curve with the features described under (a) and (b).

In contrast, in the Hippias trisectrix (Fig. 5.2), the points of the curve APQT are generated through a continuous synchronous motion of the ray DP (which rotates uniformly around D like the hand of a clock) and of the horizontal ray MP (which moves uniformly downwards, so that whilst the ray rotates from DA to DC the ray MP moves from AB to DC). In fact, it is impossible to empirically obtain the position of all the points of the curve APQT since one can only imagine them through the description of the movement, not concretely depict them as in the case of a figure drawn using straight-edge and compass.

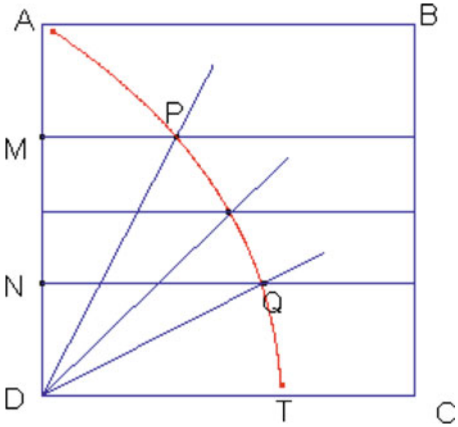


Fig. 5.2 Hippias trisectrix

As Lebesgue (1950) claims, a curve traced pointwise is obtained by approximation; it is only a graphic solution. However, if one designs a tracing instrument, the graphic solution becomes a mechanical solution. The seventeenth century mathematicians found the mechanical solution acceptable because it refers to one of the basic intuitions about the continuum: namely, the movement of an object. Descartes did not confront the question of whether the two given criteria – the mechanical and the algebraic – are equivalent or not. This problem requires more advanced algebraic tools and, more important, changing the status of the new drawing instruments from tools for solving geometric problems to objects of a theory.

In the classical age and in the seventeenth century, changing the drawing tools would clearly have changed the set of solvable problems. So, if one accepts only the straight-edge and compass (i.e., only straight lines and circles), one cannot rigorously solve the problems of cube duplication and angle trisection. If, on the contrary, other tools are admitted (e.g., the Nicomedes compass that draws a conchoid; see Heath 1956), these problems can be solved rigorously.

The previous examples highlighted a crucial dialectical relationship between practical and theoretical problems. The core of this relationship resides in the notion of *construction* as related to the specific tools available. Therefore, the practical realisation of any graphical element has a counterpart in a theoretical element, in either an axiom that states how to use a tool or a theorem that validates the construction procedure according to the stated axioms. In these terms, we can consider a geometrical construction archetypal for a theoretical approach to geometry.

However, in spite of their long tradition, geometrical constructions have recently lost their centrality and almost disappeared from the Geometry curriculum, at least in the Western world. One can rarely find any reference to 'drawing tools' when geometrical axioms are stated, and geometrical constructions no longer belong to the set of problems commonly proposed in the textbooks. This disappearance began as nineteenth century mathematicians from Pasch to Peano to Hilbert tried to eliminate the observational "intuitive" hidden hypotheses from Geometry.

Similarly, the school of Weierstrass eliminated any reference to space or motion in the geometric definition of limits through the epsilon-delta machinery (Lakoff and Núñez 2000): in fact, the new definition entails only purely logical relationships ("for all epsilon, there is a delta…") and any reference to motion and time (e.g. "whilst x approaches x_0, $f(x)$ approaches l") is eliminated. Until recently, observational components had apparently been completely banished from the geometric scene. However, philosophical criticisms by many recent scholars (for a survey, cf. Tymoczko 1998) and the development of computational techniques have produced a fresh approach to mathematical learning and discovery. They have revived epistemological stances which underlie the observational, experimental and empirical aspects of mathematical inquiry, including the use of geometric constructions (see Lovasz 2006).

Indeed, geometric constructions are rich in meaning and perfectly suitable for implementation in today's classrooms, even though the relationship between a geometrical construction and the theorem which validates it is very complex and certainly not immediate for students, as Schoenfeld (1985) discussed. As he explained, "many of the counterproductive behaviors we see in students are learned as unintended by-products of their mathematics instruction" (p. 374). Apparently, the very nature of the construction problem may make it difficult to take a theoretical perspective (cf. Mariotti 1996).

Nevertheless, the analysis above allows us to state a specific hypothesis, namely, that *geometrical construction* can serve as a *key to accessing* the meaning of proof. Different research groups have undertaken to test or apply this hypothesis, in different directions with different tools and different mathematical theories.

2.3 Constructions with Straight-Edge and Compass in the Mathematics Classroom

A recent teaching experiment in Italy has shown the potential of straight-edge and compass for developing an experimental approach with theoretical aims (Bartolini Bussi et al. in print). The project involved a group of 80 mathematics teachers (only six from primary school, the others equally divided between junior secondary and high school; see Martignone 2010) and nearly 2,000 students (scattered all over a large region of Northern Italy). Straight-edge and compass problems were set in the larger context of mathematical "machines" (Bartolini Bussi 2000, p. 343), tools that force a point to follow a trajectory or to be transformed according to a given law. A common theoretical framework (see below; also Bartolini Bussi and Mariotti 2008) structured the exploration of the tools and of the functions they served in the solution of geometrical problems by construction. Similar learning processes were implemented with the participants. First, the teachers received an in-service course of six meetings; then they instructed their students. A total of 79 teaching experiments, with detailed documentations, were collected; 25% of them concerned straight-edge and compass.

The general structure of the approach comprised:

A. Exploration and analysis of the tool (shorter for teachers; longer for students, in order to make them aware of the relationship between the physical structure of the compass and Euclid's definition of a circle).
B. Production of very simple constructions of geometrical figures (e.g., "draw an equilateral triangle with a given side") in open form, in order to allow a variety of constructions based on different known properties.
C. Comparison of the different constructions in large group discussions, to show that the "same" drawing may be based on very different processes, each drawing on either implicit or explicit assumptions and on the technical features of the tool.
D. Production of proofs of the constructions exploiting each times the underlying assumptions.

These stages were structured around three key questions concerning the compass as a tool:

1. How is it made?
2. What does it do?
3. Why does it do that?

The third question, dependent on the others, aimed at connecting the tool's practical use to the theoretical content. In fact, the justification of a construction draws on the geometrical properties of the compass, as is clearly shown in the proof of Proposition 1, Book 1 of Euclid's *Elements* (Heath 1956, p. 241), with the construction of an equilateral triangle.

2.4 Constructions in a DGS

The interest in constructions has been renewed in particular by the appearance of Dynamic Geometry Systems (DGS), where the basic role played by construction has been reinforced by the use of graphic tools available in a dynamic system, like *Cabri-géomètre*, *Sketchpad*, *Geogebra*, etc. Any DGS figure is the result of a construction process, since it is obtained after the repeated use of tools chosen from those available in the "tool bar". However, what makes DGS so interesting compared to the classic world of paper and pencil figures is not only the construction facility but also the direct manipulation of its figures, conceived in terms of the embedded logic system (Laborde and Straesser 1990; Straesser 2001) of Euclidean geometry. DGS figures possess an intrinsic logic, as a result of their construction, placing the elements of a figure in a hierarchy of relationships that corresponds to the procedure of construction according to the chosen tools and in a hierarchy of properties, and this hierarchy corresponds to a relationship of logical conditionality. This relationship is made evident in the "dragging" mode, where what cannot be dragged by varying the basic points (elements) of a built figure constitutes the results of the construction. The dynamics of the DGS figures preserves its intrinsic logic; that is, the logic of its construction. The DGS figure is the complex of these elements, incorporating various relationships which can be differently referred to the definitions and theorems of geometry.

The presence of the dragging mode introduces in the DGS environment a specific criterion of validation for the solution of the construction problems: A solution is valid if and only if the figure on the screen is stable under the dragging test. However, the system of DGS figures embodies a system of relationships consistent with the broad system of geometrical theory. Thus, solving construction problems in DGS means not only accepting all the facilities of the software but also accepting a logic system within which to make sense of them.

The DGS's intrinsic relation to Euclidean geometry makes it possible to interpret the control 'by dragging' as corresponding to theoretical control 'by proof and definition' within the system of Euclidean Geometry, or of another geometry that allows recourse to a larger set of tools. In other words, there is a correspondence between the world of DGS constructions and the theoretical world of Euclidean Geometry.

2.5 DGS Constructions in the Classroom

Mariotti (2000, 2001) carried out teaching experiments with grade 10 students attending first year in a science-oriented school (Liceo Scientifico). The design of the teaching sequence was based on the development of the field of experience (Boero et al. 1995) of geometrical constructions in a DGS (*Cabri-Géomètre*). The educational aim was to introduce students to a theoretical perspective; its achievement relied on the potential correspondence between DGS constructions and geometric theorems.

The activity started by revisiting drawings and concrete artefacts which the pupils had already experienced: for example, the compass. The students were more or less familiar with the artefacts' constraints, which determine possible actions and expected results; for instance, a compass's intrinsic properties directly affect the properties of the graphic trace it produces. Revisitation involved transferring the drawing activity into the *Cabri* environment, thus moving the external context from the physical world of straight-edge and compass to the virtual world of DGS figures and commands.

In a DGS environment, the new 'objects' available are Evocative Computational Objects (Hoyles 1993; Hoyles and Noss 1996, p. 68), characterised by their computational nature and their power to evoke geometrical knowledge. For *Cabri*, they comprise:

1. The *Cabri*-figures realising geometrical figures;
2. The *Cabri*-commands (primitives and macros), realising the geometrical relationships which characterise geometrical figures*;*
3. The dragging function, which provides a perceptual control of the construction's correctness, corresponding to a theoretical control consistent with geometric theory.

The development of the field of experience occurred through activities in *Cabri*'s world, such as construction tasks, interpretation and prediction tasks and mathematical discussions. However, that development also involved making straight-edge and compass constructions, which became both concrete referents and signs of the *Cabri* figures. Relating the drawings on paper and the *Cabri* figures gave the students a unique experience with a 'double face', one physical and the other virtual.

In the DGS environment, a construction activity, such as drawing figures through the commands on the menu, is integrated with the dragging function. Thus, a construction task is accomplished if the figure on the screen passes the dragging test.

In Mariotti's (2000, 2001) research, the necessity of justifying the solution came from the need to validate one's own construction, in order to explain why it worked and/or to foresee that it would work. Although dragging the figure might suffice to display the correctness of the solution, the second component of the teaching/learning activities came into play at this point. Namely, construction problems become part of a social interchange, where the students reported and compared their different solutions. This represented a crucial element of the experience.

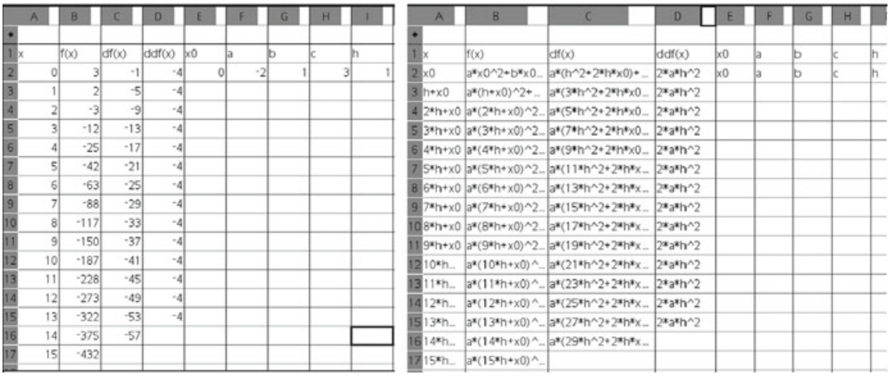
2.6 Experiments and Proofs with the Computer

Typically, current experimental mathematics involves making computations with a computer. Crucially, validating numerical solutions, which may have already been found, requires producing suitable proofs (cf. Borwein and Devlin 2009, for example). We illustrate with an example precisely how a so called CAS (Computer Algebra System: it processes not only numerical values but also algebraic expressions with letters and infinite-precision rational numbers) can be used as a tool for promoting the production of proofs for found numerical solutions.

Arzarello (2009) researched Grade 9 students, attending first year in a science-oriented higher secondary school (Liceo Scientifico), who were studying functions through tables of differences. The students had already learnt that for first-degree functions, the first differences are constant. The teacher asked them to make conjectures on which functions have the first differences that change linearly and to arrange a spreadsheet as in Fig. 5.3a, where they utilise:

1. Columns A, B, C, D to indicate respectively the values of the variable x, of the function $f(x)$ (in B_i there is the value of $f(A_i)$) and of its related first and second differences (namely in C_i there is the value $f(A_{i+1}) - f(A_i)$ and D_j there is the value $C_{j+1} - C_j$);
2. Variable numbers in cells E2, F2, …,I2 to indicate respectively: the values x_0 (the first value for the variable x to put in A2); a, b, c for the coefficients of the second degree function $ax^2 + bx + c$; the step h of which the variable in column A is incremented each time for passing to A_i to A_{i+1}.

By modifying the values of E2, F2, …, I2, the students could easily do their explorations. This practice gradually became shared in the classroom, through



a

	A	B	C	D	E	F	G	H	I
◆									
1	x	f(x)	df(x)	ddf(x)	x0	a	b	c	h
2	0	3	-1	-4	0	-2	1	3	1
3	1	2	-5	-4					
4	2	-3	-9	-4					
5	3	-12	-13	-4					
6	4	-25	-17	-4					
7	5	-42	-21	-4					
8	6	-63	-25	-4					
9	7	-88	-29	-4					
10	8	-117	-33	-4					
11	9	-150	-37	-4					
12	10	-187	-41	-4					
13	11	-228	-45	-4					
14	12	-273	-49	-4					
15	13	-322	-53	-4					
16	14	-375	-57						
17	15	-432							

b

	A	B	C	D	E	F	G	H	I
◆									
1	x	f(x)	df(x)	ddf(x)	x0	a	b	c	h
2	x0	a*x0^2+b*x0...	a*(h^2+2*h*x0)+...	2*a*h^2	x0	a	b	c	h
3	h+x0	a*(h+x0)^2+...	a*(3*h^2+2*h*x0...	2*a*h^2					
4	2*h+x0	a*(2*h+x0)^2...	a*(5*h^2+2*h*x0...	2*a*h^2					
5	3*h+x0	a*(3*h+x0)^2...	a*(7*h^2+2*h*x0...	2*a*h^2					
6	4*h+x0	a*(4*h+x0)^2...	a*(9*h^2+2*h*x0...	2*a*h^2					
7	5*h+x0	a*(5*h+x0)^2...	a*(11*h^2+2*h*x...	2*a*h^2					
8	6*h+x0	a*(6*h+x0)^2...	a*(13*h^2+2*h*x...	2*a*h^2					
9	7*h+x0	a*(7*h+x0)^2...	a*(15*h^2+2*h*x...	2*a*h^2					
10	8*h+x0	a*(8*h+x0)^2...	a*(17*h^2+2*h*x...	2*a*h^2					
11	9*h+x0	a*(9*h+x0)^2...	a*(19*h^2+2*h*x...	2*a*h^2					
12	10*h...	a*(10*h+x0)^...	a*(21*h^2+2*h*x...	2*a*h^2					
13	11*h...	a*(11*h+x0)^...	a*(23*h^2+2*h*x...	2*a*h^2					
14	12*h...	a*(12*h+x0)^...	a*(25*h^2+2*h*x...	2*a*h^2					
15	13*h...	a*(13*h+x0)^...	a*(27*h^2+2*h*x...	2*a*h^2					
16	14*h...	a*(14*h+x0)^...	a*(29*h^2+2*h*x...						
17	15*h...	a*(15*h+x0)^...							

Fig. 5.3 (**a**) Numerical finite differences. (**b**) Algebraic finite differences

the intervention of the teacher. In fact the class teacher stressed its value as an instrumented action (Rabardel 2002), to support explorations in the numerical environment. Students realised that:

1. If they changed only the value of *c*, column B changed, whilst columns C and D of the first and second differences did not change; hence, they argued, the way in which a function increased/decreased did not depend on the coefficient *c*;
2. If they changed the coefficient *b*, then columns B and C changed but column D did not; many students conjectured that the coefficient *b* determines whether a function increases or decreases, but not its concavity;
3. If they changed the coefficient *a*, then columns B, C and D changed; hence the coefficient *a* was responsible for the concavity of the function.

Here, it is difficult to understand why such relationships hold and to produce at least an argument or even a proof of such conjectures. The tables of numbers do not suggest any justification. Now, the symbolic power of the spreadsheet became useful.[2] The students' very interesting instrumented actions consisted in substituting letters for the numbers (Fig. 5.3b); in most cases, the teacher had suggested this practice, but a couple of students used it autonomously. The resulting spreadsheet shows clearly that the value of the second difference is $2ah^2$. The letters condense the symbolic meaning of the numerical explorations, so proofs can be produced (with teacher's help) because of the spreadsheet's symbolic support. In the subsequent lesson, the teacher stressed the power of the symbolic spreadsheet; a fresh practice had entered the classroom.

Finally, a typical algebraic proof, where the main steps are computations – like the proof produced through use of the spreadsheet – apparently differs from the more discursive proofs produced in elementary geometry. Such algebraic proofs

[2] They were using the TI-Nspire software of Texas Instruments.

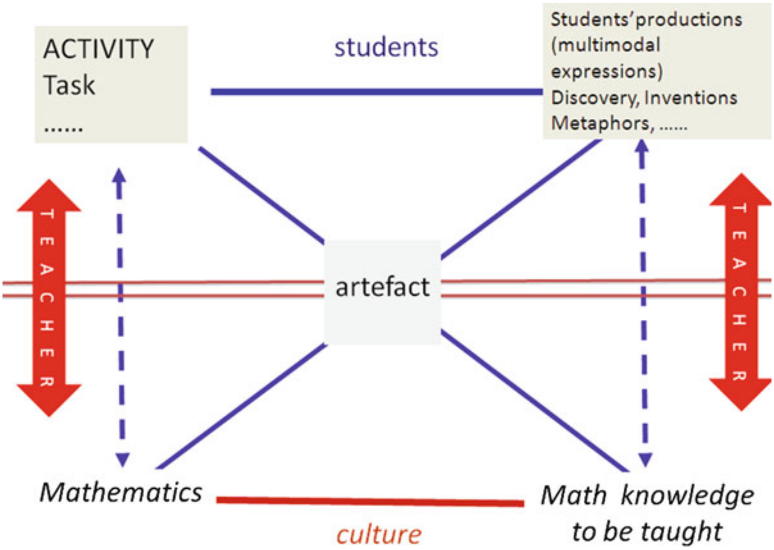


Fig. 5.4 Relationships between teachers, students, mathematics and artefacts in didactical activities

result from the "algebraisation" of geometry, which started with Descartes and improved further in the succeeding development of mathematics (e.g., the Erlangen programme by F. Klein 1872; see also Baez 2011). Consequently, so-called synthetic proofs have been surrogated by computations developed in linear algebra environments. Though students find big obstacles in learning algebra as a meaningful topic (e.g. Dorier 2000), CAS environments can support students in conceiving and producing such computational proofs, which are typically not so easy to reach in paper and pencil environments.

2.7 Implementation in Mathematics Classrooms

In all the cases above (straight-edge and compass as well as DGS or CAS), the teacher's role is crucial. The teacher not only selects suitable tasks to be solved through constructions and visual, numerical or symbolic explorations, but also orchestrates the complex transition from practical actions to theoretical argumentations. Students' argumentations rest on their experimental experiences (drawing, dragging, computing, etc.), so the transition to a validation within a theoretical system requires delicate mediation by an expert (see diagram, Fig. 5.4).

The upper part of Fig. 5.4 represents the student's space. The students are given a task (left upper vertex) to be solved with an artefact or set of artefacts. The presence of the artefact(s) calls into play experimental activities: for example, drawing with straight-edge and compass; creating DGS-figures with DGS-tools; or using numbers and letters in the symbolic spreadsheet. An observer, the teacher for instance, may monitor the process: students gesticulate, point, and tell themselves or their fellows something about their actions; from this observable behaviour one may gain insight into their cognitive processes. If the task requires giving a final report (either oral or written), traces of the experience are likely to remain in the text produced. Such reports may thus differ from the decontextualised texts typical of mathematics; nevertheless, they can evoke specific mathematical meanings.

The lower part of Fig. 5.4 represents the mathematical counterpart of the students' experience. There is the activity of mathematics in general as a cultural product, and there is the mathematical knowledge to be taught according to curricula. The link between the students' productions and the mathematics to be taught is the responsibility of the teacher, who has to construct a suitable process that connects the students' personal productions with the statements and proofs expected in the mathematics to be taught.

Hence, Fig. 5.4 highlights two important responsibilities for the teacher:

1. Choosing suitable tasks (left side);
2. Monitoring and managing of the process from students' productions to mathematical statements and proofs (right side)

The second point constitutes the core of the semiotic mediation process, in which the teacher is expected to foster and guide the students' evolution towards recognisable mathematics. The teacher acts both at the cognitive and the metacognitive levels, by fostering the evolution of meanings and guiding the pupils to awareness of their mathematical status (see the idea of mathematical norms, Cobb et al. 1993; see also chapter 5 in this volume). From a sociocultural perspective, one may interpret these actions as the process of relating students' "personal senses" (Leont'ev 1964/1976, pp. 244 ff.) to mathematical meanings, or of relating "spontaneous" to "scientific" concepts (Vygotsky 1978/1990, p. 286 ff.). The teacher, as an expert representative of mathematical culture, participates in the classroom discourse to help it proceed towards sense-making within mathematics.

Within this perspective, several investigations have focused on the teacher's contribution to the development of a mathematical discourse in the classroom, specifically in the case of classroom activities centred on using an artefact (Bartolini Bussi et al. 2005; Mariotti 2001; Mariotti and Bartolini Bussi 1998). The researchers aimed at identifying specific "semiotic games" (Arzarello and Paola 2007; Mariotti and Bartolini Bussi 1998) played by the teacher, when intervening in the discourse, in order to make the students' personal senses emerge from their common experience with the artefact and develop towards shared meanings consistent with the target mathematical meanings. Analysis of the data highlighted a recurrent pattern of interventions encompassing a sequence of different types of operations (Bartolini Bussi and Mariotti 2008; for further discussion, see Mariotti 2009; Mariotti and Maracci 2010).

Thus, artefacts have historically been fruitful in generating the idea of proof and consequently can provide strong didactical support for teaching proofs, specifically, if the teacher acts as a semiotic mediator. In the next section, we illustrate this issue from the point of view of students.

3 Part 2: A Student-Centred Analysis

Suitably designed technology can help students to face and possibly to overcome the obstacles between their empirical mathematical tasks and the discipline's theoretical nature. When integrated in the teaching of proofs, artefacts trigger a network of interactive activities amongst different components categorisable at two different epistemological levels:

1. The convincing linguistic logical arguments that explain WHY according to the specific theory of reference;
2. The artefact-dependent convincing arguments that explain WHY according to the mathematical experimentation facilitated by an artefact.

Approaching proof in school consists in promoting a *network of interactive activities* in order to connect these different components. For example, as we discuss below, abductive processes can support interactions between (1) and (2) above. Other interactive activities concern students' multimodal behaviour[3] whilst interacting within technological environments. Such activities feature in the transition to proof within experimental mathematics, a transition with novel and specific features compared to the transition to proof within more traditional approaches. Here, we scrutinise when and how the distance between arguments and formal proofs (Balacheff 1999; Pedemonte 2007) produced by students can diminish because of the use of technologies within a precise pedagogical design.

To focus the didactical and epistemological aspects of this claim, we recall four theoretical constructs taken from the current literature:

1. *Almost-empiricism* and experimental mathematics;
2. *Abductive* vs. *deductive* activities in mathematics learning;
3. *Cognitive unity* between arguments and proofs;
4. *Negation* from a mathematical and cognitive point of view.

Using these theoretical constructs, we scrutinise some studies of students asked to explore different mathematical situations with different artefacts and

[3] The notion of *multimodality* has evolved within the paradigm of *embodiment*, which has been developed in recent years (Wilson 2002). Embodiment is a movement in cognitive science that grants the body a central role in shaping the mind. It concerns different disciplines, e.g. cognitive science and neuroscience, interested with how the body is involved in thinking and learning. It emphasises sensory and motor functions, as well as their importance for successful interaction with the environment, particularly palpable in human-computer interactions. A major consequence is that the boundaries among perception, action and cognition become *porous* (Seitz 2000). Concepts are so analysed not on the basis of 'formal abstract models, totally unrelated to the life of the body, and of the brain regions governing the body's functioning in the world' (Gallese and Lakoff, 2005, p.455), but considering the *multimodality* of our cognitive performances. We shall give an example of multimodal behaviours of students when discussing the multivariate language of students who work in DGE. For a more elaborate discussion, see Arzarello and Robutti (2008).

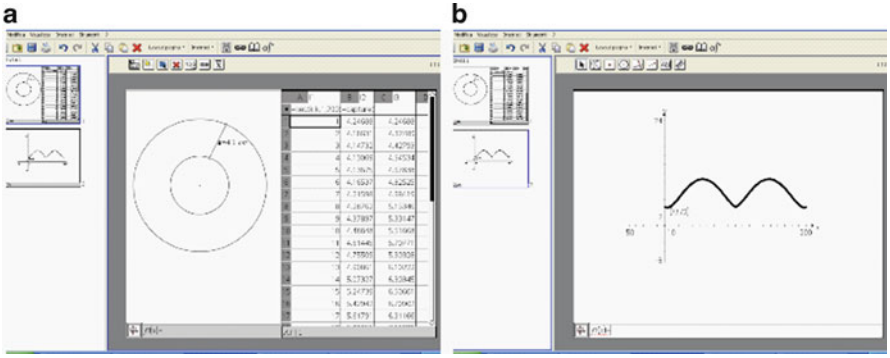


Fig. 5.5 (**a**) Solving a problem with data capture in TI-nspire. (**b**) Graph of the solution

within different pedagogical designs. Specifically, we show that a suitable use of technologies may improve the *almost-empirical* aspects in students' mathematical activities through a specific production of *abductive arguments*, which generate a *cognitive unity* in the transition from arguments to proofs. We also focus on some reasons why such a unity may not be achieved, particularly in the case of arguments and proofs by contradiction, where the *logic of negation* typically presents a major difficulty for students.

3.1 Almost-Empiricism and Experimental Mathematics

The notion of *almost-empirical actions*, introduced by Arzarello (2009), describes some instrumented actions[4] within DGS and CAS environments. It refines the usual epistemic/pragmatic dyadic structure of the instrumental approach. We provide a brief emblematic example.

In Arzarello's (2009) study, students of the 10th grade faced a simple problem, originated by the PISA test:

> *The students A and B attend the same school, which is 3 Km far from A's home and 6 Km far from B's home. What are the possible distances between the two houses?*

They produced a solution by using TI-Nspire software as illustrated in Fig. 5.5. They drew two circles, whose centre is the school and which represent the

[4] The so called *instrumentation approach* has been described by Vérillon & Rabardel (1995) and others (Rabardel 2002; Rabardel and Samurçay 2001; Trouche 2005). In our case particular ways of using an artefact, e.g. specific dragging practices in DGS or data capture in TI-Nspire, may be considered an *artefact* that is used to solve a particular *task* (e.g. for formulating a conjecture). When the user has developed particular *utilisation schemes* for the artefact, we say that it has become an *instrument* for the user.

possible positions of the two houses with respect to the school. They then created two points, say *a* and *b*, moving on each circle, constructed the segment *ab* and measured it using a software command. Successively they created a sequence of the natural numbers in column A of the spreadsheet (Fig. 5.5a) and through two animations (moving *a* and *b* respectively) they collected the corresponding lengths of *ab* in columns B and C. In the end, they built the "scattered plot" A vs. B and A vs. C (Fig. 5.5b), and drew their conclusions about the possible distances of A's and B's houses by considering the regularities of the scatter graph and discussing why it is so.

The variable points and the ways they are manipulated in the example are typical of the software, which allows a collection of data similar to those accomplished in empirical sciences. One first picks out the variables involved, then through sequence A one gets a device to reckon the time in the animation conventionally; namely, the time variable is made explicit. The instrumented actions of TI-Nspire software naturally induce students to do so. The scattered plot thus combines the time variable A versus the length variable B or C, because the TI-Nspire software enables making the time variable explicit within mathematics itself.[5]

Given a mathematical problem like that above, one can "do an experiment" very similar to those made in empirical sciences. One picks out the important variables and makes a concrete experiment using them (e.g., collecting the data in a spreadsheet through the data-capture command). On can study mutual inter-relationships between variables (e.g., using the scatter plot) and conjecture and validate a mathematical model, possibly by new experiments. In the end, one can investigate why such a model is obtained and produce a proof of a mathematical statement. All these steps follow a precise protocol: pick out variables, design the experiment, collect data, produce the mathematical model, and validate it. The protocol is made palpable by different specific commands in the (TI-Nspire) software, such as naming variables, animation or dragging, data capture, and producing a scatter plot.

Such practices within TI-Nspire are as crucial as the dragging practices within DGS. Both incorporate almost-empirical features that can support the transition from the empirical to the deductive side of mathematics. Baccaglini-Frank (in print) has suggested how this can happen when the students are able to internalise such practices and to use them as *psychological tools* (Kozulin 1998; Vygotsky 1978, p. 52 ff) for solving conjecture-generation problems.

In this sense, the practices with the software introduce new methods in mathematics. Of course, the teacher must be aware of these potentialities of the software and integrated them into a careful didactical design. Such practices consist not only in the possibility of making explorations but also in the precise protocols that students learn to follow according to the teacher's design. Similarly, external data concerning certain quantities are passed to a computer through the use of probes. In our

[5] This procedure is very similar to the way Newton introduced his idea of scientific time as a quantitative variable, distinguishing it from the fuzzy idea of time about which hundreds of philosophers had (and would have) speculated (Newton, CW, III, p. 72).

case, the measures are collected through the "data capture" from the "internal experiment" made in the TI-Nspire mathematical world by connecting the three environments illustrated in Fig. 5.5a, b (Geometrical, Numerical, Cartesian) through suitable software commands. From one side, these methods are empirical, but from the other side they concern mathematical objects and computations or simulations with the computer, not physical quantities and experiments. Hence, the term *almost-empirical* (Arzarello 2009), which recalls the vocabulary used by some earlier scholars; for example, Lakatos (1976) and Putnam (1998) claimed that mathematics has a *quasi-empirical* status (cf. Tymoczko 1998). However, "almost-empirical" stresses a different meaning: The main feature of almost-empirical methods is the precise protocol that the users follow to make their experiments, in the same way that experimental scientists follow their own precise protocols in using machines.

Almost-empirical methods also apply within DGS environments; in fact, there are strong similarities between instrumented actions produced in TI-Ns and DGS environments. In addition, almost-empirical actions made by students in either environment are not exclusively pragmatic but also have an epistemic nature. As we discuss below, they can support the production of abductions and, hence, the transition from an inductive, empirical modality to a deductive, more formal one.

3.2 Abductions in Mathematics Learning

Abduction is a way of reasoning pointed out by Peirce, who observed that abductive reasoning is essential for every human inquiry, because it is intertwined both with perception and with the general process of invention: "It [abduction] is the only logical operation which introduces any new ideas" (C.P. 5.171).[6] In short, abduction becomes part of the *process of inquiry* along with induction and deduction.

Peirce gave different definitions of abduction, two of which are particularly fruitful for mathematical education (Antonini and Mariotti 2009; Arzarello 1998; Arzarello and Sabena in print; Baccaglini-Frank 2010a), particularly when technological tools are considered:

1. The so-called *syllogistic abduction* (C.P. 2.623), according to which a *Case* is drawn from a *Rule* and a *Result*. There is a well-known Peirce example about beans:

   Rule: All the beans from this bag are white
   Result: These beans are white
   Case: These beans are from this bag

   Such an abduction is different from a *Deduction* that would have the form: the *Result* is drawn from the *Rule* and the *Case*, and it is obviously different from an *Induction*, which has the form: from a *Case* and many *Results* a *Rule* is drawn.

[6] Peirce's work is usually referred to in the form C.P. n.m., with the following meaning. C.P. = Collected Papers; n = number of volume; m = number of paragraph.

Of course the conclusion of an abduction holds only with a certain probability. (In fact Pólya 1968, called this abductive argument an *heuristic syllogism*.)

2. Abduction as "the process of forming an *explanatory hypothesis*" (Peirce, CP 5.171; our emphasis).

Along this stream of thought, Magnani (2001, pp. 17–18) proposed the following conception of abduction: the process of inferring certain facts and/or laws and hypotheses that render some sentences plausible, that explain or discover some (eventually new) phenomenon or observation. As such it is the process of reasoning in which explanatory hypotheses are formed and evaluated. A typical example is when a logical or causal dependence of two observed properties is captured during the exploration of a situation. The dependence is by all means an "explanatory hypothesis" developed to explain a situation as a whole.

As pointed out by Baccaglini-Frank (2010a, pp. 46–50), the two types of abduction correspond to two different logics of producing a hypothesis: the logic of *selecting a hypothesis* from amongst many possible ones (first type) versus the logic of *constructing a hypothesis* (second type). According to Peirce (C.P. 5.14-212), an abduction in either form should be *explanatory*, *testable*, and *economic*. It is an *explanation* if it accounts for the facts, but remains a suggestion until it is verified, which explains the need for *testability*. The motivation for the *economic* criterion is twofold: it is a response to the practical problem of having innumerable explanatory hypotheses to test, and it satisfies the need for a criterion to select the best explanation amongst the testable ones.

Abductions can be produced within DGS environments, and can bridge the gap between perceptual facts and their theoretical transposition through supporting a *structural cognitive unity* (see below) between the explorative and the proving phase, provided there is a suitable didactic design.

For example, Arzarello (2000) gave the following problem to students of ages 17–18 (Grade 11–12) who knew Cabri-géomètre very well and had already had a course in Euclidean geometry. Moreover, the students knew how to explore situations when presented with open problems (see Arsac et al. 1992) and could construct the main geometrical figures. The students were already beyond the third van Hiele level and were entering the fourth or fifth one. (For the use of van Hiele levels in DGS environments, see Govender and de Villiers 2002.) The problem read:

> *Let ABCD be a quadrangle. Consider the perpendicular bisectors of its sides and their intersection points H, K, L, M of pairwise consecutive bisectors. Drag ABCD, considering all its different configurations: What happens to the quadrangle HKLM? What kind of figure does it become?*

Many pairs of expert[7] students typically solved the problem in five "phases":

1. The students start to shape ABCD into standard figures (parallelogram, rectangle, trapezium) and check what kind of figures they get for HKLM. In some cases they see that all the bisectors pass through the same point.

[7] Students who have acquired a sufficient instrumented knowledge of dragging practices according to a precise didactical design. The word is taken from Baccaglini-Frank (2010a).

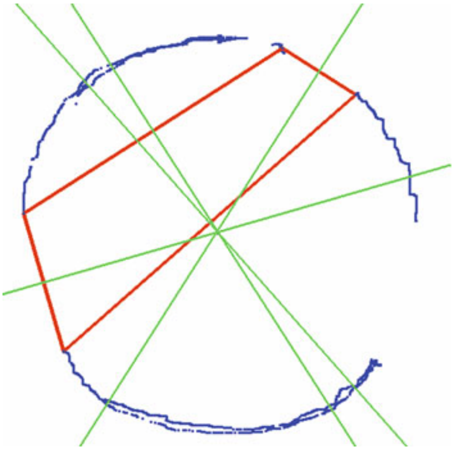


Fig. 5.6 Dragging with trace: generating a conjecture

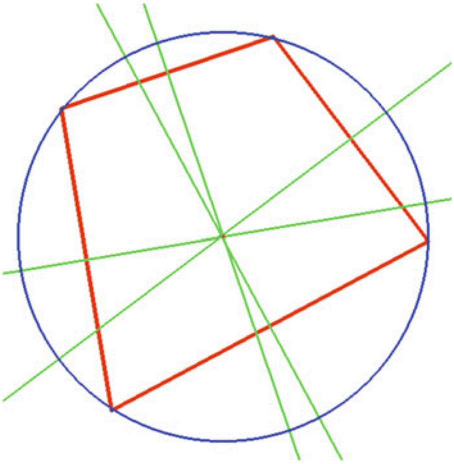


Fig. 5.7 Checking the conjecture with a construction

2. As soon as they see that HKLM becomes a point when ABCD is a square, they consider this interesting; therefore they drag a vertex of ABCD (starting from ABCD as a square) so that H, K, L, M keep on being coincident.
3. They realise that this kind of configuration is also true with quadrilaterals that apparently have no special property. Using the trace command, they find that whilst dragging a vertex along a curve that resembles a circle they can keep the four points together (Fig. 5.6). Hence they formulate the conjecture: *If the quadrilateral ABCD can be inscribed in a circle, then its perpendicular bisectors meet in one point, centre of the circle*.
4. They validate their conjecture by constructing a circle, a quadrilateral inscribed in this circle and its perpendicular bisectors, and observing that all of them meet in the same point (Fig. 5.7).

5. They write a proof of the conjecture. This process mainly consists in transforming (or eliminating) parts of the discussion held in the previous phases into a linear discourse, which is essentially developed according to the formal rules of proof.

Two major phenomena characterise the development above and are emblematic of these types of open tasks:

1. The production of an abduction: it typically marks a crucial understanding point in the process of solution;
2. The structural continuity between the conjecturing phases 1–4 and the transforming-eliminating activities of the last phase.

In producing an abduction, students first see a perceptual invariant, namely the coincidence of the four points in some cases (phases 1 & 2). So they start an exploration in order to see what conditions make the four points H, K, L, M coincide (phases 2 & 3). A particular kind of dragging (*maintaining dragging*: Baccaglini-Frank 2010b) supports this exploration: Using the trace command they carefully move the vertexes of ABCD so that the other four points remain together; finally, they realise they have thus produced a curve that resembles a circle (phase 3), namely a second invariant. At this point, they conjecture a link between the two invariants and see the second as a possible "cause" of the first; namely they produce an abduction in the form of an "explanatory hypothesis" (phase 4).

In producing a proof, (phase 5) the students write a proof that exhibits a strong continuity with their discussion during their previous explorations; more precisely, they write it through linguistic eliminations and transformations of those aforementioned utterances.

3.3 Maintaining Dragging as an Acquired Instrumented Action

The results discussed above were acquired through suitably designed teaching interventions, carefully considering the instrumented practices with the software, aimed at students' interiorising those practices as psychological tools (cf. Vygotsky 1978) they can use to solve mathematical problems. One approach to the instrumentation of dragging in DGS accords with this aim: the configuration of the '*Maintaining Dragging* Conjecturing Model' for describing a specific process of conjecture-generation, as developed by A. Baccaglini-Frank (Baccaglini-Frank 2010a) and by Baccaglini-Frank and Mariotti (2010). Enhancing Arzarello et al.'s (2002) analysis of dragging modalities (and of the consequent abductive processes), Baccaglini-Frank developed a finer analysis of dragging. She has also advanced hypotheses on the potential of dragging practices, introduced in the classroom, becoming a psychological tool, not only a list of automatic practices learnt by rote.

According to the literature (Olivero 2002), spontaneous use of some typologies of dragging does not seem to occur frequently. Consequently, Baccaglini-Frank first explicitly introduced the students to some dragging modalities, elaborated from

Arzarello et al.'s (2002) classification,[8] then asked the students to solve open tasks like the problem on quadrilaterals in the previous section. In such problem-solving activities, a specific modality of dragging appeared particularly useful to students: *maintaining dragging* (MD). Maintaining dragging consists of trying to drag a base point whilst also maintaining some interesting property observed. In the example above, the solvers noticed that the quadrilateral HKLM, part of the *Cabri*-figure, could "become" a single point; thus, they could attempt to drag a base point whilst trying to keep the four points together. In other words, MD involves both the recognition of a particular configuration as interesting and the attempt to induce the particular property to remain invariant during dragging. Healy's (2000) terminology would denote such an invariant as a soft invariant as opposed to a robust invariant, which derives directly from the construction steps. Maintaining dragging is an elaboration of *dummy locus dragging* but differs slightly: dummy locus dragging can be described as "wandering dragging that has found its path," a dummy locus that is not yet visible to the subject (Arzarello et al. 2002, p. 68), whilst MD is "the mode in which a base point is dragged, not necessarily along a pre-conceived path, with the specific intention of the user to maintain a particular property." (Baccaglini-Frank and Mariotti 2010).

In the example above, MD happened in phases 2 and 3, when the students dragged the vertices of the quadrilateral in order to keep together the four points H, K, L, M. As in phases 3–4 of the example, when MD is possible, the invariant observed during dragging may automatically become "the regular movement of the dragged-base-point along the curve" recognised through the trace mark; this can be interpreted geometrically as the property "dragged-base-point belongs to the curve" (Baccaglini-Frank in print). As pointed out by Baccaglini-Frank (2010b), the *expert* solvers proceed smoothly through the perception of the invariants and immediately interpret them appropriately as conclusion and premise in the final conjecture. However, becoming expert is not immediate, since it requires a careful didactical design that pushes the students towards a suitable instrumented use of the MD-artefact. In fact, "from the perspective of the instrumental approach, MD practices may be considered a utilization scheme for expert users of the *MD- artefact* thus making MD an *instrument* (the *MD-instrument*) for the solver with respect to the task for producing a conjecture" (Baccaglini-Frank, *ibid.*).

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[8] Arzarello and his collaborators distinguish between the following typologies of dragging:

– *Wandering dragging*: moving the basic points on the screen randomly, without a plan, in order to discover interesting configurations or regularities in the figures.

– *Dummy locus dragging*: moving a basic point so that the figure keeps a discovered property; that means you are following a hidden path even without being aware of it.

– *Line dragging*: moving a basic point along a fixed line (e.g. a geometrical curve seen during the dummy locus dragging).

– *Dragging test*: moving draggable or semi-draggable points in order to see whether the figure keeps the initial properties. If so, then the figure passes the test; if not, then the figure was not constructed according to the desired geometric properties.

Baccaglini-Frank has organised this development of instrumented maintaining dragging (MD) in a Model (Baccaglini-Frank 2010a) that serves as a precise protocol for students, who follow it in order to produce suitable conjectures when asked to tackle open problems (Arsac 1999). This protocol structurally resembles that illustrated above for data-capture with TI-Nspire software. It is divided into three main parts:

1. Determine a configuration to be explored by inducing it as a (soft) invariant. Through wandering dragging the solver can look for interesting configurations and conceive them as potential invariants to be intentionally induced. (See phases 1–2 in our DGS example).
2. Searching for a Condition through MD: students look for a condition that makes the intentionally induced invariant be visually verified through maintaining dragging from path to the geometric interpretation of the path. Genesis of a Conditional Link through the production of an abduction. (See phases 2–3).
3. Checking the Conditional Link between the Invariants and verifying it through the dragging test. (See phases 3–4).

After a conjecture has been generated through this process, the students (try to) prove their conjecture (see phase 5).

The MD-conjecturing Model relates dragging and the perception of invariants with the developing a conjecture, especially with the emergence of the premise and the conclusion. This apparently common process is well-illustrated by the MD-conjecturing protocol as a sequence of tasks a solver can engage in. Baccaglini-Frank's model allows us to "unravel" the abductive process that supports both the formulation of a conjecture and the transition from an explorative phase to one in which the conjecture is checked.[9] The *path* (in our DGS example the circle created by the students in phase 4, Fig. 5.7) plays a central role by incorporating an answer to the solver's "search for a cause" for the intentionally induced invariant (phase 3, Fig. 5.6), and thus leading to the premise of a potential conjecture. Its figure-specific component (the actual curve that can be represented on the screen) contains geometrical properties that may be used as a bridge to proof.

[9] Arzarello et al. (1998a,b, 2000, 2002) showed that the transition from the inductive to the deductive level is generally marked by an *abduction*, accompanied by a cognitive shift from *ascending to descending* epistemological modalities (see Saada-Robert 1989), according to which the figures on the screen are looked at. The modality is ascending (from the environment to the subject) when the user explores the situation, e.g., a graph on the screen, with an open mind and to see if the situation itself can show her/him something interesting (like in phases 1, 2, 3 of our example); the situation is descending (from the subject to the environment) when the user explores the situation with a conjecture in mind (as in phase 4 of our example). In the first case the instrumented actions have an explorative nature (to see if something happen); in the second case they have a checking nature (to see if the conjecture is corroborated or refuted). Epistemologically, the cognitive shift is marked by the production of an *abduction*, which also determines the transition from an inductive to a deductive approach.

When MD is used expertly, abduction seems to reside at a meta-level with respect to the dynamic exploration. However, abduction at the level of the dynamic explorations only seems to occur when MD is used as a psychological tool (Kozulin 1998; Vygotsky 1978, p. 52 ff). According to Baccaglini-Frank analysis, it seems that:

> if solvers who have appropriated the MD- instrument *also internalize it* transforming it into a psychological tool, or a fruitful “mathematical habit of mind” (Cuoco 2008) that may be exploited in various mathematical explorations leading to the generation of conjectures, a greater cognitive unity (Pedemonte 2007) might be fostered. In other words, it may be the case that when the MD instrument is used as a psychological tool the conjecturing phase is characterized by the emergence of arguments that the solver can set in chain in a deductive way when constructing a proof (Boero et al. 1996).
>
> (Baccaglini-Frank in print)

Something similar pertains to the protocol of data-capture with TI-Nspire software, which also involves almost-empirical actions (discussed above). Such almost-empirical methods seem fruitful for supporting the transition to the theoretical side of mathematics, provided their instrumentation can produce their internalisation as psychological tools and foster cognitive unity. On the contrary, when such protocols are merely used “automatically” they tend to lead to conjectures with no theoretical elements to bridge the gap between the premise and the conclusion of the conditional link; in other words, they do not encourage cognitive unity.

Since it is crucial in the transition from arguments to proofs, from the empirical to the theoretical, in the next section we discuss cognitive unity as the latest research has elaborated it.

3.4 Cognitive Unity

Boero has defined *cognitive unity* as the continuity that may exist between the argumentation of producing a conjecture and the construction of its proof (Boero et al. 1996). He hypothesises that, in some cases, “this argumentation can be exploited by the student in the construction of a proof by organizing some of the previously produced arguments into a logical chain” (Boero et al. 2010, p. 183). Pedemonte (2007) has further refined this concept, introducing the notion of *structural continuity* between argumentation and proof; that is, when inferences in argumentation and proof are connected through the same structure (abduction, induction, or deduction). For example, there is structural continuity between argumentation and proof if some abductive steps used in the argumentation are also present in the proof, as was the case in the problem of the distances of the houses from the school (see Fig. 5.5 and Boero et al. 2010).

Recently, Boero and his collaborators (Boero et al. 2010,) have integrated their analysis of *cognitive unity* with Habermas’ elaboration (Habermas 2003) of *rational behaviour in discursive practices*. They have adapted Habermas’ three components of rational behaviour (teleologic, epistemic, communicative) to the discursive practice of proving and have identified:

(A) An *epistemic aspect***,** consisting in the conscious validation of statements according to shared premises and legitimate ways of reasoning…;
(B) A *teleological aspect*, inherent in the problem-solving character of proving, and the conscious choices to be made in order to obtain the desired product;
(C) A *communicative aspect*, consisting in the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning and the conformity of the products (proofs) to standards in a given mathematical culture.

(Boero et al. 2010, pp. 188)

In this model, the expert's behaviour in proving processes can be described in terms of (more or less) conscious constraints upon the three components of rationality: "constraints of epistemic validity, efficiency related to the goal to achieve, and communication according to shared rules" (*ibid.*, p. 192). As the authors point out, such constraints result in *two levels of argumentation*:

– a level (that we call *ground level*) inherent in the specific nature of the three components of rational behaviour in proving;
– a *meta-level*, "inherent in the awareness of the constraints on the three components"

(*ibid.*, p. 192).

The two notions – cognitive unity and levels of argumentations – are important for analysing students' thought processes in the transition from argumentations to proofs within technological environments (especially DGS) and in particular very useful for analysing indirect proofs.

3.5 Indirect Proofs

Antonini and Mariotti (2008) have developed a careful analysis of indirect proofs and related argumentations from both a mathematical and a cognitive point of view, and have elaborated a model appropriate for interpreting students' difficulties with such proofs. Essentially, the model splits any indirect proof of a sentence S (principal statement) into a pair (*s,m*), where *s* is a direct proof (within a theory T, for example Euclidean Geometry) of a secondary statement S* and *m* is a meta-proof (within a meta-theory MT, generally coinciding with classical logic) of the statement $S^* \rightarrow S$. However, this meta-proof *m* does *not* coincide with Boero et al.'s (2010) meta-level considered above; rather, it is at the meta-mathematical level. As an example, they consider the (principal) statement S: "Let *a* and *b* be two real numbers. If $ab=0$ then $a=0$ or $b=0$" and the following indirect proof: "Assume that $ab=0$, $a\neq 0$, and $b\neq 0$. Since $a\neq 0$ and $b\neq 0$ one can divide both sides of the equality $ab=0$ by a and by b, obtaining $1=0$". In this proof, the secondary statement S* is: "let a and b be two real numbers; if $ab=0$, $a\neq 0$, and $b\neq 0$ then $1=0$". A direct proof is given. The hypothesis of this new statement is the negation of the original statement and the thesis is a false proposition ("$1=0$").

Antonini and Mariotti (2008) use their model to point out that the main difficulty for students facing indirect proof consists in switching from *s* to *m*. Yet the difficulty seems less strong for statements that require a proof by contrapositive; that is, to prove $B' \rightarrow A'$ (secondary statement) in order to prove $A \rightarrow B$ (principal statement). Integrating the two models, we can say that switching from *s* to *m* requires a well-established epistemic and teleological rationality in the students and in this respect does need the activation of Boero et al.'s (2010) meta-level of argumentation.

The distinction between this meta-level and the ground level in Boero et al.'s (2010) model may be very useful in investigating the argumentation and proving processes related to indirect proof. Based on this distinction, we introduce the notion of *meta-cognitive unity*: a cognitive unity between the two levels of argumentation described above, specifically between the *teleological* component at the *meta-level* and the *epistemic* component at the *ground level*.

Different from structural and referential cognitive unity (Garuti et al. 1996; Pedemonte 2007), meta-cognitive unity is not concerned with two diachronic stages in students' discursive activities (namely argumentation and proving, which are produced sequentially), rather it refers to a synchronic integration between the two levels of argumentation. We hypothesise that the existence of such a meta-cognitive unity is an important condition for producing indirect proofs. In other words, lacking the integration between the two levels of argumentation can block students' proving processes or produce cognitive breaks like those described in the literature on indirect proofs. Meta-cognitive unity may also entail structural cognitive unity at the ground level and may develop through what we call '*the logic of not*' (see Arzarello and Sabena, in print).

3.6 The Logic of Not

The '*logic of not*' is an interesting epistemological and cognitive aspect of argumentation that sometimes is produced by students who tackle a problem where a direct argument is revealed as not viable.

Their strategy is similar to that of a chemist, who in the laboratory has to detect the nature of some substance. For example, knowing that the substance must belong to one of three different categories (a, b, c), the chemist uses suitable reagents to test: if the substance reacts in a certain way to a certain reagent it may be of type a or b but *not* c, and so on. In such practices, abductive processes are usually used: if, as a *Rule*, the substance S makes blue the reagent r and if the *Result* of the experiment shows that the unknown substance X makes blue the reagent r, then the chemist reasons that X = S (*Case* of the abduction).

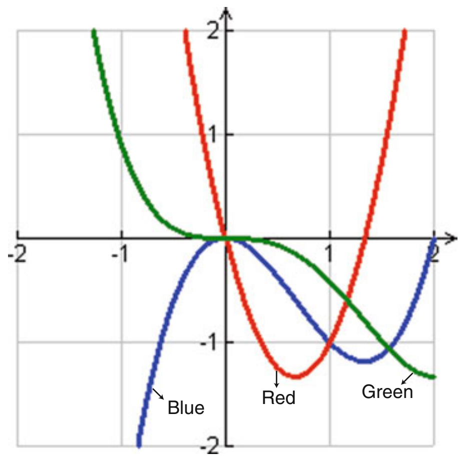

Fig. 5.8 The given task

For example, we summarise the case of a student comprehensively discussed in Arzarello and Sabena (in print). The student S (grade 9 in a science-oriented school) is solving the following task:

> *The drawing* (Fig. 5.8) *shows the graphs of: a function* f*, its derivative, one of its anti-derivatives. Identify the graph of each function, and justify your answer.*

The functions are differently coloured: the parabola red (indicated with R); the cubic (with a maximum point in the origin) is blue (B); the last (a quartic with an inflection point in the origin) is green (G). S does not know the analytic representations of the functions but has only their graphs. As such, he refers to the functions only by their colours.

In the first part of his protocol S checks which of the three functions can be *f*. He does this by looking for possible abductions, which involve the features of the given graphs.

For example, he starts supposing that *f* is the red function, probably because it is the simplest graph, and wonders whether he can apply an abductive argument with the following form to conclude that its derivative possibly is the green function:

1. Rule: "any derivative of a decreasing function is negative"
2. Result: "the green function is negative";
3. Case: "the green function is the derivative of *f*"

(ARG. 1)

Like the metaphorical chemist, S is able first to find a 'reagent' that discriminates between the substances (functions) he is analysing and then to validate his hypothesis with a further discriminating experiment, using his learnt practices with the graphs of functions. Arzarello and Sabena (in print) argue that some of S's argumentations are *teleological* and at the *meta-level*: they address S's own successive actions and his control of what is happening. The teleological component at the meta-level intertwines with the epistemological component at the ground level in a deep unity. This complex unity allows S to produce a proof by contraposition (the reasoning that logicians call "*modus tollens*": from "A implies B" to "not B implies not A"). Through this transition to a new epistemological status for his statements, S can lighten the cognitive load of the task using Arzarello and Sabena's 'logic of not', as written in part of S's protocol:

> Then I compared the "red" with the "green" function: but, the "green" function cannot be a derivative of the "red" one, because in the first part, when the "red" function is decreasing, its derivative should have a negative sign, but the "green" function has a positive sign.

Here the structure of the sentence is more complex than before: S is thinking towards a possible argument in the following form:

1. "any derivative of a decreasing function is negative"
4. "the "green" function has a positive sign"
5. "the "green" function cannot be a derivative of an increasing function"

} (ARG. 2)

Unlike the possible abduction ARG. 1 above, ARG. 2 has the form: (1) and not (2); hence not (3). Crucially, the refutation of the usual Deduction (Rule, Case; hence Result) has the same structure, because of the converse of an implication ("A implies B" is equivalent to "not B implies not A"). In other words, the refutation of an argument by abduction coincides with the refutation of an argument by deduction. Whilst abductions and deductions are structurally and cognitively different, their refutations are identical formally. So S can produce a form of deductive argument "naturally" within an abductive modality – though remarkably from an epistemological and cognitive point of view, because the apparently "natural" abductive approach of students in the conjecturing phases (Arzarello et al. 1998) often does not lead to the deductive approach of the proving phase (Pedemonte 2007). The transition from an abductive to a deductive modality requires a sort of "somersault", an inversion in the functions and structure of the argument (the Case and Result functions are exchanged) which may cognitively load the students. However, this inversion is not necessary in either the refutation of an abduction or the refutation of a deduction. An "impossible" abductive argument already has the structure of a deduction; namely, it is an argument by contraposition. Of course greater cognitive effort is required to manage the refutation of an abduction than to develop a simple direct abduction. But the coincidence between abduction and deduction in cases of refutation allows avoiding the "somersault".

De Villiers (who does not use this terminology) has pointed out another possible use of the 'logic of not'. He observes that in DGS environments it is important:

> …to sensitize students to the fact that although *Sketchpad* is very accurate and extremely useful for exploring the validity of conjectures, one could still make false conjectures with it if one is not very careful. Generally, even if one is measuring and calculating to 3 decimal accuracy, which is the maximum capacity of *Sketchpad 3*, one cannot have absolute certainty that there are no changes to the fourth, fifth or sixth decimals (or the 100th decimal!) that are just not displayed when rounding off to three decimals. This is why a logical explanation/proof, even in such a convincing environment as *Sketchpad*, is necessary for absolute certainty.
>
> (de Villiers 2002, p. 9)

One way of promoting students' sensibility is to create some cognitive conflict to counteract students' natural inclination to just accept the empirical evidence that the software provides. For example, one can use an activity where students are led to make a false conjecture; though they are convinced it is true, it turns out false: In such cases the logic of not can drive them to produce a proof.

Thus, DGS has the potential of introducing students to indirect arguments and proofs; specifically, the use of "maintaining dragging" (MD) supports producing abductions. This can be fruitfully analysed in terms of the "logic of not".

3.7 Indirect Proof Within DGS

Theorem acquisition and justification in a DGS environment is a **"**cognitive-visual dual process potent with structured conjecture-forming activities, in which dynamic visual explorations through different dragging modalities are applied on geometrical entities" (Leung and Lopez-Real, 2002, p. 149). In this duality, visualisation plays a pivotal role in the development of epistemic behaviour like the Maintaining Dragging Model (Baccaglini-Frank and Mariotti 2010). On the cognitive side, DGS facilitates experimental identification of geometrical invariants through functions of variation induced by dragging modalities which serve as cognitive-visual tools to conceptualise conjectures and DGS-situated argumentative discourse (Leung 2008). With respect to indirect proof within DGS, Leung proposed a visualisation scheme to "see a proof by contradiction" in a DGS environment (Leung and Lopez-Real 2002). The scheme's key elements were the DGS constructs of pseudo-object and locus of validity; together, they serve as the main cognitive-visual bridge to connect the semiotic controls and the theoretical controls in the argumentation process. This scheme developed out of a *Cabri* problem-solving workshop conducted for a group of Grade 9 and Grade 10 students in Hong Kong. The researcher gave the following problem to students to explore in the *Cabri* environment:

> *Let ABCD be a quadrilateral such that each pair of interior opposite angles adds up to 180°. Find a way to prove that ABCD must be a cyclic quadrilateral.*

3.7.1 The Proof

After exploration, a pair of students wrote down the following "*Cabri*-proof" (Fig. 5.9).

The labelling of the angles in their diagram was not part of the actual *Cabri* figure. The key idea in the proof was the construction of an impossible quadrilateral EBFD. However, the written proof did not reflect the dynamic variation of the impossible quadrilateral in the *Cabri* environment that promoted the argumentation. An in-depth interview with the two students on how they used *Cabri* to arrive at the proof led to the construction of the cognitive-visual scheme.

PROOF:

Assume that for a quadrilateral with each pair of interior opposite angles adding up to 180°, the four vertices can be on different circles.

From the diagram we see that it has a contradiction as the sum of the opposite angles of the blue quadrilateral (EBFD) is 360°, which is impossible.

Therefore, for a quadrilateral with each pair of interior opposite angles adding up to 180°, the four vertices must be on the same circle.

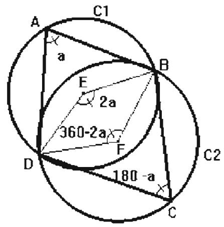

Fig. 5.9 The proof of the students

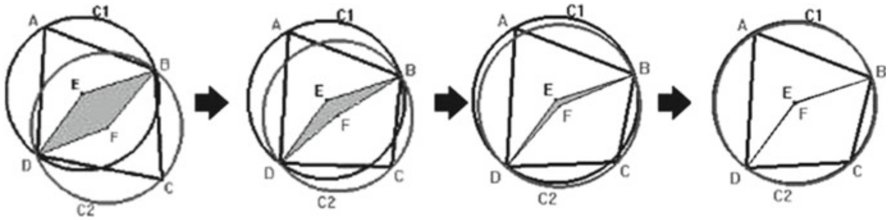

Fig. 5.10 Dragging the pseudo-quadrilateral EBFD

3.7.2 The Argumentation

The impossible quadrilateral EBFD, henceforth called a *pseudo-quadrilateral*, in Fig. 5.9 plays a critical role in organising the cognitive-visual process that leads to the construction and justification of a theorem. EBFD is a visual object that measures the degree of anomaly of a biased *Cabri* world with respect to the different positions of the vertices A, B, C and D. There are positions where the pseudo-quadrilateral EBFD vanishes when a vertex of ABCD is being dragged. Figure 5.10 depicts a sequence of snapshots in a dragging episode when C is being dragged until EBFD vanishes.

The last picture in the sequence shows that when C lies on the circumcircle C1 of the quadrilateral ABCD, E and F coincide, and at this instance, $\angle DEB + \angle DFB = 360^\circ$ (a contradiction arising from the pseudo-quadrilateral EBFD). Furthermore, this condition holds only when C lies on C1; that is, when A, B, C and D are concyclic. The pseudo-quadrilateral EBFD and the circumcircle C1 play a dual role in an argumentation process. First, they restrict the quadrilateral ABCD to a special configuration that leads to the discovery that ABCD possesses

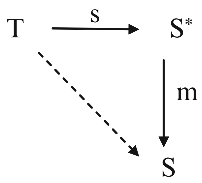

Fig. 5.11 Indirect proofs scheme

certain properties (through abductive inference). Second, they generate a convincing argument that collapses onto a Reductio ad Absurdum proof (Fig. 5.9) acceptable in Euclidean Geometry.

3.7.3 The Scheme

Suppose A is a figure (quadrilateral ABCD in the *Cabri*-proof) in a DGS environment. Assume that A satisfies a certain condition C(A) (interior opposite angles are supplementary) and impose it on all figures of type A in the DGS environment. This *forced presupposition* evokes a 'mental labelling' (the arbitrary labelling of $\angle DAB = 2a$ and $\angle DCB = 180\text{-}2a$ in the *Cabri*-proof) which leads one to *act cognitively* on the DGS environment. Thus C(A) makes an object of type A biased with extra meaning that might not necessarily be true in the actual DGS environment. This *biased DGS environment* exists as a kind of hybrid state between the *visual-true* DGS (a virtual representation of the Euclidean world) and a *pseudo-true* interpretation, C(A), insisted on by the user. In this pseudo world, the user can construct an object associated with A which inherits a local property that is not necessarily consistent with the Euclidean world because of C(A) (e.g., the impossible quadrilateral EBFD in the *Cabri*-proof): We call such an object associated a *pseudo object* and denote it by O(A).When part of A (the point C) is being dragged to different positions, O(A) might vanish (or degenerate; i.e., a plane figure to a line, a line to a point). The path or locus on which this happens gives a constraint (both semiotic and theoretical) under which the forced presupposition C(A) is "Euclidean valid"; that is, where the biased microworld is being realised in the Euclidean world. This path is called the *locus of validity* of C(A) associated with O(A) (the circle C1).

In the Indirect Proof context (Antonini and Mariotti, 2008), one can interpret this scheme as follows: S is the principle statement "If the interior opposite angles of a quadrilateral add up to 180°, then it is a cyclic quadrilateral"; S^* is the secondary statement "If the interior opposite angles of a quadrilateral add up to 180° and its vertices can lie on two circles, then there exists a quadrilateral with the property that a pair of interior opposite angles add up to 360°." (Fig. 5.11)

In this scheme, T is Euclidean Geometry; s is a direct Euclidean proof. In a DGS environment, the meta-proof m could be a kind of dragging-based visual logic. In the case discussed above, when a pseudo object and a locus of validity arise, m could be a drag-to-vanish MD visual logic. Thus the *composite proof*

$$m \circ s : T \dashrightarrow S$$

is an indirect proof that is both theoretical and DGS-mediated.

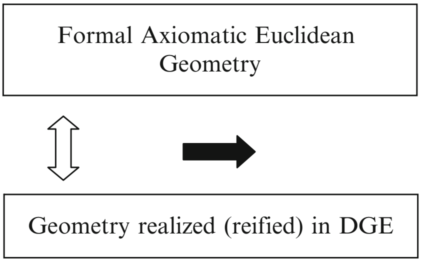

Fig. 5.12 From Lopez-Real and Leung (2006)

In relation to the Boero-Habermas model (Boero et al. 2010), the theoretical part (s) is the epistemological component (theoretical control) at the ground level where the existence of the pseudo-quadrilateral EBFD was deduced. The DGS-mediated part (m) is the teleological component (semiotic control) at the meta-level where the dragging-based argumentation took place. Hence the intertwined composite proof (m o s) can be seen as a meta-cognitive unity in which argumentation crystallises into a Reductio ad Absurdum proof.

Within the "logic of not", the DGS-mediated part (m) allows a link from the abductive modality to the deductive modality. In the previous example of S, the distance between the two modalities was annihilated because of the coincidence between the negations of the abduction and of the deduction; here the distance is shortened through m: in both cases, the cognitive effort required is reduced.

Lopez-Real and Leung (2006) suggested that Formal Axiomatic Euclidean Geometry (FAEG) and Dynamic Geometry Environment (DGE) are 'parallel' systems that are "situated in different semiotic phenomena" instead of two systems having a hierarchical relationship (Fig. 5.12).

> The vertical two-way arrow denotes the connection (networking) that enables an exchange of meaning between the systems. The horizontal arrow stands for a concurrent mediation process that signifies some kind of mathematical reality. This perspective embraces the idea that dragging in DGE is a semiotic tool (or a conceptual tool) that helps learners to form mathematical concepts, rather than just a tool for experimentation and conjecture making that doesn't seem to match the 'logical rigour' in FAEG. (Lopez-Real and Leung 2006, p.667)

In this connection, the MD dragging scheme – together with the construct of pseudo-object and locus of validity, and with the associated reasoning carried out, on the one hand, in the context of the DGS and, on the other hand, in T by the solver – may serve as channels to enable an exchange of meaning between the two systems (Fig. 5.13).

The idea of composite proof in DGS environment could possibly be expanded to a wider scope where there is a hybrid of Euclidean and DGS registers. Leung (2009) presented such a case where a student produced a written proof that intertwined Euclidean and DGS registers. The first results of Leung's analysis are promising, opening new perspectives of investigation.

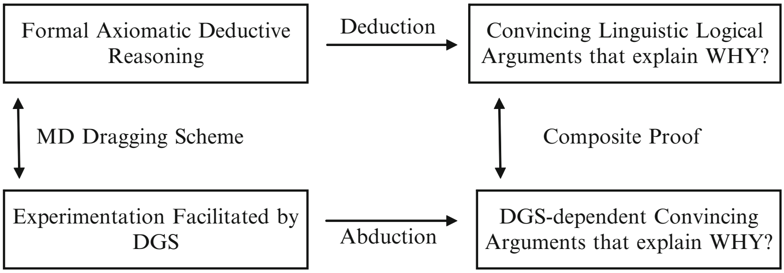

Fig. 5.13 The two semiotic systems: FAEG and DGE

4 Part 3: Towards a Framework for Understanding the Role of Technologies in Geometrical Proof

As discussed in Part 1, geometry may be split into what Einstein (1921) called "practical geometry", obtained from physical experiment and experience, and "purely axiomatic geometry" containing its logical structure. Central to learning geometry is an understanding of the relationship between the technologies of geometry and its epistemology. Technology in this context is the range of artefacts (objects created by humans) and the associated techniques which together are needed to achieve a desired outcome. Part 2 has set out in detail how the process of coordinating technologies with the development of geometric reasoning combines artefactual "know-how" with cognitive issues. In Part 3, we first provide a model, using Activity Theory, that highlights the role of technology in the process. Second, this part discusses the mediational role of digital technology in learning geometry, and the implications for developing proofs.

4.1 Modelling Proof in a Technological Context

To analyse proof in a technological context, it is useful to consider a framework derived from Activity Theory, shown in Fig. 5.14 (Stevenson 2008). The framework provides a way of describing the use of artefacts, (e.g., digital devices, straight-edges, etc.) in processes of proving. Activity Theory, a framework for analysing artefact-based social activity, is a "theory" in the sense that it claims that such activity can be described as a system using the categories shown in Fig. 5.14. In this section, we "flesh out" the epistemological, cultural and psychological dimensions of the system in relation to technology and proof.

In Fig. 5.14, the "object" of the system is the formulation of a problem which motivates and drives the proof process, with the "outcome" being the proof created from this system. "Artefacts" are any material objects used in the process, which in our case includes straight-edges, compasses and DGS. A "subject" is a person who

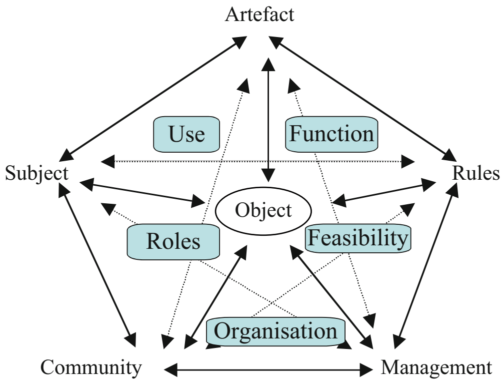

Fig. 5.14 Proving in a technological context: a framework from activity theory

is part of a "community" in which the activity is set; the community is "managed" through power structures that assign roles and status within a given context (e.g. classroom, professional mathematics community). "Rules" comprise the relationships that define different aspects of the system.

A "technology" in this system consists of the artefacts used, objects to which they can be applied, persons permitted to use the artefacts, and the sets of rules appropriate to each of the relationships which make up the system. Figure 5.14 identifies five aspects of the interactions with technology useful for the analysis. 'Function' covers the set of techniques applicable with the artefacts to a specific object, within a particular setting and social grouping. 'Use' relates to the ways in which individuals or groups actually behave with the artefacts within the social context, governed by the norms of community organisation. 'Roles' indicates the types of linguistic interactions adopted by the participants, and 'Organisation' refers to the groupings of those participating in the technology-based activities. Finally, 'Feasibility' relates to the practical constraints placed on an activity by the physical and temporal location.

Figure 5.4 (Part 1) and its associated description highlight the cultural dimension of a mathematics classroom engaging in proof activities with technology (straight-edge and compass or DGS) by linking together the five aspects of the model in Fig. 5.14 (Use, Function, Roles, Feasibility and Organisation, cf. Stevenson 2008). The model brings together both the selection of tasks related to the objectives and outcomes of activities and the teacher's use of artefacts. In particular, it expresses how teachers tailor their use of artefacts to their specific classrooms in order to mediate ideas about proof and its forms to their students. As a result, the model expresses how specific forms of activity and styles of linguistic interaction between pupils and teacher in a given physical location provide the context for studying proof.

Epistemologically, the system is defined by the rules governing the formal object of mathematical knowledge that is the context for the proof process (e.g., geometry). For "standard" proof, the technology consists of the artefacts needed to create the proof (e.g., paper and pencil) and the rules of inference that govern how statements are organised. Their rules of inference describe the syntactical dimension of the process. A major claim for using proof as a method for obtaining knowledge is that, if "applied correctly", the rules of inference preserve the semantic integrity of the argument forming the proof. Semantics deals with the meanings given to geometric statements and is concerned with the truth and knowledge claims that a proof contains. Technology, therefore, plays a key role in the relationship between the semantics and syntactical structures of a proof, raising the issue of whether syntax and semantics are separable or inextricably intertwined with the technology.

As the section "Abductions in mathematics learning" (Part 2) proposes, conjectures become proofs by applying the technology of logical inference. Abduction is, epistemologically, a non-linear process which develops over a period of time and involves iterations between facts and the conjectured rules that gradually come to explain those facts. Constructing a proof involves restructuring a conjecture to suit the linear form of logical inference so that the technique can be applied to organise the argument "on the page". Such linearisation re-interprets (or removes) the diachronic aspect of abduction as an epistemological structure. In the process of translating conjecture into a proof, constructions, false starts, and strings of informal calculations are removed. References to sensorimotor processes in geometry are suppressed by talking about "ideal" points and lines, with the paper surface acting as a kind of window on the "real" geometry. (Livingstone 2006). In terms of the model in Fig. 5.14, the role of the teacher is crucial in helping learners make this linguistic transition. The extracts of dialogue related to this process of linearisation in Part 2 indicate how the cues and leads given by the teacher aid the learner in filtering the conjecture so that the techniques of inference can be used to organise it appropriately.

Adding DGS to this situation does not change the essential dynamic tensions resulting from the need to translate from one technological setting (straight-edge and compass or DGS) to another (rules of inference). The discussions in Part 2 of the "logic of not" and "indirect proof" (the 'Use' aspects of the model in Fig. 5.14) imply that abduction arises as a strategy to deal with those dynamic tensions.

4.2 Digital Technology as a Mediational Artefact

Learning how to use geometrical equipment, whether physical or digital, is part of the instrumentation of geometry (Verillion and Rabardel 1995), the interplay between facility with artefacts and the development of psychological concepts. Physically, one has the experience of using a straight edge to draw a "straight line" and compasses to make a "curve". In Lakoff's (1988) framework, such actions can be interpreted as developing a "prototype". The motor-sensory action of using a straight-edge and pencil, combined with the word "straight" and the Gestalt perception of the resulting mark on a surface, embodies the concept of "straightness".

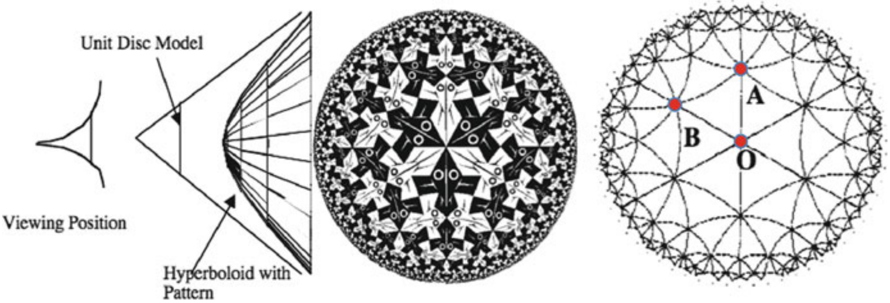

Fig. 5.15 A genesis of the Poincaré disc for hyperbolic geometry

This account raises the question of how far technology, in general, mediates the understanding of geometrical concepts, and how that mediation relates to proof in the formal sense. For example, rather than simply motivating proofs of results, does/can/should DGS play an integral part in forming the conceptual structures that constitute geometry? Much of the interest in DGSs lies in the representation of Euclidean geometry, but as Part 2 implies, DGSs provide a different kind of geometry from that obtained by paper and pencil construction, or by the axiomatic version of Euclidean geometry. Consequently, different versions of geometry emerge with these different technologies: "pencil geometry", "digital geometry", and "axiomatic geometry". The question is not whether technology mediates knowledge, but how different technologies mediate different kinds of knowledge. Proof, as a means of establishing knowledge claims, should therefore take account of the mediational role that artefacts play in epistemology.

Learning non-Euclidean geometry, for example, has a number of complexities when compared to the Euclidean case. On the one hand, spherical geometry is relatively straightforward, since learners may have everyday opportunities to develop visual intuitions. Being "smaller" than Euclidean space, spherical surfaces are both closed and bounded, and allow both physical and digital manipulation. On the other hand, hyperbolic space poses a different problem; it is "larger" than Euclidean space, so learners have difficulty defining a complete physical surface to manipulate (Coxeter 1969). The learning process also lacks opportunities for visual intuition and suffers from difficulties in finding appropriate artefacts to support instrumentation. However, digital technologies offer possibilities for engaging with hyperbolic geometries that cannot be found otherwise (Jones et al. 2010). Figure 5.15 shows how a two-dimensional Euclidean model can be obtained by projecting a hyperbolic surface, and illustrates the geometry associated with the projection.

Imagine that an Escher tessellation is spread isometrically across the hyperboloid on the left-hand side of Fig. 5.15. Viewing it as a projection onto a flat disc gives the image in the centre of Fig. 5.15. The grid for the tessellation is shown on the right of Fig. 5.15, together with the basic hyperbolic triangle OAB used to tessellate the disc. Triangle OAB shows one of the key differences between hyperbolic geometry and Euclidean: the angle sum of the hyperbolic triangles is less than 180°. The edge of the circle represents infinity (it is called the "horizon"), which can be

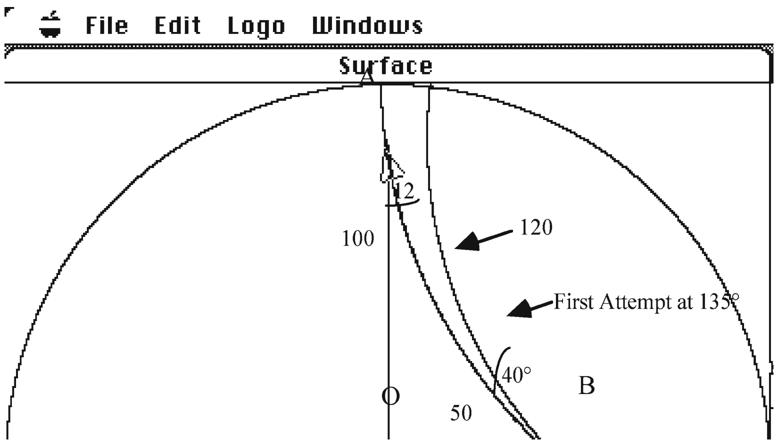

Fig. 5.16 The turtle within the hyperbolic world

approached but never reached, as indicated by the "bunching" of the tessellations at the circumference of the central image and of the grid lines on the right. Further, "straight" lines in the hyperbolic world can be either Euclidean straight lines (in fact diameters of the horizon) or circular arcs (orthogonal to the horizon).

Using Turtle geometry, it is possible to animate the two-dimensional projective model of the hyperbolic surface to provide an artefact for exploring the geometry (Jones et al., *ibid*). Taken from Stevenson (2000), the following snippet shows the work of two adults (S and P) using the non-Euclidean Turtle microworld to illustrate the role of artefacts in mediating understanding of the geometry. In Fig. 5.16, S and P first draw the lines OA and OB; then they attempt to find the line (AB) to close the triangle. Starting with the Turtle at B, pointing to the right of the screen, they turn it left through 135°, and use a built-in procedure called "Path" which indicates how the Turtle would travel if moved along that heading. They see that line does not close the triangle, so they turn the Turtle by a further 5° to the left. This time the path goes through A, and they reflect on the screen results (Fig. 5.16).

S picks up a hyperbolic surface provided for them and reminds himself about the projection process (the left-hand side of Fig. 5.15). S comments on the diagram in Fig. 5.16:

S: We haven't got 180, but it's walking a straight-line path (reflectively).
(S here refers to the metaphor that, in order to trace a straight line on a curved surface – in this case the hyperboloid, the Turtle must take equal strides, hence "walking a straight-line path".)

P: Yeah, you've probably got to turn.
(Instinctively, P thinks that a Turtle walking on the curved surface must turn to compensate for the curvature. S is clear that this is not happening.)

S: No, you don't have to turn. It's actually drawing a triangle on the surface.
(Looking at the hyperboloid, S imagines the Turtle marking out the triangle on the surface.)

S: The projection defies Pythagoras. No! Hang on, walking on the surface is defying it, isn't it! Because we walk straight lines on the surface; we just see them as curves on the projection.

For S, the projection clearly preserves the geometric properties of the Turtle's path on the surface. S believes that what they see is a fact about the geometry, not a result of the software or the projection. Two points are significant: First, S's insight would be impossible without both the material artefacts (physical surface and digital application) and the metaphor that turtles walk straight lines on curved surfaces by taking "equal-strides without turning" (Abelson and diSessa 1980, p. 204). Second, S is convinced by what he sees, and provides an explanation about why the image preserves something about the geometry of the hyperbolic surface.

There remains the problem of what might constitute a proof that the angle sum of any hyperbolic triangle is always less than 180°. Given that the artefacts mediate the object of study (hyperbolic geometry), one can convert S's insight into the technology of logical inference with its associated linearisation and the re-interpretation of the diachronic aspect. Proofs of the angle sum do exist, but they rely on the axiomatic approach described by Einstein and on the technology of inference (e.g. Coxeter 1969, p. 296 ff.). However, the proofs are abstract and lack visual intuition; the digital and physical artefacts described here offer learners a more concrete and visual situation.

In the not too distant future, learners may use electronic media, rather than paper, to develop their work, which would enable them to embed digital applications. Effectively, this process will separate the technology of inference from the need to lay out arguments on paper as some kind of final statement. As for the model presented in Fig. 5.15, its value lies in being able to provide the cultural and pedagogic context for these activities; it embeds technologies in social relationships and human motivation. It also shows how dynamic tensions arise in reasoning due to conflicts between technologies. Coupled with Balacheff's analysis of proof types (2008), the model identifies how the assumptions and expectations of those engaging in proof generate contradictions in their practices (Stevenson 2011).

In closing, in this chapter we have discussed some strands of experimental mathematics from both an epistemological and a didactical point of view. We have introduced some past and recent historical examples in Western culture in order to illustrate how the use of tools has driven the genesis of many abstract mathematical concepts.

The intertwining between concrete tools and abstract ideas introduces both an "experimental" dimension in mathematics and a dynamic tension between the *empirical nature* of the activities with the tools –which encompass perceptual and operational components– and the *deductive nature* of the discipline –which entails a rigorous and sophisticated formalisation. This *almost empirical* aspect of mathematics was hidden in the second half of the nineteenth and the first half of the twentieth century because of a prevailing formalistic attitude. More recently, the perceptual and empirical aspects of the discipline have come again on the scene. This is mainly due to the heavy use of the new technology, which is deeply and quickly changing both research and teaching in mathematics (Lovasz 2006).

We have illustrated the roles both perception and empiricism now play in proving activities within the classroom and have introduced some theoretical frameworks which highlight the dynamics of students' cognitive processes whilst working in CAS and DGS environments. The learners use those technologies to explore problematic situations, to formulate conjectures and finally to produce proofs. We have pointed out the complex interplay between inductive, abductive, and deductive modalities in the delicate transition from the empirical to the theoretical side in the production of proofs. This dynamic can be strongly supported by a suitable use of technologies, provided the students learn some practices in their use, for example the *maintaining dragging* scheme in DGS. We have also shown how the induced instrumental genesis can help learners in producing indirect proofs.

Finally, we have used Activity Theory to model the dynamic tension between empiricism and deduction as a consequence of *translating between different technologies*, understood in the broadest sense as something that can mediate between different ontologies and epistemologies.

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5 Response to "Experimental Approaches to Theoretical Thinking: Artefacts and Proofs"

Jonathan M. Borwein, and Judy-anne Osborn

An overview of the chapter. The material we review focuses on the teaching of proof, in the light of the empirical and deductive aspects of mathematics. There is emphasis on the role of technology, not just as a pragmatic tool but also as a shaper of concepts. Technology is taken to include ancient as well as modern tools with their uses and users. Examples, both from teaching studies and historical, are presented and analysed. Language is introduced enabling elucidation of mutual relations between tool-use, human reasoning and formal proof. The article concludes by attempting to situate the material in a more general psychological theory.

We particularly enjoyed instances of the 'student voice' coming through, and would have welcomed the addition of the 'teacher's voice' as this would have further contextualised the many descriptions the authors give of the importance of the role of the teacher. We are impressed by the accessibility of the low-tech examples, which include uses of straight-edge and compass technology, and commonly available software such as spreadsheets. Other more high-tech examples of computer geometry systems were instanced and it would of interest to know how widely available these technologies are to schools in various countries (examples in the text were primarily Italian with one school from Hong Kong) and how much time-investment is called upon by teachers to learn the tool before teaching with it. The general principles explicated by the authors apply equally to their low-tech and high-tech examples, and are thus applicable to a broad range of environments including both low and high-resourced schools.

The main theoretical content of the chapter is in the discussion of why and how tool-use can lighten cognitive load, making the transition from exploring to proving easier. On the one hand tool-use is discussed as it relates to the discovery of concepts, both in the practical sense of students coming to a personal understanding, and in the historical sense of how concepts make sense in the context of the existence of a given tool. On the other hand the kinds of reasoning used in the practice of mathematics are made explicit – deduction, induction and a third called 'abduction' – with their roles in the stages of mathematical discovery, as well as how tool-use can facilitate these kinds of reasoning and translation between them.

Frequent use of the term 'artefact' is made in the writing, thus it is pertinent to note that the word has different and opposing meanings in the educational and the science-research literature. In the educational context the word means a useful purposely human-created tool, so that a straight-edge with compass or computer-software is an artefact in the sense used within the article. In the science-research literature, an artefact is an accidental consequence of experimental design which is misleading until identified, so for instance a part of a graph which a computer gets wrong due to some internal rounding-error is an artefact in this opposite sense.

The structure of the chapter. There is an introduction: essentially a reminder that mathematics has its empirical side as well as the deductive face which we see in formal proofs. Then there are three parts. Part 1 begins with a discussion of the history of mathematics, with reference to the sometimes less-acknowledged aspect of empiricism. In its second half, Part 1 segues from history into modern teaching examples. Part 2 is the heart of the article. It deals with the kind of reasoning natural to conjecture-forming, 'abduction', the concept of 'instrumentation' and cognitive issues relating to 'indirect proofs'; all through detailed examples and theory. Part 3 reads as though the authors are trying to express a large and fledgling theory in a small space. A general psychological paradigm called 'Activity Theory', is introduced, which deals with human activities and artefacts. An indication of digital technology as a means of translating between different ways of thinking is given in this context. We now discuss each Part in more detail.

Part 1. The historical half of Part 1 deals with geometric construction as a paradigmatic example, with the authors showing that since Euclid, tools have shaped concepts. For example, the straight-edge and compass is not just a practical technology, but helps define what a solution to a construction problem means. For instance cube duplication and angle trisection are impossible with straight-edge and compass (i.e. straight lines and circles) alone, but become solvable if the Nicomedes compass which draws a conchoid are admitted. A merely approximate graphic solution becomes a mechanical solution with the new tool. The moral is that changing the set of drawing tools changes the set of theoretically solvable problems, so that practical tools become theoretical tools. Another theme is the ambiguity noted in Descartes' two methods of representing a curve, by either a continuous motion or an equation; and subsequent historical developments coming with Pasch, Peano, Hilbert and Weierstrass, in which the intuition of continuous motion is suppressed in favour of purely logical relations. The authors perceive that historical suppression as having a *cost* which is only beginning to be counted, and rejoice that the increased use of computers is accompanying a revived intuitive geometric perspective.

This revival also offers the prospect of teachers who better understand mathematics in its historical context. Ideally, their students will gain a better appreciation of the lustrous history of mathematics. It is not unreasonable that students find hard concepts which took the best minds in Europe decades or centuries to understand and capture.

Part 1 is completed by examples from three educational studies followed by a discussion of the importance of the role of the teacher. Each example uses a tool to explore some mathematical phenomenon, with the teaching aim being that students develop a theoretical perspective. The first study involved over 2,000 students in various year-groups and 80 teachers, setting straight-edge and compass in the wider context of mathematical "machines". The second study, of Year 10 students, sits in the context of a particular DGS (Dynamical Geometry System), specifically the software called *'Cabri'*. Students first revised physical straight-edge and compass work, then worked in the virtual *Cabri* world, in which their drawings become what are termed 'Evocative Computational Objects' | no longer just shapes but shapes with associated *Cabri* commands and the capacity to be ‘dragged’ in interesting ways whose stability relates to the in-built hierarchical structure of the object. Interesting assertions made by the authors are that drag-ability relates to prove-ability, and that the original pencil drawings become signs for the richer *Cabri* objects. As in the previous study, a central aspect was student group discussion and comparison of solutions. The third study was of the use Year 9 students made of a CAS (Computer Algebra System) to explore the behaviour of functions. The students initially made numerical explorations, from which they formulated conjectures. Then, largely guided by a suggestion from their teacher, they substituted letters for numbers, at which point the path to a proof became evident.

The way in which the role of the teacher is crucial, in all three studies, is described with reference to a model expressed in Fig. 5.4. On the left of the diagram, activities and tasks chosen by the teacher sit above and relate to mathematics as a general entity within human culture. On the right side of the diagram, student’s productions and discoveries from carrying out the tasks sit above and relate to the mathematical knowledge required by the school curriculum. The artefact (purpose-created tool) sits in the middle. Reading the picture clockwise in an arc from bottom left to bottom right neatly captures that teachers need to choose suitable tasks, students carry them out, and teachers help the students turn their discovered personal meanings into commonly understood mathematics. It is pointed out that as students discuss their use of artefacts, teachers get an insight into students’ thought-processes.

Part 2. In Part 2, we get to the core of the article’s discussion of proving as the mental process of transitioning between the exploratory phase of understanding a mathematical problem to the formal stage of writing down a deductive proof. The central claim is that this transition is assisted by tools such as Dynamic Geometry System (DGS) softwares and Computer Assisted Algebra (CAS) softwares, provided these tools are used within a careful educational design. The concept of abduction is central to the authors’ conceptual framework. The term is used many times before it is defined – a forward reference to the definition in about the tenth paragraph of Part 2 would have been useful to us. It is worth quoting the definition (due to Peirce) verbatim:

> The so-called syllogistic abduction (C.P.2.623), according to which a Case is drawn from a Rule and a Result. There is a well-known Peirce example about beans:
>
> Rule: All beans from this bag are white
>
> Result: These beans are white Case:
>
> These beans are from this bag

Clearly this kind of reasoning is not deduction. The conclusion doesn’t necessarily hold. But it might hold. It acts as a potentially useful conjecture. Nor, as the authors note, is this kind of reasoning induction, which requires one case and many results from which to suppose a rule.

The authors’ naming and valuing of abduction sits within their broader recognition and valuing of the exploratory and conjecture-making aspects of mathematics which can be hidden in final-form deductive proofs. Their purpose is to show how appropriate abductive thinking arises in experimentation and leads to deductive proofs, when the process is appropriately supported.

The role of abductive reasoning in problem solving strikes these reviewers as a very useful thing to bring to educator's conscious attention. One of us personally recalls observing an academic chastise a student for reasoning which the academic saw as incorrect use of deduction, but which we now see as correct use of abduction in the early part of attempting to find a proof.

An example of 10th grade students faced with a problem about distances between houses and armed with a software called TI-Nspire is presented in detail in this section. The empirical aspect of mathematical discovery is described in an analogy with a protocol for an experiment in the natural sciences. We note that this example could be usefully adapted to a non-computerised environment. A point which the authors make, specific to the use of the computer in this context, is that the software encourages/requires useful behaviour such as variable-naming; which can then assist students in internalising these fundamental mathematical practices as psychological tools.

A teaching/learning example regarding a problem of finding and proving an observation about quadrangles, presented to 11th and 12th grade students is given. The authors give a summary of the steps most students used to solve the problem, and then interpret the steps in terms of the production of an abduction followed by a proof. The authors write

> In producing a proof, (Phase 5) the students write a proof that exhibits a strong continuity with their discussion during their previous explorations; more precisely, they write it through linguistic eliminations and transformations of those aforementioned utterances.

This statement is in the spirit of a claim at the start of Part 2 that empirical behaviour using software appropriately in mathematics leads to abductive arguments which supports cognitive unity in the transition to proofs.

The next main idea in Part 2 after 'abduction' is that of 'instrumentation'. The special kind of 'dragging' which has been referred to during discussions of DGS softwares is recognised as maintaining dragging (MD), where what is being maintained during the dragging is some kind of visible mathematical invariant. Furthermore the curve that is traced out during dragging is key to conjecture-formation and potentially proof.

The third main concept dealt with in Part 2 is that of 'indirect proof' and the difficulties that students often have with it. The authors usefully describe how software-mediated abductive reasoning may help, which they support with two plausible arguments. First, the authors note that indirect proofs can be broken up into direct proofs of a related claim (they use the term 'ground level') together with a proof of the relationship between the two claims (they use the term 'meta level'). Thus an argument for software-mediated reasoning says that use of software helps students keep track of the two levels of argument.

Second, the authors argue that abduction is useful to students partly because of what happens to formal (mathematical) claims when they are negated. The authors state that in some sense the cognitive distance between the conjecture and the proof is decreased in the negation step. They give two examples. The first is a study of a Year 9 student presented with a delightful problem about functions and their derivatives and anti-derivatives. In this case, refutation of an argument by abduction turns out to coincide with refutation of an argument by deduction. The second example is of a study of two students from Years 9 and 10 in Hong Kong given a problem about cyclic quadrilaterals in a *Cabri* environment. In this case, ‘dragging’ behaviour led the students to an argument which collapsed to a formal ‘reductio ad absurdum’.

To summarise one stream of thought from Part 2 relating to the practice of learning and teaching: (a) tool-use facilitates exploration, especially visual exploration; (b) exploration (in a well-designed context) leads to conjecture-making; (c) practical tool-use forces certain helpful behaviours such as variable-naming; (d) this ‘instrumentation’ can lead to internalising tools psychologically; (e) for indirect proofs, the way negation works helps bridge the distance between kinds of reasoning used in conjecture-making and proof.

In the closing section of Part 2, the authors go beyond the claim that abduction supports proof and become more speculative. They quote Lopez-Real and Leung to claim that deduction and abduction are parallel processes in a pair of ‘parallel systems’, Formal Axiomatic Euclidean Geometry on the one hand, and Geometry realised in a Dynamic Geometry Environment (DGE) on the other hand, and that interaction between the two ways of knowing and storing information could be productive in ways not yet fully elucidated. It would be fascinating to see these ideas fleshed out.

Part 3. Part 3 introduces ‘Activity Theory’, a general framework concerned with the know-how that relates to artefacts, and attempts to situate the discussions of Parts 1 and 2 in this context, however as readers we found it difficult to gain insight from this formulation lacking as we do previous detailed knowledge of Activity Theory. The attempted translation between languages is scanty, although there are some illuminating examples, for instance an interesting use of ‘turtle geometry’ to explore hyperbolic geometry is presented in this section, where the turtle geometry is regarded as an artefact within Activity Theory. This part also expands upon the idea of instrumentation, linking ideas about concept-development, Gestalt perception and embodiment. There is much here which could be further developed; and which assuredly will be.

Conclusions. We first highlight a notion which is implicit throughout the chapter, which is the valuing of the teaching of proof in schools. Proof is a central component of mathematics however the valuing of the teaching of proof is not always taken for granted. For instance in the Australian context we know of instances of stark contrast, where the current state-based curricula does not emphasise proof (it is mentioned in the context of upper level advanced classes only), although we know of cases in which teacher-training does emphasise it. In short, we believe that there is often not enough teaching of proof in schools and that the chapter under review may help by providing a conceptual and practical bridge for students and their teachers between the activities of exploring mathematics and of creating and understanding proofs.

We also highlight the authors' own advisements about implementation in practice of the theory they have articulated. The authors emphasise that the role of the teacher is crucial both in lesson design and classroom interaction; as is neatly captured by Fig. 5.4 near the end of Part 1. They observe, for instance in their discussion of "maintaining dragging" in Part 2, that desired student understandings and behaviours often do not arise spontaneously. Further, they warn early in Part 1 (quoting Schoenfeld) that counterproductive student behaviour can arise as unintended by-products of teaching. At the end of Part 1 the authors give references to studies in which the kinds of *useful* interventions that teachers repeatedly make are analysed. It is helpful to have the centrality of the mathematics teacher made so clear. The importance of design and interaction are emphasised in quotations such as "The teacher not only *selects suitable tasks* to be solved through constructions and visual, numerical or symbolic explorations, but also *orchestrates* the complex transition from practical actions to theoretic arguments"; and "The teacher, as an expert representative of mathematical culture, *participates in the classroom discourse* to help it proceed towards sense-making in mathematics" (our emphasis).

In summary, this work repays the effort to read it. The historical perspective at the beginning brings the duality between empiricism and deductive reasoning usefully to mind. The examples, language and theory developed in Part 2 are likely to be clarifying and inspiring to both educators and theorists. The more speculative aspects at the end of Part 2 and in Part 3 call for further elucidation and development to which we look forward.