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ROBUST SUPERLINEAR KRYLOV CONVERGENCE FOR COMPLEX NONCOERCIVE COMPACT-EQUIVALENT OPERATOR PRECONDITIONERS*

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Owe Axelsson passed away in June 2022. It was a privilege that we could work with him on this and previous papers, and we will never forget his unique inspiration and presence.

Abstract. Preconditioning for Krylov methods often relies on operator theory when mesh independent estimates are looked for. The goal of this paper is to contribute to the long development of the analysis of superlinear convergence of Krylov iterations when the preconditioned operator is a compact perturbation of the identity. Mesh independent superlinear convergence of GMRES and CGN iterations is derived for Galerkin solutions for complex non-Hermitian and noncoercive operators. The results are applied to noncoercive boundary value problems, including shifted Laplacian preconditioners for Helmholtz problems.

Key words. Krylov iteration, preconditioning, noncoercive operators, mesh independence, shifted Laplace

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1. Introduction.

Preconditioned Krylov methods are a central tool in the solution of discretized boundary value problems. In this paper we aim to contribute to the long development of the analysis of superlinear convergence of Krylov iterations based on operator theory, most of which was published in SIAM journals. Robustness, that is, mesh independent convergence estimation, is the focus of our investigations.

Superlinear convergence is often a typical second stage in the convergence history of a Krylov iteration; see, e.g., [1, 47]. This notion (called more precisely R-superlinear convergence; see [31]) expresses, roughly speaking, that the number of iterations required to achieve a new correct digit will be decreasing in the course of the iteration.

When mesh independent estimates are looked for, operator preconditioners are often involved. Then the preconditioning matrix is obtained as the discretization of a suitable operator in the function space. This approach naturally invokes the use of Hilbert space theory and of operator versions of Krylov iterations. For linear convergence, such a theory of so-called equivalent operators was elaborated in [17, 27, 41], which has given a solid and organized framework to derive mesh independent

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convergence results, and has lead to various applications; see also [23, 24, 40, 46] for some recent developments.

Superlinear convergence is related to more special situations: when the preconditioned operator is a compact perturbation of the identity. Since the classical papers [49, 50], it has become a general principle to expect superlinear convergence in this case. On the other hand, as we briefly summarize below, it has required a long development to gradually extend this principle to more and more general settings, involving new ideas and techniques. We only indicate the main steps and papers; for further details see the references therein.

- The well-known first results [30, 49] to yield superlinear convergence were obtained for self-adjoint positive operators, later also analyzed in [7, 34].
- In the nonsymmetric case, a first classical result in real Hilbert spaces is due to [50]. A general convergence estimate is given in [42] on the operator level.
- Regarding mesh independence, in the nonsymmetric case for coercive problems, robust superlinear convergence estimates for Galerkin discretizations have been established in [2] and further described in an organized way in [3] by the authors; see also the more recent application [5]. This was based on the elaboration of a theory of so-called compact-equivalent operators for the coercive case, also allowing mesh independent bounds of the estimates.
- The next step was a more recent analysis [31] for symmetric operators, where superlinear estimates were extended to the indefinite case and a detailed analysis was given in Hilbert space.

As shown by the above, an important missing stage of the existing results is to establish mesh independent superlinear convergence for the complex non-Hermitian and noncoercive case. Such operators arise in important applications, such as in acoustics or electromagnetics, and mesh independent convergence results are desirable in order to provide robust estimates for the finite element solution of such problems.

The goal of this paper is to establish robust superlinear convergence estimates of Krylov iterations for Galerkin solutions when the underlying Hilbert space is complex and the operators (both the original one and the preconditioner) are in general non-Hermitian and noncoercive. The theoretical challenge is to develop the required substantial changes, mainly due to lack of coercivity, in the techniques compared to our previous work in [3]. The proofs have to be redone: instead of coercivity, they must be based just on the invertibility of the operators, using proper inf-sup conditions in the Babuška–Aziz framework, and suitable projections have to be introduced connected with theoretical background from the singular values of compact operators. In particular, in contrast to coercive problems, an inf-sup condition is not inherited by Galerkin discretizations for a general operator equation, which might cause a major difficulty. However, in our case we are able to prove it in an asymptotic sense using the compact perturbation property in the problem. Altogether, compact-equivalence is shown to result in mesh independent superlinear convergence in the lack of coercivity as well. Finally, in addition, it was a shortcoming of [3] that only CGN iterations were considered. Now we also include the practically more relevant GMRES iteration in the results.

In this paper, after brief preliminaries, the Hilbert space results are presented in sections 3 and 4, which contain the operator framework and the derivation of robust superlinear estimates in Galerkin subspaces, respectively. Then section 5 provides some applications in H^1 spaces. Here detailed attention is paid to interior Helmholtz equations and shifted Laplace preconditioners, to which a lot of recent research has been devoted (e.g., [13, 25, 20, 37]), however only focused on the aspects of linear convergence. We also indicate other applications and further directions. In the last section the theoretical results are illustrated with experiments for two acoustic model problems, providing numerical results in accordance with the theoretical estimates.

2. Preliminaries.

2.1. Singular values of compact operators. Here, and in what follows, the notations $\langle.,.\rangle_H$ and $\|.\|$ stand for the inner product and the corresponding norm of a complex Hilbert space H. We summarize briefly some required facts on compact linear operators to be used later.

DEFINITION 2.1.

(i) *We call $\lambda_j(F)$ $(j=1,2,\ldots)$ the ordered eigenvalues of a compact self-adjoint linear operator $F: H \to H$ if each of them is repeated as many times as its multiplicity and $|\lambda_1(F)| \geq |\lambda_2(F)| \geq \ldots$.*

(ii) *The singular values of a compact operator $C: H \to H$ are*

$$s_j(C) := \lambda_j(C^*C)^{1/2} \qquad (j=1,2,\ldots),$$

where $\lambda_j(C^*C)$ are the ordered eigenvalues of C^*C. In particular, if C is self-adjoint, then $s_j(C) = |\lambda_j(C)|$.

A basic property of compact operators is that $s_j(C) \to 0$ as $j \to \infty$. The following statements are the consequences of Corollary 3.3 and Propositions 1.3 and 1.4, respectively, in [26, Chapter VI].

PROPOSITION 2.2. *Let $C: H \to H$ be a compact operator. Then*

(a) *for any $k \in \mathbf{N}^+$ (positive integer) and any orthonormal vectors $u_1, \ldots, u_k \in H$,*

$$\sum_{j=1}^{k} |\langle Cu_j, u_j\rangle_H| \leq \sum_{j=1}^{k} s_j(C);$$

(b) *if B is a bounded linear operator in H, then*

$$s_j(BC) \leq \|B\|\, s_j(C) \qquad (j=1,2,\ldots);$$

(c) *if P is an orthogonal projection in H with range* ImP, *then*

$$s_j(PC_{|\mathrm{Im}P}) \leq s_j(C) \qquad (j=1,2,\ldots).$$

We note that compact operators from a Hilbert space H into its dual H' can be handled via the Riesz isomorphism between H and H'; this will be addressed in Definition 3.5 later.

2.2. Superlinear convergence of Krylov type methods. We briefly summarize some basic facts on the iterative solution of linear systems

$$Au = b \tag{2.1}$$

with a nonsingular matrix $A \in \mathbf{C}^{n\times n}$, with a focus on superlinear convergence rates.

When $A \in \mathbf{R}^{n\times n}$ is a real-valued symmetric positive definite matrix, then the widespread iterative solution is the standard CG method; see, e.g., [1, 49]. One is generally interested in the energy norm $\|e_k\|_A = \langle Ae_k, e_k\rangle^{1/2}$ of the error vector $e_k := u_k - u$. A well-known superlinear convergence estimate is expressed in terms of the decomposition

$$A = I + E \tag{2.2}$$

(where I is the identity matrix). Then (see, e.g., [1, section 13.2.3]),

$$\left(\frac{\|e_k\|_A}{\|e_0\|_A}\right)^{1/k} \leq \frac{2\|A^{-1}\|}{k} \sum_{j=1}^{k} |\lambda_j(E)| \qquad (k=1,2,\ldots). \tag{2.3}$$

Here the moduli of eigenvalues $|\lambda_j(E)|$ are in nonincreasing order, hence the right-hand side (r.h.s.) of (2.3) is nonincreasing, and numerically approaches zero. This is the main contrast to the linear convergence estimates, where the convergence factor should only be bounded.

For non-Hermitian matrices $A \in \mathbf{C}^{n\times n}$, several Krylov algorithms exist; see, e.g., [1, 11, 44]. In particular, GMRES and its variants are widely used. There exist similar superlinear convergence estimates for the GMRES as in (2.3), using singular values and the residual vectors $r_k := Au_k - b$. In fact, the sharpest one, proved in [42] on the Hilbert space level for an invertible operator $A \in B(H)$, uses the product of singular values, which implies (using the inequality between the geometric and arithmetic means) that

$$\left(\frac{\|r_k\|}{\|r_0\|}\right)^{1/k} \leq \frac{\|A^{-1}\|}{k} \sum_{j=1}^{k} s_j(E) \qquad (k=1,2,\ldots), \tag{2.4}$$

which is a proper analogue of (2.3) and tends to zero again as k tends to infinity. Further, note that by denoting the r.h.s. of (2.4) by q_k, we can rewrite it as $\|r_k\| \leq \|r_0\|(q_k)^k$, and then the "error equation" $Ae_k = r_k$ implies the following estimates for e_k itself:

$$\|e_k\| \leq \|A^{-1}\|\|r_0\|(q_k)^k \leq cond(A)\|e_0\|(q_k)^k .$$

Another possible way to solve (2.1) with non-Hermitian $A \in \mathbf{C}^{n\times n}$ is to consider

$$A^*Au = A^*b \tag{2.5}$$

(the "normal equation") and apply the symmetric CG algorithm for the latter [11, 17]. The resulting CGN method (i.e., "CG for the normal equation") is often avoided due to a larger condition number, but it involves a very simple recursion, in contrast to the GMRES method; thus for many non-Hermitian problems it has proved to be efficient [10, 17], and it has also been used in the study of equivalent operators to produce mesh independent linear [17] and superlinear [3] convergence. In terms of the decomposition (2.2), using the relations $\|e_k\|_{A^*A} = \|Ae_k\| = \|r_k\|$ for the residual error vectors, further, that $\|(A^*A)^{-1}\| = \|A^{-1}\|^2$ and $A^*A = I + (E^* + E + E^*E)$, the analogue of the superlinear estimate (2.3) for (2.5) becomes

$$\left(\frac{\|r_k\|}{\|r_0\|}\right)^{1/k} \leq \frac{2\|A^{-1}\|^2}{k} \sum_{j=1}^{k} \left(|\lambda_j(E^*+E)| + \lambda_j(E^*E)\right) \qquad (k=1,2,\ldots). \tag{2.6}$$

The above estimates also hold in a Hilbert space level for a proper compact operator E in (2.2).

In what follows, we will formulate our estimates for both the GMRES and CGN.

3. Hilbert space setting. Let $(H, \langle .,. \rangle_H)$ be a complex Hilbert space with dual H' and pairing $\langle .,. \rangle$. Let us consider an operator equation

$$L_w u = g \tag{3.1}$$

with a given bounded linear operator $L_w : H \to H'$ and r.h.s. $g \in H'$. The index w follows the terminology of [41], since typical realizations of such bounded linear operators are the weak forms of elliptic operators from a Sobolev space into its dual; see, e.g., (5.1) and (5.11) later.

The operator L_w may be non-Hermitian and noncoercive. We only require well-posedness of (3.1), using the following.

DEFINITION 3.1. *The bounded linear operator $L_w : H \to H'$ is regular if it possesses a bounded inverse.*

In fact, if L_w is bijective, then it is regular owing to Banach's theorem. Moreover, this regularity is equivalent to the following well-known conditions, called by various names (Babuška–Aziz, Babuška–Nečas, or generalized Lax–Milgram theorem [6, 14]).

Assumption 3.2. The linear operator $L_w : H \to H'$ satisfies

(i) $$M := \sup_{\substack{u\in H\\ \|u\|=1}} \sup_{\substack{v\in H\\ \|v\|=1}} |\langle L_w u, v\rangle| < \infty;$$

(ii) $$m := \inf_{\substack{u\in H\\ \|u\|=1}} \sup_{\substack{v\in H\\ \|v\|=1}} |\langle L_w u, v\rangle| > 0;$$

(iii) for any $u \in H \setminus \{0\}$: $\sup_{v\in H} |\langle L_w v, u\rangle| > 0.$

Remark 3.3. Such an L_w is regular due the Babuška–Aziz theorem, and moreover, $\|L_w^{-1}\| = 1/m$. Hence, in particular, (3.1) has a unique solution.

Condition (ii) is sometimes called weak coercivity, as opposed to standard coercivity requiring $m := \inf_{\|u\|=1}\langle L_w u, u\rangle > 0$. Thus, altogether, the Babuška–Aziz conditions impose boundedness, weak coercivity, and adjoint injectivity.

Equation (3.1) will be solved numerically using a Galerkin discretization: let

$$V_h = span\{\varphi_1, \dots, \varphi_n\} \subset H$$

be a given finite-dimensional subspace, where φ_i are linearly independent vectors, and

$$\mathbf{L}_h := \{\langle L_w \varphi_j, \varphi_i\rangle\}_{i,j=1}^n$$

the stiffness matrix. Finding the discrete solution $u_h \in V_h$ requires solving the $n \times n$ system

$$\mathbf{L}_h \, \mathbf{c} = \mathbf{b} \tag{3.2}$$

with $\mathbf{b} = \{\langle g, \varphi_j\rangle\}_{j=1}^n$.

We have introduced the notion of compact-equivalent operators in [3] for the coercive case. Roughly speaking, such operators differ only up to a compact perturbation, hence preconditioning yields a compact perturbation of the identity. Such compact-equivalence can be similarly defined in our more general situation.

DEFINITION 3.4. *The regular operators L_w and N_w from H to H' are called compact-equivalent if*

$$L_w = \mu N_w + Q_w \tag{3.3}$$

for some constant $\mu > 0$ and compact operator $Q_w : H \to H'$.

Clearly, the property compact-equivalence is an equivalence relation.

Later we will need the singular values of such compact operators. This can be reduced to Definition 2.1 via the Riesz isomorphism

$$S : H \to H' , \qquad \langle Su, v \rangle = \langle u, v \rangle_H \tag{3.4}$$

as follows.

DEFINITION 3.5. *Let $Q_w : H \to H'$ be a compact operator, and let $Q_S := S^{-1}Q_w$. (Then Q_S maps from H to H and is also compact.) The singular values of Q_w are defined as*

$$s_j(Q_w) := s_j(Q_S) \qquad (j = 1, 2, \ldots).$$

Note that by definition

$$\langle Q_w u, v \rangle = \langle Q_S u, v \rangle_H . \tag{3.5}$$

Remark 3.6. As described in this section, operators will be considered as mapping from H to H', which is usual for elliptic problems. An alternative setting is to consider operators mapping from H into H itself, which is widespread in a large portion of classical Hilbert space theory [43], e.g., when the identity operator is involved such as in spectral theory, eigenvalues, compact perturbations of the identity, etc. Clearly, (3.1) can be recast to this setting with the same idea as used above for singular values. Let S denote the Riesz isomorphism (3.4), define the operator $L_S := S^{-1}L_w : H \to H$, and let $g_S := S^{-1}g \in H$ be the Riesz representant of the functional $g \in H'$. Then (3.1) is equivalent to the following equation in H:

$$L_S u = g_S . \tag{3.6}$$

4. Iterative solution and robust superlinear convergence in Hilbert space. Based on the previous section, let us consider the Galerkin discretization (3.2) of the operator equation (3.1) for a regular operator L_w. We apply a preconditioned Krylov method, GMRES or CGN, to solve (3.2), and our goal is to establish robust superlinear convergence independently of V_h.

4.1. Compact-equivalent operator preconditioning. Let N_w be a(n in general non-Hermitian) regular operator, which is compact-equivalent to L_w in the sense of Definition 3.4. For simplicity we may consider compact-equivalence with $\mu = 1$ in (3.3), which is clearly no restriction. (Indeed, if a preconditioner N_w satisfies (3.3) with some $\mu \neq 0$, then we can consider the preconditioner μN_w instead.) Thus (3.3) becomes

$$L_w = N_w + Q_w . \tag{4.1}$$

PROPOSITION 4.1. *The preconditioned operator $N_w^{-1}L_w : H \to H$ is a compact perturbation of the identity.*

Proof. We have $N_w^{-1}L_w = I + N_w^{-1}Q_w$ from (4.1). Since N_w is regular, we have N_w^{-1} bounded; thus we obtain that $N_w^{-1}Q_w$ is also compact. ∎

Using the preconditioning operator N_w, we introduce its stiffness matrix

$$\mathbf{N}_h := \{\langle N_w \varphi_j, \varphi_i \rangle\}_{i,j=1}^n$$

as preconditioner for the discretized system (3.2). Conditions to ensure regularity of $\mathbf{N}_h$ will be discussed in the next subsection. We wish to solve

$$\mathbf{N}_h^{-1}\mathbf{L}_h\,\mathbf{c} = \hat{\mathbf{b}} \tag{4.2}$$

(with $\hat{\mathbf{b}} = \mathbf{N}_h^{-1}\mathbf{b}$) using a Krylov iteration. Here, using (4.1) and letting

$$\mathbf{Q}_h := \{\langle Q_w\varphi_j, \varphi_i\rangle\}_{i,j=1}^n ,$$

we have the decomposition $\mathbf{L}_h = \mathbf{N}_h + \mathbf{Q}_h$; thus system (4.2) takes the form

$$(\mathbf{I}_h + \mathbf{N}_h^{-1}\mathbf{Q}_h)\,\mathbf{c} = \tilde{\mathbf{b}}, \tag{4.3}$$

where $\mathbf{I}_h$ is the $n \times n$ identity matrix, i.e., we have a counterpart of (2.2). In order to define an energy inner product on $\mathbf{R}^n$ we use the Gram matrix corresponding to the inner product of H,

$$\mathbf{S}_h = \{\langle\varphi_j, \varphi_i\rangle_H\}_{i,j=1}^n , \tag{4.4}$$

and we endow $\mathbf{R}^n$ with the $\mathbf{S}_h$-inner product $\langle \mathbf{c}, \mathbf{d}\rangle_{\mathbf{S}_h} := \mathbf{S}_h\,\mathbf{c}\cdot\overline{\mathbf{d}}$. Then the GMRES and CGN algorithms for the matrix $A = \mathbf{N}_h^{-1}\mathbf{L}_h$, and with the $\mathbf{S}_h$-inner product, provide the following counterpart of estimates (2.4) and (2.6).

For the GMRES, the matrix $E := \mathbf{N}_h^{-1}\mathbf{Q}_h$ and its $\mathbf{S}_h$-adjoint $E^* = \mathbf{S}_h^{-1}\mathbf{Q}_h^*\mathbf{N}_h^{-*}\mathbf{S}_h$ yield the singular values $s_j(E) = \lambda_i(\mathbf{S}_h^{-1}\mathbf{Q}_h^*\,\mathbf{N}_h^{-*}\mathbf{S}_h\mathbf{N}_h^{-1}\mathbf{Q}_h)^{1/2}$; hence (2.4) implies

$$\left(\frac{\|r_k\|_{\mathbf{S}_h}}{\|r_0\|_{\mathbf{S}_h}}\right)^{1/k} \le \frac{\|(\mathbf{N}_h^{-1}\mathbf{L}_h)^{-1}\|_{\mathbf{S}_h}}{k}\sum_{i=1}^k \lambda_i(\mathbf{S}_h^{-1}\mathbf{Q}_h^*\,\mathbf{N}_h^{-*}\mathbf{S}_h\mathbf{N}_h^{-1}\mathbf{Q}_h)^{1/2} \tag{4.5}$$

$(k = 1, 2, \ldots, n)$. Similarly, for the CGN, (2.6) implies

$$\left(\frac{\|r_k\|_{\mathbf{S}_h}}{\|r_0\|_{\mathbf{S}_h}}\right)^{1/k} \le \frac{2\|(\mathbf{N}_h^{-1}\mathbf{L}_h)^{-1}\|_{\mathbf{S}_h}^2}{k}$$

$$\times \sum_{i=1}^k \left(|\lambda_i(\mathbf{S}_h^{-1}\mathbf{Q}_h^*\,\mathbf{N}_h^{-*}\mathbf{S}_h + \mathbf{N}_h^{-1}\mathbf{Q}_h)| + \lambda_i(\mathbf{S}_h^{-1}\mathbf{Q}_h^*\,\mathbf{N}_h^{-*}\mathbf{S}_h\mathbf{N}_h^{-1}\mathbf{Q}_h)\right). \tag{4.6}$$

Our goal is to give bounds on the above estimates that are independent of V_h.

4.2. The discrete inf-sup condition. By Assumption 3.2(ii), the regular operator L_w satisfies the inf-sup condition. The same holds for N_w. For the discrete case we formulate the analogous property below as an assumption. Then we show that this assumption is asymptotically always satisfied for compact perturbations of the identity.

Assumption 4.2. The operators L_w and N_w satisfy the following discrete inf-sup conditions w.r.t. the considered family of subspaces V_h $(h > 0)$:

$$\inf_{\substack{u_h\in V_h\\ \|u_h\|=1}} \sup_{\substack{v_h\in V_h\\ \|v_h\|=1}} |\langle L_w u_h, v_h\rangle| =: m_0 > 0, \qquad \inf_{\substack{u_h\in V_h\\ \|u_h\|=1}} \sup_{\substack{v_h\in V_h\\ \|v_h\|=1}} |\langle N_w u_h, v_h\rangle| =: m_1 > 0, \tag{4.7}$$

where the constants $m_0, m_1 > 0$ are independent of V_h.

COROLLARY 4.3. *If Assumption 4.2 holds, then for each subspace V_h, system (3.2) has a unique solution.*

Based on Remark 3.6 and (3.5), the inf-sup conditions can be reformulated with the inner product of H and the operator L_S, because $\langle L_w u_h, v_h\rangle = \langle L_S u_h, v_h\rangle_H$ and similarly for N_S. In this context we can now show that the inf-sup condition for an operator implies asymptotically the same discrete inf-sup conditions under the compact perturbation property, for sufficiently fine discretizations.

PROPOSITION 4.4. *Let $A \in B(H)$ be a regular operator in a Hilbert space H, and in particular,*

$$m := \inf_{\substack{u\in H\\ \|u\|=1}} \sup_{\substack{v\in H\\ \|v\|=1}} |\langle Au, v\rangle_H| > 0, \tag{4.8}$$

and let the compact perturbation property (2.2) *hold for some compact operator E, that is, $A = I + E$. Let $(V_n)_{n\in\mathbf{N}^+}$ be a sequence of closed subspaces of H such that the following approximation property holds:*

$$\text{for any } u \in H, \quad \operatorname{dist}(u, V_n) := \min\{\|u - v_n\| : \ v_n \in V_n\} \to 0 \qquad (\text{as } \ n \to \infty). \tag{4.9}$$

Then the sequence of real numbers

$$m_n := \inf_{\substack{u_n\in V_n\\ \|u_n\|=1}} \sup_{\substack{v_n\in V_n\\ \|v_n\|=1}} |\langle Au_n, v_n\rangle_H| \qquad (n \in \mathbf{N}^+)$$

satisfies

$$\liminf m_n \ \geq m \quad (> 0). \tag{4.10}$$

Proof. Let the operators $P_n \in B(H)$ denote the orthogonal projections to V_n $(n \in \mathbf{N}^+)$. That is, for any $x \in H$, let x_{V_n} and $x_{V_n^\perp}$ denote the components in V_n and in its orthocomplement, respectively: then

$$x = x_{V_n} + x_{V_n^\perp} \quad \Rightarrow P_n x := x_{V_n}.$$

Further, we have

$$m = \inf_{\substack{z\in H\\ \|z\|=1}} \|Az\| \qquad \text{and} \qquad m_n = \inf_{\substack{z_n\in V_n\\ \|z_n\|=1}} \|P_n A z_n\|. \tag{4.11}$$

Now, assume for contradiction that (4.10) is false, i.e., $\liminf m_n < m$. Then, for some $\delta > 0$, there exists a subsequence $(e_{k_n}) \subset H$ such that

$$e_{k_n} \in V_{k_n}, \quad \|e_{k_n}\| = 1 \quad \text{and} \qquad \|P_{k_n} A e_{k_n}\| \leq m - \delta \qquad (\forall k_n \in \mathbf{N}^+). \tag{4.12}$$

We may assume without loss of generality that $k_n = n \quad (n \in \mathbf{N}^+)$. Here $Ae_n = (Ae_n)_{V_n} + (Ae_n)_{V_n^\perp}$, hence $P_n A e_n := (Ae_n)_{V_n} = Ae_n - (Ae_n)_{V_n^\perp}$. Since $\|e_n\| = 1$, (4.11) implies $\|Ae_n\| \geq m$, hence $\|P_n A e_n\| \geq m - \|(Ae_n)_{V_n^\perp}\|$. Then (4.12) implies

$$\|(Ae_n)_{V_n^\perp}\| \geq \delta \qquad (\forall n \in \mathbf{N}^+). \tag{4.13}$$

Here, by (2.2),

$$(Ae_n)_{V_n^\perp} = (e_n)_{V_n^\perp} + (Ee_n)_{V_n^\perp} = (Ee_n)_{V_n^\perp}$$

since $e_n \in V_n$ has no component in $V_n^\perp$. Now, the sequence (e_n) is bounded, hence it has a weakly convergent subsequence. We may assume again without loss of generality that (e_n) itself is weakly convergent, i.e., $e_n \rightharpoonup y \in H$. Since E is compact, we have $Ee_n \to Ey$ (in norm). Then

$$(Ae_n)_{V_n^\perp} = (Ee_n)_{V_n^\perp} = (Ey)_{V_n^\perp} + (E(e_n - y))_{V_n^\perp}.$$

Here $\|(Ey)_{V_n^\perp}\| = \operatorname{dist}(Ey, V_n) \to 0$ by (4.9), and $\|(E(e_n - y))_{V_n^\perp}\| \le \|E(e_n - y)\| \to 0$, hence

$$(Ae_n)_{V_n^\perp} \to 0,$$

which contradicts (4.13). ∎

4.3. Robust superlinear convergence results. In order to find uniform bounds for the sequences in (4.5)–(4.6), the following proposition relates the singular values of the arising matrices to those of the proper continuous operators. Recall the connection of the operators L_w, N_w, and Q_w with the stiffness matrices $\mathbf{L}_h$, $\mathbf{N}_h$, and $\mathbf{Q}_h$, respectively: for any $i, j = 1, \ldots, n$,

$$(\mathbf{L}_h)_{i,j} = \langle L_w \varphi_j, \varphi_i \rangle, \quad (\mathbf{N}_h)_{i,j} = \langle N_w \varphi_j, \varphi_i \rangle, \quad (\mathbf{Q}_h)_{i,j} = \langle Q_w \varphi_j, \varphi_i \rangle, \tag{4.14}$$

and similarly, from (4.4), the matrix $\mathbf{S}_h$ corresponds to the H-inner product.

PROPOSITION 4.5. *Let N_w be a compact-equivalent preconditioner for L_w such that* (4.1) *holds for some compact operator Q_w, and let $s_j(Q_w)$ $(j = 1, 2, \ldots)$ denote the singular values of Q_w. Further, let Assumption* 4.2 *hold with $m_0, m_1 > 0$ defined in* (4.7)*, and let $M := \|N_w\|$. Then the following relations hold:*

(a) $$\lambda_j(\mathbf{S}_h^{-1}\mathbf{Q}_h^* \mathbf{N}_h^{-*}\mathbf{S}_h\mathbf{N}_h^{-1}\mathbf{Q}_h) \le \frac{1}{m_1^2} s_j(Q_w)^2 \qquad (j = 1, 2, \ldots, n),$$

(b) $$\sum_{j=1}^k \left|\lambda_j(\mathbf{S}_h^{-1}\mathbf{Q}_h^* \mathbf{N}_h^{-*}\mathbf{S}_h + \mathbf{N}_h^{-1}\mathbf{Q}_h)\right| \le \frac{2}{m_1} \sum_{j=1}^k s_j(Q_w) \qquad (k = 1, 2, \ldots, n),$$

(c) $$\|(\mathbf{N}_h^{-1}\mathbf{L}_h)^{-1}\|_{\mathbf{S}_h} \le \frac{M}{m_0}.$$

Proof. First, using (3.5), we can rewrite (4.14) as

$$(\mathbf{L}_h)_{i,j} = \langle L_S \varphi_j, \varphi_i \rangle_H, \quad (\mathbf{N}_h)_{i,j} = \langle N_S \varphi_j, \varphi_i \rangle_H, \quad (\mathbf{Q}_h)_{i,j} = \langle Q_S \varphi_j, \varphi_i \rangle_H . \tag{4.15}$$

(a) Let $\lambda_j := \lambda_j(\mathbf{S}_h^{-1}\mathbf{Q}_h^* \mathbf{N}_h^{-*}\mathbf{S}_h\mathbf{N}_h^{-1}\mathbf{Q}_h)$ $(j = 1, \ldots, n)$ and let $\mathbf{c}^j = (c_1^j, \ldots, c_n^j) \in \mathbf{C}^n$ be corresponding eigenvectors, i.e.,

$$\mathbf{S}_h^{-1}\mathbf{Q}_h^* \mathbf{N}_h^{-*}\mathbf{S}_h\mathbf{N}_h^{-1}\mathbf{Q}_h\mathbf{c}^j = \lambda_j \mathbf{c}^j \tag{4.16}$$

with the additional normalization property

$$\mathbf{S}_h\, \mathbf{c}^j \cdot \overline{\mathbf{c}}^l = \delta_{jl} \qquad (j, l = 1, \ldots, n), \tag{4.17}$$

where $\cdot$ denotes the ordinary Euclidean inner product. Let $u_j := \sum_{s=1}^n c_s^j \varphi_s \in V_h$. Further, let $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{C}^n$ be arbitrary and let $v := \sum_{s=1}^n p_s \varphi_s \in V_h$. Then, multiplying (4.16) with $\mathbf{S}_h$ from the left and $\overline{\mathbf{p}}$ from the right, we obtain $\mathbf{Q}_h^* \mathbf{N}_h^{-*}\mathbf{S}_h\mathbf{N}_h^{-1}\mathbf{Q}_h\mathbf{c}^j \cdot \overline{\mathbf{p}} = \lambda_j\, \mathbf{S}_h\mathbf{c}^j \cdot \overline{\mathbf{p}} = \lambda_j \langle u_j, v \rangle_H$. Denote

$$\mathbf{e}^j := \mathbf{N}_h^{-*}\mathbf{S}_h\mathbf{N}_h^{-1}\mathbf{Q}_h\mathbf{c}^j, \tag{4.18}$$

and let $\mathbf{e}^j = (e_1^j, \dots, e_n^j)$ and $y_j := \sum_{s=1}^n e_s^j \varphi_s \ \in V_h$; then

$$\begin{aligned}\mathbf{Q}_h^* \mathbf{N}_h^{-*} \mathbf{S}_h \mathbf{N}_h^{-1} \mathbf{Q}_h \mathbf{c}^j \cdot \overline{\mathbf{p}} &= \mathbf{Q}_h^* \mathbf{e}^j \cdot \overline{\mathbf{p}} = \mathbf{e}^j \cdot \overline{\mathbf{Q}_h \mathbf{p}} \\ &= \overline{\mathbf{Q}_h \mathbf{p} \cdot \overline{\mathbf{e}}^j} = \overline{\langle Q_S v, y_j \rangle_H} = \langle y_j, Q_S v \rangle_H = \langle Q_S^* y_j, v \rangle_H .\end{aligned}$$

Equating the above two expresssions, we obtain $\langle Q_S^* y_j, v\rangle_H = \langle \lambda_j u_j, v\rangle_H$ (for all $v \in V_h$). Hence, denoting by P_h the orthogonal projection of H into the subspace V_h, we have

$$P_h Q_S^* y_j = \lambda_j u_j . \tag{4.19}$$

Now, multiplying (4.18) with $\mathbf{N}_h^*$ from the left and with an again arbitrary $\overline{\mathbf{p}} \in \mathbf{C}^n$ from the right, we obtain $\mathbf{N}_h^* \mathbf{e}^j \cdot \overline{\mathbf{p}} = \mathbf{S}_h \mathbf{N}_h^{-1} \mathbf{Q}_h \mathbf{c}^j \cdot \overline{\mathbf{p}}$. Denoting $\mathbf{d}^j := \mathbf{N}_h^{-1} \mathbf{Q}_h \mathbf{c}^j$, we thus have $\mathbf{e}^j \cdot \overline{(\mathbf{N}_h \mathbf{p})} = \mathbf{S}_h \mathbf{d}^j \cdot \overline{\mathbf{p}}$. Let $z_j := \sum_{s=1}^n d_s^j \varphi_s \in V_h$; then we obtain $\langle y_j, N_S v\rangle_H = \langle z_j, v\rangle_H$ for all $v \in V_h$, i.e., $\langle N_S^* y_j, v\rangle_H = \langle z_j, v\rangle_H$ $(v \in V_h)$, which yields

$$P_h N_S^* y_j = z_j . \tag{4.20}$$

Finally, similarly as above, the definition of $\mathbf{d}^j$ yields $\mathbf{N}_h \mathbf{d}^j \cdot \overline{\mathbf{p}} = \mathbf{Q}_h \mathbf{c}^j \cdot \overline{\mathbf{p}}$ (for all $\overline{\mathbf{p}} \in \mathbf{C}^n$), hence $\langle N_S z_j, v\rangle_H = \langle Q_S u_j, v\rangle_H$ $(v \in V_h)$ and thus

$$P_h N_S z_j = P_h Q_S u_j . \tag{4.21}$$

Consider now the mapping $(P_h N_S)_{|V_h} : V_h \to V_h$. Here, from (4.7), for all $z \in V_h$

$$m_1 \|z\| \le \sup_{\substack{v \in V_h \\ v \neq 0}} \frac{|\langle N_w z, v\rangle|}{\|v\|} = \sup_{\substack{v \in V_h \\ v \neq 0}} \frac{|\langle N_S z, v\rangle_H|}{\|v\|} = \sup_{\substack{v \in V_h \\ v \neq 0}} \frac{|\langle P_h N_S z, v\rangle_H|}{\|v\|} = \|P_h N_S z\|,$$

hence $(P_h N_S)_{|V_h}$ is regular and

$$\|(P_h N_S)_{|V_h}^{-1}\| \le \frac{1}{m_1}. \tag{4.22}$$

It follows by definition that

$$\big((P_h N_S)_{|V_h}\big)^* = (P_h N_S^*)_{|V_h}, \tag{4.23}$$

hence $(P_h N_S^*)_{|V_h}$ is also regular and

$$(P_h N_S^*)_{|V_h}^{-1} = \Big((P_h N_S)_{|V_h}^{-1}\Big)^* . \tag{4.24}$$

Here, (4.20)–(4.21) yield $y_j = (P_h N_S^*)_{|V_h}^{-1} z_j$ and $z_j = (P_h N_S)_{|V_h}^{-1} P_h Q_S u_j$. Substituting these into (4.19), and using that $u_j, y_j \in V_h$, we altogether obtain

$$(P_h Q_S^*)_{|V_h} (P_h N_S^*)_{|V_h}^{-1} (P_h N_S)_{|V_h}^{-1} (P_h Q_S)_{|V_h} u_j = \lambda_j u_j . \tag{4.25}$$

Let us define the operator

$$F := (P_h N_S)_{|V_h}^{-1} (P_h Q_S)_{|V_h} .$$

By virtue of properties (4.23)–(4.24), the operator on the left of (4.25) equals $F^* F$ and thus λ_j is an eigenvalue of $F^* F$ (with eigenvector u_j), i.e., λ_j is the square of a singular value of F. Moreover, due to (4.17), the eigenvalues λ_j $(j = 1, \dots, n)$ possess orthonormal eigenvectors. Using the ordering as in Definition 2.1, let λ_j be the square of the k_jth singular value of F, i.e., $\lambda_j = s_{k_j}(F)^2$. Moreover, since the ordered values λ_j $(j = 1, \ldots, n)$ are present in the ordered nonincreasing sequence s_j^2 $(j \in \mathbf{N}^+)$, we have

$$\lambda_j \le s_j(F)^2. \tag{4.26}$$

Here, using statements (b)–(c) of Proposition 2.2 and estimate (4.22), respectively, we obtain

$$\begin{aligned} s_j(F) &= s_j\left((P_h N_S)^{-1}_{|V_h} (P_h Q_S)_{|V_h}\right) \le \|(P_h N_S)^{-1}_{|V_h}\| \; s_j\left((P_h Q_S)_{|V_h}\right) \\ &\le \frac{1}{m_1}\; s_j\left((P_h Q_S)_{|V_h}\right) \le \frac{1}{m_1}\, s_j(Q_S), \end{aligned}$$

hence $\lambda_j(\mathbf{S}_h^{-1}\mathbf{Q}_h^*\,\mathbf{N}_h^{-*}\mathbf{S}_h\mathbf{N}_h^{-1}\mathbf{Q}_h) =: \lambda_j \le s_j(F)^2 \le \frac{1}{m_1^2}\, s_j(Q_S)^2 = \frac{1}{m_1^2}\, s_j(Q_w)^2\,.$

(b) Now let $\lambda_j := \lambda_j(\mathbf{S}_h^{-1}\mathbf{Q}_h^*\,\mathbf{N}_h^{-*}\mathbf{S}_h + \mathbf{N}_h^{-1}\mathbf{Q}_h)$ and let $\mathbf{c}^j = (c_1^j, \ldots, c_n^j) \in \mathbf{C}^n$ be corresponding eigenvectors, i.e.,

$$(\mathbf{S}_h^{-1}\mathbf{Q}_h^*\,\mathbf{N}_h^{-*}\mathbf{S}_h + \mathbf{N}_h^{-1}\mathbf{Q}_h)\,\mathbf{c}^j = \lambda_j\,\mathbf{c}^j$$

with the orthonormality property (4.17). Then multiplying with $\mathbf{S}_h$ and $\overline{\mathbf{c}}^j$,

$$\begin{aligned} \lambda_j &= \lambda_j\,\mathbf{S}_h\,\mathbf{c}^j\cdot\overline{\mathbf{c}}^j = \mathbf{Q}_h^*\,\mathbf{N}_h^{-*}\mathbf{S}_h\,\mathbf{c}^j\cdot\overline{\mathbf{c}}^j + \mathbf{S}_h\mathbf{N}_h^{-1}\mathbf{Q}_h\,\mathbf{c}^j\cdot\overline{\mathbf{c}}^j \\ &= 2\,Re\;\mathbf{S}_h\mathbf{N}_h^{-1}\mathbf{Q}_h\,\mathbf{c}^j\cdot\overline{\mathbf{c}}^j = 2\,Re\,\mathbf{Q}_h\,\mathbf{c}^j\cdot\overline{\mathbf{e}}^j, \end{aligned} \tag{4.27}$$

where $\mathbf{e}^j := \mathbf{N}_h^{-T}\mathbf{S}_h\,\mathbf{c}^j$ for all j. Here for all $v = \sum_{s=1}^n p_s\varphi_s \in V_h$, with notation $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{C}^n$, we obtain $\mathbf{e}^j \cdot \overline{(\mathbf{N}_h\,\mathbf{p})} = \mathbf{S}_h\,\mathbf{c}^j \cdot \overline{\mathbf{p}}$, which means $\langle y_j, N_S v\rangle_H = \langle u_j, v\rangle_H$ for all $v \in V_h$, where $z_j = \sum_{s=1}^n e_s^j\varphi_s$ and $u_j = \sum_{s=1}^n c_s^j\varphi_s$, or

$$\langle N_S^* z_j, v\rangle_H = \langle u_j, v\rangle_H \qquad (v \in V_h). \tag{4.28}$$

Hence

$$u_j = P_h N_S^* z_j. \tag{4.29}$$

We have seen before that the linear mapping $(P_h N_S)_{|V_h} : V_h \to V_h$ is regular and satisfies (4.22), and further that $(P_h N_S^*)^{-1}_{|V_h} = ((P_h N_S)^{-1}_{|V_h})^*$. It follows that the latter inherits the same bound for its norm:

$$\|(P_h N_S^*)^{-1}_{|V_h}\| \le \frac{1}{m_1}. \tag{4.30}$$

Now, first, using (4.29), we obtain

$$\mathbf{Q}_h\,\mathbf{c}^j\cdot\overline{\mathbf{e}}^j = \langle Q_S u_j, z_j\rangle_H = \langle Q_S u_j, (P_h N_S^*)^{-1}_{|V_h} u_j\rangle_H = \langle u_j, Q_S^*(P_h N_S^*)^{-1}_{|V_h} u_j\rangle_H. \tag{4.31}$$

Here the operator $(P_h N_S^*)^{-1}_{|V_h}$ has a norm-preserving extension $\hat{N}_S$ from V_h onto H (namely, with $\hat{N}|_{(V_h)\perp} := 0$), and from (4.30) we have $\|\hat{N}\| \le \frac{1}{\hat{m}}$. Altogether, using (4.27), (4.31) and statements (a)–(b) of Proposition 2.2, respectively, we obtain

$$\begin{aligned} \sum_{j=1}^k |\lambda_j| &\le 2\sum_{j=1}^k |\mathbf{Q}_h\,\mathbf{c}^j\cdot\overline{\mathbf{e}}^j| = 2\sum_{j=1}^k \left|\langle Q_S^*(P_h N_S^*)^{-1}_{|V_h} u_j,\, u_j\rangle\right| = 2\sum_{j=1}^k \left|\langle Q_S^*\hat{N} u_j,\, u_j\rangle\right| \\ &\le 2\sum_{j=1}^k s_j\left(Q_S^*\hat{N}\right) \le \frac{2}{m_1}\sum_{j=1}^k s_j\left(Q_S^*\right) = \frac{2}{m_1}\sum_{j=1}^k s_j\left(Q_S\right) = \frac{2}{m_1}\sum_{j=1}^k s_j\left(Q_w\right). \end{aligned}$$

(c) Let $\mathbf{c} \in \mathbf{C}^n$ be arbitrary, $\mathbf{d} := \mathbf{N}_h^{-1}\mathbf{L}_h\mathbf{c}$. Let $u = \sum_{s=1}^n c_s\varphi_s \in V_h$ and $z = \sum_{s=1}^n d_s\varphi_s \in V_h$. Then for all $v = \sum_{s=1}^n p_s\varphi_s \in V_h$, with notation $\mathbf{p} = (p_1, \dots, p_n) \in \mathbf{C}^n$, we have $\langle L_S u, v\rangle_H = \mathbf{L}_h\,\mathbf{c}\cdot\overline{\mathbf{p}} = \mathbf{N}_h\,\mathbf{d}\cdot\overline{\mathbf{p}} = \langle N_S z, v\rangle_H$, hence from (4.7)

$$m_0\|u\| \le \sup_{\substack{v\in V_h\\ v\neq 0}} \frac{|\langle L_w u, v\rangle|}{\|v\|} = \sup_{\substack{v\in V_h\\ v\neq 0}} \frac{|\langle L_S u, v\rangle_H|}{\|v\|} = \sup_{\substack{v\in V_h\\ v\neq 0}} \frac{|\langle N_S z, v\rangle_H|}{\|v\|} \le \|N_S z\| = \|N_w z\|$$

and thus

$$\frac{\|\mathbf{N}_h^{-1}\mathbf{L}_h\mathbf{c}\|_{\mathbf{S}_h}^2}{\|\mathbf{c}\|_{\mathbf{S}_h}^2} = \frac{\mathbf{S}_h\,\mathbf{d}\cdot\overline{\mathbf{d}}}{\mathbf{S}_h\,\mathbf{c}\cdot\overline{\mathbf{c}}} = \frac{\|z\|^2}{\|u\|^2} \ge m_0^2\,\frac{\|z\|^2}{\|N_w z\|^2} \ge \frac{m_0^2}{M^2}\,.$$

This implies statement (c), which concludes the proof. ◻

THEOREM 4.6. *Under the assumptions of Proposition* 4.5*, the GMRES and CGN iterations for the $n\times n$ preconditioned system* (4.2) *provide mesh independent superlinear convergence estimates, i.e., we have*

$$\left(\frac{\|r_k\|_{\mathbf{S}_h}}{\|r_0\|_{\mathbf{S}_h}}\right)^{1/k} \le\ \varrho_k \qquad (k = 1, 2, \dots, n), \tag{4.32}$$

where $(\varrho_k)_{k\in\mathbf{N}^+} \to 0$ and it is a sequence independent of n and V_h. Namely, in the case of GMRES,

$$\varrho_k \le \frac{M}{m_0 m_1}\cdot\ \frac{1}{k}\ \sum_{j=1}^k s_j(Q_w) \ \ \to 0 \qquad (\textit{as } k\to\infty), \tag{4.33}$$

and in the case of CGN

$$\varrho_k = \frac{2M^2}{m_0^2}\cdot\ \frac{1}{k}\ \sum_{j=1}^k \left(\frac{2}{m_1}\,s_j(Q_w) + \frac{1}{m_1^2}\,s_j(Q_w)^2\right) \ \to 0 \qquad (\textit{as } k\to\infty). \tag{4.34}$$

Proof. The estimates follow directly from (4.5)–(4.6) and Proposition 4.5. The convergence of ϱ_k to zero follows from the compactness of Q_w. ◻

5. Applications to noncoercive boundary value problems.

In this section we illustrate the applicability of the obtained superlinear convergence results to various problems arising for elliptic PDEs. We discuss in detail the case of interior Helmholtz equations (an important model in wave propagation and acoustics) using shifted Laplace preconditioners in the Krylov iteration. Then we indicate other applications and further directions.

5.1. Interior Helmholtz equations in acoustics.

Let us consider the following interior Helmholtz equation with impedance boundary condition on a bounded domain $\Omega \subset \mathbf{R}^d$:

$$\begin{cases} Lu := -\Delta u - \kappa^2 u &= g \qquad \text{in } \Omega,\\ \qquad\quad \frac{\partial u}{\partial n} - i\kappa u &= 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $\kappa > 0$ is the wave-number, i is the imaginary unit, and $g \in L^2(\Omega)$ is given. Such problems serve as a basic model of great importance in the formulation of time-harmonic wave propagation, arising in cavities in electromagnetics, acoustics, etc.

We look for the weak solution, i.e., $u \in H^1(\Omega)$, for which

$$\int_\Omega (\nabla u \cdot \nabla \overline{v} - \kappa^2 u\overline{v}) - i\kappa \int_{\partial\Omega} u\overline{v} = \int_\Omega g\overline{v} \qquad (\forall v \in H^1(\Omega)), \tag{5.2}$$

where $\overline{v}$ denotes the complex conjugate of v. Here we endow the complex space $H^1(\Omega)$ with the usual inner product $\langle u, v\rangle_{H^1} := \int_\Omega (\nabla u \cdot \nabla \overline{v} + u\overline{v})$. Owing to the imaginary coefficient in the boundary condition, the positive real number κ^2 cannot attain an eigenvalue, i.e., the homogeneous problem with $g \equiv 0$ has only the trivial solution $u \equiv 0$; see, e.g., [33]. The Fredholm alternative then ensures that problem (5.1) has a unique weak solution in $H^1(\Omega)$. Moreover, the Fredholm well-posedness result involves the invertibility of the corresponding operator on the l.h.s. of (5.2), which in particular implies that the *inf-sup condition* holds:

$$\inf_{\substack{u\in H^1(\Omega)\\ u\neq 0}} \sup_{\substack{v\in H^1(\Omega)\\ v\neq 0}} \frac{\left|\int_\Omega (\nabla u \cdot \nabla \overline{v} - \kappa^2 u\overline{v}) - i\kappa \int_{\partial\Omega} u\overline{v}\right|}{\|u\|_{H^1}\|v\|_{H^1}} =: \hat{m}_0 > 0. \tag{5.3}$$

5.1.1. Shifted Laplace operator preconditioners. Finite element approximations of problem (5.1) are widely used in acoustics, automotive applications, and electromagnetics, and their solution is still a challenging task [15, 38]. Therefore, efficient iterative solvers for the resulting linear algebraic systems are of great interest in practice. Recently a lot of research has been devoted to preconditioners arising as the discretization of the so-called complex shifted Laplace problems of the form

$$\begin{cases} Nu := -\Delta u - (\kappa^2 + i\varepsilon)u &= g \qquad \text{in } \Omega, \\ \qquad\qquad \frac{\partial u}{\partial n} - i\mu u &= 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{5.4}$$

where $\varepsilon > 0$ is a properly chosen "absorption" parameter, and $\mu > 0$ is a suitably chosen constant; see, e.g., [12, 13, 16, 18, 20, 37] in the context of multigrid solvers and [18, 28, 39] involving domain decomposition methods. The behavior of such preconditioners and choices of shifting parameters have also been analyzed [15, 16, 19]; typically $\varepsilon = O(\kappa^2)$ or $\varepsilon = O(\kappa)$ are used. Mesh independent linear convergence follows theoretically from [17, Theorem 3.11] in the framework of equivalent operators and was numerically observed in [25]. Operator equivalence was also used in a first-order least-squares approach [35]. The basic idea in the preconditioner (5.4) is that the complex shift improves significantly the spectral properties by moving the spectrum further away from zero.

Therefore, as summarized in the paper [37], the shifted Laplacian preconditioner has become a basic tool in the most successful multigrid approach for solving highly indefinite Helmholtz equations. The main point is that such preconditioning significantly accelerates Krylov iterations, and coupled with MG, the method is more efficient than MG that directly addresses the original Helmholtz equation. Hence the shifted Laplacian approach has a prominent place among modern solution methods for interior Helmholtz equations [9, 13].

However, all the above works were focused on the aspects of linear convergence of Krylov iterations. In what follows, our goal is to complete the above studies with establishing and estimating superlinear convergence.

We will use the fact that the inf-sup condition holds for the shifted Laplace operator as well:

$$\inf_{\substack{u\in H^1(\Omega)\\ u\neq 0}} \sup_{\substack{v\in H^1(\Omega)\\ v\neq 0}} \frac{\left|\int_\Omega (\nabla u \cdot \nabla \overline{v} - (\kappa^2 + i\varepsilon) u\overline{v}) - i\mu \int_{\partial\Omega} u\overline{v}\right|}{\|u\|_{H^1}\|v\|_{H^1}} =: \hat{m}_1 > 0, \tag{5.5}$$

since the complex shift in (5.4) implies readily that no eigenvalue is attained and the problem is uniquely solvable, as was the case for the original Helmholtz operator.

5.1.2. FEM discretization, iterative solution, and preconditioning. We consider finite element discretization of problem (5.1): let $V_h \subset H^1(\Omega)$ $(h > 0)$ be a family of FEM subspaces, depending on some mesh parameter h.

Assumption 5.1. The discrete inf-sup conditions for operators L_w and N_w, that is, the analogues of (5.3) and (5.5) in V_h, are satisfied for suitable constants m_0 and $m_1 > 0$ independent of h. (In other words, (4.7) holds in the space $H := H^1(\Omega)$.)

On the other hand, the above assumption always holds for fine enough meshes. Namely, it is well-known that the weak form of the Helmholtz problem leads to a compact perturbation of the identity in $H^1(\Omega)$. In fact, one can write the l.h.s. of (5.2) as the perturbation of the H^1 inner product with lower order terms that generate a compact operator, which leads to an operator equation of the form $(I + E)u = b$ in the space $H = H^1(\Omega)$ in the setting of Remark 3.6. The same applies to the shifted operator. Altogether, we can apply Proposition 4.4.

COROLLARY 5.2. *The discrete inf-sup conditions, imposed in Assumption* 5.1*, are satisfied for both the original and the shifted Helmholtz operators if* $h < h_0$ *for some suitable* $h_0 > 0$*, that is, for fine enough meshes.*

For a fixed FEM subspace $V_h = span\{\varphi_1, \dots, \varphi_n\}$, the discrete solution requires solving the $n \times n$ system

$$\mathbf{L}_h\,\mathbf{c} = \mathbf{b}, \tag{5.6}$$

where

$$(\mathbf{L}_h)_{i,j} = \int_\Omega \left(\nabla\varphi_j \cdot \nabla\overline{\varphi}_i - \kappa^2 \varphi_j \overline{\varphi}_i\right) - i\kappa \int_{\partial\Omega} \varphi_j \overline{\varphi}_i \qquad (i, j = 1, \dots, n) \tag{5.7}$$

and $\mathbf{b}_j = \int_\Omega g\varphi_j$ $(j = 1, \dots, n)$. We wish to apply a preconditioned Krylov iteration to solve (5.6). The preconditioner is based on the complex shifted Laplacian operator (5.4), that is, we introduce the stiffness matrix $\mathbf{N}_h$ of N_w as preconditioner for system (5.6):

$$(\mathbf{N}_h)_{i,j} = \int_\Omega \left(\nabla\varphi_j \cdot \nabla\overline{\varphi}_i - (\kappa^2 + i\varepsilon)\varphi_j \overline{\varphi}_i\right) - i\mu \int_{\partial\Omega} \varphi_j \overline{\varphi}_i \qquad (i, j = 1, \dots, n). \tag{5.8}$$

Remark 5.3. Regarding the invertibility of $\mathbf{L}_h$ and $\mathbf{N}_h$, it follows from Corollary 5.2 that both matrices are nonsingular if h is small enough. In fact, much more can be said.

First, the nonsingularity of $\mathbf{L}_h$ has been widely studied; see, e.g., [8, 45] and the references therein. Various conditions of the type $h \le C_{res}\kappa^{-\gamma}$ have been found, where κ is the wave-number and $\gamma = 2$, $3/2$, or 1, depending on conditions on the domain and the used FEM. However, a more practical result than these is given in the recent paper [8], where a computable criterion and a simple checking algorithm are given to guarantee in an a posteriori way that a given mesh provides a well-posed Helmholtz discretization.

Second, $\mathbf{N}_h$ is nonsingular for any mesh, due to the complex shift. Namely, (5.8) yields that the skew-Hermitian part of $\mathbf{N}_h$ is $-i\,(\varepsilon\mathbf{M}_h + \mu\mathbf{B}_h)$, where $\mathbf{M}_h$ and $\mathbf{B}_h$ denote the bulk and boundary mass matrices, respectively, hence the sum in the brackets is positive definite.

Henceforth, we can precondition system (5.6) with $\mathbf{N}_h$:

$$\mathbf{N}_h^{-1}\mathbf{L}_h\,\mathbf{c} = \tilde{\mathbf{b}} \tag{5.9}$$

(with $\tilde{\mathbf{b}} = \mathbf{N}_h^{-1}\mathbf{b}$). We will solve (5.9) using the GMRES or CGN iteration. Our analysis will use the discrete H^1-inner product on $\mathbf{C}^n$, generated by the matrix

$$(\mathbf{S}_h)_{i,j} := \int_\Omega (\nabla\varphi_j \cdot \nabla\overline{\varphi}_i + \varphi_j\overline{\varphi}_i) \qquad (i,j = 1,\dots,n). \tag{5.10}$$

5.1.3. Mesh independent superlinear convergence in the FEM subspaces. Our goal is to apply Theorem 4.6 to the above situation in the space $H := H^1(\Omega)$. Here the weak forms of the operators (5.1)–(5.4) are as follows: $L_w : H^1(\Omega) \to H^{-1}(\Omega)$ and $N_w : H^1(\Omega) \to H^{-1}(\Omega)$ are the bounded linear operators defined by the identities

$$\langle L_w u, v\rangle := \int_\Omega (\nabla u \cdot \nabla\overline{v} - \kappa^2 u\overline{v}) - i\kappa\int_{\partial\Omega} u\overline{v} \qquad (\forall v \in H^1(\Omega)), \tag{5.11}$$

$$\langle N_w u, v\rangle := \int_\Omega (\nabla u \cdot \nabla\overline{v} - (\kappa^2 + i\varepsilon)u\overline{v}) - i\mu\int_{\partial\Omega} u\overline{v} \qquad (\forall v \in H^1(\Omega)), \tag{5.12}$$

respectively. Further, let us define the operator $Q_w : H^1(\Omega) \to H^{-1}(\Omega)$ by the identity

$$\langle Q_w u, v\rangle := i\varepsilon\int_\Omega u\overline{v} \;\; + i(\mu-\kappa)\int_{\partial\Omega} u\overline{v} \qquad (v \in H^1(\Omega)). \tag{5.13}$$

PROPOSITION 5.4. *The operator Q_w is compact.*

Proof. The operators Q_1 and Q_2, defined by

$$\langle Q_1 u, v\rangle := \int_\Omega u\overline{v} \qquad \text{and} \qquad \langle Q_2 u, v\rangle := \int_{\partial\Omega} u\overline{v} \qquad (v \in H^1(\Omega)),$$

are known to be compact (see, e.g., [27]; this essentially follows from the compact embeddings of $H^1(\Omega)$ into $L^2(\Omega)$ and of $H^1(\Omega)_{|\partial\Omega}$ into $L^2(\partial\Omega)$). We have $Q_w = i\varepsilon Q_1 + i(\mu-\kappa)Q_2$, hence Q_w is also compact. ∎

We will also use the constant

$$M := \|N_w\|, \tag{5.14}$$

which can be easily bounded, e.g., as $M \le 1 + \sqrt{\kappa^2+\varepsilon^2}C_\Omega^2 + \mu C_{\partial\Omega}^2$, where C_Ω and $C_{\partial\Omega}$ are the embedding constants of $H^1(\Omega)$ into $L^2(\Omega)$ and $L^2(\partial\Omega)$, respectively.

THEOREM 5.5. *Let $s_j(Q_w)$ $(j = 1,2,\dots)$ denote the singular values of the operator Q_w in* (5.13). *Further, let Assumption* 5.1 *hold with inf-sup constants $m_0, m_1 > 0$, and let M be defined by* (5.14).

Then the GMRES and CGN iterations for the $n \times n$ preconditioned system (5.9) *provide mesh independent superlinear convergence estimates, i.e., we have*

$$\left(\frac{\|r_k\|_{\mathbf{S}_h}}{\|r_0\|_{\mathbf{S}_h}}\right)^{1/k} \le \; \varrho_k \qquad (k = 1,2,\dots,n), \tag{5.15}$$

where $(\varrho_k)_{k\in\mathbf{N}^+} \to 0$, and it is a sequence independent of n and V_h. Namely, in the case of GMRES,

$$\varrho_k \le \frac{M}{m_0 m_1} \cdot \frac{1}{k}\sum_{j=1}^{k} s_j(Q_w) \;\; \to 0 \qquad (\textit{as } k \to \infty), \tag{5.16}$$

and in the case of CGN

$$\varrho_k = \frac{2M^2}{m_0^2} \cdot \frac{1}{k} \sum_{j=1}^{k} \left(\frac{2}{m_1} s_j(Q_w) + \frac{1}{m_1^2} s_j(Q_w)^2 \right) \to 0 \qquad (as\ k \to \infty). \tag{5.17}$$

Proof. We wish to apply Theorem 4.6. The definitions (5.11)–(5.13) imply the decomposition $L_w = N_w + Q_w$, where Q_w is compact by Proposition 4.1. Assumption 4.2 coincides with Assumption 5.1 for the operators (5.11)–(5.12). Hence the conditions of Theorem 4.6 are satisfied in the space $H := H^1(\Omega)$. ∎

In addition to the fact of superlinear convergence, its magnitude and the presence of parameters can also be derived from the asymptotic behavior of $s_j(Q_w)$. We illustrate this below for the case of GMRES.

THEOREM 5.6. *Under the conditions of Theorem* 5.5, *the sequence* (5.16) *satisfies*

$$\varrho_k \le (\varepsilon\, c(\Omega) + (\mu - \kappa)\, c(\partial\Omega)) \frac{\log k}{k} = O\left(\frac{\log k}{k}\right) \qquad if\ \ d = 2, \tag{5.18}$$

$$\varrho_k \le \varepsilon\, c(\Omega) \frac{1}{k^{2/d}} + (\mu - \kappa)\, c(\partial\Omega) \frac{\log k}{k} = O\left(\frac{1}{k^{2/d}}\right) \qquad if\ \ d \ge 3 \tag{5.19}$$

for some constants $c(\Omega)$, $c(\partial\Omega) > 0$ *independent of* h.

Proof. Using the notations of Proposition 4.1, we have $Q_w = i\varepsilon Q_1 + i(\mu - \kappa) Q_2$. Using [26, Chapter II, Corollary 3.2] and the equality $s_j(cA) = |c| s_j(A)$, and since Q_1, Q_2 are self-adjoint positive operators, we obtain

$$\sum_{j=1}^{k} s_j(Q_w) \le \sum_{j=1}^{k} s_j(i\varepsilon Q_1) + \sum_{j=1}^{k} s_j(i(\mu-\kappa) Q_2) = \sum_{j=1}^{k} \varepsilon \lambda_j(Q_1) + \sum_{j=1}^{k} (\mu - \kappa) \lambda_j(Q_2),$$

where λ_j denotes the jth eigenvalue of the given operator. Here Q_1 and Q_2 correspond to the embeddings of $H^1(\Omega)$ into $L^2(\Omega)$ and of $H^1(\Omega)_{|\partial\Omega}$ into $L^2(\partial\Omega)$, respectively. Using the variational characterization of the eigenvalues, it follows that the asymptotics of $\lambda_j(Q_1)$ and $\lambda_j(Q_2)$ are inversely related to those of the Neumann Laplacian eigenvalues on Ω and to the Steklov eigenvalues on $\partial\Omega$, which are known to be $O(k^{2/d})$ and $O(k)$, respectively [36, 48]. Taking reciprocals and summation, elementary analysis implies the desired estimates. ∎

5.2. Helmholtz problems with more general conditions. Our above results remain valid for the related problems where the Robin boundary condition in (5.1) is replaced by a more general mixed one:

$$u = 0 \quad \text{on } \Gamma_D, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_N, \qquad \frac{\partial u}{\partial n} - i\kappa u = 0 \quad \text{on } \Gamma_R,$$

where $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_R$ is a proper decomposition of $\partial\Omega$. Then the underlying Sobolev space to be used in (5.2) is

$$H^1_D(\Omega) := \{u \in H^1(\Omega) : \ u = 0 \quad \text{on } \Gamma_D\}, \tag{5.20}$$

and $\int u\overline{v}$ on $\partial\Omega$ is replaced by the one on Γ_R. The operator N_w is then defined with the same decomposition of boundary conditions, and κ is replaced by μ on Γ_R.

Similarly, it is expected that our results can be extended to Helmholtz equations with variable coefficients, which arise, e.g., in seismic applications.

5.3. Elliptic systems at nonresonance. Let us consider the coupled elliptic system

(5.21)
$$-\nu\Delta u_i - \sum_{j=1}^{s} K_{ij}u_j = g_i, \qquad u_{i\,|\Gamma_D} = 0, \qquad \tfrac{\partial u_i}{\partial n}{}_{|\Gamma_N} = 0 \qquad (i = 1,\dots,s),$$

where $\partial\Omega = \Gamma_D \cup \Gamma_N$, for some constant $\nu > 0$, given matrix-valued function $K \in L^\infty(\Omega, \mathbf{C}^{s\times s})$ and r.h.s. $g_i \in L^2(\Omega)$ $(i = 1,\dots,s)$. The weak form reads as

$$\langle(\nu I - \mathcal{K})u, v\rangle_{H^1_D(\Omega)^s} \equiv \int_\Omega (\nu\,\nabla\mathbf{u} : \nabla\overline{\mathbf{v}} - K\mathbf{u}\cdot\overline{\mathbf{v}}) = \int_\Omega \mathbf{g}\cdot\overline{\mathbf{v}} \qquad (\forall \mathbf{v} \in H^1_D(\Omega)^s),$$

where $H^1_D(\Omega)^s$ is the product space of $H^1_D(\Omega)$ in (5.20). We consider nonresonance, that is, when ν is not an eigenvalue of the compact operator $\mathcal{K}$. Then, by the Fredholm theory, $\nu I - \mathcal{K}$ is regular.

The preconditioner can be defined as the discretization of separate (uncoupled) problems

$$-\nu\Delta u_i - \varrho_i u_i = g_i, \qquad u_{i\,|\Gamma_D} = 0, \qquad \tfrac{\partial u_i}{\partial n}{}_{|\Gamma_N} = 0 \qquad (i = 1,\dots,s),$$

where ϱ_i/ν are not eigenvalues of $-\Delta$, hence the preconditioning operator is also regular. One can choose the constants ϱ_i such that the matrix $diag(\varrho_1,\dots,\varrho_s)$ is a uniform diagonal approximation of K, obtained, e.g., by lumping or by spectral averaging. The solution with these uncoupled operators is much cheaper than for the original one, in particular when s is large.

It is straightforward that the decomposition (4.1) holds for these operators. Together with the inf-sup conditions, we obtain that Theorem 4.6 can be applied again to ensure mesh-independent superlinear convergence for the Krylov iterations in FEM subspaces.

5.4. Boundary integral operators, further applications and directions. Further directions where the above results might be extended arise for boundary integral equations forming Fredholm equations of the second kind. For the method of fundamental solutions, it was found that the monopole and dipole formulations lead to matrices that approximate linear operators being compact perturbations of the identity [21, 22]. For electric field integral equations on screens, compact-equivalent preconditioning operators have been constructed in [32] as solution operators of variational problems set in low-regularity standard trace spaces. For such problems, it requires future work to define proper discretizations and find their relation to the operators in order to extend the robust superlinear results of the present paper.

6. Numerical experiments. We illustrate the robust superlinear convergence properties for the complex shifted preconditioner of the Helmholtz equation. We first consider a three-dimensional (3D) interior acoustic problem, arising from the automotive industry, in an acoustic cavity of a car compartment. The air within the car compartment is meshed with hexahedral elements. Three different meshes are considered with 1,208 finite elements and 1,727 nodes (mesh L1), 9,664 finite elements and 11,637 nodes (mesh L2), and 77,312 finite elements and 85,001 nodes (mesh L3), respectively. These meshes are represented in Figure 6.1. We have run our algorithm for the wave-numbers $\kappa = 16$, 20, and 32. In what follows, we introduce the "absolute wave-number"

$$\kappa_a := \kappa a$$

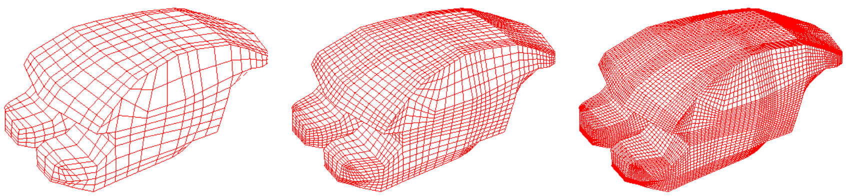

FIG. 6.1. *Finite element meshes of the interior car compartment.*

(where a is the characteristic length of the domain), which is used in acoustic practice. For the car we thus have $\kappa_a = 80$, 100, and 160, respectively. These parameters (wave-numbers and mesh widths) cover real-life interest, since the corresponding noise is never exceeded in practice in a car. Accordingly, the mesh parameter is bounded by the standard criterion of acoustic tests, that is, the resolution of six elements per wavelength must be satisfied for the three meshes.

Q_1 stabilized finite elements [29] are considered for the discretization of the Helmholtz equation. Such stabilized elements have the properties of improving the numerical stability by appending terms to the basic Galerkin formulation in order to reduce the pollution and dispersion effect, for which Galerkin finite element solutions with low-order piecewise polynomials differ significantly from the best approximation.

For the iterative solution of the linear system, we have applied the shifted Laplacian preconditioner with the choice $\varepsilon = \kappa$ proposed in [15, 16], and we have also set $\mu = \kappa$. In practice, the iterative solution of the preconditioned system is often executed using one of the simpler algorithms that are mathematically equivalent to the full GMRES; accordingly, we have used the generalized conjugate residual (GCR) method [44]. In addition, we have also tested the CGN method, since this method is often used in the community as mentioned in subsection 2.2. To follow the theoretical estimates, the error is measured using the discrete Sobolev norm of the residual using (5.10), that is, $\|r_j\|_{\mathbf{S}_h} := (\mathbf{S}_h r_j \cdot \overline{r}_j)^{1/2}$. The superlinear convergence behavior then means that the convergence factor

$$\varrho_j := \left(\frac{\|r_j\|_{\mathbf{S}_h}}{\|r_0\|_{\mathbf{S}_h}} \right)^{1/j} \qquad (j = 1, 2, \dots)$$

approaches zero as j is increased, in contrast to the case of linear convergence, where ϱ_j is only expected to stay bounded away from one.

We present the behavior of ϱ_j on meshes L2 and L3. First, on mesh L2, Figure 6.2 plots the values of ϱ_j using GCR, and Figure 6.3 plots the same using CGN.

We may observe that the convergence of GCR behaves superlinearly throughout all the iterations, whereas CGN enters the superlinear phase only after some iterations. Figure 6.4 plots the values of ϱ_j on mesh L3 using GCR. Altogether, the expected decreasing behavior of ϱ_j is seen in these tests, and moreover, we observe that this superlinear behavior is only slightly affected by the value of κ.

To illustrate mesh independence of the bounds of convergence, in Table 6.1 we give the discrete Sobolev norms of the residuals, i.e., $\|r_j\|_{\mathbf{S}_h}$, on three meshes, using GCR and CGN, respectively. Here the fixed wave-number has a smaller value $\kappa_a = 40$

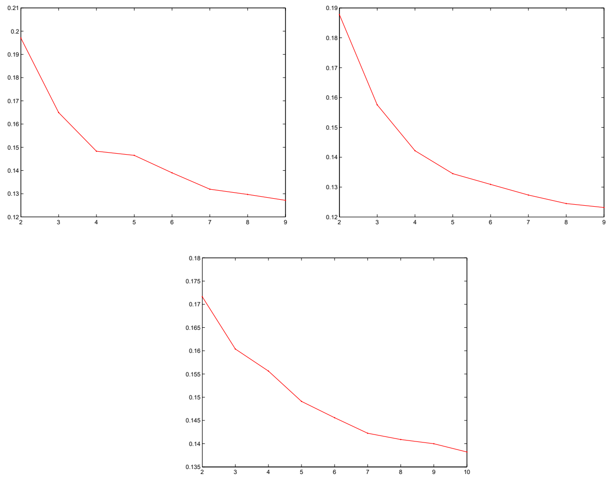

FIG. 6.2. *The convergence factors* ϱ_j *on car mesh L2 using GCR for wave-numbers* $\kappa_a = 80$, 100, *and* 160.

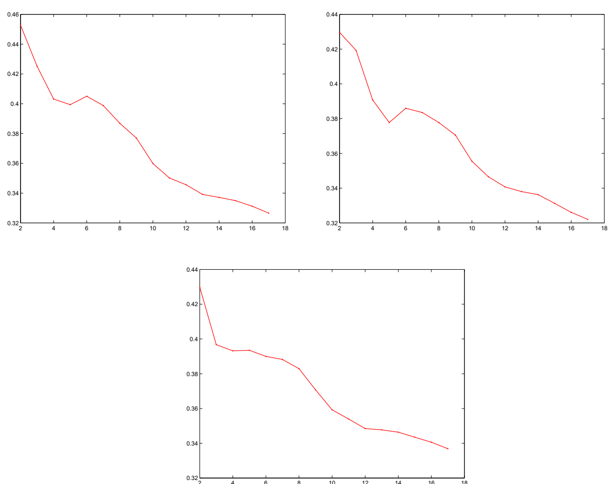

FIG. 6.3. *The convergence factors* ϱ_j *on car mesh L2 using CGN for wave-numbers* $\kappa_a = 80$, 100, *and* 160.

to fit the coarser meshes as well. The results show both a fast convergence of the iteration and that this indeed does not deteriorate when we use finer meshes.

Second, we consider another 3D interior acoustic problem in an auditorium, which has a more complex geometry due to the stairs. Here the realistic wave-numbers correspond to human voice. We plot the results for $\kappa_a = 5$, 60, and 90 in Figure 6.5. We observe that the iteration produces a general superlinear behavior again, but less smoothly than for the car test, which may be due to the complex geometry of the domain.

The above tests for acoustic problems well illustrate the theoretical results of this paper, both on superlinear behavior and on mesh independence. For other problems

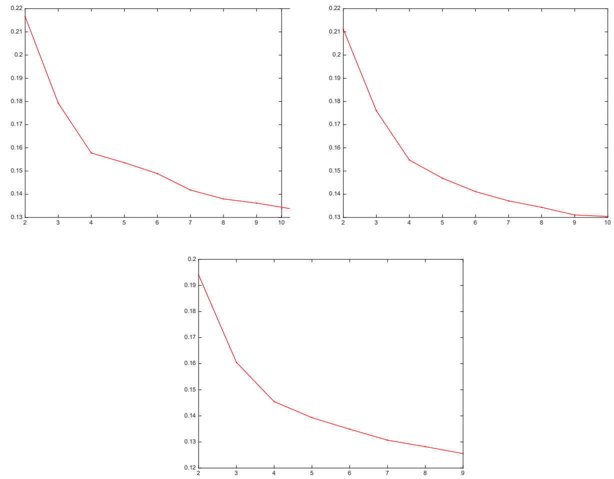

FIG. 6.4. *The convergence factors* ϱ_j *on car mesh L3 using GCR for wave-numbers* $\kappa_a = 80$, 100, *and* 160.

TABLE 6.1
The norms of the residuals using GCR and CGN on three meshes for $\kappa_a = 40$.

	GCR			CGN		
#iter.	Mesh L1	Mesh L2	Mesh L3	Mesh L1	Mesh L2	Mesh L3
1	0.0035961	0.0041012	0.0041681	0.0076460	0.0083094	0.0082863
2	0.0005941	0.0007629	0.0008109	0.0031451	0.0035655	0.0035947
3	6.0945e-05	8.4985e-05	9.3098e-05	0.0013514	0.0016036	0.0016429
4	7.1500e-06	9.3725e-06	1.0211e-05	0.0005009	0.0006099	0.0006315
5	1.0685e-06	1.5614e-06	1.6957e-06	0.0001863	0.0002224	0.0002299
6	1.1117e-07	1.8910e-07	2.2744e-07	7.2979e-05	8.7221e-05	9.0232e-05
7	1.3598e-08	2.3904e-08	2.7538e-08	2.7728e-05	3.6400e-05	3.8383e-05
8	1.2342e-09	2.8738e-09	3.7767e-09	8.0275e-06	1.1401e-05	1.2324e-05
9	1.5992e-10	3.2629e-10	4.0991e-10	2.4680e-06	3.8882e-06	4.2731e-06

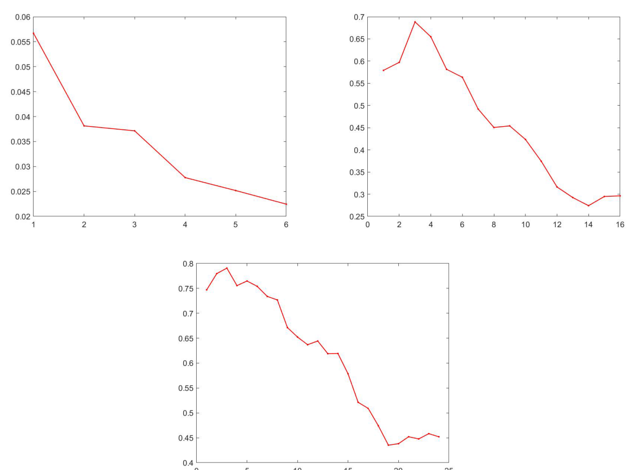

FIG. 6.5. *The convergence factors* ϱ_j *in the auditorium for wave-numbers* $\kappa_a = 5$, 60, *and* 90.

with even finer meshes or higher wave-numbers, entering the superlinear phase might arise in a later stage or depend more strongly on the parameters. Such investigations are beyond the scope of this paper and might be the subject of further research.

7. Conclusions. We have derived mesh independent superlinear convergence estimates of the GMRES and CGN iterations for Galerkin solutions of the operator equation in complex Hilbert space, when the preconditioned operator is a compact perturbation of the identity, for the case of non-Hermitian and noncoercive operators. The results apply to various noncoercive boundary value problems. We have detailed the case of complex shifted Laplacian preconditioners for interior Helmholtz problems, where also the asymptotic magnitude of superlinear convergence has been given. The theoretical results have been reinforced by experiments, first for a 3D interior acoustic cavity problem arising from the automotive industry: the superlinear convergence behavior has been demonstrated using different meshes and wave-numbers, and the mesh independent bounds have been illustrated on three meshes of the same geometry. A similar (somewhat less smooth) superlinear behavior has been obtained for tests in an auditorium.

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