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Corporate financing policies, financial leverage, and stock returns ☆

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ABSTRACT

We examine the interaction between equity returns and firms' financing policies in a stochastic Ramsey model with heterogeneous firms. Motivated by empirical evidence, firms maintain stationary financial leverage ratios by issuing debt. We present a novel closed-form solution to this class of models and, subsequently, use this solution to show that restricting firms' financing policies can explain various stylized facts of both the dynamic as well as the cross-sectional behavior of equity returns. Our restrictions induce persistent and countercyclical heteroskedasticity in returns, predictability by dividend–price ratios, a security market line that is "too flat", and mean-reverting CAPM betas that correlate with leverage. Our model explains both established as well as recent empirical evidence, but challenges recent theoretical macro-finance explanations of the link between capital structure and equity returns.

1. Introduction

We develop a stochastic Ramsey model with heterogeneous firms in which we connect firms' stock returns to their financing policies to explain the stylized behavior of asset prices. Motivated by stylized facts, we incorporate two features into our model: the supply of equity shares is inelastic and the firms' financial leverage ratios are stationary. We obtain a closed-form solution to this model, which we use to study the subsequent implications for asset prices. The model features firms that are heterogeneous in both their financing policies and the degree of uncertainty in their future cash flows. The firm heterogeneity allows us to characterize the asset pricing implications for both the dynamic properties of returns as well as their cross-sectional behavior. We show that the assumptions of inelastic equity supply and stationary leverage ratios give rise to a set of equations of price dynamics that can explain many documented properties of equity returns. The pricing equation for excess returns provides a unifying framework for the literature on return predictability, the cross-section of expected returns, and volatility clustering.

To investigate the effects of the two main stylized financing decisions, we introduce corporate debt and corporate equity in an otherwise standard stochastic Ramsey model. Our principle goal is to highlight the qualitative, rather than the quantitative, implications of these financing decisions. Therefore, we require a tractable model to scrutinize the consequences of said financing decisions and dissect the various mechanisms that elicit the observed behavior of stock returns. On the one hand, we therefore do not explicitly model the underlying mechanisms that beget these financing decisions, even though the financing choices we consider can

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be linked to the theoretical corporate finance literature — for example, the pecking order theory, the trade-off theory, and frictions in financial markets. The motivation of our two main assumptions simply comes from empirical observations, and our goal is to investigate the consequences, rather than the causes, of these observed decisions. Nevertheless, we argue that our specification of firms' financing policies indeed summarizes various identified mechanisms that induce stylized corporate financing decisions.

On the other hand, we make standard assumptions about the structures of preferences, production, and investment. The upshot of this approach, and one of our main contributions, is that we can identify exactly to what extent the effects of financing decisions on equity returns are purely mechanical. For example, Choi (2013) documents a correlation between asset betas and financial leverage in the cross-section, and provides a corporate finance theory explanation for this observation. This correlation arises endogenously in our model, simply by restricting the capital structure. Since there are no externalities in the *real* economy of the model, both Modigliani–Miller irrelevance propositions hold; neither the financing decisions nor dividend policy have real effects on the total value of the firm and therefore also not on the real side of the economy, i.e. real investments, the real capital stock, and consumption. However, these corporate financing decisions do have an effect on both stock price behavior as well as the behavior of financial ratios, such as the dividend–price ratio and earnings–price ratio. For example, the two stylized financing decisions restrict the flexibility of dividend policies; given an optimal investment level and that equity is inelastic, firms face a trade-off between retaining earnings and issuing debt in pursuit of the target leverage ratio. This trade-off induces a slow-moving dividend–price ratio, which is a requisite for replicating return predictability.

We derive explicit solutions for the variables of interest and study both the time-series behavior as well as the cross-sectional behavior of returns. The closed-form solution to the model is obtained by a flexible parameter restriction which, to the best of our knowledge, has not yet been considered in the literature. Specifically, the parameter restriction allows us to derive explicit expressions of firm dynamics in stochastic differential equations. The closed-form solution and continuous-time setting allow us to thoroughly inspect the relevant mechanisms that connect variation in risk premia to the firms' financing decisions. First, we evaluate the time series of returns, and find that the variation in the leverage ratio affects both the expected returns and volatility of returns when equity supply is inelastic. As a result, we find both a high level of as well as high volatility in excess stock returns. We also find variation in expected returns; in particular, low prices (i.e. low price–dividend ratios) predict high expected excess returns. In addition, we find persistent time-varying volatility of stock returns, highly persistent dividend–price ratios, and highly persistent price–earnings-ratios, as one should expect when expected returns can be predicted by dividend–price ratios (e.g. see Cochrane, 2009).

Next, we analyze the cross-section of stock returns. The Capital Asset Pricing Model (CAPM) holds conditionally in our model. However, both the drift and diffusion terms of the return processes depend on the inverse of the portfolio weights of the assets. This mechanism arises endogenously and generates time-varying betas that regress to the mean value of one.[1] On the one hand, this can explain why, with long time series, it is difficult to find variation in unconditional CAPM betas and variation in average risk premia for particular sorts of test portfolios. On the other hand, these results might explain why Choi (2013), Dam and Qiao (2020), and Doshi et al. (2019) find that "unlevering" returns increases the performance of the CAPM and that this (increase in) performance is robust against different sorts of test portfolios.

Our paper is closely related to the theoretical literature on market-clearing effects on asset prices and, in particular, the equity premium. The market-clearing effects arise when prices have to adjust to make markets clear because asset supply is inelastic. As pointed out by, amongst others, Rosenberg and Ohlson (1976), Fernholz and Shay (1982), He and Leland (1993), Raimondo (2005), Cochrane et al. (2008), and Martin (2013), independent and identically distributed (i.i.d) returns are *not* compatible with a (perfectly) inelastic supply of assets. If returns are i.i.d., the two-fund separation principle indicates that the main reason for investors to change their shares of wealth in various assets disappears. Therefore, over time, the investor's desired shareholdings remain in fixed proportions. If prices are to change, that must be a result of changes in equity supply. However, when equity supply is inelastic, investors cannot all rebalance; it follows that returns cannot be i.i.d. and are subject to equilibrium restrictions, except for the degenerate case when they are perfectly correlated.

Only more recently, the importance of market clearing on the dynamic behavior of asset prices has been analyzed while allowing for intermediate consumption in an endowment economy with multiple assets. Most notable are the contributions by Cochrane et al. (2008), Branger et al. (2011), Han et al. (2019), Hansen (2015), and Martin (2013), who extend the (Lucas, 1978) tree approach. In general equilibrium models that specify an endowment economy, there are no investments, and therefore the role of how investments are financed cannot be studied. These models implicitly assume a fixed supply of equity (and usually debt is in zero net supply), but as this is implicit, the role of financing is by construction disregarded. We add to this literature by explicitly modeling the production side of the macroeconomy and allowing for real investments. Thereby, we can introduce a clear distinction between equity financing and debt financing on the firm level and in the aggregate. In our model, the supply of debt adjusts such that the debt-equity ratio slowly reverts to a target level. This setup allows us to explain stylized empirical asset price dynamics – such as return predictability and the lack of spread in both average returns of long time series of returns as well as in estimated CAPM betas – and connect these patterns to firms' financing policies.

Because the market-clearing effects arise due to inelastic equity supply, our paper complements the literature that examines the impact of firm-specific financial frictions on corporate investment and asset pricing. Gomes and Schmid (2010) develop a dynamic

[1] There is some discussion about whether a conditional CAPM could hold, and the literature is not conclusive. For example, Lewellen and Nagel (2006) and Ang and Kristensen (2012) reject the conditional CAPM, however, Ang and Chen (2007) and Adrian and Franzoni (2009) do not reject the conditional CAPM. The discussion of whether or not the conditional CAPM holds seems to center around value portfolios, for which the pricing errors are found to be the largest.

investment model to couple equity betas and firms' financing decisions. They argue why high book-to-market firms tend to be highly levered and that highly levered firms are also more mature firms, with (relatively safe) book assets. Therefore, Gomes and Schmid (2010) suggest that "cross-sectional studies that fail to control for the interdependence of leverage and investment decisions are unlikely to be very informative". Bolton et al. (2013), Bianchi et al. (2017), Belo et al. (2018), and Gomes and Schmid (2021) endogenize the supply of financial assets by incorporating issuance costs. Bianchi et al. (2017) estimate a business cycle model with ambiguity-averse investors and show that these, jointly with corporate financing frictions, account for many observed post-war asset price dynamics. Bolton et al. (2013), Belo et al. (2018), and Gomes and Schmid (2021) calibrate dynamic q-theoretic models and show that variation in external equity financing costs generate plausible variation in firm's asset prices. Additionally, Belo et al. (2018) show that a two-factor model with the market risk premium and a proxy for time-varying equity issuance costs as the two factors, performs well in the cross-section. In contrast to these papers, our principal focus is on the qualitative, rather than quantitative, implications of corporate financing decisions in the sense that we would like to identify mechanisms that explain such relations. Our specification of the firms' corporate financing policies summarizes many of the financing frictions, among which the frictions considered by Gomes and Schmid (2010), Bolton et al. (2013), Bianchi et al. (2017), and Belo et al. (2018).

Our paper contributes to the empirical asset pricing literature on leverage and equity premia, in the sense that our model goes a long way in explaining well-established empirical patterns. Choi (2013) finds that variation in leverage drives variation in equity betas. Moreover, Choi (2013) finds that the effect for value firms is stronger than for other firms, such as growth firms. Choi and Richardson (2016) find that leverage induces a lot of variation in equity premia as well as volatility in equity risk. They additionally find that variation in leverage obfuscates the relationship between the market risk premium and the equity risk premium because they show that adjusting for leverage adjusts the cross-sectional fit of the CAPM. Dam and Qiao (2020) and Doshi et al. (2019) show that adjusting for leverage improves the cross-sectional fit of the CAPM on the firm level as well as for various portfolio sorts. Moreover, they show that adjusting for leverage is robust against various portfolio sorts. Next to that, Dam and Qiao (2020) show that leverage explains variation in the cross-section of equity returns beyond the value premium. The exact mechanisms that drive these results, however, remain opaque. Our model can explain many of the patterns that are documented by Choi (2013), Choi and Richardson (2016), Dam and Qiao (2020), and Doshi et al. (2019).

The paper is organized as follows: In the next section we discuss the stylized facts of corporate financing decisions and equity returns. We present our model in Section 3 and discuss the implications in Section 4. Section 5 concludes.

2. Patterns in financial leverage and stylized facts of equity returns

Because we include and attempt to explain various stylized facts on financial leverage, equity returns, and their interaction, we list the relevant observations and stylized facts here. We discuss and present empirical evidence on each stylized fact. In Section 4 we discuss the model's implications for asset returns, and we refer back to the stylized facts on equity returns we have listed below.

2.1. Financial leverage and corporate financing decisions

Observation 1. *The supply of equity is highly inelastic both in the aggregate as well as on the firm level, and particularly in tranquil economic times.*

Observation 2. *Firms' most capital structure relevant decisions are debt issuances and repurchases.*

Previous studies suggest that firms primarily resort to debt financing when they seek external financing for new projects or when they change their capital structures. Fig. 1 shows the net percentage yearly in- and outflows of corporate equity and corporate debt in the United States for 1946–2019, and we see that the supply of corporate debt is much more elastic. The yearly change in corporate equity is −0.6% on average, with a standard deviation of 2%, while the yearly change in corporate debt is 7.9% on average with a standard deviation of 4.3%. This is only anecdotal evidence in the aggregate, but Dichev (2007) shows on the firm level that the historical net in- and outflows of equity (including both dividends and stock repurchases) are close to zero on average and hardly ever exceed 5% of total market capitalization. Moreover, Hovakimian et al. (2001) show that seasoned equity issuances were widespread in"hot market" periods, such as in 1983 and 1991–1996, but rare in other years — this pattern also prevails in Fig. 1. Additionally, Hovakimian et al. (2001) document that long-term straight debt issues are the most frequent way of raising capital and that debt reductions are far more common than equity repurchases. Welch (2004) shows on the firm level that issuing long-term corporate debt is the most capital structure-relevant corporate activity on the firm level, and Collin-Dufresne and Goldstein (2001) argue that firms adjust outstanding debt levels in response to changes in firm value.

Observation 3. *Financial leverage ratios are stationary in the aggregate, on the industry level, and on the firm level.*

At the same time, studies document that financial leverage ratios are stationary on the aggregate level, the industry level, *and* the firm level. Among others, Frank and Goyal (2008) and Wright (2004) show that market leverage ratios are stationary over long horizons. Collin-Dufresne and Goldstein (2001) show that on the industry level "leverage ratios have remained within a fairly narrow band even as equity indices have increased 10-fold over the past 20 years", and Hovakimian et al. (2001) and Elsas and Florysiak (2011) provide empirical support for this using firm-level data. Lemmon et al. (2008) show that firms' leverage ratios exhibit a similar persistence level as the market. Graham and Harvey (2001) provide survey evidence that 81% of firms actively

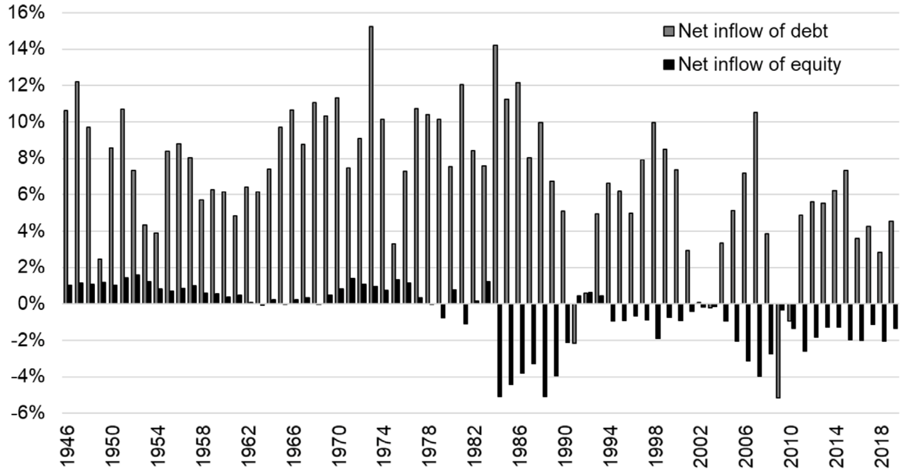

Fig. 1. Annual net in- and outflows of U. S. Corporate Equity and Corporate Debt (%), 1946–2019.
Source: Flow of Funds Accounts, Federal Reserve. The net inflow of debt is computed by dividing the flow *Nonfinancial corporate business; debt securities and loans; liability* (Z1.FU104104005 A) by its level (Z1.FL104104005 A). The net inflow of equity is computed by dividing the flow of *Nonfinancial corporate business; corporate equities; liability* (Z1.FU103164103 A) by its level (Z1.FL103164103 A). Negative numbers are associated with net outflows.

pursue target leverage ratios. Additionally, Elsas and Florysiak (2011) show that firms' speed of adjustment to their target ratio is positively related to default risk, expected bankruptcy costs, and the level of the opportunity costs from deviating from the target.[2]

Based on these observations, we conclude (i) that firms aim for a target leverage ratio, or at least act in such a way that the ratio remains within certain bounds, and (ii) that firms primarily reach their targets by issuing debt, rather than equity. Even though we consider an economy in which the Modigliani-Miller irrelevance theorems hold, target leverage ratios are consistent with firms following a capital structure to maximize firm value (e.g. see Goldstein et al., 2001). For the sake of model tractability, we assume that the supply of equity is fixed.

2.2. Equity returns, the Capital Asset Pricing Model, and the leverage effect

Stylized Fact 1. *Expected returns and volatilities exhibit a so-called leverage effect; a large price decline accompanies a period of high expected returns and high volatility. Lagged returns have a negative effect on volatility, implying that volatility goes up after a price drop.*

A number of empirical studies have reported that volatility of stock prices increases when prices fall, which is attributed to the so-called "leverage effect". Black (1976) explains that leverage induces future stock price volatility to vary inversely with the stock price. Black (1976) and Cheung and Ng (1992) find that this patterns emerges on the firm level and Nelson (1991) documents evidence for the leverage effect in market indices. However, more recent empirical literature on the leverage effect is inconclusive. Bekaert and Wu (2000) extend the GARCH-in-mean approach and conclude that the leverage effect is not driven by financial leverage, and Hasanhodzic and Lo (2019) show that firms without debt also exhibit a leverage effect. For many, the leverage effect has become a synonym for asymmetric response of volatility to return shocks. Based on this notion, Wu (2001) estimates a model with asymmetric volatility and allows for both the leverage effect and the volatility feedback effect. Wu (2001) finds that both effects account for an asymmetric volatility. Based on these observations, Ait-Sahalia et al. (2013) conclude that the leverage effect appears to be too large to be explained by variations in financial leverage alone. Christensen et al. (2015) finds that the leverage effect is larger during financial crises compared to NBER recessions. Choi and Richardson (2016) find that leverage and asset volatility have, respectively, permanent and transitory effects on equity volatility. Additionally, Choi and Richardson (2016) find that financial leverage, operating leverage, *and* the volatility feedback effect all contribute to the asymmetry between equity returns and volatility.

Stylized Fact 2. *Financial leverage induces heteroskedasticity in the cross-section of returns.*

Doshi et al. (2019) plot monthly returns against leverage and find a diverging pattern; the cross-sectional variance in returns for highly levered returns is higher than the cross-sectional variance for returns with low leverage. That is, they find evidence for the fact that leverage induces multiplicative heteroskedasticity. They adjust for leverage using the (Merton, 1974) approach, and show that this adjustment removes the diverging pattern.

[2] Welch (2011) shows, however, that the empirical correlation between equity-issuing activity and capital structure changes is weak. From this observation, he concludes that the capital issuing literature and capital structure literature are distinct.

Stylized Fact 3. *Over long horizons, CAPM betas cluster around one.*

Stylized Fact 4 (*Conditional*)**.** *CAPM betas have the tendency to mean revert (to one) over successive time periods.*

Keim (1983), Choi (2013), and Dam and Qiao (2020) show that when one uses long horizons, the unconditional estimates for betas cluster around one. Blume (1975, 1979) shows that an asset with a high (low) beta tends to have a lower (higher) beta in the subsequent period. In fact, all betas have the tendency to regress to the mean of grand betas, namely one.

Bali et al. (2009) show that conditional betas, however, can vary a lot over time. Bali et al. (2009) adopt several specifications of conditional betas, and show that these explain the cross-section of expected returns much better than unconditional betas. Additionally, they find that stocks that have a low (high) beta have on average low (high) expected returns, for each specification of the conditional betas.

2.3. The interaction between financial leverage and equity returns

Stylized Fact 5. *Firms' asset betas and financial leverage are correlated in the cross-section.*

Stylized Fact 6. *Firms' financial leverage goes a long way in explaining the value effect, but the expected return-beta relationship of equity in unconditional tests of the CAPM is more generally distorted by leverage.*

Choi (2013) reports that it indeed may not be unsurprising that leverage explains the value premium. Asset betas are correlated with leverage ratios in the cross-section. Leverage ratios of highly levered firms rise in bad times, their stocks become riskier and investors demand higher expected returns. High book-to-market firms tend to be highly levered – because they have low asset risk – and the equity betas of high book-to-market firms are higher in bad times. Choi and Richardson (2016) show that by decoupling asset risk and financial leverage, the cross-sectional fit of the CAPM improves. Dam and Qiao (2020) and Doshi et al. (2019) show that adjusting for leverage improves the cross-sectional fit of the CAPM on the firm level as well as that of portfolio sorts. Additionally, Dam and Qiao (2020) and Doshi et al. (2019) show that leverage induces additional heteroskedasticity in equity returns and, subsequently, show that adjusting for leverage lets the book-to-market factor disappear and weakens the size effect.

3. The model

We first characterize the real economy, which is represented by a standard stochastic Ramsey model in continuous time. We specify optimal consumption, savings, investments and real capital accumulation, and production. Thereafter, we characterize the financial policies that we have discussed in the introduction; real investments are financed by retained earnings and/or debt, while making sure that dividend payout policy is such that the leverage of the firm converges to its target level. We propose a closed-form solution to the model by imposing a highly flexible constraint on the parameters. To the best of our knowledge, this parameter constraint has not been considered in the literature.

3.1. The real economy

3.1.1. Firms and technology

We consider N firms. Our simple setup can also be applied to industries, but we refer to firms for the sake of parsimony. Each firm, indexed by $i \in \{1, \dots, N\}$, uses the same constant returns to scale Cobb–Douglas production technology $Y_i(t) = K_i(t)^{\alpha}(A(t)L_i(t))^{1-\alpha}$, using labor $L_i(t)$ and capital $K_i(t)$, with $0 < \alpha < 1$, and $A(t) = A(0)\exp(\mu_A t)$ represents an index of aggregate productivity, with $\mu_A > 0$ the growth rate of productivity. The capital stock of firm i evolves according to:

$$dK_i(t) = I_i(t)dt - \delta_i K_i(t)dt + \sigma_i K_i(t)dz_i(t). \tag{1}$$

where the depreciation rate, $\delta_i \geq 0$, and volatility parameter, $\sigma_i > 0$, are constants, and the vector $\boldsymbol{z}(t) := (z_1(t), \dots, z_N(t))'$ is a collection of standard Brownian motions defined on $(\Omega, \mathbb{P}, \mathcal{F}_t)$, where $\mathbb{P}$ comprises real (objective) probabilities. A shock dz_i reflects changes in the productivity of a firm's capital stock, which are learned over time. The stochastic shocks are correlated, and the correlation between the stochastic shocks $d\boldsymbol{z}(t) = (dz_1(t), \dots, dz_N(t))'$ are given by $\rho_{ij}dt := \mathbb{E}\left[dz_i(t)dz_j(t)\right]$, with $\rho_{ii} = 1$, $-1 < \rho_{ij} < 1$ for each $i \neq j$, and the matrix $\{\rho_{ij}\}$ is positive semi-definite.

The firms' objectives are to maximize their values. Each period they decide how much labor $L_i(t)$ to hire at wage rate $W(t)$ and how much to invest, $I_i(t)$, in new real capital, $K_i(t)$. The maximum value of the firm $V(K_i(t))$ is given by:

$$V(K_i(t)) = \max_{\{L_i, I_i\}} \mathbb{E}_t \left[\int_t^{\infty} \frac{\Lambda(s)}{\Lambda(t)} CF_i(s)ds \right], \tag{2}$$

where $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_t]$, $\Lambda(t)$ is the stochastic discount factor at time t, and the cash flows $CF_i(t)$ generated during period t by firm i are given by:

$$CF_i(t) = \left[K_i(t)^{\alpha}(L_i(t)A(t))^{1-\alpha} - W(t)L_i(t)\right] - I_i(t). \tag{3}$$

The stochastic discount factor follows the following dynamics: $d\Lambda = \mu_\Lambda(\cdot)\Lambda dt + \sum_i \sigma_{i\Lambda}(\cdot)\Lambda dz_i$, with $\mu_\Lambda, \sigma_{i\Lambda}$ to be determined functions. The firms' optimal policies are characterized by the following set of Hamilton–Jacobi–Bellman (HJB) equations:

$$\forall i \in \{1,\dots,N\}: \quad 0 = \max_{L_i, I_i} K_i^\alpha (AL_i)^{1-\alpha} - WL_i - I_i + V_t^i + V^i \mu_\Lambda(\cdot) + V_i^i \left(I_i - \delta_i K_i + \sum_j \rho_{ij}\sigma_{j\Lambda}\sigma_i K_i \right) + \frac{1}{2} V_{ii}^i \sigma_i^2 K_i^2, \tag{4}$$

where we define $V^i = V(K_i)$, $V_t^i = \partial V^i/\partial t$, $V_i^i = \partial V^i/\partial K_i$, and $V_{ii}^i = \partial^2 V^i/\partial K_i^2$.[3] We have dropped the time argument, and we will continue to do so in instances where no confusion can arise from doing so.

The partial derivatives with respect to I_i and L_i yield the following first-order conditions:

$$I_i: \quad V_i^i = 1, \tag{5}$$

$$L_i: \quad W = (1-\alpha)A\left(\frac{K_i}{AL_i}\right)^\alpha. \tag{6}$$

Since there are no adjustment costs for capital, the value of the firm is equal to its capital stock, i.e. $V^i = K_i$. Substituting the first-order conditions into (4) and dividing both sides by K_i simplifies the HJB equations (4) to:

$$0 = \alpha\left(\frac{K_i}{AL_i}\right)^{\alpha-1} + \mu_\Lambda(\cdot) + \sum_j \rho_{ij}\sigma_{j\Lambda}(\cdot)\sigma_i - \delta_i. \tag{7}$$

Because the risk-free rate, $r(t)$, satisfies $r(t)dt = -\mathbb{E}_t\left[d\Lambda(t)/\Lambda(t)\right]$, we simply have:

$$\alpha\left(\frac{K_i}{AL_i}\right)^{\alpha-1} - r - \delta_i = -\sum_j \rho_{ij}\sigma_{j\Lambda}(\cdot)\sigma_i. \tag{8}$$

This last equation relates the SDF to the optimal capital stock of firm i. The functional form of these functions will be pinned down by the equilibrium conditions.

3.1.2. Consumers and preferences

The representative consumer exhibits constant relative risk aversion (CRRA) with risk parameter τ, and wants to maximize expected lifetime utility $U(t)$ at time t:

$$U(t) = \mathbb{E}_t\left[\int_t^\infty e^{-\rho(s-t)}\frac{c(s)^{1-\tau}-1}{1-\tau}ds\right], \tag{9}$$

where $c(t)$ is the consumption rate at time t and ρ denotes the rate of time preference, with $c(t) \geq 0$ and $\rho > 0$. The dynamics of the consumption rate are governed by

$$\frac{dc(t)}{c(t)} = \mu_c(\cdot)dt + \sum_i \sigma_{ci}(\cdot)dz_i(t), \tag{10}$$

where μ_c and $\{\sigma_{ci}\}$ are to be determined functions of the model's state variables; log-consumption growth, thus, does not have to be independently and identically distributed (i.i.d.). Because the representative consumer is the single marginal investor in this economy, the SDF, Λ, obeys $\Lambda(t) = e^{-\rho t}c(t)^{-\tau}$. By applying Itô's lemma, we find that the SDF evolves according to:

$$\frac{d\Lambda(t)}{\Lambda(t)} = \left(-\rho - \tau\mu_c(\cdot) + \frac{1}{2}\tau(\tau+1)\sum_{i,j}\rho_{ij}\sigma_{ci}(\cdot)\sigma_{cj}(\cdot)\right)dt - \tau\sum_i \sigma_{ci}(\cdot)dz_i. \tag{11}$$

The functional forms of μ_c and $\{\sigma_{ci}\}$ are pinned down by equilibrium conditions, which are discussed next.

3.1.3. Equilibrium

Since we assume that all firms have the same technology and face the same wage rate (which is implied by a frictionless labor market), the first-order conditions associated with labor imply that all firms use the same capital–effective labor ratio; $\forall i,j: \frac{K_i(t)}{A(t)L_i(t)} = \frac{K_j(t)}{A(t)L_j(t)}$. Furthermore, we assume the total supply of labor is fixed and, without loss of generality, we normalize the size of the labor force to one, so that in equilibrium $\sum_i L_i = 1$ at each point in time. In this case, aggregation is straightforward:

$$\sum_i Y_i(t) = A(t)\left(\sum_i K_i(t)\right)^\alpha. \tag{12}$$

[3] We have used the Itô product rule:

$$d(xy) = ydx + xdy + dxdy.$$

We characterize the risk-free rate and consumption dynamics next. The risk-free rate is given by $r(t)dt = -\mathbb{E}_t\left[d\Lambda(t)/\Lambda(t)\right]$. Using the drift term of (11) we thus have:

$$r(t) = \rho + \tau\mu_c(\cdot) - \frac{1}{2}\tau(\tau+1)\sum_i\sum_j \rho_{ij}\sigma_{ci}(\cdot)\sigma_{cj}(\cdot). \tag{13}$$

In equilibrium we must have that the national accounting identity holds, consumption is output minus investment, $c(t) = \sum_i Y_i(t) - \sum_i I_i(t)$. As a result, for the drift term of the capital stocks we have $I_i(t) - \delta_i K_i(t) = Y_i(t) - CF_i(t) - W(t)L_i(t)$. We apply Itô's lemma, to pin down the drift and diffusion terms of consumption:

$$\mu_c = \frac{c_t + \sum_i c_{ki}\left(Y_i(t) - CF_i(t) - W(t)L_i(t)\right)}{c} + \frac{1}{2}\sum_i\sum_j \frac{c_{kikj}}{c}\rho_{ij}\sigma_i\sigma_j k_i k_j, \quad \sigma_{ci} = \frac{c_{ki}}{c}\sigma_i k_i. \tag{14}$$

Finally, we note from (11) that the diffusion term of consumption growth is proportional to the diffusion of the discount factor: $\sigma_{i\Lambda}(\cdot) = -\tau\sigma_{ci}(\cdot)$. We obtain a system of second-order partial differential equations by combining Eq. (14) with (13) and (8).

The model does not have an explicit solution in general, but we can get an explicit solution of the system of partial differential equations if we restrict the parameters (see Appendix). Let $\boldsymbol{\Sigma}$ be a symmetric and positive definite $N \times N$ matrix whose elements i, j are equal to $\rho_{ij}\sigma_i\sigma_j$ and $\boldsymbol{\delta}$ be a N-vector whose elements are equal to δ_i. The parameters of the model are $\rho, \tau, \alpha, \{\delta_i\}, \mu_A, \{\rho_{ij}\}$ and $\{\sigma_i\}$, and they are constrained by the following equation:

$$\rho = (\tau\alpha - 1)B - \tau(1-\alpha)\mu_A, \tag{15}$$

where

$$B = \left[\left(\frac{1}{\tau\alpha}\boldsymbol{h}' - \frac{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}}{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}\right)\boldsymbol{\delta} - \left(\frac{1}{2\tau\alpha}\boldsymbol{h}'\boldsymbol{\Sigma}\boldsymbol{h} - \frac{\boldsymbol{h}'\boldsymbol{\iota} - \tau\alpha/2}{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}\right)\right], \tag{16}$$

$$\boldsymbol{h} = \left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\delta} - \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\delta}}{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}\right), \tag{17}$$

and $\boldsymbol{\iota}$ a vector of ones. In this case, the equilibrium consumption rate is a fixed fraction of output:

$$c(t) = \frac{\tau-1}{\tau}\sum_i Y_i(t) = \frac{\tau-1}{\tau}A(t)\left(\sum_i k_i(t)\right)^{\alpha}. \tag{18}$$

To ensure that the model is non-degenerate we have the mild restriction that $\tau > 1$.

We further show in Appendix A that the weights of the relative capital stocks satisfy:

$$\frac{\boldsymbol{K}}{\boldsymbol{\iota}'\boldsymbol{K}} = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}} - \frac{\boldsymbol{h}}{\tau\alpha},$$

where $\boldsymbol{K}$ is the vector of capital stocks K_i. The total capital stock of firm i should be a fixed fraction of the total capital stock, using mean–variance efficient portfolio weights. Asset returns are not constant but shift up and down with the interest rate. However, their risk premium is constant, and so the optimal portfolio weights of capital stocks are constant.

In equilibrium, we thus have that total wealth $K = \sum_i K_i$, and all capital stocks K_i evolve according to:

$$\frac{dK}{K} = \frac{dK_i}{K_i} = \left(\frac{k^{\alpha-1}}{\tau} - \delta\right)dt + \sum_i w_i\sigma_i dz_i, \tag{19}$$

with $k = \frac{K}{LA} = \frac{K_i}{L_i A}, \forall i$, and $w_i = K_i/(\sum_i K_i)$.

To the best of our knowledge, the parameter restriction (15) has not yet been considered in the literature, and we would like to highlight its flexibility. Most of the literature considers a single representative firm. Then, a linear production function, $\alpha = 1$, corresponds to a version of the (Merton, 1990) ICAPM, and Chang (2004) and Smith (2007), among others, consider the case where $\tau = \alpha$ to find an explicit solution. Our solution is less restrictive. Let us consider a representative firm set-up by defining $\delta = \boldsymbol{w}'\boldsymbol{\delta}$, $\sigma^2 = \boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}$, with $\boldsymbol{w}$ the vector of portfolio weights w_i, and $dz(t) = \sum_i w_i(\sigma_i/\sigma)dz_i(t)$. Then, we see that the dynamics of aggregate production, aggregate consumption, and the aggregate capital stock are identical to a single-asset economy. When we consider one representative firm, i.e. $N = 1$, this parameter restriction reduces to $\rho = (\alpha\tau - 1)(\frac{1}{2}\alpha\tau\sigma^2 + \delta) - \tau(1-\alpha)\mu_A$. This particular parametrization is sufficiently flexible to allow for calibration with realistic parameter values.[4] We think that the benefits of explicit solutions outweigh this restriction; in particular for this model since our restriction is flexible and mild. Because we have a closed-form solution, we can explicitly analyze the consequences of the firms' financing decisions.

We also observe that all capital stocks are perfectly correlated in the cross-section, as in the Cox, Ingersoll, and Ross (1985) general equilibrium model and ICAPM model of Merton (1990). This perfect correlation has been subject to criticism and regarded as an undesirable feature of these models by, among others, Rosenberg and Ohlson (1976). However, in the next section we introduce the financial structure of the firm and show that by explicitly modeling the supply side of equity and debt, asset prices are no longer perfectly correlated in the cross-section.

[4] For example, choosing risk aversion $\tau = 7$, a quarterly rate of depreciation $\delta = 0.025$, capital share in output $\alpha = 0.33$, average quarterly GDP per capita growth $\mu_A = 0.005$, and quarterly volatility of the real capital stock $\sigma = 0.02$, imply a quarterly rate time preference of $\rho = 0.01$.

3.2. Financial markets

We now introduce the financing decisions. Investments are financed through retained earnings, first, and if needed by debt. Additionally, firms adopt a target debt-to-equity ratio and adjust their dividend policies and debt issuance accordingly, which implies stationary leverage ratios.

3.2.1. The financing of real investments

We denote the total value of debt of firm i at time t by $B_i(t)$ and the total value of total equity by $P_i(t)$. Naturally, the total value of the firm, $V^i(t) := V(K_i(t))$, is equal to the value of debt plus the value of equity $V^i(t) = B_i(t) + P_i(t)$. Since there are no adjustments costs of capital, the market value of the firm equals its book value, i.e. $V^i(t) = K_i(t)$, and so the dynamics of a firm's value are characterized by

$$\frac{dK_i}{K_i} = \frac{B_i}{K_i}\frac{dB_i}{B_i} + \frac{P_i}{K_i}\frac{dP_i}{P_i}. \tag{20}$$

We define $P_i^E(t)$ and $P_i^B(t)$ as, respectively, the price of a unit of equity and a unit of debt for firm i at time t, and $s_i(t)$ and $m_i(t)$ as, respectively, the number of shares and the number of debt instruments outstanding. We can decompose the changes in the total value of debt and the total value of equity into price changes and quantity changes, and substitute the expression for capital accumulation on the left-hand side:

$$\begin{aligned}
&\underbrace{\frac{I_i(t)dt}{K_i(t)} - \delta_i dt}_{\text{Net investments}} + \underbrace{\sigma_i dz_i(t)}_{\text{Exogenous shock}} \\
&= \left[\underbrace{\frac{dP_i^B(t)}{P_i^B(t)}}_{\text{Price change of debt}} + \underbrace{\frac{dm_i(t)}{m_i(t)}\left(1 + \frac{dP_i^B(t)}{P_i^B(t)}\right)}_{\text{Debt issuances/repurchases}}\right]\frac{B_i(t)}{K_i(t)} \\
&+ \left[\underbrace{\frac{dP_i^E(t)}{P_i^E(t)}}_{\text{Price change of equity}} + \underbrace{\frac{ds_i(t)}{s_i(t)}\left(1 + \frac{dP^E(i)}{P_i^E(t)}\right)}_{\text{Equity issuances/repurchases}}\right]\frac{P_i(t)}{K_i(t)}.
\end{aligned} \tag{21}$$

Since we assume changes in outstanding shares are equal to zero, that is $\frac{ds_i(t)}{s_i(t)} = 0$, investments are financed by debt or retained earnings, and debt is risk free, (21) can be simplified to:

$$\frac{I_i dt}{K_i} - \delta_i dt + \sigma_i dz_i = \left[rdt + \frac{dm_i}{m_i}\right]\frac{B_i}{K_i} + \left(\mu_{Pi}dt + \sigma_{Pi}dz_i\right)\frac{P_i}{K_i}, \tag{22}$$

where $dP_i^E(t) = \mu_{Pi}(\cdot)P_i^E(t)dt + \sigma_{Pi}(\cdot)P_i^E(t)dz_i(t)$ defines $\mu_{Pi}(\cdot)$ and $\sigma_{Pi}(\cdot)$.

The functional forms of $\mu_{Pi}(\cdot)$ and $\sigma_{Pi}(\cdot)$ are characterized next. Because investments are financed with debt, we have $\frac{I_i dt}{K_i} - \delta_i dt = \frac{dm_i(t)}{m_i(t)}\frac{B_i}{K_i}$. The remaining exogenous shock $\sigma_i dz_i(t)$ affects the price of equity, which implies that $\sigma_{Pi}(t) = \frac{K_i(t)}{P_i(t)}\sigma_i$. From now on we assume (without loss of generality) that the number of outstanding shares is equal to one, and so the total value of equity is equal to the share price; $P_i = P_i^E$. The stochastic discount factor and the price of equity jointly satisfy (e.g. see Cochrane, 2009, p. 74):

$$\left(\mu_{P_i} + \frac{D_i}{P_i} - r\right)dt = -\mathbb{E}_t\left[\frac{d\Lambda}{\Lambda}\frac{dP_i}{P_i}\right], \tag{23}$$

which, in our case, reduces to

$$\mu_{Pi} = r - \frac{D_i}{P_i} + \tau\alpha\sigma^2 b_i\frac{K_i}{P_i}, \quad b_i = \frac{\sum_j \rho_{ij}w_j\sigma_i\sigma_j}{\sigma^2}. \tag{24}$$

So we find an expression for the excess return process of equity:

$$dR_i - rdt = \frac{dP_i}{P_i} + \frac{D_i}{P_i}dt - rdt = \tau\alpha\sigma^2 b_i\frac{K_i}{P_i}dt + \frac{K_i}{P_i}\sigma_i dz_i. \tag{25}$$

The leverage ratio of the firm determines both the equity premium and the volatility. Nonetheless, the key feature of the multiple asset model is that the portfolio weights of equity are cross-sectionally not perfectly correlated in equilibrium (as the ICAPM implies), since these only depend on the direct shocks, $\sigma_i dz_i$. See Section 4.2.1 for a more detailed discussion on this. Although the capital stocks are also perfectly correlated in our model, each firm exhibits idiosyncratic risk in its equity returns. The conditional expected returns are in line with the CAPM, where the covariance with the market portfolio determines expected returns, but market clearing with (perfectly) inelastic supply of equity implies that means, covariances, and variances vary over time. The implications of the endogenous time variation in covariances and variances will be discussed in detail in Section 4.

3.2.2. Target leverage ratio and the dividend policy

We propose how individual firms target a certain leverage ratio by adopting a simple dividend policy. It follows from an application of Itô's lemma that the dynamics of the leverage ratio, as measured by the equity–total assets ratio, P_i/K_i, are governed by:

$$d\left(\frac{P_i}{K_i}\right) = \left[\tau\alpha\sigma^2 b_i - \sigma_i^2 - \left(\frac{D_i}{P_i} + \frac{I_i}{K_i} - \delta_i - r - \sigma_i^2\right)\frac{P_i}{K_i}\right]dt + \sigma_i\left(1 - \frac{P_i}{K_i}\right)dz_i. \tag{26}$$

This expression conveys two important points already. First, the equity–total assets ratio is mean reverting if dividends are large enough, and dividend policy is fairly smooth.[5]

Second, this leverage ratio reverts to a value that depends on the particular dividend policy. That is, if the firm wants to target a leverage ratio without issuing or repurchasing equity, the dividend–price ratio needs to be large enough and the dividend–price ratio needs to be rather stable to ensure that this inequality holds. Put differently, dividend policy is the only instrument left for the firm if it wants to target a leverage ratio without issuing new equity while financing investments with debt. Thus, a target leverage ratio and inelastic equity supply predicate dividend smoothing. This result aligns with the finding of Belo et al. (2015) that stationary (not constant) leverage ratios are inherently associated with dividend smoothing. In this sense, the model generates an endogenous dividend policy that is linked to financial leverage, similar to the mechanism described in Belo et al. (2015) and Bianchi et al. (2017).

At this point there is not much more we can deduce about what the specific dividend policy needs to be. Since the Modigliani-Miller dividend irrelevance theorem holds in our model – in a frictionless market dividends are irrelevant to the value of the firm – we can simply postulate a dividend policy, such that this specific dividend policy makes the leverage ratio mean reverting. Together with the considerations above, this leads us to postulate a simple specification, namely that dividends are a linear combination of total capital and the price of equity. It turns out that such a specification indeed implies a mean-reverting leverage ratio. More specifically, we postulate the following dividend policy:

$$D_i = \left[(\tau\alpha - 1)\sigma^2 b_i - \kappa_i\phi_i\right]K_i + \left(\frac{r+\delta_i}{\tau\alpha}(\tau\alpha - 1) + \phi_i\right)P_i. \tag{28}$$

where, as we will show below, ϕ_i and κ_i are, respectively, the firm-specific speed of adjustment and target leverage ratio (cf. (27)). Substituting this dividend policy in the expression for the dynamic behavior of leverage of the firm, we have:

$$d\left(\frac{P_i}{K_i}\right) = \phi_i\left(\kappa_i - \frac{P_i}{K_i}\right)\frac{P_i}{K_i}dt + \sigma_i\left(1 - \frac{P_i}{K_i}\right)dz_i. \tag{29}$$

We define the target leverage ratio as the value of P_i/K_i for which $\mathbb{E}[d(P_i/K_i)] = 0$, which happens when $P_i/K_i = \kappa_i$. Thus, κ_i is the target leverage ratio of firm i. Now, if the leverage ratio is to be stationary, the speed of adjustment ϕ_i has to be positive and $\kappa_i \in (0,1)$.[6] Because $P_i/K_i \leq 1$, leverage is counter-cyclical; we can deduce from (29) that after a sequence of negative productivity shocks, $\sigma_i dz_i$, the ratio P_i/K_i falls. We would like to highlight that this process closely resembles the Cox–Ingersoll–Ross specification of leverage that Dai and Piccotti (2020) adopt to empirically link deviations from the target leverage ratio to stock returns.

We do neither explicitly model adjustment costs of equity nor debt to keep the model tractable, and yet it could be argued that our model mimics costly adjustment of the capital structure. Naturally, firms will adjust gradually to target levels if swift and large adjustments are costly (e.g. see Belo et al., 2018; Bianchi et al., 2017). That is to say, one can interpret ϕ_i as a parameter that governs adjustment costs of financial capital, where low values of ϕ_i reflect that capital structure adjustments are costly. We summarize the model in Table 1.

4. Implications for asset returns

4.1. Dynamic properties of asset returns

The process for the excess returns of equity shows that the mean and volatility of excess equity returns depend on the total assets–equity ratio, K/P. This result is fairly robust to alternative model assumptions. Using only the identity $V(K) = P + B$ and imposing that equity P bears the risk associated with the shocks in K will generate this leverage effect in equity returns for most variations of the model. The leverage effect will play a more important role when equity supply is inelastic, as the leverage ratio, $V(K)/P$, is likely to vary more and more persistently when equity supply is inelastic.

[5] We consider a process $x(t)$ to be mean-reverting when its SDE is of the following form:

$$dx(t) = \left[a_1(\cdot) - a_2(\cdot)x(t)\right]dt + \sigma_x(\cdot)dz_x(t), \tag{27}$$

where a_1 and σ_x are known functions (and possibly constant), a_2 a given function that is strictly positive (and possibly a constant), and $z_x(t)$ a Brownian motion. Since a_2 is strictly positive, $x(t)$ has the tendency to revert to $a_1(\cdot)/a_2(\cdot)$.

[6] Note that the process is bounded from above so that it is guaranteed that $P_i/K_i < 1$. However, there is a small probability that eventually $P/K < 0$, because the diffusion term dominates the drift in $P/K = 0$.

Table 1
Main model equations.

Variable	Dynamics
Financial markets	
Excess stock returns	$dR_i^e = \tau\alpha\sigma^2 b_i \frac{K_i}{P_i} dt + \frac{K_i}{P_i}\sigma_i dz_i$
	$b_i = (\sum_j \rho_{ij} w_j \sigma_i \sigma_j)/\sigma^2$
Excess return	$dR^e = \tau\alpha\sigma^2 \frac{K}{P} dt + \frac{K}{P}\sigma dz$
Dividends	$D_i = \left((\tau\alpha - 1)\sigma^2 b_i - \kappa_i \phi_i\right) K_i + \left(\frac{r+\delta_i}{\tau\alpha}(\tau\alpha - 1) + \phi_i\right) P_i$
Firm i financial leverage	$d\left(\frac{P_i}{K_i}\right) = \phi_i\left(\kappa_i - \frac{P_i}{K_i}\right) dt + \sigma_i\left(\frac{K_i}{P_i} - 1\right) dz_i$
Interest rate	$r = \alpha k^{\alpha-1} - \delta - \tau\alpha\sigma^2$
Real economy	
Output	$Y_i = K_i^\alpha A^{1-\alpha}$
	$Y = A(\sum_i K_i)^\alpha$
Consumption	$c = \frac{\tau-1}{\tau} Y$
	$\frac{dc}{c} = \left(\frac{r-\rho}{\tau} + (\tau+1)\frac{(\alpha\sigma)^2}{2} + \mu_A\right) dt + \alpha\sigma dz$
Capital	$\frac{dK}{K} = \frac{dK_i}{K_i} = \left(\frac{k^{\alpha-1}}{\tau} - \delta\right) dt + \sum_i w_i \sigma_i dz_i$
Wage rate	$w = (1-\alpha)Ak^\alpha$

Variables: c is consumption, K_i is the capital stock of firm i, $A \equiv A_0 \exp(\mu_A t)$ is an index of productivity, L is labor (in fixed supply; $L = 1$), $f(k) \equiv k^\alpha$ is total output in effective labor units, r is the interest rate, w is the wage rate, P_i is equity, D_i is dividend, $dR_i^e = \frac{dP_i}{P_i} + \frac{D_i}{P_i}dt - rdt$ are excess stock returns, $\boldsymbol{z} = (z_1, \dots, z_N)'$ is a N-dimensional Brownian motion. *Parameters*: τ is relative risk aversion, μ_A is the rate of technological progress, α is the share of capital in production, $\boldsymbol{\delta} = (\delta_1, \dots, \delta_N)'$ are the rates of depreciation, $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N)'$ are the firm-specific volatilities of capital productivity, the elements of $\boldsymbol{z}$ have correlation ρ_{ij}, Σ has elements $\rho_{ij}\sigma_i\sigma_j$, ρ is rate of time preference, ϕ_i is adjustment speed of leverage ratio of firm i, κ_i is target leverage ratio of firm i. To get tractable solutions, the restriction $\rho = (\alpha\tau - 1)(\frac{1}{2}\alpha\tau\sigma^2 + \delta) - \tau(1-\alpha)\mu_A$ is imposed. *Auxiliary variables*: $K = \sum_i K_i$, $\sigma^2 = \boldsymbol{w}'\Sigma\boldsymbol{w}$, $\delta = \boldsymbol{w}'\boldsymbol{\delta}$, $dz = \sum_i w_i(\sigma_i/\sigma)dz_i$, $k = K/(AL)$.

4.1.1. *Return predictability*

Even though the parameters of the model are fixed over time, our return process exhibits a time-varying equity premium that arises endogenously. Indeed, other authors have stressed the importance of incorporating a varying equity premium in asset pricing models to match the stylized facts of stock returns (e.g. Campbell & Cochrane, 1999), but this variation in the equity premium is usually exogenous to those models. That is, counter-cyclical leverage increases risk premia in downturns, which are associated with concurrent low dividend–price ratios.

The resulting drift term for returns implies that low prices predict high expected returns and vice-versa (SF1). After a negative shock, the price or supply of the stock should decline correspondingly. However, investors will also want to move away from risky stocks. Since the average investor should hold the market portfolio and supply is inelastic, the price of the risky stock must decrease even further until the expected return has increased to the point where the average investor is willing to hold the risky stock. Market clearing under inelastic equity supply, thus, generates a time-varying equity premium and movements in returns that might be been interpreted as "excess volatility". Nonetheless, *excess* volatility is equivalent to return predictability, because the former implies that prices inevitably have to recoil. This is exactly what happens in our model.

In addition, our return process exhibits other well-known time-series properties of stock returns. Since the leverage ratio does not only predict conditional stock returns but is also a very persistent time series, returns will exhibit "momentum" (Balvers & Wu, 2006; Carhart, 1997), in the short run. At the same time, stock returns cannot stay high for an extended period, since a series of high returns implies lower values K/P. In time the leverage ratio will then lower the expected return, so we have "mean reversion" in expected returns, but only in the long run as reported by DeBondt and Thaler (1985), Poterba and Summers (1988), Balvers and Wu (2006), and Bali et al. (2008).

4.1.2. *Volatility dynamics of asset returns*

Generalized Auto-Regressive Conditional Heteroskedasticity (GARCH) models and other econometric methods have shown that volatility in stock returns changes over time, and exhibits autocorrelation. Time-varying volatility in these models is taken as exogenous, whereas in our model it also arises endogenously. Because the diffusion term of expected returns is increasing in the leverage ratio and the leverage ratio is persistent, the model can account for periods of high volatility and low volatility (SF1 and SF2). This is indeed the "leverage effect"; a large price decline accompanies a period of high volatility (Nelson, 1991; Schwert, 1989)[7] Additionally, note that a stationary leverage ratio ensures that dividends are not excessively volatile at short horizons. The low elasticity of equity supply, thus, generates prolonged periods of high volatility even when dividends are rather stable.

[7] Admittedly, Schwert (1989) mentions that leverage cannot fully account for the variation in stock market volatility. We believe that this is in part because under limited liability of equity, leverage has a non-linear effect on volatility. We do not model this feature of equity here, since the non-linear effect only becomes economically significant when the value of equity is rather low.

4.2. Cross-sectional properties of asset returns

4.2.1. Distribution of portfolio weights in the market portfolio of equity

Fernholz and Shay (1982) have argued that the distribution of portfolio weights of individual assets in the market portfolio should be stable. Their finding reformulates the fundamental issue raised by Rosenberg and Ohlson (1976), that returns cannot be i.i.d when equity supply is fixed. In short, the argument runs as follows. Suppose betas of assets are constant. This implies that the asset with the highest beta should on average have the highest return. Suppose this asset has a beta larger than one. With a sufficiently inelastic supply of equity, the growth rate of this asset dominates the other assets, and so in the long run this asset grows exponentially faster than the other assets. Consequently, it will eventually dominate the portfolio in terms of the portfolio weight. But if the market portfolio is dominated by one large asset, it implies that the beta of this asset should converge to one. This result implies a theoretical contradiction, and hence returns cannot be i.i.d. Additionally, this result also violates the empirical finding by Fernholz et al. (1998) that portfolio weights appear to be stable over time. In light of this result, we present an important result regarding the equilibrium distribution of portfolio weights which illustrates how our return process resolves this fundamental contradiction.

We analyze the portfolio weights with price processes according to Eq. (25) but, for exposition's sake, consider a period without dividend payments. During a dividend-free period, the dynamic process of the market portfolio weight $w_i^P(t)$ of firm i's equity in the aggregate portfolio of equity $P(t) = \sum_i P_i(t)$, $w_i^P(t) = P_i(t)/P(t)$, is given by:

$$d w_i^P(t) = \sigma^2 \left(\tau\alpha - \frac{K(t)}{P(t)} \right) \frac{K(t)}{P(t)} \left[w_i b_i - w_i^P(t) \right] dt + w_i^P(t) \left(\frac{K_i(t)}{P_i(t)} \sigma_i d z_i(t) - \frac{K(t)}{P(t)} \sigma d z(t) \right) \tag{30}$$

In addition, w_i^P is mean-reverting to $w_i b_i$ if the parameter of risk aversion $\tau\alpha > \frac{K}{P}$.[8] In that case, the long-term equilibrium portfolio weight of firm i's equity in the portfolio of equity is given by $w_i b_i$.

Eq. (30) essentially tells us that portfolio weights converge to their "asset betas", $b_i w_i$; the regression coefficient of firm-specific shocks, $\sigma_i d z_i$, on the portfolio of aggregate shocks, $\sigma d z$. It also implies that asset betas are likely to be correlated with leverage ratios (SF5). This correlation is purely mechanical and does not necessarily require a corporate finance explanation as in Choi (2013). The rationale is simple. Eq. (25) shows that the drift term of individual equity returns is inversely related to the price of equity. As the price of the asset increases, its expected return will go down relative to the other assets. This result is driven by the market clearing mechanism. If there is news about one or more firms (or industries) investors want to rebalance. When the supply of equity is inelastic, and investors want to collectively rebalance, equity prices will adjust to induce investors to hold all shares to clear the market in equilibrium. This reversion in betas has imperative implications for mean–variance analysis and the estimation of CAPM betas, which are discussed next.

4.2.2. Mean-reverting betas and a dynamic security market line

The price process given by (25) can be rewritten in a more familiar form, so that the excess returns of firm-specific equity, $d R_i^e(t) = d R_i(t) - r(t)dt$, are related to the excess returns of the market portfolio of equity, $d R^e(t) = d R(t) - r(t)dt$:

$$d R_i^e(t) = b_i w_i \frac{P(t)}{P_i(t)} d R^e(t) + \left(\sigma_i \frac{K_i(t)}{P_i(t)} d z_i(t) - b_i \frac{K(t)}{P(t)} \sigma d z(t) \right). \tag{31}$$

We observe that the conventional beta, i.e. the coefficient in front of the excess return of the market portfolio, varies over time and is given by:

$$\beta_i(t) = b_i w_i \frac{P(t)}{P_i(t)} = b_i \frac{w_i}{w_i^P(t)}. \tag{32}$$

Stocks thus have periods of low and high betas, depending on their relative market value, i.e. their portfolio weight. We have shown that the market portfolio weight P_i/P will converge to a value of $b_i w_i$ in dividend-free periods, which implies that the unconditional betas, β_i, have a tendency to converge to a value of one over long horizons (SF3 and SF4). This mean reversion in betas is well-documented (see e.g. Blume, 1975) and, moreover, individual betas are on average equal to one over longer time periods (see e.g. Choi & Richardson, 2016). The implication of our results is that when one sorts portfolios on betas and calculates the average returns for, say, the next twelve months, the betas and the associated returns for these next twelve months will typically have converged to their equilibrium values, yielding a security market line that is too flat, as is documented by Fama and French (2004) and Dam and Qiao (2020).

We summarize the results as follows. The conditional "beta", $\beta_i(t)$, defined as the conditional covariance of the excess return $d R_i^e(t)$ of firm i's equity with the excess return of the market portfolio of equity, $d R^e(t)$; and the conditional expected excess return of firm i's equity are given by:

$$\mathbb{E}_t[d R_i^e(t)] = \beta_i(t) \tau\alpha\sigma^2 \frac{K(t)}{P(t)} dt. \tag{33}$$

We observe that betas and expected returns should be on a straight line *conditionally*. However, the slope of this line varies over time, depending on the current market equity risk premium. In turn, this market premium depends on the current aggregate leverage in

[8] Based on historical U.S. data (e.g. see Fig. 2) the lower bound for $\tau\alpha \approx 2$, and the historical average of $K/P \approx 1.6$.

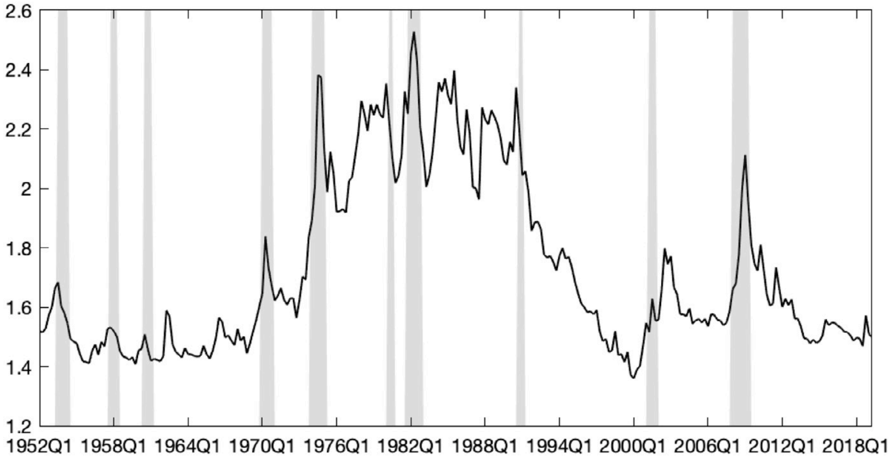

Fig. 2. Leverage ratio of U.S. Total Market Value of Corporate Assets and Total Market Value of Corporate Equity, 1952Q1-2019Q2.
Source: Flow of Funds Accounts, Federal Reserve. The total market value of assets is the sum of the total market value of debt and the total market value of equity. The market value of debt is calculated by taking *Nonfinancial business; debt securities and loans; liability, Level* (Z1.FL144104005Q) net of *Nonfinancial business; debt securities and loans; asset, Level* (Z1.FL144004005Q) and *Nonfinancial business; money market fund shares; asset, Level* (Z1.FL143034005Q). The total market value of equity is calculated by taking *Nonfinancial business; equity and investment fund shares excluding mutual fund shares and money market fund shares; liability (IMA), level* (Z1.FL143181105Q) net of *Nonfinancial business; other equity; liability (IMA), Level* (Z1.FL143181115Q) and *Nonfinancial business; mutual fund and money market fund shares; assets, Level* (Z1.FL143081205Q).

the market $K(t)/P(t)$. As said, betas also vary over time and are related to the inverse of the portfolio weight of the asset. If the price of an asset is relatively low (which happens when the portfolio weight is lower compared to the long-run "equilibrium" portfolio weight) the beta of this asset will be high, but the expected return can be extraordinarily high, depending on the current slope of the security market line (SF6). Naturally, this mechanism only has substance if the market leverage ratio is persistent but varies a lot over longer horizons. Indeed, the market leverage ratio is slow-moving in both our model as well as in the data — as we can see in Fig. 2 , which exhibits the evolution of the historical leverage ratio, measured as corporate debt to total assets, in the United States for 1952Q1–2019Q2.

We would like to highlight that both the time-varying betas and the time-varying trade-off between expected returns and beta arise endogenously since all the fundamental parameters of the model are assumed to be fixed over time. Naturally, our assumption of fixed equity supply exacerbates the aforementioned effects, since shocks are now fully absorbed by equity prices. Notwithstanding, with imperfectly inelastic equity supply, the model's implications for (unconditional) estimation of CAPM betas naturally persist, as shocks will remain to be largely absorbed by prices and the leverage ratios will remain persistent.

The practical implication is that one needs to account for both time-varying betas as well as varying risk premiums to describe the beta-return relationship. This issue is also pointed out by Cochrane (2011) and to some extent by Belo et al. (2018). Nonetheless, our model provides us with guidelines on how one could proceed. Fig. 3 gives a summary of these results and an overview of how the security market line dynamically behaves according to our model. Fig. 3 shows that the slope of the security market line is varying over time. In bad times, when prices are low, expected returns are relatively high, for each value of the conditional beta $\beta_i(t)$. The slope will always converge back to some equilibrium value, which we labeled "Normal Times". Not only the market portfolio has a beta equal to one but, in principle, all risky assets have an equilibrium value of one for the beta. Stocks that have relatively low values (that is, a low portfolio weight w^p), will have a higher beta, and hence a higher return. However, in the long run, the (conditional) betas revert to a value of one, while the fundamental betas, b_i do not exhibit this mean reversion, and nor do they have means equal to 1. Since both the slope of the security market line and the position of individual assets on the security market line vary over time, we may observe various empirical patterns regarding historical betas and historical average returns.

4.2.3. Implications for the estimation of CAPM betas

As mentioned, studies that plot betas of buy-and-hold portfolios find a large cluster around beta equal to one (see e.g. Dam & Qiao, 2020; Keim, 1983), with usually similar expected returns. This empirical regularity is obviously in line with what our model predicts. It is common practice to do yearly sorts of portfolios to ensure that sufficient variation in both conditional betas and average returns is preserved — in line with, for example, the (Fama & French, 1993) sorts on size and book-to-market values. Indeed, it is not surprising that the Fama–French approach requires constructing the portfolios with an annual frequency, as the betas and returns will converge in the long run, leaving little cross-sectional variation to explain. Looking at it the other way, our analysis suggests that sorting on high and low market-to-book values with an annual frequency is actually a good idea, as this will very likely preserve variation in the average returns and estimated CAPM betas. These so-called "managed sorts", thus, are adequate to empirically validate an asset pricing model.

Our analysis suggests that one can deal with this clustering of betas and average returns by correcting for time-varying leverage directly. Since the variation and reversion in returns and betas are partly driven by the time-varying and mean-reverting leverage

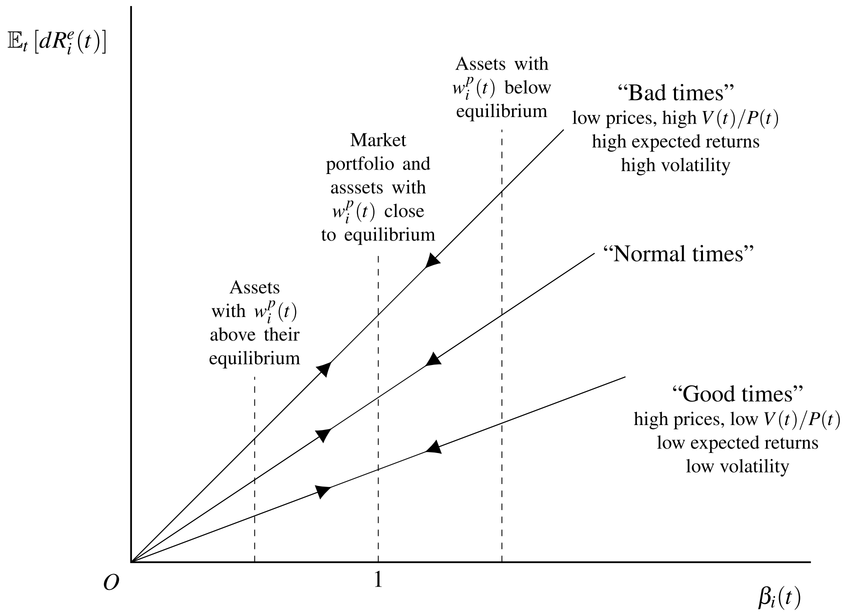

Fig. 3. Dynamic behavior of the security market line.

ratio, the connection between the market risk premium and the firm-specific equity risk premium is obfuscated. Consequently, it is not necessary to explicitly model corporate financing decisions to estimate the CAPM unconditionally. On the one hand, variation and reversion in leverage might explain why Belo et al. (2018) find that a two-factor model with the market risk premium and a proxy for time-varying equity issuance costs as the two factors, performs well in the cross-section. Specifically, they find that investors require a higher risk premium for holding assets that are highly exposed to equity issuance costs, i.e. assets that do poorly just when it is more costly to issue equity. In Belo et al. (2018), equity issuance costs go a long way in explaining variation in leverage, i.e. deviations from the target leverage ratio. Moreover, Belo et al. (2018) document that value firms in particular are sensitive to the equity issuance costs factor. Choi (2013) shows that value firms "suffer" more from the leverage effect, as value firms are typically highly levered.[9] Given the convexity and the inverse relation of the drift and diffusion terms of excess returns in leverage, deviations from the target leverage ratio for highly levered (value) firms generate more "excess volatility" indeed. Thus, this leverage effect obfuscates the relation between the market risk premium and the equity risk premium more for leverage firms and, hence, accounting for variation in leverage due to variation in equity issuance costs is particularly beneficial for highly levered (value) firms.

On the other hand, this notion might explain why Choi and Richardson (2016), Dam and Qiao (2020), and Doshi et al. (2019) find that "unlevering" returns (i.e. dividing a firm's equity return by its leverage ratio) improves the cross-sectional fit of the CAPM. Dam and Qiao (2020) and Doshi et al. (2019) show that unlevered betas provide a better explanation of cross-sectional unlevered returns, and they also find that unlevering the returns is robust against various portfolio sorts. Furthermore, Dam and Qiao (2020) find that firms' financial leverage goes a long way in explaining the value effect, but that the expected return-beta relationship of equity in unconditional tests of the CAPM is more generally distorted by leverage. Even more importantly in light of our results, they find indeed that unlevering performs well on the firm level.

4.3. A comparison of statistics from simulated and historical data

One cannot expect, nor require, the model to match historical data — this is not the purpose of our study. Yet an indication as to how far a stylized model can take us can be informative. To this end, we simulate 1000 years of data of a single (representative) firm version of the model to calculate a variety of statistics and compare these to statistics of historical U.S. data. We calibrate the free parameters of the model such that they match to a large extent the moments of the macro-economic and financial variables. Rather than trying to match the data perfectly, we choose parameters that are "reasonable", and examine how the moments generated by the model relate to historical moments. We set the capital share in production to one-fifth, $\alpha = 0.2$. We choose the volatility parameter, $\sigma = 11\%$, so that it will match the volatility of unlevered returns. Economic growth, reflected by μ_A, is equal to 2.2%, and we set the risk aversion rather high, at $\tau = 12.5$. Since we have imposed a restriction on the parameters, the range of values for δ is limited if we want a positive rate of time preference ρ. We therefore choose $\delta = 0\%$.

[9] In particular, he finds that in economic downturns (when the risk premium is high), the leverage ratios of value portfolios increase sharply, increasing equity betas. Notwithstanding, he also finds that growth portfolios are both less levered as well as less sensitive to economic conditions.

Table 2
Moments of simulated and historical data.
Source: NIPA tables for consumption growth (non-durables and services) scaled by population and the consumer price index (CPI). Shiller for aggregate dividends, aggregate earnings, and stock returns. We take the annualized yield on 3-month T-bills as a proxy for the risk-free rate from the Federal Reserve Economic Database (FRED). Leverage we have computed by the Flow of Funds accounts, Federal Reserve. The stock returns, risk-free rate, and dividend growth rate have been deflated by the CPI as well.

	US 1952–2019			Simulated data		
	Mean	St.D.	A.Cor.	Mean	St.D.	A.Cor.
Consumption Growth ($dln(c)$)	1.8%	1.6%	0.22	2.2%	2.2%	−0.20
Dividend Growth ($dln(D)$)	2.1%	5.8%	0.47	2.2%	8.7%	0.04
Returns (Rp)	7.0%	16.8%	−0.01	9.1%	19.2%	−0.02
Excess Returns ($R_p - r$)	6.1%	16.5%	−0.04	5.7%	19.3%	−0.05
Dividend-Price (D/P)	3.1%	1.2%	0.88	5.3%	2.1%	0.95
Leverage (P/K)	59.7%	9.2%	0.89	67%	15.2%	0.97
Price-Earnings (P/E)	20.92	7.0	0.76	14.8	8.1	0.91
Dividend Payout Ratio (D/E)	49.8%	20%	0.10	64%	4.6%	0.97
Interest Rate (r)	0.9%	2.1%	0.73	3.4%	2.4%	0.97

This table shows means, standard deviations, and autocorrelations for historical and simulated data. Parameter choices: $\tau = 12.5, \alpha = 0.2, \sigma = 0.11, \mu_A = 0.022, \delta = 0, \phi = 0.01, \kappa = 0.52$.

The parameters mentioned above affect the macroeconomy, while the remaining two parameters, ϕ and κ do not. We choose the target level of equity to total assets equal to $\kappa = 0.52$. Estimates of autocorrelation in the leverage ratio of the firm are hard to measure, but they are usually 0.85 or higher for annual data (e.g. see Welch, 2004). Keeping in mind that these two parameters also affect dividend policy, we choose a value of $\phi = 0.01$ that is rather low, to ensure that dividends stay strictly positive. The leverage ratio will thus be close to a random walk in the short run, while it is mean reverting in the long run, in line with the results by Welch (2004).

We compare our simulated data to annual U.S. data for 1952–2019. We proxy consumption growth by taking the sum of personal consumption expenditures of non-durables and services, and we adjust this value for population growth and inflation by the consumer price index (CPI). The consumption and population data are obtained from the U.S. National Income and Product Accounts (NIPA) Tables. Aggregate dividends, aggregate earnings, the stock price index, and the CPI are taken from Robert Shiller's website. Stock returns are computed by taking the cum-dividend returns on the stock price index. To proxy the risk-free rate, we have taken the annualized yield on 3-month U.S. T-bills. Aggregate dividends, aggregate earnings, the stock price index, and the interest rate are deflated by the CPI as well. The aggregate leverage is computed by using the Flow of Funds Accounts from the Federal Reserve. The total market value of assets is the sum of the total market value of debt and the total market value of equity. The market value of debt is calculated by taking *Nonfinancial business; debt securities and loans; liability, Level* (Z1.FL144104005Q) net of *Nonfinancial business; debt securities and loans; asset, Level* (Z1.FL144004005Q) and *Nonfinancial business; money market fund shares; asset, Level* (Z1.FL143034005Q). The total market value of equity is calculated by taking *Nonfinancial business; equity and investment fund shares excluding mutual fund shares and money market fund shares; liability (IMA), level* (Z1.FL143181105Q) net of *Nonfinancial business; other equity; liability (IMA), Level* (Z1.FL143181115Q) and *Nonfinancial business; mutual fund and money market fund shares; assets, Level* (Z1.FL143081205Q).

4.3.1. Means, standard deviations and autocorrelations

We report the means, standard deviations, and autocorrelations of historical and simulated data in Table 2. Comparing historical and simulated data, we observe that most statistics are in the same ballpark. For the macroeconomic statistics, the biggest concern is the interest rate. Our macroeconomic set-up is standard so we do not resolve the interest rate puzzle and we need implausible levels of either the pure time preference and/or risk-aversion. Again, we adopt this tractable model set-up to inspect the mechanisms, not to resolve the equity premium and interest rate puzzles. Regarding the financial ratios, all values seem reasonable as well. Even though we choose the parameters in particular to match the average leverage ratio, we observe in the simulated data a persistent dividend yield of 5.3% with fairly low volatility, a price–earnings ratio of about 14.8, and a dividend payout ratio of 64%. These values are reasonable. However, the standard deviation for the leverage ratio is somewhat large in the simulated data, while the standard deviation of the dividend–payout ratio is somewhat low in the simulated data, and all simulated financial variables (except for the returns) are rather persistent too.

4.3.2. Varying equity premium

In our model, the dependence of equity returns on the leverage ratio shows that low prices should predict high returns; if leverage as measured by the price–total assets ratio is low, expected returns are high. Various authors have stressed the importance of incorporating a varying equity premium in asset pricing models to match the stylized facts of stock returns (Campbell & Cochrane, 1999; Cochrane, 2011, e.g. see). The question remains whether the variation in conditional expected returns (and thus the slope of the conditional security market line) generated by the model matches the data. We check this by regressing returns on our main predictor variable, the inverse leverage ratio, (K/P). Several other variables that are deflated with the stock price, e.g. dividend–price (D/P) ratios, or earnings-price (E/P) ratios have also been shown to predict expected returns, in particular for longer horizons (e.g. see Campbell & Shiller, 1988; Cochrane, 2008; Fama & French, 1988). These findings are in line with our return

Table 3
Equity premium prediction regressions for various horizons.
Source: NIPA tables for consumption growth (non-durables and services) scaled by population and the consumer price index (CPI). Shiller for aggregate dividends, aggregate earnings, and stock returns. We take the annualized yield on 3-month T-bills as a proxy for the risk-free rate. Leverage we have computed by the Flow of Funds accounts, Federal Reserve. The stock returns, risk-free rate, and dividend growth rate have been deflated by the CPI as well.

U.S. 1952–2019						
	K/P		D/P		E/P	
Horizon (k)	Coeff.	R^2	Coeff.	R^2	Coeff.	R^2
1	0.09	0.02	3.33	0.05	1.02	0.03
3	0.17	0.03	7.73	0.08	2.28	0.03
5	0.48	0.09	18.94	0.11	3.67	0.04
7	0.88	0.17	23.45	0.18	6.92	0.08
Simulated Data						
	K/P		D/P		E/P	
Horizon (k)	Coeff.	R^2	Coeff.	R^2	Coeff.	R^2
1	0.08	0.05	1.64	0.03	0.91	0.04
3	0.29	0.15	5.97	0.09	3.31	0.10
5	0.64	0.27	13.26	0.16	7.36	0.17
7	1.10	0.37	23.27	0.23	12.86	0.25

This table shows results for regressions of the form $R_{t+h} = c_0 + c_1 X_t + \epsilon_{t+k}$, where R_{t+h} is the cumulative cum-dividend return on equity at time t for a horizon of h years, and X_t is either the dividend–price ratio (D/P), earnings–price ratio (E/P), or total assets–total equity ratio (K/P) at time t; c_0 and c_1 are estimated coefficients and ϵ_{t+h} is the error term. The table reports the estimated coefficients c_1 and the R^2's of the regression.

process. If D/P or E/P ratios proxy for the total assets–price ratio in our return process, they should predict conditional expected returns as well.

We run standard return regressions for various horizons, as is common in the return predictability literature, to determine how much variation there is in the slope of the conditional security market line. We compare the regression results based on simulated data with those based on historical data. Table 3 reports long-horizon return regressions and shows that all "deflated price" ratios are able to forecast stock returns, both in the historical and simulated data. Indeed, the estimated coefficients are in the same ballpark and, more importantly, together with the R^2s they exhibit an upward-sloping term structure.

The model captures the dynamics of the conditional security market line fairly well since the estimated coefficients of the return predictions and historical moments of the predictors are in the same ballpark. We do indeed replicate an upward-sloping term structure for both the regression coefficients as well as the R^2s. The R^2s for the simulated data are somewhat higher than in the real data, especially at the longer horizons. We attribute to the fact that in the simulated data, both the dividend–price ratio as well as the price–earnings ratio are more persistent (cf. the autocorrelations in Table 2).

5. Conclusion

We present a continuous-time stochastic Ramsey model with firms that are heterogeneous in their target leverage ratio. The model is able to generate various established patterns in equity returns in both the time series and the cross-section. The main innovation is that we assume an inelastic supply of equity and a target leverage ratio, and both are motivated by empirical evidence. Our specification of the firms' financing policies summarizes various mechanisms that drive the stylized financing decisions. We study the qualitative implications of the financing decisions by coupling them to the process of excess returns. To that end, we propose a new parameter restriction that is very flexible. We solve for equilibrium prices and characterize the resulting price processes.

Our model is able to generate various observed patterns in equity returns. Variation in the equity premium arises endogenously in our model, and our pricing equation for excess returns provides a unifying framework for the literature on return predictability, the cross-section of average stock returns, and volatility clustering. The model contains only a few time-invariant parameters and yet is able to endogenously generate phenomena such as a varying risk premium, time-varying CAPM betas, volatility clustering, and return predictability. We show how the conventional security market line should incorporate changes in the equity-premium and time-varying betas. Many of these properties can be linked to the role the leverage ratio plays in the conditional mean and volatility of stock returns. Indeed, the pivotal role of the leverage ratio arises due to market-clearing effects. The model explains why various researchers find that adjusting for leverage improves the cross-sectional fit of the Capital Asset Pricing Model.

Declaration of competing interest

None.

Appendix A. Derivation of equilibrium

Since there are no frictions and externalities in the model, the first welfare theorem holds and we can calculate the centrally planned solution. The objective is to maximize the lifetime utility of the representative investor:

$$U(t) = \mathbb{E}_t \left[\int_t^{\infty} e^{-\rho(s-t)} \frac{c(s)^{1-\tau} - 1}{1-\tau} ds \right], \tag{34}$$

subject to $\sum_i dK_i(t) = \left[\sum_i \left(F(K_i, L_i) - \delta_i K_i(t)\right) - c(t)\right] dt + \sum_i \sigma_i K_i(t) dz_i(t)$ and $\sum_i L_i = 1$. Because every firm faces the same technology and there are no dynamics for L_i, each firm adopts the same capital–labor ratio, i.e. $\forall i, j : \frac{K_i}{L_i} = \frac{K_j}{L_j}$. Consequently, we can easily aggregate individual firm production to aggregate production and write total output, Y, as follows:

$$Y = \sum_i Y_i = \left(\sum_i K_i\right)^{\alpha} (A(t) \sum_i L_i)^{1-\alpha}. \tag{35}$$

We define total wealth $K = \sum_i K_i$, and so we can write aggregate output as $Y = K^{\alpha}(A(t)\sum_i L_i)^{1-\alpha}$. To remove the time-dependency of the system, we write all variables in effective labor terms; $k = K/AL$, $\tilde{c} = c/AL$, and $y = Y/AL = f(k)$. Since $K_i/AL_i = K/AL = k, \forall i$, and $L = \sum_i L_i = 1$, we have $L_i = K_i/K$. Consequently, we can write total output per effective labor unit as $y = k^{\alpha} \frac{A \sum_i L_i}{AL} = \boldsymbol{a}'\boldsymbol{\iota}(k)^{\alpha}$, where we have defined portfolio weights $\boldsymbol{a} = \boldsymbol{K}/K$, the elements of the vector $\boldsymbol{K}$ are K_i, and $\boldsymbol{\iota}$ is a vector of ones. Since all firms adopt the same capital/labor ratio, we can reduce the system to one state variable, k, which evolves according to: $dk = (\boldsymbol{a}'\boldsymbol{\iota}(k^{\alpha} - \mu_A k) - \boldsymbol{a}'\boldsymbol{\delta}k - \tilde{c})dt + k\boldsymbol{a}'\boldsymbol{\sigma}d\boldsymbol{z}$, where $\boldsymbol{\delta}$ is a vector containing the elements δ_i, $\boldsymbol{\sigma}$ is a diagonal matrix with the σ_i's associated with the dz_i's, and the elements of the vector $d\boldsymbol{z}$ are dz_i.

Because maximizing expected utility with respect to c is equivalent to maximizing expected utility with respect to $\tilde{c}$, though using a discount factor equal to $\rho + (\tau - 1)\mu_A$, the Hamilton–Jacobi–Bellman (HJB) equation is:

$$0 = \max_{\tilde{c}, \boldsymbol{a}} u(\tilde{c}) - (\rho + (\tau - 1)\mu_A)V + V_k(\boldsymbol{a}'\boldsymbol{\iota}(k^{\alpha} - \mu_A k) - \boldsymbol{a}'\boldsymbol{\delta}k - \tilde{c}) + \frac{1}{2}\boldsymbol{a}'\boldsymbol{\Sigma}\boldsymbol{a}V_{kk}k^2, \tag{36}$$

subject to $\boldsymbol{a}'\boldsymbol{\iota} = 1$, where $\boldsymbol{\Sigma}$ is a symmetric matrix whose elements i, j are equal to $\rho_{ij}\sigma_i\sigma_j$. The first-order conditions for optimality are $u'(\tilde{c}) = V_k$ and

$$\boldsymbol{h} := \left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\delta} - \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\delta}}{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}\right) \tag{37a}$$

$$\boldsymbol{a} := \frac{V_k}{V_{kk}k}\boldsymbol{h} + \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}. \tag{37b}$$

Substituting the first-order conditions into the HJB equation gives:

$$\begin{aligned} 0 = {} & \frac{\tau V_k^{1-\frac{1}{\tau}}}{1-\tau} - (\rho + (\tau - 1)\mu_A)V + V_k(k^{\alpha} - \mu_A k) - \boldsymbol{h}'\boldsymbol{\delta}\frac{V_k^2}{V_{kk}} - \frac{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\delta}}{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{1}\boldsymbol{\iota}}V_k k \\ & + \frac{1}{2}\boldsymbol{h}'\boldsymbol{\Sigma}\boldsymbol{h}\frac{V_k^2}{V_{kk}} + \frac{\boldsymbol{h}'\boldsymbol{\iota}}{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}V_k k + \frac{1}{2}\frac{1}{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}V_{kk}k^2. \end{aligned}$$

We use the guess-and-verify method to solve the HJB Equation. We propose the solution $V(k) = Ck^{1-\tau\alpha}$, with C a coefficient to be determined. Substituting the proposed solution in the HJB gives:

$$\begin{aligned} & \frac{\tau(C(1-\tau\alpha))^{1-\frac{1}{\tau}}k^{\alpha-\tau\alpha}}{1-\tau} - (\rho + (\tau - 1)\mu_A)Ck^{1-\tau\alpha} + C(1-\tau\alpha)k^{\alpha-\tau\alpha} \\ & + \left[\left(\frac{1}{\tau\alpha}\boldsymbol{h}' - \frac{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}}{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}\right)\boldsymbol{\delta} - \left(\frac{1}{2\tau\alpha}\boldsymbol{h}'\boldsymbol{\Sigma}\boldsymbol{h} - \frac{\boldsymbol{h}'\boldsymbol{\iota} - \tau\alpha/2}{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}\right)\right]C(1-\tau\alpha)k^{1-\tau\alpha} = 0. \end{aligned} \tag{38}$$

We rearrange this equation:

$$\begin{aligned} & \left(\frac{\tau(C(1-\tau\alpha))^{1-\frac{1}{\tau}}}{1-\tau} + C(1-\tau\alpha)\right)k^{\alpha-\tau\alpha} \\ & + Ck^{1-\tau\alpha}\left(-(\rho + \tau(1-\alpha)\mu_A) + (1-\tau\alpha)B\right) = 0, \end{aligned} \tag{39}$$

where

$$B = \left[\left(\frac{1}{\tau\alpha}\boldsymbol{h}' - \frac{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}}{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}\right)\boldsymbol{\delta} - \left(\frac{1}{2\tau\alpha}\boldsymbol{h}'\boldsymbol{\Sigma}\boldsymbol{h} - \frac{\boldsymbol{h}'\boldsymbol{\iota} - \tau\alpha/2}{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}}\right)\right].$$

If we impose the restriction $\rho = (\tau\alpha - 1)B - \tau(1-\alpha)\mu_A$, we see that the second term is equal to zero.

The HJB is solved by $C = \frac{((\tau-1)/\tau)^{-\tau}}{\tau\alpha - 1}$. Subsequently, the first-order condition of consumption, $u'(\tilde{c}) = V_k$, implies that optimal consumption is equal to $c = \frac{\tau-1}{\tau}Y$. We note that the solution is sensible if and only if $C > 0$ and $\forall t : 0 < c(t) < Y(t)$. The first restriction requires that $\tau > 1$ and $\tau\alpha - 1 > 0$, and the second restriction requires that $\tau > 1$. Since $\alpha \in (0, 1)$, it is sufficient to impose that $\tau > \alpha^{-1}$. Substituting the solution for the value function in the first-order condition (37) gives the relative capital stocks $\boldsymbol{a}$ in the text.

Appendix B. Derivation of the evolution of market portfolio weights

By Itô's lemma, we know that the portfolio weight $w_i^P = P_i/P$ evolves according to:

$$dw_i^P = w_i^P\frac{dP_i}{P_i} - w_i^P\frac{dP}{P} + w_i^P\left(\frac{dP}{P}\frac{dP}{P}\right) - \left(\frac{dP_i}{P}\frac{dP}{P}\right). \tag{40}$$

In a dividend-free period, we have

$$\frac{dP_i}{P_i} = dR_i^e + rdt = \tau\alpha\sigma^2 b_i \frac{K_i}{P_i} dt + \sigma_i \frac{K_i}{P_i} dz_i + rdt, \tag{41}$$

for individual stocks and

$$\frac{dP}{P} = dR^e + rdt = \tau\alpha\sigma^2 \frac{K}{P} dt + \sigma \frac{K}{P} dz + rdt. \tag{42}$$

for the market portfolio. Using these last two equations we can write:

$$dw_i^P = -w_i^P \tau\alpha\sigma^2 \left(\frac{K}{P} - \frac{K_i}{P_i} b_i\right) dt + w_i^P \left(\frac{K}{P}\right)^2 \sigma^2 dt - \frac{K_i}{P}\frac{K}{P} \underbrace{\sum_j w_j \rho_{ij} \sigma_i \sigma_j}_{b_i\sigma^2} dt + w_i^P \left(\frac{K_i}{P_i}\sigma_i dz_i - \frac{K}{P}\sigma dz\right) \tag{43}$$

$$= -\sigma^2 \left(\tau\alpha\left(w_i^p - w_i b_i\right) + w_i b_i \frac{K}{P} - w_i^P \frac{K}{P}\right) \frac{K}{P} dt + w_i^P \left(\frac{K_i}{P_i}\sigma_i dz_i - \frac{K}{P}\sigma dz\right) \tag{44}$$

$$= -\sigma^2 \left(\tau\alpha - \frac{K}{P}\right)\left(w_i^P - w_i b_i\right) \frac{K}{P} dt + w_i^P \left(\frac{K_i}{P_i}\sigma_i dz_i - \frac{K}{P}\sigma dz\right). \tag{45}$$

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