Singular Switched Systems in Discrete Time: Solvability, Observability, and Reachability Notions

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DOI:
10.33612/diss.812183357

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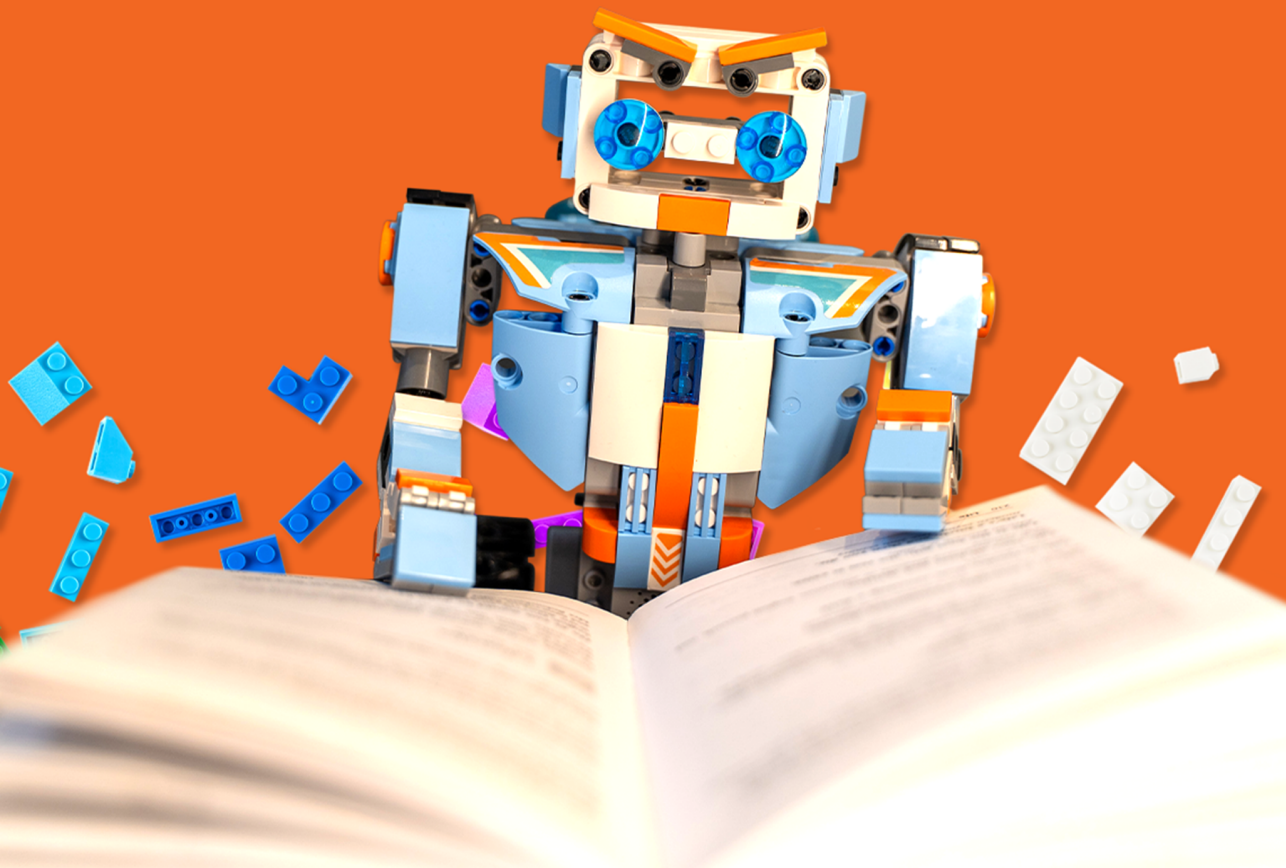

Singular Switched Systems in Discrete Time: Solvability, Observability, and Reachability Notions

by
Sutrisno

This work has been carried out at the Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence, Faculty of Science and Engineering, University of Groningen, The Netherlands.

This work has been completed in partial fulfillment of the requirements of the Dutch Institute of Systems and Control (DISC) for graduate study.

The work presented in this dissertation is supported by Universitas Diponegoro, Semarang, Indonesia. Sutrisno received PhD scholarship from Universitas Diponegoro.

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PhD Thesis University of Groningen

ISBN: 978-94-93353-32-9

university of groningen

Singular Switched Systems in Discrete Time: Solvability, Observability, and Reachability Notions

PhD Thesis

to obtain the degree of PhD
at the University of Groningen
on the authority of the
Rector Magnificus Prof. J.M.A. Scherpen
and in accordance with
the decision by the College of Deans.

This thesis will be defended in public on
Thursday 16 November 2023 at 9.00 hours

by
Sutrisno

born on 1 September 1986
in Cilacap, Central Java, Indonesia

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to my parents, my wife Nisa, and my son Raffasya

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Acknowledgments

Embarking on the journey to obtain a PhD has been an extraordinary experience, and achieving this milestone would not have been possible without the invaluable assistance and encouragement from numerous individuals. Therefore, I would like to seize this chance to convey my heartfelt thanks, deep appreciation, and sincere gratitude to everyone who has inspired, supported, and assisted me throughout this endeavor.

First, I would like to express my gratitude to my first supervisor Prof. Stephan Trenn. I thank him for his endless support, supervision, and guidance during my studies and for his patience in helping me to perfect my mathematical thinking skills. His feedback played a crucial role in enhancing and honing my research, and the findings presented in my thesis would have been unattainable without his guidance, direction, and supervision.

Second, I would like to thank Prof. Bayu Jayawardhana, my second supervisor, for guiding me at various moments in these last four years. I also would like to thank him for inspiring me and enlightening my knowledge about nonlinear system analysis. The input he provided was instrumental in refining and improving my research, especially for those that are related to nonlinear parts, and the discoveries presented in my thesis would have been beyond reach without his advice and guidance.

I would also like to express my gratitude to Prof. Do Duc Thuan, Prof. Nima Monshizadeh Naini, and Prof. Zhendong Sun for assessing my thesis and for providing me with valuable comments and suggestions to improve my thesis. Their feedback was of immense value in enhancing my thesis.

Next, I would like to thank my colleagues in the SCO (formerly SCAA) research group, (current and former) professors, Postdocs, Guests, and PhD students: Alden, Arjan, Bart, Harry, Henk, Kanat, Juan, Anne-Men (also thanks for the dutch translation of my thesis summary), Amir, Armin, Atiyeh, Azka, Brayan, Cheng, Di, Hamin, Huayuan, Jaap, Jiaming, Jiajia, Jiaming, Jiwei, Junjie, Koorosh, Marieke, Mark, Melvin, Paul, Radu, Sumon, Teke, Yahao, Yanxin, Yongzhang, and Zepeng. SCO is a wonderful group that has an outstanding environment for conducting mathematical research, especially for me

during the past four years. My gratitude also goes to all secretaries of the Bernoulli Institute for their assistance regarding all kinds of office supplies and administrations.

My gratitude also goes to Mechatronics and Nonlinear Control members: Agung, Arijit, Bahadir, Hao, Jin, Ridho, Santiago, Yulyan, Zaki, Zhiyuan, and all other members of ENTEG. The bi-weekly meetings in this group gave me more insights into nonlinear systems both in theory and applications.

I would like to extend my heartfelt appreciation to my colleagues from the Mathematics Department, Universitas Diponegoro, my home university, for their invaluable contributions and unwavering teamwork throughout my career. Your dedication and collaborative spirit have significantly enriched our collective efforts, leading to our successful outcomes and resulting in the advancement of our department. And, I am truly grateful for the privilege of working alongside such dedicated individuals. My appreciation also goes to Universitas Diponegoro for providing me with a scholarship to pursue my PhD, as well as the team from *BPSDM* for their help with a range of administrative tasks both prior to and during my journey toward obtaining my PhD.

Many thanks also go to my badminton playmates at GSBC Amor, especially players from Team 6 in the Federal Competition 2021-2022 and Team 5 in the Federal Competition 2022-2023, and our coaches. Training at Amor was wonderful, and playing competition and tournament matches was enjoyable. Also, thanks to members of *BL Garis Keras* and the CSC badminton group. All of you made my badminton time in Groningen, as Oma Gill usually says, sensational.

I would also like to express my gratitude to Indonesian communities in Groningen: deGromiest (*Komunitas Muslim Indonesia di Groningen)*, PPIG (*Perhimpunan Pelajar Indonesia Groningen*), and Diaspora Indonesia. Every event, moment, occasion, and experience within the communities I participated in was consistently pleasant. Moreover, the communities consistently added cheerfulness and enjoyment to my life in Groningen.

Lastly, I would like to express my very profound gratitude to my parents, my wife, Nisa, and my son, Raffasya, for their never-ending, unwavering, and unfailing support and continuous encouragement throughout my life, as well as for infusing vibrancy into my life. You are all my guardian angels, and I also thank you for bringing joy and liveliness into my world.

Sutrisno
Groningen
October 1, 2023

List of Publications

Journal Papers

1. **Sutrisno** and Stephan Trenn, "Nonlinear Switched Singular Systems in Discrete Time: The One-step Map and Stability Under Arbitrary Switching Signals," *European Journal of Control*, in press. https://doi.org/10.1016/j.ejcon.2023.100852

2. **Sutrisno** and Stephan Trenn, "Switched linear singular systems in discrete time: solution theory and observability notions," *submitted to journal*.

3. **Sutrisno** and Stephan Trenn, "Inhomogeneous singular linear switched systems in discrete time: solvability and reachability for restricted switching signals", *under preparation*.

Conference Proceedings

4. **Sutrisno** and Stephan Trenn, "Observability of Singular Linear Switched Systems in Discrete Time: Single Switch Case," in *Proc. European Control Conference (ECC)*, 2021. https://doi.org/10.23919/ECC54610.2021.9654844

5. **Sutrisno** and Stephan Trenn, "Observability and Determinability Characterizations for Linear Switched Systems in Discrete Time," in *Proc. 60th IEEE Conference on Decision and Control (CDC)*, 2021. https://doi.org/10.1109/CDC45484.2021.9682894

6. **Sutrisno** and Stephan Trenn, "Reachability and Controllability Characterizations for Linear Switched Systems in Discrete Time: A Geometric Approach," in *Proc. European Control Conference (ECC)*, 2022. https://doi.org/10.23919/ECC57647.2023.10178124

7. **Sutrisno** and Stephan Trenn, "Discrete-time Singular Linear (Switched) Systems: Solvability, Reachability, and Controllability Characterizations," in *Proc. 62nd IEEE Conference on Decision and Control (CDC)*, 2023. (*to appear)*.

8. **Sutrisno**, Hao Yin, Stephan Trenn and Bayu Jayawardhana, "Nonlinear singular switched systems in discrete-time: solution theory and incremental stability under restricted switching signals," in *Proc. 62nd IEEE Conference on Decision and Control (CDC)*, 2023. (*to appear)*.

Peer Reviewed Extended Abstracts

9. **Sutrisno**, Hao Yin, Stephan Trenn, and Bayu Jayawardhana, "Nonlinear singular switched systems in discrete-time: solution theory and (incremental) stability under fixed switching signals," in *42nd Benelux meeting on Systems and Control 2023*, Elspeet, Netherlands, 21-23 March 2023.

10. Md Sumon Hossain, **Sutrisno**, Stephan Trenn, "A time-varying approach for model reduction of singular linear switched systems in discrete time," in *25th International Symposium on Mathematical Theory of Networks and Systems (MTNS)*, Bayreuth, Germany, 12-16 September 2022.

11. **Sutrisno** and Stephan Trenn, "The one-step function for discrete-time nonlinear switched singular systems," in *41st Benelux meeting on Systems and Control 2022*, Brussels, Belgium, 5-7 July 2022.

12. **Sutrisno** and Stephan Trenn, "Observability and Determinability of Discrete-Time Switched Linear Singular Systems: Multiple Switches Case," in *Benelux Workshop on Systems and Control*, Rotterdam, Netherlands, 29 June 2021.

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List of Notations

Basic sets

$\mathbb{N}$	set of natural numbers $\{0, 1, 2, \ldots\}$
$\mathbb{R}$	set of all real numbers
$\mathbb{R}^n$	n-dimensional linear space
$\mathbb{R}^{n\times m}$	space of $n \times m$ real constant matrices

Basic operators

$\in$	element of
$\subseteq$, $\subsetneq$	subset, proper subset
$\cap$	intersection
$\cup$	union
$\|\cdot\|$	Euclidian norm

Matrices and subspaces

I_n or I	$n \times n$ identity matrix or identity matrix with appropriate dimension
$M^\top$	transpose of matrix M
M^+	generalized inverse or Moore-Penrose pseudoinverse of matrix M
$\operatorname{rank} M$	rank of matrix M
$\operatorname{im} M$	image or range of the linear map $M : \mathbb{R}^n \to \mathbb{R}^n$, i.e., $\operatorname{im} M = \{\ Mx \mid x \in \mathbb{R}^n\ \}$
$\ker M$	kernel of the linear map $M : \mathbb{R}^n \to \mathbb{R}^n$, i.e., $\ker M = \{\ x \in \mathbb{R}^n \mid Mx = 0\ \}$
$\dim \mathcal{V}$	dimension of the subspace $\mathcal{V}$
$M^{-1}\mathcal{V}$	preimage or inverse image of a subspace $\mathcal{V} \subsetneq \mathbb{R}^n$ under the linear map $M : \mathbb{R}^n \to \mathbb{R}^n$, i.e., $M^{-1}\mathcal{V} = \{\ x \in \mathbb{R}^n \mid Mx \in \mathcal{V}\ \}$
$\mathcal{V} + \mathcal{W}$	sum of subspaces $\mathcal{V}, \mathcal{W} \in \mathbb{R}^n$, i.e., $\mathcal{V} + \mathcal{W} = \{\ x \in \mathbb{R}^n \mid x = v + w, v \in \mathcal{V}, w \in \mathcal{W}\ \}$
$\mathcal{V} \oplus \mathcal{W}$	direct sum of subspaces $\mathcal{V}, \mathcal{W} \in \mathbb{R}^n$, i.e., $\mathcal{V} \oplus \mathcal{W} = \mathcal{V} + \mathcal{W}$ and $\mathcal{V} \cap \mathcal{W} = \{0\}$

Dynamical systems

(E, A)	matrix pair of matrices E and A, shorthand notation for a homogeneous singular linear system of the form $Ex(k+1) = Ax(k)$
(E, A, B)	matrix triplet of matrices E, A, and B, shorthand notation for an inhomogeneous singular linear system of the form $Ex(k+1) = Ax(k) + Bu(k)$
$(E_\sigma, A_\sigma, B_\sigma)$	shorthand notation for an inhomogeneous singular linear switched system of the form $E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k)$

Abbrevations

QWF	quasi-Weierstrass form
DTLS	discrete-time linear system
LSS	linear switched systems
SLS	singular linear systems
SLSS	singular linear switched systems
SNS	singular nonlinear systems
SNSS	singular nonlinear switched systems

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Chapter 1 Introduction

Contents

1.1 Background and Motivation

Physical processes can be modeled as dynamical systems which describe their behavior. Dynamical systems for different physical systems generally have different characteristics which result in many classes of dynamical systems including discrete-time linear and nonlinear (ordinary/normal) systems.

Algebraic constraints can appear on some dynamical systems such as the Leontief economic model, which describes the growth of the inter-industry economy proposed by W. Leontief in the early 1950s [1]. Such systems are called singular systems. In existing studies, some other terms have also been used to call this system class namely descriptor systems, difference algebraic equations, generalized systems, implicit systems, strong coupling systems, and incomplete state systems [2, 3].

Furthermore, some physical systems can switch from one dynamical system equation to another one due to, for instance, structural changes during operation; its dynamical system is commonly referred to as a switched system, and each dynamical system equation involved in the system is called mode or subsystem. The signal ruling the switching between modes is referred to as the switching signal.

The study in this thesis is mainly concerned with the geometric analysis utilized to investigate the solvability of singular (switched) systems in discrete time. Some new solvability notions based on the consideration of switching

signal classes are established. For solvable singular systems, the so-called surrogate systems–ordinary systems that have equivalent behavior are established. By utilizing these surrogate systems, further analyses including observability, determinability, reachability, controllability, and stability are also studied.

1.2 Related Studies, Research Gaps, and Contribution

To be precise, the literature review of existing studies, research gaps, and the corresponding contributions provided in this thesis are expressed separately for each of the following three system classes.

(Ordinary) Linear Switched Systems

Solvability for linear switched systems has been fully characterized in existing studies. Every system in this system class has always a unique solution for arbitrary initial states, and the solution at any time instant can be calculated by iterating the equation over time instants [4, p. 27]. Additionally, fundamental properties including observability, reachability, controllability, and stability have been extensively studied.

The observability, which concerns the state estimation from output measurements, is commonly characterized in a Gramian form by checking the kernel of the corresponding Gramian observability matrix which is a huge matrix and is not a computational-friendly form (see e.g. [5, 6]).

Reachability analysis studies whether a state can be reached from an initial state whereas controllability studies whether an initial state can be brought to zero. Reachability was initially studied in [7] in which the set of points reachable from the origin was investigated; this set was called the controllable set in this study, and it has been pointed out that this set is a subspace under some hypothesis. This study was extended in [8] where the reachable set was formulated as the union of its maximal components; some structural properties such as the bound of the number of time steps necessary to reach a state were also presented. Nevertheless, no necessary and sufficient condition for reachability characterization was formulated in those reports. Another study related to reachability was reported in [9] for the case that a zero-nonzero structure of the system's matrices is known, which is rather more restrictive. Necessary and sufficient conditions for reachability as well as controllability have been proposed in [10] using geometric approaches. The reachable set was presented as a subspace derived from a calculation using the system's matrices obtained from each time step for the whole time interval of the characterization, which may demand high computational resources.

Those existing characterizations for observability, reachability, and controllability were developed under arbitrary switching signals. In this thesis, the characterizations are formulated with fixed switching signals or with fixed mode sequences (with arbitrary switching times). The condition for fixed switching signals can basically be extracted from the condition under arbitrary switching signals, however, they can be simplified by utilizing some algebraic properties; this is the main novel aspect concerning the observability, reachability, and controllability characterizations studied in this thesis. Besides those properties, the determinability, which concerns the final state estimation from output measurements over a finite time interval, is also studied in this thesis. Apart from studies in continuous time, no one has investigated this notion for systems in discrete time.

Furthermore, an important point in discrete-time systems is that, in general, it cannot possess those four properties within too short observation time and hence a sufficiently large dwell time for each mode is needed. However, this is not enough to preserve these properties if the switching time(s) is changed and thus it becomes dependent on the switching time(s). In other words, in general, observability, reachability, controllability, and determinability depend on the switching times, meaning that a system may not possess those properties for a certain switching signal, however, under some other switching signal, it may possess them. The dependency of those properties on (multiple) switching times seems not to be discussed yet in any existing study and therefore is a novel aspect studied in this thesis. This part of the study will answer the question of when the characterization of those properties stays the same if the switching times are changed (the mode sequence is still the same). It is important if it is known whether the system always has the same characterization result under different switching times. For instance, if it is known whether the system is always reachable or unreachable for any choice of switching times then only one characterization with certain switching times is needed to perform.

Linear Singular Systems

The study of (non-switched) singular linear systems in discrete time case initially appeared in [11] which investigated the solvability analysis whereas some primary analysis and control methods for this system class have been widely addressed in numerous studies, see e.g. [12, 13, 14]. For solvability, under the regularity assumption of the system's matrices, this system class is always well-posed, and the solution can be derived via a state transformation. A recent study in [15] discovered that under some assumptions, an ordinary linear system that has the same solutions with a corresponding singular linear system can be found via a projector and a state transformation which transforms the system

into the so-called quasi Weierstrass form. In this thesis, a new approach is introduced to finding such equivalent ordinary systems for singular systems by utilizing some geometric properties of subspaces including properties of generalized inverse matrices. In this new approach, no state transformation is needed so that the system's matrices are directly utilized, and furthermore, it makes the proof more straightforward.

Meanwhile, solvability for singular linear switched systems has been investigated also in [15] under arbitrary switching signals. The corresponding equivalent ordinary linear switched systems for solvable singular systems have also been established in that study, also via a state transformation that transforms the system into the quasi Weierstrass form. In this thesis, the solvability of this system class is studied under fixed switching signals, and the corresponding equivalent ordinary switched systems for solvable singular systems are also established. Again, it uses geometric properties of subspaces including properties of generalized inverse matrices, and no state transformation is needed. Furthermore, by utilizing those equivalent ordinary systems, the observability, determinability, reachability, and controllability of the singular switched systems are also studied in this thesis.

In particular, studies about the stability analysis for this system class have been extensively studied, see e.g. [16, 17, 18, 19, 20, 21, 22, 23], and some control methods have been proposed for some purposes such as trajectory tracking via iterative learning [24] and filtering & state feedback [25]. Other related studies observed the reachable set estimation [26], and admissibility property [27]. All of those studies utilized the quasi-Weierstrass form of the singular systems. The equivalent ordinary systems established in this thesis can be utilized to study further properties such as stability, and can also be exploited for control designs. However, those topics are out of the scope of this thesis and can be considered as future research directions.

Nonlinear Singular Systems

Investigation for nonlinear singular systems was initiated by D.G. Luenberger in [28] in which the set of solutions of this system class is, in general, a manifold. Furthermore, a local solvability notion was introduced together with the corresponding sufficient condition for the system to be solvable under the full rank assumption on the coefficient matrix of the linearized system. Nevertheless, there was no establishment regarding an equivalent ordinary system for solvable systems.

Singular systems containing both linear and nonlinear terms have also been studied in some reports. In [29], a sufficient condition has been proposed for the existence of a unique solution. Meanwhile, in [30, 31], the stability analysis has

been carried out. However, in existing studies, the nonlinearity was considered and treated as a disturbance-like term so that the solution theory for singular linear systems still applies.

In this thesis, the solution theory is then further extended for nonlinear singular systems both for systems without switching and with switching. Similar techniques used in the study for linear singular systems are utilized here for studying solvability and for establishing the corresponding equivalent ordinary nonlinear systems. Furthermore, by utilizing those equivalent ordinary systems, the study continued with Lyapunov and incremental stability analyses, which are intriguing in the nonlinear system field.

1.3 Thesis Outline

As preliminary contents, in Chapter 2 (Preliminaries), theories related to the regularity of matrix pairs and their corresponding quasi-Weierstrass forms are recalled. Some fundamental theories for linear dynamical systems in discrete time are also recalled here. Moreover, the switching signal and system classes considered in this thesis are introduced in the last part of the chapter.

The main content of this thesis is divided into three parts based on the system class. Part I focuses on ordinary linear switched systems, and contains Chapter 3 (Linear Switched Systems). In this chapter, the characterizations for observability, determinability, reachability, and controllability properties are presented. Furthermore, studies regarding the dependency of those properties on the switching times are also presented in this chapter.

Part II focuses on singular linear (switched) systems and includes the next three chapters. Chapter 4 (Solvability) contains the results for the solvability characterizations, which are divided into two sections: one section for homogeneous systems and another section for inhomogeneous systems. Each section incorporates nonswitched and switched cases. In this arrangement, the solution theory for each type of system can be easier to follow, and the generalization process from a specific case to a more general case can also be compared and seen directly from their proofs and observations. In Chapter 5 (Observability and Determinability), results about the observability and determinability characterizations for linear singular systems are presented. It is also divided into two sections: nonswitched case and switched case. Furthermore, for the switched case, the characterization is first considered only with single switching signals and later with general switching signals. In the single switch case, the characterization is more straightforward since the indices of the modes are put directly into the condition. In this way, it is easier to understand what is happening in the characterization. The characterization for the multiple switch

case is then developed by generalizing results from the single switch case. In the last part of this chapter, the study results of the dependency of observability and determinability on switching times are presented. With a similar flow as in Chapter 5, Chapter 6 (Reachability and Controllability) contains results for the reachability and controllability characterizations. The dependency of reachability and controllability on switching times are also discussed in this chapter.

Part II covers Chapter 7 (Singular Nonlinear (Switched) Systems) in which the solvability characterizations for nonlinear singular systems are presented both without switching and with switching; this is an extension of the study for linear systems. Furthermore, Lyapunov and incremental stability analyses are also investigated for this nonlinear system class.

The thesis is closed by Chapter 8 (Conclusions and Outlooks) in which the outcomes obtained in the study are summarized. Possible future research directions are also discussed at the end of the chapter to expose open problems inferred from this thesis.

Chapter 2 Preliminaries

Contents

In this chapter, some theoretical foundations, which will be used in the subsequent chapters, are presented. It includes regular matrix pairs , the quasi-Weierstrass form of regular matrix pairs, index-1 regular matrix pairs, and strictly index-1 matrix triplets. Basic analysis for linear time-invariant systems in discrete time will also be briefly discussed to provide some basic knowledge about linear systems in the discrete-time domain. This includes the explicit solution formula, observability, reachability, and controllability. The precise class of linear and nonlinear systems considered throughout the thesis will also be described together with three different classes of switching signals. In some parts of the analysis, the dwell-time notion will also be used; this will also be introduced at the end of this chapter.

2.1 Regular Matrix Pairs

Consider the matrix pair (E, A) with $E, A \in \mathbb{R}^{n\times n}$. The following definition provides the notion of regularity of a matrix pair, which plays a crucial role in studying the solvability of singular linear systems.

Definition 2.1 (Regular matrix pairs). The matrix pair (E, A), $E, A \in \mathbb{R}^{n\times n}$, is said to be **regular** if $\det(sE - A)$ is not the zero polynomial. $\Diamond$

The crucial role of the regularity property in singular linear systems is based

on the observation of the so-called Wong sequences together with their properties and the Quasi-Weierstrass form of the regular matrix pairs.

Let $A\mathcal{M}$ be the image of a set $\mathcal{M} \subseteq \mathbb{R}^n$ under $A \in \mathbb{R}^{n\times n}$, i.e., $A\mathcal{M} = \{ Ax \mid x \in \mathcal{M} \}$, and $A^{-1}\mathcal{M}$ be the pre-image of the set $\mathcal{M} \subseteq \mathbb{R}^n$ over A, i.e. $A^{-1}\mathcal{M} = \{ x \in \mathbb{R}^n \mid Ax \in \mathcal{M} \}$. Note that if A is invertible and $\mathcal{M}$ is a subspace with $\mathcal{M} = \operatorname{im} M$ for some matrix M of suitable size, then $A^{-1}\mathcal{M} = \operatorname{im} A^{-1}M$.

Definition 2.2 (Wong sequences). For a matrix pair (E, A), the sequences of subspaces

$$\mathcal{V}_0 := \mathbb{R}^n, \qquad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i),\ i = 0, 1, \ldots, \tag{2.1a}$$

$$\mathcal{W}_0 := \{0\}, \qquad \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i),\ i = 0, 1, \ldots \tag{2.1b}$$

are called Wong sequences. ◊

By its definition and properties of pre-images, images, and kernels, the following properties hold [32]:

$$\exists k^* \in \mathbb{N} : \quad \mathcal{V}_0 \supsetneq \mathcal{V}_1 \supsetneq \cdots \supsetneq \mathcal{V}_{k^*} = \mathcal{V}_{k^*+j} =: \mathcal{V}^* = A^{-1}(E\mathcal{V}^*) \supseteq \ker A \quad \forall j \in \mathbb{N},$$

$$\exists \ell^* \in \mathbb{N} : \mathcal{W}_0 \subseteq \ker E = \mathcal{W}_1 \subsetneq \cdots \mathcal{W}_{\ell^*} = \mathcal{W}_{\ell^*+j} =: \mathcal{W}^* = E^{-1}(A\mathcal{W}^*) \quad \forall j \in \mathbb{N},$$

$$A\mathcal{V}^* \subseteq E\mathcal{V}^* \text{ and } E\mathcal{W}^* \subseteq A\mathcal{W}^*.$$

In the following lemma, some basic properties related to dimension and intersection of subspaces in Wong sequences are presented.

Lemma 2.3 (Dimension and intersection properties of Wong sequences, [32]). For a regular matrix pair (E, A), the subspaces $\mathcal{V}_i$, $\mathcal{W}_i$, $\mathcal{V}^*$ and $\mathcal{W}^*$ of its corresponding Wong sequences satisfy

(i) $\dim \mathcal{V}_i + \dim \mathcal{W}_i = n \ \forall i \in \mathbb{N}$

(ii) $k^* = \ell^*$ and $\mathcal{V}^* \oplus \mathcal{W}^* = \mathbb{R}^n$,

(iii) $\ker E \cap \mathcal{V}^* = \{0\}$, $\ker A \cap \mathcal{W}^* = \{0\}$ and $\ker E \cap \ker A = \{0\}$. ◊

2.2 Quasi-Weierstrass Form

The subspaces in Wong sequences can then be used to transform a regular matrix pair (E, A) into the so-called Quasi-Weierstrass Form (QWF); this is presented in the following lemma.

Lemma 2.4 (QWF of regular matrix pairs, [32]). For any regular matrix pair (E, A), there exist invertible matrices $S, T \in \mathbb{R}^{n\times n}$ satisfying

$$(SET, SAT) = \left(\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix} \right) \tag{2.2}$$

where $N \in \mathbb{R}^{n_2 \times n_2}$ is a nilpotent with nilpotency index, $J \in \mathbb{R}^{n_1 \times n_2}$ with $n_1 + n_2 = n$, and such matrices are given by $S = [EV, AW]^{-1}$ and $T = [V, W]$ where $\operatorname{im} V = \mathcal{V}^*$ and $\operatorname{im} W = \mathcal{W}^*$ with k^*, $\mathcal{W}^*$, and $\mathcal{W}^*$ are as in Lemma 2.3. $\Diamond$

Matrix form (2.2) is called a Quasi-Weierstrass form of (E, A). Note that the nilpotency index of N is independent of the choices for V and W [32]. Now, the index of a regular matrix pair (E, A) is defined as the nilpotency index of N of its QWF. This leads to the following definition of index-1 regular matrix pairs.

Definition 2.5 (Index-1 matrix pair). A regular matrix pair (E, A) is called **index-1** if its corresponding matrix N as in (2.2) satisfies $N = 0$. $\Diamond$

In the case of index-1 i.e. $N = 0$, Lemma 2.4 together with the properties of Wong sequences in Lemma 2.3 yields the following corollary.

Corollary 2.6 (QWF of regular and index-1 matrix pairs, cf. [15]). A matrix pair (E, A) is regular and index-1 if and only if

$$\ker E \oplus \mathcal{S} = \mathbb{R}^n \tag{2.3}$$

where $\oplus$ denotes the direct sum. In this case, matrices S and T in Lemma 2.4 transform (E, A) into

$$(SET, SAT) = \left(\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix} \right). \tag{2.4}$$

Furthermore, the matrices $S = [V, W]$ and $T = [EV, AW]^{-1}$ can be chosen in such a way that

$$\operatorname{im} V = \mathcal{S} := A^{-1}(\operatorname{im} E) := \{ \ \xi \in \mathbb{R}^n \ | \ A\xi \in \operatorname{im} E \ \}, \tag{2.5a}$$

$$\operatorname{im} W = \ker E. \tag{2.5b}$$

In particular, for regular matrix pairs (E, A), the matrix pair is index-1 if and only if $\ker E \cap \mathcal{S} = \{0\}$. $\Diamond$

The corollary above is a direct consequence of the properties of Wong-sequences in which (2.3) is derived from Lemma 2.3, and QWF (2.4) and formula (2.5) are derived from Lemma 2.4 with $N = 0$. Moreover, note that for regular matrix pairs, the index-1 condition $\ker E \oplus \mathcal{S} = \mathbb{R}^n$ is equivalent to $\ker E \cap \mathcal{S} = \{0\}$ due to Lemma 2.3, i.e., for regular matrix pairs, $\ker E \cap \mathcal{S} = \{0\}$ implies index-1; the converse is also true, i.e., for regular matrix pairs, index-1 implies $\ker E \cap \mathcal{S} = \{0\}$. However, note that in general (the matrix pairs are possibly irregular), $\ker E \cap \mathcal{S} = \{0\}$ does not always imply regularity; this is illustrated by the matrix pair $(E, A) = \left(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \right)$ in which $\ker E \cap \mathcal{S} = \{0\}$ but it is not regular.

Remark 2.7. One should distinguish the term *index* that corresponds to a matrix pair (E, A) to the index of a single matrix M. The latter is defined to

be the smallest nonnegative integer k such that $\operatorname{rank} M^k = \operatorname{rank} M^{k+1}$. The index of M is only positive if M is singular, and in that case, the index of M is the maximal grade of 0-vectors of M or the maximal size of the Jordan block corresponding to the zero eigenvalue of M [33, Page 34-35]. Furthermore, it is easily seen that the index of M is equal to the index of the matrix pair (M, I). ◊

Consider now a matrix triplet (E, A, B), $E, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$. Assume (E, A) is regular. Then, by utilizing the QWF of (E, A) in (2.2), the corresponding QWF of the matrix triplet (E, A, B) is

$$(SET, SAT, SB) = \left(\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix}, \begin{bmatrix} B^J \\ B^N \end{bmatrix} \right) \tag{2.6}$$

where $B^J \in \mathbb{R}^{n_1\times m}$ and $B^N \in \mathbb{R}^{n_2\times m}$. The following defines the strictly index-1 notion of a matrix triplet.

Definition 2.8 (Strictly index-1 of a matrix triplet). A matrix triplet (E, A, B), $E, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ with (E, A) regular is called **strictly index-1** if (E, A) is index-1 and $B^N = 0$ where B^N is as in (2.6). ◊

This strictly index-1 notion, together with the nonstrict index-1 notion, will be used in the solvability theory of singular systems in the forthcoming Chapter 4. The following lemma characterizes the strictly index-1 notion of a matrix triplet.

Lemma 2.9 (Strictly index-1 condition). A matrix triplet (E, A, B), $E, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, (E, A) regular, is strictly index-1 if and only if either $\ker E \cap \widehat{\mathcal{S}} = \{0\}$ or $\ker E \oplus \widehat{\mathcal{S}} = \mathbb{R}^n$ where $\widehat{\mathcal{S}} := A^{-1}(\operatorname{im}[E, B]) = \{\xi \in \mathbb{R}^n : A\xi \in \operatorname{im}[E, B]\}$. ◊

Proof. **Part 1: the condition** $\ker E \cap \widehat{\mathcal{S}} = \{0\}$.
Sufficiency: Since $A^{-1}(\operatorname{im} E) \subseteq A^{-1}(\operatorname{im}[E, B])$ which implies $\mathcal{S} \subseteq \widehat{\mathcal{S}}$, $\ker E \cap \widehat{\mathcal{S}} = \{0\}$ implies $\ker E \cap \mathcal{S} = \{0\}$, i.e., (E, A) is index-1. Now, from the QWF representation (2.6), it is derived that

$$\ker E \cap \widehat{\mathcal{S}} \simeq (\{0\} \times \ker N) \cap (J^{-1}(\operatorname{im}[I_r, B^J]) \times \operatorname{im}[N, B^N]).$$

From (E, A) being index-1, it follows that $N = 0$. Hence,

$$\ker E \cap \widehat{\mathcal{S}} \simeq (\{0\} \times \mathbb{R}^{n_N}) \cap (\operatorname{im} J \times \operatorname{im} B^N), \tag{2.7}$$

and $\ker E \cap \widehat{\mathcal{S}} = \{0\}$ implies $B^N = 0$.
Necessity: For index-1 (E, A), $\ker E \cap \mathcal{S} = \{0\}$ holds. From (2.7), it follows that $B^N = 0$ implies $\ker E \cap \widehat{\mathcal{S}} = \{0\}$.
Part 2: the alternative condition $\ker E \oplus \widehat{\mathcal{S}} = \mathbb{R}^n$.
Now, it will be shown that the condition $\ker E \cap \widehat{\mathcal{S}} = \{0\}$ can be replaced by $\ker E \oplus \widehat{\mathcal{S}} = \mathbb{R}^n$. The sufficiency is clear since $\ker E \oplus \widehat{\mathcal{S}} = \mathbb{R}^n$ implies

$\ker E \cap \widehat{\mathcal{S}} = \{0\}$ and thus (E, A, B) is strictly index-1 (from Part 1 of the proof). For the necessity, the knowledge $\ker E \oplus \mathcal{S} = \mathbb{R}^n$ implies $\dim \ker E + \dim \mathcal{S} = n$ which yields $\dim \ker E + \dim \widehat{\mathcal{S}} = n$ (due to $\mathcal{S} \subseteq \widehat{\mathcal{S}}$). Thus, $\ker E \oplus \widehat{\mathcal{S}} = \mathbb{R}^n$. $\square$

This section is closed by the following remark which discusses the relationship between index-1 (E, A) and strictly index-1 (E, A, B).

Remark 2.10 (Relationship between index-1 and strictly index-1). By definition, strictly index-1 implies index-1 The converse is not always true; this is confirmed by the matrix triplet

$$(E, A, B) = \left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] \right)$$

with $\mathcal{S} = \operatorname{span}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\ker E = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. It is index-1 since $\ker E \cap \mathcal{S} = \{0\}$ but not strictly index-1 since $\widehat{\mathcal{S}} = \mathbb{R}^2$ and thus $\ker E \cap \widehat{\mathcal{S}} \neq \{0\}$. $\Diamond$

2.3 Discrete-time Linear Systems

Consider the linear time-invariant system class in discrete-time of the following general form represented by the matrix quadruplet (A, B, C, D) :

$$x(k+1) = Ax(k) + Bu(k) \tag{2.8a}$$

$$y(k) = Cx(k) + Du(k) \tag{2.8b}$$

where $k \in \mathbb{N}$ representing time instant/step, $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$, $m \in \mathbb{N}$ is the input, $y(k) \in \mathbb{R}^p$, $p \in \mathbb{N}$ is the output, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, and $D \in \mathbb{R}^{p \times m}$ are constant matrices. Solving system (2.8) is done by iterating the state equation over time steps $k = 0, 1, \ldots$,i.e.

$$\begin{aligned} x(1) &= Ax(0) + Bu(0) \\ x(2) &= Ax(1) + Bu(1) = A^2x(0) + ABu(0) + Bu(1) \\ &\vdots \\ x(k) &= A^k x(0) + \sum_{j=0}^{k-1} A^{k-j-1} Bu(j). \end{aligned}$$

By defining $A^k =: \psi(k)$ as the transition matrix satisfying $\psi(k+1) = A\psi(k)$ with $\psi(0) = I_n$, the solution can be rewritten as

$$x(k) = \psi(k)x(0) + \sum_{j=0}^{k-1} \psi(k-j-1)Bu(j). \tag{2.9}$$

The solution for the output y can then be derived by substituting the solution

x into the output equation (2.8b), i.e.

$$y(k) = C\psi(k)x(0) + C\sum_{j=0}^{k-1}\psi(k-j-1)Bu(j) + Du(k).$$

System (2.8) is said to be (completely) observable on the time domain $[0, K]$, $K \in \mathbb{N}$ if every initial state $x(0)$ can be determined from the observation of output $y(0)$, $y(1)$, ..., $y(K)$ and input signal $u(0), u(1), \ldots, u(K)$. The concept of observability is useful in solving the problem of reconstructing unmeasurable state variables [34]. Mathematically, observability means that different output sequences $(y(0), y(1), ..., y(K))$ correspond to different state sequences $(x(0), x(1), ..., x(K))$ under the same input signal. Due to linearity, the observability is independent of how the input influences the state dynamics; this can also be seen from the output equation (2.8b). Thus, only the matrix pair (A, C) or its corresponding system without inputs is considered in observability characterizations. From the output equations over the time domain $[0, K]$:

$$\begin{aligned} y(0) &= Cx(0) \\ y(1) &= CAx(0) \\ &\vdots \\ y(K) &= CA^K x(0) \end{aligned}$$

the output sequence $(y(0), ..., y(K))$ completely determines $x(0)$ if and only if $\operatorname{rank}\left[\,C^\top, A^\top C^\top, ..., (A^K)^\top C^\top\,\right] = n$; this is known as the Kalman observability criteria, and the matrix $\left[\,C^\top, A^\top C^\top, ..., (A^K)^\top C^\top\,\right]$ is known as the Kalman observability matrix on the time interval $[0, K]$, see e.g. [34].

System (2.8) is said to be (completely state) reachable on $[0, K]$ if any final state $x(K) \in \mathbb{R}^n$ can be reached from any initial state $x(0) \in \mathbb{R}^n$ with some input sequence $(u(0), u(1), ..., u(K-1))$. The solution of (2.8a) at $k = K$ can be written as

$$x(K) - A^K x(0) = [B, AB, ..., A^{K-1}B] \begin{bmatrix} u(K-1) \\ u(K-1) \\ \vdots \\ u(0) \end{bmatrix}. \tag{2.10}$$

Then, the system is observable on $[0, K]$ if and only if $\operatorname{rank}[B, AB, ..., A^{K-1}B] = n$. The matrix $[B, AB, ..., A^{K-1}B]$ is then called the reachability matrix on the time interval $[0, K]$.

Meanwhile, system (2.8) is called (completely) controllable to zero (null controllable) on $[0, K]$ if any initial state can be brought to zero within this time interval with some input sequence $(u(0), u(1), ..., u(K-1))$. From the solution equation (2.10), by setting up $x(K) = 0$, then the system is controllable if and

only if $\operatorname{im} A^K \subseteq \operatorname{im}[B, AB, ..., A^{K-1}B]$.

In general, controllability is indeed not equivalent to reachability (with equivalency if A invertible); this can be seen from system $x(k+1) = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right] x(k) + \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] u(k)$ which is unreachable but controllable.

2.4 Classes of Switching Signals

In switched systems, switching signals determine how the system switches from one mode (an individual system such as (2.8)) to another mode. The switching signals used for switched systems throughout the thesis are defined as follows.

Consider individual systems or modes indicated by indices $0, 1, ..., \mathtt{p},\ \mathtt{p} \in \mathbb{N}$. The set of modes is denoted by $\mathtt{P} := \{0, 1, ..., \mathtt{p}\}$, and it is assumed that, unless stated otherwise, this set of modes is known with a finite number of modes and every mode is active for at least one time instant. A switching signal σ is defined as a map $\sigma : \mathbb{N} \to \mathtt{P}$ indicating which mode $\sigma(k)$ is active at time instant k. Based on the on-hand available information, switching signals can be classified into three classes namely fixed switching signals, fixed mode sequences, and arbitrary switching signals.

Fixed Switching Signals

A fixed switching signal σ is uniquely determined by its mode sequence (σ_0, σ_1, ...) and the sequence of mode durations $(k^s_{j+1} - k^s_j)_{j=0,1,...}$ (see Figure 2.1 for an illustration) defined as[1]

$$\sigma(k) = \sigma_j \text{ if } k \in [k^s_j, k^s_{j+1}),\ j = \{0, 1, 2, ...\}. \tag{2.11}$$

Mode σ_0 is referred to as the initial mode, and the switching times k^s_j are assumed to be strictly increasing, i.e. $k^s_{j+1} > k^s_j\ \forall j$, which means that each mode is active at least one time instant when it is active. The switching signal (2.11) also indicates that the switching is triggered only by the time and neither by the state nor the input nor the output.

When a finite time interval $[0, K],\ K \in \mathbb{N}$ is considered, the last index is $j = J$ and define $k^s_{J+1} := K + 1$, i.e., there are J switches on $[0, K]$ (finitely many switches occur). If the switching signal of a switched system is known (or fixed), analysis can be focused on this known switching signal, and characterizations for properties being investigated can be made with respect to only this fixed switching signal without knowing whether the investigation results also hold for other switching signals.

[1] The standard interval notation for natural numbers is used here, in particular, $[k, \ell) := \{k, k+1, \ldots, \ell - 1\}$ for any $k, \ell \in \mathbb{N}$ with $k < \ell$.

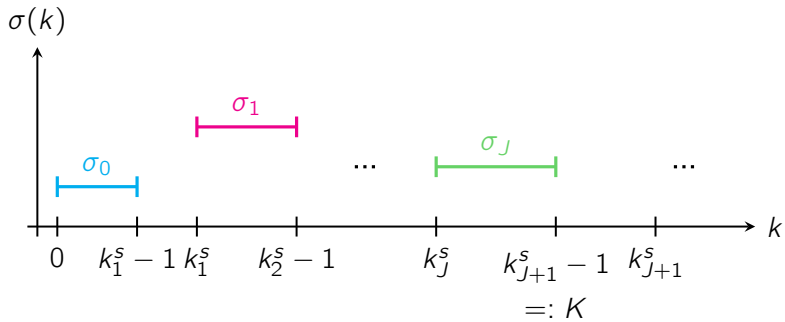

Figure 2.1: Mode sequence (2.11)

Fixed Mode Sequences

A fixed mode sequence, denoted by $(\sigma_0, \sigma_1, \ldots) =: (\sigma_j)_{j=0,1,\ldots}$ (for short just (σ_j)), has the information of the initial mode which actives at the initial time $k = 0$ and its subsequent modes in the future, however, the switching times are unknown. If a finite time interval $[0, K]$, $K \in \mathbb{N}$ is considered, then a fixed (finite) mode sequence on this time interval refers to $(\sigma_0, \sigma_1, \ldots, \sigma_J) =: (\sigma_j)_{j=0,1,\ldots,J}$ (the short notation (σ_j) can also be used together with the information of a finite time interval being considered). Therefore, with respect to a fixed mode sequence, investigations are done under a known mode sequence but with arbitrary mode durations. This implies that results are valid for all switching signals with the same mode sequence. However, in general, it is not known whether the results are also valid for other switching signals with a different mode sequence.

Arbitrary Switching Signals

The term arbitrary switching signals means only the set of modes is known, and both mode sequences and switching times are unknown. Thus, study results under arbitrary switching signals are also valid for specific or constrained switching signals (fixed switching signals or fixed mode sequences). However, the characterization of some properties such as observability, reachability, and stability, is often not necessary when considered for a restricted class of switching signals, and thus studies under restricted switching signals are also crucial for the switched systems class.

Dwell Time

The term "dwell-time", which was initially introduced in [35], is also used in this thesis. It is defined as follows: for a positive integer τ_D, let $\mathbb{S}^{[\tau_D]}$ be the

set of all switching signals of the form (2.11) with the interval between two consecutive switching times no smaller than τ_D. With the form (2.11), the initial time is also considered as an (initial) switching time, and furthermore, if the switching signal is defined on the finite time interval $[0, K]$, then k^s_{J+1} is the (final) switching time. The positive integer τ_D is referred to as the (fixed) dwell time. Furthermore, let $\mathbb{S}^{[\tau_D]}_{(\sigma_j)}$ be the set of all switching signals $\sigma \in \mathbb{S}^{[\tau_D]}$ with the same mode sequence (σ_j).

2.5 Classes of Switched Systems

The following three switched systems classes are studied in this thesis. All of them are considered in the discrete-time domain. Note that nonswitched systems are a particular case of switched systems in which the system is being studied with a constant switching signal.

(Ordinary) Linear Switched Systems

The first system class is switched systems composed of modes of the form (2.8). The system is considered with the switching signal of the form (2.11). This system class is called the Inhomogeneous Linear Switched System (InhLSS) and has the following general form:

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) \tag{2.12a}$$

$$y(k) = C_{\sigma(k)}x(k) + D_{\sigma(k)}u(k) \tag{2.12b}$$

where $k \in \mathbb{N}$ is the time instant, $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$, $m \in \mathbb{N}$ is the input, $y(k) \in \mathbb{R}^p$, $p \in \mathbb{N}$ is the output, σ is the switching signal of the form (2.11) determining which mode $\sigma(k)$ is active at time instant k, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{p\times n}$, and $D_i \in \mathbb{R}^{p\times m}$ are constant matrices for each mode i.

Studies for solvability, controllability, and observability of this system class are well established [10, 5, 6]. In this thesis, alternative characterizations will be presented (Chapter 3).

Singular Linear Switched Systems

The second system class is Singular Linear Switched Systems (SLSSs) of the form

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) \tag{2.13a}$$

$$y(k) = C_{\sigma(k)}x(k) + D_{\sigma(k)}u(k) \tag{2.13b}$$

where $E_i \in \mathbb{R}^{n\times n}$ may be singular and the rest of the notations stand for the same information as in (3.1). The presence of singular matrices E_i occurs in some dynamical processes which are subject to algebraic constraints, see e.g. [11]. If all E_i are invertible, then this system belongs to the system class (2.8). Note that some authors denote the state of a singular system as an internal variable [36, 37] or semi-state [38]. In many references, singular systems are also known as difference-algebraic equations, strong coupling systems, incomplete state systems, generalized systems, algebro-differential systems, descriptor systems, and implicit differential equations [2, 3].

Singular Nonlinear Switched Systems

In this switched system class, the modes are nonlinear singular systems. In this thesis, the following general form is studied:

$$E_{\sigma(k)}x(k+1) = F_{\sigma(k)}(x(k)) \tag{2.14}$$

where $F_i : \mathbb{R}^n \to \mathbb{R}^n$ is some nonlinearity. In this system class, E_i are (constant) matrices, and nonlinearities only appear on F.

Part I

Ordinary Linear Switched Systems

Contents of this part are based on the following papers:

- Sutrisno and Stephan Trenn, "Reachability and Controllability Characterizations for Linear Switched Systems in Discrete Time: A Geometric Approach," in *Proc. 2022 European Control Conference (ECC)*, 2022. DOI: 10.23919/ECC57647.2023.10178124
- Sutrisno and Stephan Trenn, "Observability and Determinability Characterizations for Linear Switched Systems in Discrete Time," in *Proc. 60th IEEE Conference on Decision and Control (CDC)*, 2021. DOI: 10.1109/CDC45484.2021.9682894

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Chapter 3 Linear Switched Systems

"Geometrical analysis makes it possible to
look into deeper parts."

Contents

Recall the class of Inhomogeneous Linear Switched Systems (InhLSSs)

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) \tag{3.1a}$$

$$y(k) = C_{\sigma(k)}x(k) + D_{\sigma(k)}u(k) \tag{3.1b}$$

where $k \in \mathbb{N}$ is the time instant, $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m, m \in \mathbb{N}$ is the input, $y(k) \in \mathbb{R}^p, p \in \mathbb{N}$ is the output, σ is the switching signal of the form (2.11), $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{p\times n}$, and $D_i \in \mathbb{R}^{p\times m}$ are constant matrices for each mode i.

System (3.1) is considered with a fully known and fixed switching signal σ. This assumption means that (3.1) can be seen as a specially structured time-varying linear system, and there are many physical switched systems that can be modeled in that framework (see e.g. [39, 40, 41, 42, 43, 44]) and this

system class has been attracting many researchers to study its properties such as stability [45, 46, 47, 48] and control designs [49, 50, 51, 52, 53].

In this chapter, observability, reachability, and their counterpart determinability and controllability of system (3.1) under fixed switching signals are studied. Moreover, dependencies of those properties on switching times are also discussed.

3.1 Solution Theory

First, recall the solution theory for system (3.1) which will be used in the analysis later. From (3.1a), for all initial condition $x(0) = x_0 \in \mathbb{R}^n$, all input sequence $u(0), u(1), \ldots$, and all switching signals, $x(k)$ at any time instant k exists and is unique. By iterative computations over the time instant $k = 0, 1, \ldots$, its solutions satisfy $\forall k, h \in \mathbb{N}$ with $k \geq h$,

$$x(k) = \Phi(k, h)x(h) + \sum_{i=h}^{k-1} \Phi(k, i+1)B_{\sigma(i)}u(i),$$

where $\Phi(k, h) = A_{\sigma(k-1)}A_{\sigma(k-2)} \cdots A_{\sigma(h)}$ is the so-called state transition matrix. With the initial condition $x(0) = x_0 \in \mathbb{R}^n$, its solution at time instant $k \in \mathbb{N}$ can be written as

$$x(k) = \Phi(k, 0)x(0) + \sum_{i=0}^{k-1} \Phi(k, i+1)B_{\sigma(i)}u(i). \tag{3.2}$$

Using this solution, the output is then can be derived as

$$y(k) = C\Phi(k, 0)x(0) + \sum_{i=0}^{k-1} C\Phi(k, i+1)B_{\sigma(i)}u(i) + D_{\sigma(k)}u(k). \tag{3.3}$$

For a fixed switching signal on a finite time interval $[0, K]$, $K \in \mathbb{N}$ where both the mode sequence and switching times are fixed and known (see also Fig. 2.1), the solution at any switching time k_j^s, $j = 0, 1, \ldots$ is given in the following lemma.

Lemma 3.1 (Solutions of LSSs at switching times). Under a fixed switching signal (2.11), the solution of linear switched systems (3.1) at any switching

time k_j^s is given by

$$\begin{aligned} x(k_j^s) = \psi_\sigma(j,0)x(0) + \psi_\sigma(j,1)R_{\sigma_0}(k^s)\begin{bmatrix} u(k^s-1) \\ \vdots \\ u(0) \end{bmatrix} \\ +\psi_\sigma(j,2)R_{\sigma_1}(k_2^s - k^s)\begin{bmatrix} u(k_2^s-1) \\ \vdots \\ u(k^s) \end{bmatrix} + \cdots \\ +\psi_\sigma(j,j)R_{\sigma_{j-1}}(k_j^s - k_{j-1}^s)\begin{bmatrix} u(k_j^s-1) \\ \vdots \\ u(k_{j-1}^s) \end{bmatrix} \end{aligned} \tag{3.4}$$

where for $i, j, h \in \mathbb{N},\ j \geq h \geq 0,\ k = 1, 2, ...,$

$$R_i(k) = [B_i, A_iB_i, \cdots, A_i^{k-1}B_i] \tag{3.5}$$

$$\psi_\sigma(j,h) = A_{\sigma_{j-1}}^{k_j^s - k_{j-1}^s} A_{\sigma_{j-2}}^{k_{j-1}^s - k_{j-2}^s} \cdots A_{\sigma_h}^{k_{h+1}^s - k_h^s}, \tag{3.6}$$

and $\psi_\sigma(j,h)$ is called the state transition matrix from the switching time k_h^s to the switching time k_j^s. Moreover, the matrix $\psi_\sigma(j,h)$ in (3.6) can be rewritten in a recursive form as

$$\psi_\sigma(j,h) = A_{\sigma_{j-1}}^{k_j^s - k_{j-1}^s}\psi_\sigma(j-1,h) \tag{3.7}$$

with $\psi_\sigma(h,h) = I_n$. ◊

Proof. Extracting the solution from (3.2) at the switching time k_j^s yields (3.4). □

The formula (3.7) is more computational-friendly than (3.6) as we can compute a state transition matrix at a switching time based on the state transition matrix at the previous switching time. One must note that the solution at $k = K$ is obtained from the last mode equation $x(K) = A_{\sigma_J}x(K-1) + B_{\sigma_J}u(K-1)$, i.e. the equation (3.1) is considered only up to $k = K-1$.

The homogeneous version, i.e. without inputs, of system (3.1) is given by the Homogeneous Linear Switched System (HomLSS)

$$x(k+1) = A_{\sigma(k)}x(k), \quad k = 0, 1, ... \tag{3.8a}$$

$$y(k) = C_{\sigma(k)}x(k). \tag{3.8b}$$

The family of matrix pairs $\{(A_0, C_0), (A_1, C_1), \ldots (A_\mathrm{p}, C_\mathrm{p})\}$ denotes the family of the system's matrix pairs of all modes involved in (3.8). All solutions of (3.8) satisfy

$$x(k) = \Phi_\sigma(k,h)x(h), \quad \forall k, h \in \mathbb{N} \text{ with } k \geq h,$$

where $\Phi_\sigma(k,h) = A_{\sigma(k-1)} \cdots A_{\sigma(h)}$ is the state transition matrix. With initial condition $x(0) = x_0 \in \mathbb{R}^n$, system (3.8) has the unique solution

$$x(k) = \Phi_\sigma(k,0)x_0, \quad k \in \mathbb{N}. \tag{3.9}$$

3.2 Observability and Determinability

For observability, the study focuses on the state observability notion, i.e. the ability to reconstruct the initial state (the final state for determinability) from output measurement for a fixed and fully known switching signal over a finite time interval. In literature, there are several other observability notions for switched systems, for example, path-wise observability that requires observability for every path (switching signal) with some length [54], and mode observability that recovers a certain number of first modes in a switching signal [55]; however, characterizations for those notions are not studied in this thesis.

3.2.1 Definitions

Consider the InhLSS (3.1) with a fixed switching signal of the form (2.11) over a finite observation time interval $[0, K]$ where the number of switches is as many as J, the initial value $x(0)$ is unknown and $y(k)$ is the output measurement, and the number of the output measurements available to reconstruct the state is as many as $K + 1$. Intuitively, system (3.1) is called observable on $[0, K]$ with respect to a fixed switching signal σ if the knowledge of the output measurements $\{y(0), y(1), ..., y(K)\} =: y_{[0,K]}$ is sufficient to determine the state on this interval $\{x(0), x(1), ..., x(K)\} =: x_{[0,K]}$. This is defined mathematically as follows:

Definition 3.2 (Observability of LSSs). Linear switched system (3.1) is called observable on $[0, K]$ w.r.t. a fixed switching signal σ given by (2.11) if the following implication holds:

$$y'_{[0,K]} \equiv y''_{[0,K]} \implies x'_{[0,K]} \equiv x''_{[0,K]} \tag{3.10}$$

where $y'_{[0,K]}$ and $y''_{[0,K]}$ are two arbitrary outputs, and $x'_{[0,K]}$ and $x''_{[0,K]}$ are two arbitrary states of (3.1) under σ. ◊

Under a given and fixed switching signal, the ability to recover the state of the InhLSS (3.1) from the values of the external signals (input and output) is independent of how the input influences the state dynamics. This is formally stated in the following lemma.

Lemma 3.3 (Observability independent of inputs). Linear switched system (3.1) is observable on $[0, K]$ w.r.t. a fixed switching signal given by (2.11) if, and only if, its homogeneous form (3.8) is observable on $[0, K]$ w.r.t. σ. ◊

Proof. In the output equation (3.3), the matrices Φ, B_i, and D_i for all i and inputs $u(k)$ for all k are known and thus the last two terms in (3.3) are known values. Then, the state $x_{[0,K]}$ is completely determined by the observation of

the output $y_{[0,K]}$ if and only if $x(0)$ is completely determined by $y_{[0,K]}$. This means that the observability is independent of inputs, and thus the InhLSS (3.1) is observable if and only if (3.8) is observable. □

Consequently, the observability of the InhLSS (3.1) can be characterized via the HomLSS (3.8). The forthcoming simplified observability condition, therefore, is presented without input influences. Now, due to linearity, the observability condition (3.10) can be reduced as zero-observability as proved in the following proposition.

Proposition 3.4 (Zero-observability of LSSs). Linear switched system (3.8) is observable on $[0, K]$ w.r.t. a fixed switching signal given by (2.11) if, and only if, for all solutions on $[0, K]$ the following implication holds:

$$y_{[0,K]} \equiv 0 \implies x_{[0,K]} \equiv 0. \tag{3.11}$$

◊

Proof. The necessity is obvious as (3.10) implies (3.11). For the sufficiency, assume (3.8) is not observable i.e. there exists an output measurement $y_{[0,K]}$ for which there exist at least two different solutions $x'_{[0,K]}$ and $x'_{[0,K]}$ of (3.8). By linearity, $x'_{[0,K]} - x''_{[0,K]} = x_{[0,K]}$ solves (3.8) and $C_\sigma x' - C_\sigma x'' = y - y = 0$ i.e. $y_{[0,K]} \equiv 0$ does not imply $x_{[0,K]} = 0$. □

By the fact that using (3.8), the knowledge of the state on $[0, K]$ can be derived recursively if we know $x(0)$, then we can rewrite the observability definition as follows: system (3.8) is observable on $[0, K]$ w.r.t. a fixed switching signal σ if the knowledge of the output sequence $\{y(0), y(1), ..., y(K)\}$ is sufficient to determine $x(0)$. Furthermore, since $x \equiv 0$ on $[0, K]$ if, and only if, $x(0) = 0$, then the observability condition (3.11) can be reduced to

$$y_{[0,K]} \equiv 0 \implies x(0) = 0. \tag{3.12}$$

Finally, we say system (3.8) is (globally) observable if there exists such positive integer K.

In a situation where the state on the time interval $[0, K]$ cannot be reconstructed by the given output measurement, one may want to reconstruct the state at the final time instant K. Once we know the state at K, we could iterate the system's model to obtain the solutions at the future time instants, for instance to design a state feedback. Based on this motivation, we study in the following the determinability concept which was initially introduced in [56] for continuous time. Note that in this study, the switching signal is already known and fixed. If the switching signal is fully or partially unknown, one may refer to the switch observability/determinability studied in [57, 58].

Once a time instant $K > 0$ at which $x(K)$ determined from the output measurement is found, the system is called determinable on $[0, K]$. To be precise, the linear switched system (3.1) is called determinable on $[0, K]$ w.r.t. a fixed switching signal given by (2.11) if the knowledge of the output measurements $\{y(0), y(1), ..., y(K)\}$ is sufficient to determine $x(K)$. From this intuition, this determinability notion can be brought into the following final-state-sustainability notion, which is defined in a mathematically intuitive form as follows:

Definition 3.5 (Determinability of LSSs). The linear switched system (3.1) is said to be **determinable** on $[0, K]$ w.r.t. a fixed switching signal σ of the form (2.11) if the following implication holds:

$$y'_{[0,K]} \equiv y''_{[0,K]} \Rightarrow x'(K) = x''(K). \tag{3.13}$$

where $y'_{[0,K]}$ and $y''_{[0,K]}$ are two arbitrary outputs, and $x'(K)$ and $x''(K)$ are two arbitrary states at $k = K$ of (3.1) under σ. ◊

Similar to observability, by the same arguments in the proof of Lemma 3.3 in which the output is independent of the input due to the fact that the output's dependence on the input can be computed a priori, the determinability of the InhLSS 3.1 can be characterized via its homogeneous form (3.8). Furthermore, by similar arguments as in the proof of Proposition 3.4, the determinability condition (3.13) can be simplified into the zero-determinability condition. Then, we have the following corollary:

Corollary 3.6 (Zero-output-zero-final-state determinability of LSSs). The linear switched system (3.1) is determinable on $[0, K]$ w.r.t. a fixed switching signal σ of the form (2.11) if and only if (3.8) is determinable on $[0, K]$ w.r.t. σ. In particular, this system is determinable on $[0, K]$ w.r.t. σ if, and only if, the following implication holds:

$$y_{[0,K]} \equiv 0 \Rightarrow x(K) = 0. \tag{3.14}$$

under the given switching signal σ. ◊

Finally, the system (3.8) is called (globally) determinable if, and only if there exists $K \in \mathbb{N}$ such that (3.13) or (3.14) holds. In the characterizations, the latter condition is used.

Remark 3.7 (Observability and determinability are in general not equivalent). If the matrices A_i for all i in (3.8) are invertible, then determinability and observability are equivalent since (3.8) can be rewritten in a backward dynamical system. However, in general, observability implies determinability on the same time interval but the converse is not always true. For a counter-example, see the system in Example 3.13. Furthermore, if the switched system is de-

terminable, i.e. $x(K)$ can be reconstructed, then the state of the future time instants can be determined. This is why it is also called "forward observable" in some references, see e.g. [59]. ◊

3.2.2 Characterizations

The characterizations for the observability and determinability notions introduced in the previous subsection are presented in the following.

Theorem 3.8 (Observability of LSSs). The linear switched system (3.8) is observable on $[0, K], K = k^s_{J+1} - 1$ w.r.t. to the fixed switching signal (2.11) if, and only if,

$$\bigcap_{j=0}^{J} [\psi_\sigma(j, 0)]^{-1} \left(\mathcal{O}_{\sigma_j}^{k^s_{j+1}-k^s_j-1}\right) = \{0\} \tag{3.15}$$

where $\psi_\sigma(j, 0)$ is given by (3.7) and $\mathcal{O}_i^k := \ker \left[C_i^\top, (C_iA_i)^\top, \ldots, (C_iA_i^k)^\top\right]^\top$. ◊

Proof. Taking the kernel of the observability matrix over the time interval $[0, K]$ and using the fact that $\ker \begin{bmatrix} M_1 \\ M_2M_3 \end{bmatrix} = \ker M_1 \cap [M_3]^{-1} \ker M_2$ for any matrices M_i (see Appendix A) proves the observability condition (3.15). □

The subspace on the left-hand side of (3.15) is the unobservable space for system (3.8). Note that $[*]^{-1}$ denotes the preimage and not the inverse. The observability condition (3.15) above for any sufficient slow switching signal $\sigma \in \mathbb{S}^{[n]}_{[0,K]}$ can be reduced to

$$\bigcap_{j=0}^{J} [\psi_\sigma(j, 0)]^{-1} \left(\mathcal{O}_{\sigma_j}\right) = \{0\} \tag{3.16}$$

where $\mathcal{O}_{\sigma_j} := \mathcal{O}_{\sigma_j}^{n-1}$. This means that if each mode is active long enough (at least n time steps) and there is no switch after K then the observability will depend only on the switching time and thus (3.16) is the condition for global observability.

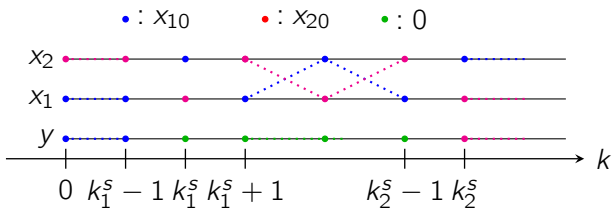
Remark 3.9 (Observability on $[0, \infty)$). In general, system (3.8) is defined on $[0, \infty)$. In this situation, the observability condition is not dependent on K anymore and it is dependent only on the switching times k^s_j. Moreover, it is clear that an observable initial mode on $[0, k^s_1 - 1]$ implies global observable but an observable subsequent mode on the corresponding time interval doesn't always imply global observable. ◊

The following example illustrates the observability characterizations of system (3.8) and in particular shows that the observability property depends in general on the switching times.

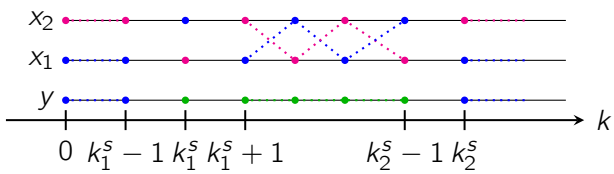
Example 3.10. Consider the linear switched system (3.8) composed of the following two modes

$$(A_0, C_0) = \left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]\right), \ (A_1, C_1) = \left(\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \end{smallmatrix}\right]\right).$$

The system starts from mode-0 and switches to mode-1 at time instant k_1^s and switches again to mode-0 at time instant k_2^s. When $k_2^s - k_1^s$ is odd then mode-1 is active for an odd number of time instants. In this situation, all information can be completely deduced from the output measurements (see Fig. 3.1a). On the other hand, if $k_2^s - k_1^s$ is even, some information will be lost (see Fig. 3.1b). This means that when $k_2^s - k_1^s$ is even, the switched system is unobservable because the initial value $x_2(0) = x_{20}$ will never be visible in the output. The observability characterization results for various switching times can be seen in Fig. 3.2a.



(a) $k_2^s - k_1^s$ is odd

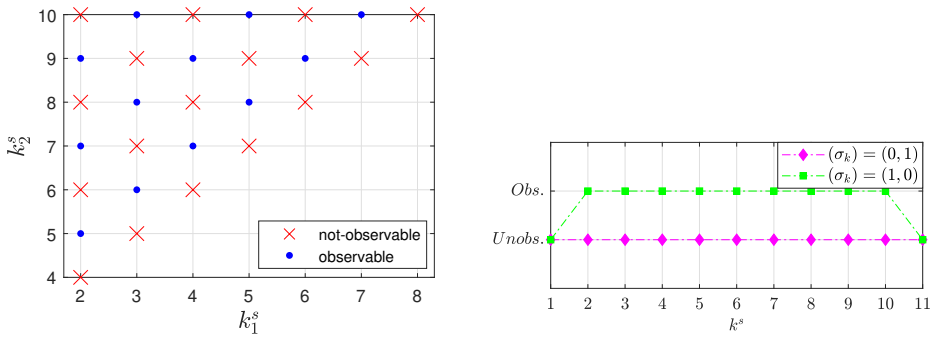


(b) $k_2^s - k_1^s$ is even

Figure 3.1: Solution of the switched system in Example 3.10

Moreover, when the system starts from mode-1 and switches to mode-0 and switches again to mode-1, the switched system is always unobservable for arbitrary switching times k_1^s and k_2^s i.e. independent from switching times. In particular, this example also shows that different mode sequences may end up in different observability results. ◊

Example 3.10 showed that the switching time dependence in the observability characterization (3.16) even for dwell-time switching signals cannot be removed in general. For multiple-switching this switching time-dependence is



(a) (b)

Figure 3.2: (a) Switching time vs observability Example 3.10 (b) Observability characterization results of Example 3.11

also present in the continuous time case and we believe that it is possible to derive sufficient or necessary conditions for observability in a similar way as in [56, Sec. IV], however, for switching signals without a dwell time, these conditions may be more complicated or may not exist at all.

Therefore, from the observability condition (3.15) and confirmation derived from the Example 3.10, indeed, in general, the observability of the LSS (3.8) depends on the switching times and on how long each mode is active. This is similar to the result for LSSs in continuous time as discussed in [60]. Furthermore, this dependency occurs even for LSSs with only two modes as illustrated by the following example.

Example 3.11. Consider the linear switched system (3.8) composed by two modes with

$$(A_0, C_0) = \left(\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \right), \ (A_1, C_1) = \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} \right).$$

We observe here for the time interval $[0, 12]$. As individual systems, both modes are not-observable since

$$\mathcal{O}_0 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ and } \mathcal{O}_1 = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

The observability property of the switched system w.r.t. the single switch switching signal given by the mode sequence $(\sigma_j) = (0, 1)$ or $(1, 0)$ with varying switching times $k_s \in [1, 11]$ are illustrated in Fig. 3.2b. For the first mode sequence, $(\sigma_j) = (0, 1)$ the switched system remains unobservable independently of the switching time. However, for the reversed switching sequence $(\sigma_j) = (1, 0)$ it turns out the switched system is observable if the switching times are sufficiently far away from the time-interval boundaries. In fact, if the

switching time is too early so the initial mode is active too short, or too late (close enough to the end time K), the switched system gets unobservable also for the switching sequence $(1, 0)$. $\Diamond$

For determinability, consider first the following characterization for the HomLSS (3.8) under single switch switching signals. The understanding of this simple case is used to characterize the cases with multiple switches. Consider the single switch switching signal given by the mode sequence $(0, 1)$ and define the following sequence of subspaces on $[0, K]$ with the initial subspace

$$\begin{aligned} \mathcal{Q}^0 &= \ker C_0 \\ \mathcal{Q}^k &= \ker C_{\sigma(k)} \cap A_{\sigma(k-1)}\mathcal{Q}^{k-1}, \quad k = 1, 2, \ldots, k^s, \ldots K. \end{aligned} \tag{3.17}$$

The interpretation of this sequence is that $x_k \in \mathcal{Q}^k$ if, and only if, there exists a solution with $y(i) = 0$ for $i \in [0, k]$ and $x(k) = x_k$. By utilizing this, the determinability characterization is presented in the following lemma.

Lemma 3.12 (Determinability Characterization of Single Switch LSSs). The linear switched system (3.8) is determinable on $[0, K], K \geq k^s$ w.r.t. the mode sequence $(0, 1)$ if, and only if,

$$\mathcal{Q}^K = \{0\} \tag{3.18}$$

where $\mathcal{Q}^K$ is given by (3.17). $\Diamond$

Proof. Necessity: By construction of $\mathcal{Q}^k$ it follows that for all $x_k \in \mathcal{Q}^k$ we have $C_{\sigma(k)}x_k = 0$ and that there exists $x_{k-1} \in \mathcal{Q}^{k-1}$ with $x_k = A_{\sigma(k-1)}x_{k-1}$. Hence by assuming that $\mathcal{Q}^K \neq \{0\}$ we can pick $x_K \in \mathcal{Q}^K \setminus \{0\}$ and a sequence $x_K, x_{K-1}, \ldots, x_2, x_1, x_0$ with $x_k \in \mathcal{Q}^k$ such that $x(\cdot)$ given by $x(k) := x_k$ is a solution of (3.8) on $[0, K]$ with $y(k) = C_{\sigma(k)}x_k = 0$. This shows that (3.8) is not determinable.
Sufficiency: Consider a solution $x(\cdot)$ of (3.8) and assume that $y(k) = 0$ for all $k \in [0, K]$. We will show that then $x(k) \in \mathcal{Q}^k$ for all $k \in [0, K]$ and hence determinability follows from $\mathcal{Q}^K = \{0\}$. It is clear that $y(k) = 0$ implies $x(k) \in \ker C_{\sigma(k)} \forall k$. Next, from $y(0) = 0$ it follows that $x(0) \in \ker C_0 = \mathcal{Q}^0$. Inductively, assume that $x(k) \in \mathcal{Q}^k$, then for $k < K$ we have that $x(k+1) = A_{\sigma(k)}x(k) \in A_{\sigma(k)}\mathcal{Q}^k$ and hence $x(k+1) \in \ker C_{\sigma(k)} \cap A_{\sigma(k)}\mathcal{Q}^k = \mathcal{Q}^{k+1}$. Thus we can conclude that $x(K) \in \mathcal{Q}^K = \{0\}$ as desired. $\square$

The subspace $\mathcal{Q}^K$ is the undeterminable space for system (3.8). If $K < k^s$ then the condition for the determinability on $[0, K]$ is equivalent to non-switched systems since we have only the initial mode that is active on that time interval. The following example illustrates determinability under a single switch switching signal.

Example 3.13. Consider the linear switched system (3.8) with the following system's matrices

$$(A_0, C_0) = \left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \end{smallmatrix}\right]\right), \ (A_1, C_1) = \left(\left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right]\right).$$

With the mode sequence $\sigma = (0, 1)$, the switched system is unobservable on $[0, 12]$ for $2 \leq k^s \leq 9$. Surprisingly, it is always determinable. This shows that even though the switched system is not-observable, it could be determinable. Moreover, the smallest time instant K such that the switched system is determinable on $[0, K]$ depends on the switching time; it needs to be two time instants after the switching time in order to be determinable. ◊

The determinability characterization for the multiple switch case can be done through straightforward generalization from the single switch case above. First, the sequence of subspaces (3.17) is now defined for $k = 1, 2, \ldots, K$ and for $k \in (k^s_J, k^s_{J+1})$. The characterization is presented in the following theorem.

Theorem 3.14 (Determinability Characterization of Multiple Switches LSSs). The linear switched system (3.8) is determinable on $[0, K], K \in [k^s_J, k^s_{J+1})$ w.r.t. the fixed switching signal (2.11) if, and only if,

$$\mathcal{Q}^K = \{0\} \tag{3.19}$$

where $\mathcal{Q}^k$ is given by (3.17). ◊

Proof. Generalizing the proof of the single switch case in Lemma 3.12 proves that system (3.8) is determinable if and only if $\mathcal{Q}^K = \{0\}$. □

As in observability, the determinability, in general, depends on the switching time, and furthermore, also depends on the number of the modes occurring on $[0, K]$, this can also be seen from the length of the nested image on the subspace formula $\mathcal{Q}^k$ in (3.17). The following example illustrates the determinability characterization using the theorem above and in particular shows that determinability is indeed dependent on switching times.

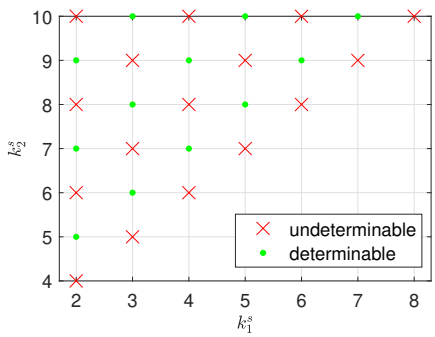


Figure 3.3: Switching time vs determinability Example 3.15

Example 3.15. Recall the Example 3.10. We check the determinability on $[0, 20]$ and the result with some various switching times is shown in Fig. 3.3. In this example, the determinability characterization result is just the same to the result in the observability characterization (see Fig. 3.2a). Compared to the result in Example 3.13, in contrast, the determinability property in this Example depends on the switching times. $\Diamond$

3.3 Reachability and Controllability

Recall the InhLSS (3.1). This section focuses on the reachability and controllability properties of the state of this system.

3.3.1 Definitions

Consider the InhLSS (3.1) on a finite time interval $[0, K]$ with a fixed and known switching signal σ of the form (2.11). Intuitively, the reachability notion considered in this study is the ability to reach a certain final state within a finite number of time instants. This is precisely defined as follows.

Definition 3.16 (Reachability of LSSs). A state $x_f \in \mathbb{R}^n$ of the InhLSS (3.1) is called **reachable from zero** on $[0, K]$ w.r.t. the switching signal σ if with $x(0) = 0$ there exists an input sequence $(u(0), u(1), .., u(K-1))$ such that $x(K)$ of (3.1) under σ satisfies $x(K) = x_f$. $\Diamond$

The reachability notion defined above is reachability from zero. However, note that a reachable final state from zero is also reachable from any initial state, see the forthcoming Remark 3.22. The reachability from zero definition above is then considered to simplify the definition and the corresponding characterization. For further analysis, the set of reachable final states is defined together with the notion of complete reachability as follows.

Definition 3.17 (Reachable set and complete reachability of LSSs). The set of all final states $x_f \in \mathbb{R}^n$ that are reachable from zero on $[0, K]$ w.r.t. the switching signal σ is called the **reachable set from zero** and denoted by $\mathcal{R}^{\sigma}_{[0,K]}$. Furthermore, the InhLSS (3.1) is called **completely reachable from zero** on $[0, K]$ w.r.t. the switching signal σ if $\mathcal{R}^{\sigma}_{[0,K]} = \mathbb{R}^n$. $\Diamond$

Meanwhile, the intuition definition for the controllability notion is the ability to bring an initial state to zero within a finite number of time instants. This is precisely defined as the controllability to zero as follows.

Definition 3.18 (Controllability of LSSs). An initial state $x_0 \in \mathbb{R}^n$ of (3.1) is called **controllable to zero** on $[0, K]$ w.r.t. the switching signal σ if with

$x(0) = x_0$ there exists an input sequence $(u(0), u(1), .., u(K-1))$ such that $x(K)$ of (3.1) under σ satisfies $x(K) = 0$. $\Diamond$

Similar to reachability, for further analysis, the following definition introduces the set of the controllable initial state together with the notion of complete controllability.

Definition 3.19 (Controllable set and complete controllability of LSSs). The controllable set of system (3.1) on $[0, K]$ w.r.t. the switching signal σ is the set of all initial states $x_0 \in \mathbb{R}^n$ that are controllable to zero on $[0, K]$ under σ and denoted as $\mathcal{C}^{\sigma}_{[0,K]}$. Furthermore, the InhLSS (3.1) is called **completely controllable to zero** on $[0, K]$ w.r.t. σ if $\mathcal{C}^{\sigma}_{[0,K]} = \mathbb{R}^n$. $\Diamond$

In addition, we introduce here the notion of deadbeat controllable i.e. a InhLSS is called **deadbeat controllable** if it is completely controllable on $[0, 1]$ i.e. within one time step. In other words, deadbeat controllable means that a single control action is enough to make the state zero. Note that using the solution from the initial mode is enough to characterize deadbeat controllability; thus, it is independent of the switching signal. Clearly, deadbeat controllable implies completely controllable by setting the input to zero for the subsequent time steps. If we know that a switched system is deadbeat controllable, then it is completely controllable on any time interval and for any switching signal.

Remark 3.20 (Reachability vs Controllability of LSSs). The same phenomenon as in non-switched systems happens here i.e. reachability and controllability are equivalent if all of A_is are nonsingular; however, in general, reachability only implies controllability because the zero final state $x_f = 0$ is always reachable from any initial state, but controllability does not imply reachability. The latter is illustrated by the following simple single switch switched system:

$$\begin{array}{r|l} k < k_1^s: & k \geq k_1^s: \\ x_1(k+1) = x_1(k) & x_1(k+1) = 0 \\ x_2(k+1) = u(k) & x_2(k+1) = x_2(k) \end{array}$$

which is easily seen to be controllable (by setting $u(k_1^s - 1) = 0$) but not reachable on $[0, K]$ for any $K > k_1^s$. This example also illustrates that for controllability of the overall switched system it is not necessary, that any of the individual modes is controllable. Moreover, by slightly changing the system above to:

$$\begin{array}{r|l} k < k_1^s: & k \geq k_1^s: \\ x_1(k+1) = x_1(k) & x_1(k+1) = u(k) \\ x_2(k+1) = u(k) & x_2(k+1) = x_2(k) \end{array}$$

we obtain a reachable switched system on $[0, K]$ composed of unreachable modes. ◊

3.3.2 Characterizations

Let $\mathcal{R}_i(k) = \operatorname{im} R_i(k) = \operatorname{im}[B_i, A_i B_i, \cdots, A_i^{k-1} B_i]$ be the "local" or "individual" reachable space of mode-i on $[0, k]$, and define the following sequence of subspaces for $j = 1, 2, \ldots, J$

$$\mathcal{M}_0 = \mathcal{R}_{\sigma_0}(k_1^s), \tag{3.20a}$$

$$\mathcal{M}_j = A_{\sigma_j}^{k_{j+1}^s - k_j^s} \mathcal{M}_{j-1} + \mathcal{R}_{\sigma_j}(k_{j+1}^s - k_j^s). \tag{3.20b}$$

The following theorem reveals that in fact, the reachable set on $[0, K]$ is equal to the subspace $\mathcal{M}_J$ defined above.

Theorem 3.21 (Reachability characterization of LSSs). Consider the InhLSS (3.1) under a fixed and known switching signal σ of the form (2.11). Let $\mathcal{R}_{[0,K]}^{\sigma}$ be its **reachable set** on $[0, K]$ w.r.t. σ. Then

$$\mathcal{M}_J = \mathcal{R}_{[0,K]}^{\sigma} \tag{3.21}$$

where $\mathcal{M}_J$ is given by (3.20). In particular, (3.1) is completely reachable if, and only if, $\mathcal{M}_J = \mathbb{R}^n$. ◊

Proof. First, note that $\mathcal{M}_J$ can be rewritten as

$$\begin{aligned} \mathcal{M}_J = \psi_\sigma(J+1, 1)\mathcal{R}_0(k_1^s) + \psi_\sigma(J+1, 2)\mathcal{R}_1(k_2^s - k_1^s) + \cdots \\ + \mathcal{R}_J(k_{J+1}^s - k_J^s). \end{aligned} \tag{3.22}$$

For any reachable state $x(K) \in \mathcal{R}_{[0,K]}^{\sigma}$, there exists an input sequence ($u(0)$, $u(1)$, ..., $u(K-1)$) such that (3.4) is satisfied with $x(k_{J+1}^s) = x(K)$ i.e. $x(K) \in \mathcal{M}_J$ and thus $\mathcal{R}_{[0,K]}^{\sigma} \subseteq \mathcal{M}_J$. From (3.22), any vector $x_f \in \mathcal{M}_J$ can be rewritten as the summation of vectors of the form (3.4) with $x_f = x(k_{J+1}^s) = x(K)$ i.e. there exists an input sequence $(u(0), u(1), ..., u(K-1))$ such that $x_f = x(K)$ and thus x_f is reachable from zero. Hence, $\mathcal{M}_J \subseteq \mathcal{R}_{[0,K]}^{\sigma}$. □

Remark 3.22. From (3.22), in fact, the reachable set is indeed a subspace in $\mathbb{R}^n$, and thus it is also called as reachable space. In particular, completely reachable from zero on $[0, K]$ is equivalent to **completely reachable** on $[0, K]$ i.e. any $x_f \in \mathcal{R}_{[0,K]}^{\sigma}$ is reachable from any initial state $x_0 \in \mathbb{R}^n$. This can be seen from the fact that the term containing the (nonzero) initial state x_0 i.e. $\psi(J+1, 0)x_0$ in (3.4) does not affect the proof of Theorem 3.21 and yields the same result. ◊

The following example illustrates the implementation of the reachability

characterization presented in Theorem 3.21. Moreover, some further observations are also revealed in this example.

Example 3.23. Consider the InhLSS (3.1) composed of two modes with

$$(A_0, B_0) = \left(\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\right) \text{ and } (A_1, B_1) = \left(\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\right).$$

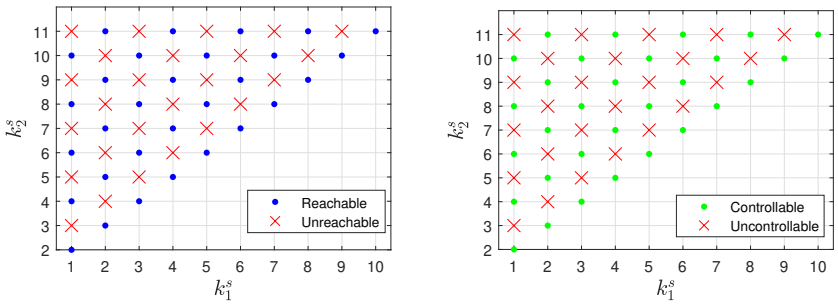
As individual systems, both modes are unreachable with their corresponding reachable spaces $\mathcal{R}_0 = \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\mathcal{R}_1 = \{0\}$ respectively. Consider now the switched systems under switching signals with the mode sequence $(\sigma) = (0, 1, 0)$ on time interval $[0, K]$ with $K = 12$ and with switching times $1 \leq k_1^s \leq k_2^s - 1$ and $k_1^s + 1 \leq k_2^s \leq K - 1$. These switched systems are reachable when $k_2^s - k_1^s$ is odd, however, they are unreachable when $k_2^s - k_1^s$ is even; this is explained as follows. The local reachable space corresponds to mode-0 and mode-1 is $\mathcal{R}_0(k) = \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $\forall k \in \mathbb{N}$ and $\mathcal{R}_1(k) = \{0\}$, $\forall k \in \mathbb{N}$ respectively. The sequence of subspaces (3.20) for the switched systems under the mode sequence $(0, 1, 0)$ is then given by

$$\mathcal{M}_0 = \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],$$

$$\mathcal{M}_1 = A_1^{k_2^s - k_1^s} \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],$$

$$\mathcal{M}_2 = A_0^{K-k_2^s}\mathcal{M}_1 + \mathcal{R}_0(K - k_2^s) = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]^{K-k_2^s} \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]^{k_2^s-k_1^s} \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right].$$

If $k_2^s - k_1^s$ is odd, then $\mathcal{M}_2 = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right] \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] = \mathbb{R}^2$, i.e. the InhLSS is (completely) reachable. If $k_2^s - k_1^s$ is even, then $\mathcal{M}_2 = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right] \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] = \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ i.e. the InhLSS is non-reachable. The characterization results for all possible switching times within the time interval $[0, 12]$ are shown in Fig. 3.4a. In particular, this example also shows that even though all individual modes are unreachable, switched systems composed of those modes under certain switching signals can be reachable. ◊



(a) (b)

Figure 3.4: (a) Switching times vs reachability of the switched system in Example 3.23 (b) Switching times vs controllability Example 3.26

Theorem 3.21 can also be used to characterize the subspace of points from nonzero initial states as follows.

Proposition 3.24 (Reachable space from nonzero initial states of LSSs). Consider the InhLSS (3.1) with $x(0) = x_0 \in \mathbb{R}^n$ under a fixed and known switching signal σ of the form (2.11). Let $\mathcal{R}^{\sigma}_{[0,K]}(x_0)$ be its reachable space on $[0, K]$ from x_0 w.r.t. σ. Then the following structure applies:

$$\mathcal{R}^{\sigma}_{[0,K]}(x_0) = \psi_\sigma(J+1, 0)x_0 + \mathcal{M}_J. \tag{3.23}$$

where $\mathcal{M}_J$ is given by (3.20). ◊

Proof. Let $x(K, x_0, u(\cdot))$ be the solution of (3.1) at time step $k = K$ with $x(0) = x_0$ and input sequence $u(\cdot) = (u(0), ..., u(K-1))$. Then, from (3.4)

$$x(K, x_0, u(\cdot)) = x(K, x_0, 0) + x(K, 0, u(\cdot))$$

i.e. $x(K, x_0, u(\cdot))$ can be decomposed as the sum of the solution with zero inputs and the solution with zero initial state. Since $x(K, x_0, 0) = \psi_\sigma(J+1, 0)x_0$ and $x(K, 0, u(\cdot)) \in \mathcal{M}_J$ then $\mathcal{R}^{\sigma}_{[0,K]}(x_0) = \psi_\sigma(J+1, 0)x_0 + \mathcal{M}_J$. □

The characterization for controllability is presented in the following theorem. First, define the sequence of subspaces

$$\mathcal{N}_{J+1} = \{0\}, \tag{3.24a}$$

$$\mathcal{N}_j = \left[A_{\sigma_j}^{k^s_{j+1}-k^s_j}\right]^{-1} \left[\mathcal{N}_{j+1} + \mathcal{R}_j(k^s_{j+1} - k^s_j)\right],\ j = J, J-1, \ldots, 0. \tag{3.24b}$$

Theorem 3.25 (Controllability characterization of LSSs). Consider the InhLSS (3.1) under a fixed and known switching signal σ of the form (2.11). Let $\mathcal{C}^{\sigma}_{[0,K]}$ be its **controllable space** on $[0, K]$ w.r.t. σ. Then

$$\mathcal{N}_0 = \mathcal{C}^{\sigma}_{[0,K]} \tag{3.25}$$

where $\mathcal{N}_j$ is defined by (3.24). In particular, (3.1) is completely controllable if, and only if, $\mathcal{N}_0 = \mathbb{R}^n$. ◊

Proof. For any controllable initial state $x_0 \in \mathcal{C}^{\sigma}_{[0,K]}$, there exists an input sequence $u(\cdot) = (u(0),\ u(1),\ ...,\ u(K-1))$ such that with $x(0) = x_0$ we obtain $x(K) = 0$. Thus, by backward iteration we have that the solution $x(k^s_j),\ j = J, J-1, ..., 0$ satisfies

$$x(k^s_J) \in [A_{\sigma_J}^{K-k^s_J}]^{-1}[\{0\} + \mathcal{R}_{\sigma_J}(K - k^s_J)] = \mathcal{N}_J,$$

$$x(k^s_{J-1}) \in \left[A_{\sigma_{J-1}}^{k^s_J-k^s_{J-1}}\right]^{-1} [\{x(k^s_J)\} + \mathcal{R}_{\sigma_{J-1}}(k^s_J - k^s_{J-1})]$$

$$\subseteq \left[A_{\sigma_{J-1}}^{k^s_J-k^s_{J-1}}\right]^{-1} \left[\mathcal{N}_J + \mathcal{R}_{\sigma_{J-1}}(k^s_J - k^s_{J-1})\right] = \mathcal{N}_{J-1},$$

$$\vdots$$

$$x(0) \in \left[A_{\sigma_0}^{k_1^s}\right]^{-1} [\{x(k_1^s\} + \mathcal{R}_{\sigma_0}(k_1^s)] \subseteq [A_{\sigma_0}^{k_1^s}]^{-1}[\mathcal{N}_1 + \mathcal{R}_{\sigma_0}(k_1^s)] = \mathcal{N}_0$$

i.e. $x_0 \in \mathcal{N}_0$. Hence, $\mathcal{C}_{[0,K]}^{\sigma} \subseteq \mathcal{N}_0$. Now, for any vector $n_0 \in \mathcal{N}_0 = [A_{\sigma_0}^{k_1^s}]^{-1}[\mathcal{N}_1 + \mathcal{R}_0(k_1^s)]$, there exist $n_1 \in \mathcal{N}_1$ and $r_0 \in \mathcal{R}_0(k_1^s)$ satisfying $n_1 = A_{\sigma_0}^{k_1^s} n_0 + r_0$. In particular there exists an input sequence $(u(0), u(1), ..., u(k_1^s - 1))$ such that with $x(0) = n_0$ we have $x(k_1^s) = n_1$. By forward iteration, there exist $n_J \in \mathcal{N}_J$ and $r_J \in \mathcal{R}_{\sigma_J}(K - k_J^s)$ which corresponds to an input sequence $(u(k_J^s), u(k_J^s + 1), ..., u(K - 1))$ satisfying

$$0 = A_{\sigma_J}^{K-k_J^s} n_J + r_J = x(K).$$

Altogether, we have found an input sequence, which controls the initial state $n_0 \in \mathcal{N}_0$ to zero on the time interval $[0, K]$, hence $\mathcal{N}_0 \subseteq \mathcal{C}_{[0,K]}^{\sigma}$. □

The following example illustrates a switched system that is controllable for some switching signals, however, uncontrollable for some other switching signals. Some further analysis regarding the connection between the controllability of the individual modes composing the switched system with the controllability of the switched system.

Example 3.26. Recall the switched system in Example 3.23. The local controllable sets corresponding to mode-0 and mode-1 are $\mathcal{C}_{[0,K]}^0 = \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\mathcal{C}_{[0,K]}^1 = \{0\}$ respectively for all $K \in \mathbb{N}$. Thus, both individual systems are not (completely) controllable. The sequence of subspaces (3.24) for the switched systems under the mode sequence $(0, 1, 0)$ is then given by

$$\begin{aligned}
\mathcal{N}_3 &= \{0\} \\
\mathcal{N}_2 &= \left[A_0^{K-k_2^s}\right]^1 [\mathcal{N}_3 + \mathcal{R}_0(K - k_2^s)] = \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \\
\mathcal{N}_1 &= \left[A_1^{k_2^s - k_1^s}\right]^{-1} [\mathcal{N}_2 + \mathcal{R}_1(k_2^s - k_1^s)] = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]^{k_2^s - k_1^s} \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \\
\mathcal{N}_0 &= \left[A_0^{k_1^s}\right]^{-1} [\mathcal{N}_1 + \mathcal{R}_0(k_1^s)] = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]^{k_2^s - k_1^s} \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right].
\end{aligned}$$

If $k_2^s - k_1^s$ is odd, then $\mathcal{N}_0 = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right] \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] = \mathbb{R}^n$ i.e. the InhLSS is (completely) controllable. If $k_2^s - k_1^s$ is even, then $\mathcal{N}_0 = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right] \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] = \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ i.e. the InhLSS is not (completely) controllable. The characterization results for all possible switching times within the time interval $[0, 12]$ are shown in Fig. 3.4b. Moreover, a similar phenomenon is derived here as in reachability where even though all individual modes are uncontrollable, the switched system composed of those modes under certain switching signals can be controllable. ◊

Example 3.27. Consider InhLSS (3.1) composed of the following two modes

$$(A_0, B_0) = \left(\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right) \text{ and } (A_0, B_0) = \left(\begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right).$$

First, we observe the switched system on $[0, K]$ with $K = 20$ under the mode sequence $(0, 1, 0)$ and switching times $k_1^s = 5$ and $k_2^s = 10$. The sequence of subspaces (3.24) for this switched system is given by

$$\begin{aligned}
\mathcal{N}_3 &= \{0\}, \\
\mathcal{N}_2 &= \left[A_0^{10}\right]^{-1} \left[\mathcal{N}_3 + \mathcal{R}_0(10)\right] = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \\
\mathcal{N}_1 &= \left[A_1^{5}\right]^{-1} \left[\mathcal{N}_2 + \mathcal{R}_1(5)\right] = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
\mathcal{N}_0 &= \left[A_0^{5}\right]^{-1} \left[\mathcal{N}_1 + \mathcal{R}_0(5)\right] = \mathbb{R}^3,
\end{aligned}$$

i.e. the switched system is (completely) controllable. If the second switching time is changed with $k_2^s = 11$, then we have $\mathcal{N}_0 = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ i.e. the switched system is uncontrollable. The characterization results including the reachability for all possible switching times within the time interval $[0, 20]$ are shown in Fig. 3.5. It can be seen that in general, the switched system is reachable and controllable for some switching times, however, it is unreachable and uncontrollable for some other switching times. ◊

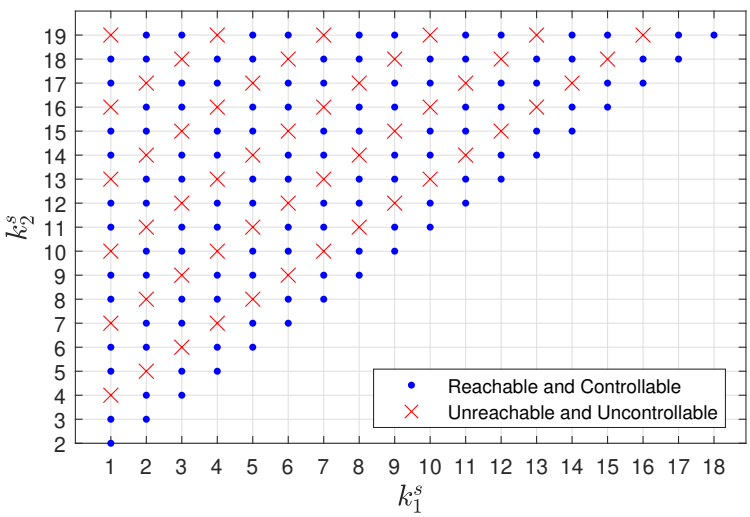


Figure 3.5: Switching times vs reachability/controllability Example 3.27

Finally, the condition for deadbeat controllable can be derived from the characterization of controllability given in Theorem 3.25 considered for $K = 1$:

Corollary 3.28 (Deadbeat controllability characterization of LSSs). The InhLSS (3.1) is **deadbeat controllable** if, and only if,

$$A_{\sigma_0}^{-1}[\operatorname{im} B_{\sigma_0}] = \mathbb{R}^n. \qquad \Diamond$$

Note that $A_{\sigma_0}^{-1}[\operatorname{im} B_{\sigma_0}] = \mathbb{R}^n$ is equivalent to $\operatorname{im} A_{\sigma_0} \subseteq \operatorname{im} B_{\sigma_0}$, which just means that the input can compensate any value $A_{\sigma_0} x_0$ resulting from an arbitrary initial state x_0.

Remark 3.29 (Dwell Time Simplification). By Cayley-Hamilton theorem, if all modes are active for at least n time steps (dwell time), then the local/individual reachable space can be simplified as $\mathcal{R}_i = \operatorname{im}[B_i, A_i B_i, \cdots, A_i^{n-1} B_i]$. In this case, the constructions for reachable and controllable sets (3.20) and (3.24) can be simplified as

$$\begin{aligned} \mathcal{M}_0 &= \mathcal{R}_{\sigma_0}, \\ \mathcal{M}_j &= A_{\sigma_j}^{k_{j+1}^s - k_j^s} \mathcal{M}_{j-1} + \mathcal{R}_{\sigma_j},\ j = 1, 2, ..., J \end{aligned}$$

and

$$\begin{aligned} \mathcal{N}_{J+1} &= \{0\}, \\ \mathcal{N}_j &= \left[A_{\sigma_j}^{k_{j+1}^s - k_j^s}\right]^{-1} [\mathcal{N}_{j+1} + \mathcal{R}_{\sigma_j}],\ j = J, J-1, ..., 0 \end{aligned}$$

respectively. If fact, the reachability space under the above dwell time condition can then also be written as

$$\mathcal{R}_{[0,K]}^{\sigma} = \sum_{j=0}^{J} \psi_\sigma(J+1, j+1) \mathcal{R}_{\sigma_j}. \tag{3.26}$$

The controllability space can however not be written in a similar way, but we can still conclude that

$$\mathcal{N}_{[0,K]}^{\sigma} \supseteq \sum_{j=0}^{J} \psi_\sigma(j+1, 0)^{-1} \mathcal{R}_{\sigma_j}.$$

This difference occurs because for general matrices M and subspaces $\mathcal{P}$ and $\mathcal{Q}$ we have $M(\mathcal{P}+\mathcal{Q}) = M\mathcal{P} + M\mathcal{Q}$ but only $M^{-1}(\mathcal{P}+\mathcal{Q}) \supseteq M^{-1}\mathcal{P} + M^{-1}\mathcal{Q}$. $\Diamond$

Remark 3.30 (Reachable set inclusions of LSSs). It is well known that for nonswitched systems, the following nice inclusion holds:

$$\mathcal{R}_{[0,1]} \subseteq \mathcal{R}_{[0,2]} \subseteq \cdots \subseteq \mathcal{R}_{[0,n]} = \mathcal{R}_{[0,n+1]} = \cdots. \tag{3.27}$$

The subspace inclusion also holds for the controllability spaces, because once an initial state is controllable to zero it can also be controlled to zero on a larger time interval (by simply choosing $u(k) = 0$ after zero was reached) and this property remains true also for the switched case, i.e.

$$\mathcal{C}_{[0,k_1]}^{\sigma} \subseteq \mathcal{C}_{[0,k_2]}^{\sigma} \quad 0 < k_1 < k_2 < K,$$

which implies that (complete) controllability on $[0, K]$ implies (complete) con-

trollability on $[0,k], \forall k > K$. In contrast, for the reachable spaces of switched systems, there is no general relationship between $\mathcal{R}^{\sigma}_{[0,k_1]}$ and $\mathcal{R}^{\sigma}_{[0,k_2]}$ with two different time steps k_1 and k_2 that correspond to two different mode activation time intervals i.e. $k_1 \in [k_i^s, k_i^s+1)$ and $k_2 \in [k_j^s, k_j^s+1)$ with $i \neq j$; a simple example for this is when we have $\mathcal{R}^{\sigma}_{[0,k_i^s]} = \mathbb{R}^n$ and the system switches at $k = k_i^s$ to the mode $(A_{i+1}, B_{i+1}) = (0,0)$ which yields $\mathcal{R}^{\sigma}_{[0,k]} = \{0\}$ for all $k > k_i^s$.

The inclusion (3.27) may even not be valid within a single mode interval; this is illustrated by the following single switch system:

$$\begin{array}{c|c} k < k_1^s: & k \geq k_1^s: \\ x(k+1) = x(k) + \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] u(k) & x(k+1) = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right] x(k) \end{array}$$

where $\mathcal{R}_{[0,K]}$ with $K > k_1^s$ is

$$\mathcal{R}_{[0,K]} = \begin{cases} \operatorname{im}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] & \text{if } K - k_1^s \text{ is odd} \\ \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] & \text{if } K - k_1^s \text{ is even.} \end{cases}$$

However, the following nice observation still applies after the mode-j is active for at least n time steps: (complete) (un)reachability on $[0,k_1], k_1 \in (k_j^s, k_{j+1}^s]$ with $k_1 - k_j^s \geq n$ implies (complete) (un)reachability on $[0, k_1+k], \forall k \leq k_{j+1}^s - k_1$; however, it does not imply (complete) (un)reachability on $[0,k_2],\ k_2 > k_{j+1}^s$ i.e. once the switched system is (un)reachable at some time step after the current mode is active for at least n time steps, it stays (un)reachable within the current mode time interval, and once it switches to another mode, it may be no longer (un)reachable. This is explained as follows; w.l.o.g., we consider here a single switch system with the mode sequence $(0,1)$ and switching time k_1^s. From (3.27), for $k > k_1^s$ we have that $\mathcal{R}_1(1) \subseteq \mathcal{R}_j(2) \subseteq \cdots \subseteq \mathcal{R}_j(n) = \mathcal{R}_j(n+1) = \cdots$, and from (3.20)

$$\mathcal{R}^{\sigma}_{[0,k]} = A_1^{k-k_1^s}\mathcal{R}_0 + \mathcal{R}_1 \text{ and } \mathcal{R}^{\sigma}_{[0,k+1]} = A_1^{k+1-k_1^s}\mathcal{R}_0 + \mathcal{R}_1.$$

Note that $\mathcal{R}_1$ is invariant under A_1, then

$$\mathcal{R}^{\sigma}_{[0,k+1]} = A_1^{k+1-k_1^s}\mathcal{R}_0 + A_1\mathcal{R}_1 = A_1(A_1^{k-k_1^s}\mathcal{R}_0 + \mathcal{R}_1) = A_1\mathcal{R}^{\sigma}_{[0,k]}.$$

If A_1 is nonsingular then clearly $\dim \mathcal{R}^{\sigma}_{[0,k+1]} = \dim \mathcal{R}^{\sigma}_{[0,k]}$ i.e. the system's reachability property remains the same. Now, if A_1 is singular then by applying the state transformation $\widetilde{x} = Px$ to the mode-1 with the nonsingular P and nilpotent N satisfying $PA_1P^{-1} = \begin{bmatrix} \widetilde{A}_1 & 0 \\ 0 & N \end{bmatrix}$, the reachable space is now

$$\mathcal{R}^{\sigma}_{[0,k+1]} = \begin{bmatrix} \widetilde{A}_1^{k+1-k_1^s} & 0 \\ 0 & 0 \end{bmatrix} \operatorname{im}\begin{bmatrix} R_0^1 \\ R_0^2 \end{bmatrix} + \operatorname{im}\begin{bmatrix} \widetilde{R}_1^1 \\ \widetilde{R}_1^2 \end{bmatrix} = \operatorname{im}\begin{bmatrix} \widetilde{A}_1^{k+1-k_1^s} R_0^1 \\ 0 \end{bmatrix} + \operatorname{im}\begin{bmatrix} \widetilde{R}_1^1 \\ \widetilde{R}_1^2 \end{bmatrix}$$

with $\operatorname{im}\begin{bmatrix} R_0^1 \\ R_0^2 \end{bmatrix} = \mathcal{R}_0$ and

$$\operatorname{im}\begin{bmatrix} \widetilde{R}_1^1 \\ \widetilde{R}_1^2 \end{bmatrix} = \operatorname{im}[PB_1, PA_1P^{-1}PB_1, \ldots, (PA_1P^{-1})^{n-1}PB_1],$$

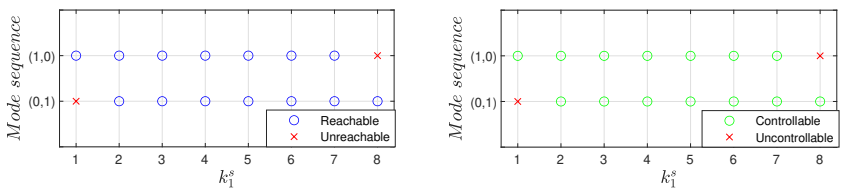
and thus $\dim \mathcal{R}^{\sigma}_{[0,k+1]} = \dim \mathcal{R}^{\sigma}_{[0,k]}$. ◊

From the reachable and controllable sets constructions (3.20) and (3.24), and the confirmation inferred from Example 3.23 and Example 3.26, the reachability and controllability properties depend on the switching times and on how long the individual modes are active. In those examples, this dependency happens due to the rotation matrix A_1 which rotates the reachable/controllable set so that the reachable/controllable set of the switched system is equal to the whole space for some switching times and is unequal to the whole space for some other switching times. Moreover, in those examples, the dependency happens both with switching times that are closed and far away from the time interval boundaries. However, under single switch switching signals, this dependency seems to happen only with switching times that are closed enough to the time interval boundaries as illustrated by the forthcoming Example 3.31. We will show that this dependency indeed only happens when the switching times are closed enough to the time interval boundaries, see the forthcoming Proposition 3.34.

Example 3.31. Consider the InhLSS (3.1) composed of the following two mode:

$$(A_0, B_0) = \left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}\right) \text{ and } (A_1, B_1) = \left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}\right).$$

We observe here the switched system with mode sequences $(0,1)$ and $(1,0)$ on the time interval $[0,K]$ with $K = 9$. With short enough (only one time step) activation time for the first/second mode i.e. when $k_1^s = 1$ or 8, the switched system is unreachable and uncontrollable, however, with longer activation times, the switched system is always reachable and controllable, see Fig. 3.6. ◊



(a) Reachability (b) Controllability

Figure 3.6: Reachability and controllability characterizations for Example 3.31

3.4 Independencies on Switching Times

For non-switched systems, it is well known that even if the pair (A, C) is observable (i.e. the corresponding Kalman observability matrix has full rank) the initial state may not be observable if the output is not measured long enough. And, never more than n output measurements are necessary to extract all the information about the n-dimensional state. For switched systems, this means that if the dwell time is smaller than the state dimension, the observability may be lost just because of the fact, that the switched system does not remain long enough in a mode to extract all available information of the state. A novel aspect of switched systems is however, that even if each mode remains active long enough (i.e. no more information about the state can be obtained by staying longer in that mode) the observability may still depend on how long each mode remains active. Switched systems where this dependence does **not** occur are of special interest and justify the following definition; the motivation for determinability, reachability, and controllability is based on a similar observation as in observability.

Definition 3.32 (Constant Observability/Determinability/Reachability/-Controllability). The observability/determinability/reachability/controllability of the InhLSS (3.1) is called **constant** w.r.t. the mode sequence (σ_j) (under slow switching) if it is either observable/determinable/reachable/controllable or unobservable/undeterminable/unreachable/uncontrollable on $[0, K]$ for all $\sigma \in \mathbb{S}^{[n]}_{(\sigma_j)}$ and all $K \geq Jn$. $\Diamond$

In other words, those constant properties mean that the properties do not depend on the switching times, and changing the switching times does not change the properties (provided each mode remains active long enough). In particular, constant properties indicate a certain robustness of these properties with respect to the switching times. Note that the constant properties defined above do not require that the properties remain the same for all mode sequences, i.e., constant properties do not eliminate the dependence of the properties on the mode sequence. This is already illustrated in Example 3.11 in which the system is (constantly) unobservable w.r.t. the mode sequence $(\sigma_j) = (0, 1)$, however, it is (constantly) observable w.r.t. the mode sequence $(\sigma_j) = (1, 0)$; even though the system in this example was investigated only with a finite number of switching times, by the forthcoming Proposition 3.34, it indeed has constant observability. Furthermore, from the definition, if $K = Jn$ i.e. there is only one possible switching signal with a dwell time of at least n on $[0, K]$, then observability, determinability, reachability, and controllability are trivially constant; however, for $K > Jn$ those properties depend in general

on the specific matrices (A_i, C_i) and the switching times.

Although the observability/determinability/reachability/controllability properties of the previous Examples 3.2-3.31 clearly depend on the switching times, it does satisfy the definition of constant observability/determinability/reachability/controllability because the properties remain constant when each mode is active long enough.

First, obvious situations for constant properties can be derived by direct observation from the corresponding condition. For observability, if the first subspace in (3.16) equals $\{0\}$ then clearly the observability is constant i.e. it is always observable for any k_j^s. For determinability, if $\ker C_i$ for some i equals $\{0\}$ then the determinability is constant. For reachability and controllability, if A_i are idempotent then powers to A_i matrices in the reachability and controllability conditions do not affect the characterization results. Two other situations that yield constant observability/determinability/reachability/controllability are one-dimensional systems and single-switch systems; this is presented in the forthcoming Propositions 3.33 and 3.34.

3.4.1 One-dimensional Systems

Proposition 3.33 (Four properties of one-dimensional LSSs are constant). The observability/determinability/reachability/controllability of (3.8) with one-dimensional states is constant w.r.t. all mode sequences. ◊

Proof. Let (a_i, c_i) be the individual mode, a_i and c_i are scalars.
Part 1: observability and determinability.
Take any switching signal σ, then its unobservable space is

$$\begin{aligned}&\ker[c_{\sigma(0)}, c_{\sigma(1)}a_{\sigma(0)}^{k_1^s-1}, c_{\sigma(2)}a_{\sigma(1)}^{k_2^s-k_1^s-1}a_{\sigma(0)}^{k_1^s-1}, \dots]^\top\\ &= \ker[c_{\sigma(0)}] \cap \ker[c_{\sigma(1)}a_{\sigma(0)}^{k_1^s-1}] \cap \ker[c_{\sigma(2)}a_{\sigma(1)}^{k_2^s-k_1^s-1}a_{\sigma(0)}^{k_1^s-1}] \cap \dots.\end{aligned}$$

For any k_j^s, the unobservable space is equal to $\mathbb{R}$ if $c_i = 0\ \forall i$, or is equal to $\{0\}$ if $c_i \neq 0$ for some i i.e. it is either unobservable or observable for any given k_j^s. Since σ is arbitrary, the observability is constant.
Now, for the determinability, at any time instant k, the subspace $\mathcal{Q}^k$ of (3.17) is either 0 or $\mathbb{R}$ independently of the switching times and the scalar $A_{\sigma(k)}$ is either 0 or a nonzero independently of the switching times. Altogether, $\mathcal{Q}^k$ does not depend on the switching times.
Part 2: reachability and controllability.
Take any mode sequence (σ_j) and any $K \geq Jn$. The local reachable spaces $\mathcal{R}_{\sigma_j}(k_{j+1}^s - k_j^s) = \operatorname{im} B_{\sigma_j}$ are either 0 or $\mathbb{R}$ independently of the switching times and the scalar $A_{\sigma_j}^{k_{j+1}^s - k_j^s}$ is either 0 or a nonzero independently of the switching

times. Altogether, the subspaces $\mathcal{M}_J$ and $\mathcal{N}_0$ do not depend on the switching signals. □

Proposition 3.33 can be explained intuitively by the fact that, in one-dimensional space, it is impossible to have different characterization results e.g. unobservable spaces with different switching signals since we will always get either $\{0\}$ or $\mathbb{R}$ from (3.15), (3.17), (3.20) and (3.24).

3.4.2 Single-switch Systems

For single-switch systems, without any further assumptions, only constant observability, reachability, and controllability are proven in the following proposition. It is still unclear whether determinability is also constant in general. Nevertheless, with some further strict assumptions, determinability is constant; this is discussed in the forthcoming Remark 3.35.

Proposition 3.34 (Three properties of single-switch LSSs are constant). Consider the InhLSS (3.1) with the number of switches is one, i.e., $J = 1$. Then, its observability, reachability, and controllability are constant w.r.t. all mode sequences. ◊

Proof. W.l.o.g. consider the mode sequence $(0, 1)$ with the switching time $k^s \geq n$.
Part 1: Observability.
Assume first A_0 is nonsingular, the observability condition (3.16) is equivalent to

$$A_0^{k^s}\mathcal{O}_0 \cap \mathcal{O}_1 = \{0\}. \tag{3.28}$$

where $A_0^k\mathcal{O}_0 = \mathcal{O}_0 \ \forall k \in \mathbb{N}$ with $k_s \geq n$ (by Cayley-Hamilton), i.e. the observability does not depend on k^s.

Assume now A_0 is singular. Then we can rewrite A_0 in the Jordan canonical form $A_0 = S_0 \begin{bmatrix} N_0 & 0 \\ 0 & \widetilde{A}_0 \end{bmatrix} S_0^{-1}$ where $S_0 \in \mathbb{R}^{n\times n}$ is invertible, $N_0 \in \mathbb{R}^{n_0\times n_0}$ is a nilpotent with nilpotency index at most n, and $\widetilde{A}_0 \in \mathbb{R}^{(n-n_0)\times(n-n_0)}$ is invertible. By state transformation $\widetilde{x}(k) = S_0^{-1}x(k)$ the observability condition becomes

$$\ker \begin{bmatrix} C_0S_0 \\ C_0S_0\begin{bmatrix} N_0 & 0 \\ 0 & \widetilde{A}_0 \end{bmatrix} \\ \vdots \\ C_0S_0\begin{bmatrix} N_0 & 0 \\ 0 & \widetilde{A}_0 \end{bmatrix}^{n-1} \end{bmatrix} \cap \ker\left(O_1 \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{A}_0^{k^s} \end{bmatrix}\right) = \{0\} \tag{3.29}$$

An arbitrary vector $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \mathbb{R}^n$, $x_1 \in \mathbb{R}^{n_0}$ satisfies

$$\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \ker\left(O_1 \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{A}_0^{k^s} \end{bmatrix}\right) \Leftrightarrow O_1 \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{A}_0^{k^s} \end{bmatrix} \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = 0$$

$$\Leftrightarrow \begin{pmatrix} 0 \\ \widetilde{A}_0^{k^s} x_2 \end{pmatrix} \in \ker O_1 = \ker[O_1^1, O_1^2]$$

$$\Leftrightarrow x_1 \text{ is arbitrary and } O_1^2 \widetilde{A}_0^{k^s} x_2 = 0 \Leftrightarrow x_1 \text{ is arbitrary and } x_2 \in \widetilde{A}_0^{-k^s} \ker O_1^2.$$

The vector $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$ also satisfies $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \ker \widetilde{O}_0 = \ker[\widetilde{O}_0^1, \widetilde{O}_0^2]$ if, and only if,

$$[\widetilde{O}_0^1, \widetilde{O}_0^2] \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \Leftrightarrow \widetilde{O}_0^1 x_1 = -\widetilde{O}_0^2 x_2.$$

Case 1: $\ker \widetilde{O}_0^1 \neq \{0\}$. In this case, every $x_1 \in \ker \widetilde{O}_0^1$ satisfies for any k^s

$$\left(\begin{smallmatrix} x_1 \\ 0 \end{smallmatrix}\right) \in \ker \widetilde{O}_0 \cap \ker \left(O_1 \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{A}_0^{k^s} \end{bmatrix} \right)$$

which means that the switched system is not-observable for every k^s.

Case 2: $\ker \widetilde{O}_0^1 = \{0\}$. In this case, we have that

$$\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \ker[\widetilde{O}_0^1, \widetilde{O}_0^2] \cap \ker \left([O_1^1, O_1^2] \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{A}_0^{k^s} \end{bmatrix} \right)$$

$$\Leftrightarrow x_2 \in [\widetilde{O}_0^2]^{-1}(\operatorname{im} \widetilde{O}_0^1) \cap \widetilde{A}_0^{-k^s} \ker O_1^2 \text{ and } x_1 = [\widetilde{O}_0^1]^{-1}(\widetilde{O}_0^2 x_2),$$

and the observability condition that corresponds to x_2 becomes

$$[\widetilde{O}_0^2]^{-1}(\operatorname{im} \widetilde{O}_0^1) \cap \widetilde{A}_0^{-k^s} \ker O_1^2 = \{0\} \Leftrightarrow \widetilde{A}_0^{k^s} [\widetilde{O}_0^2]^{-1}(\operatorname{im} \widetilde{O}_0^1) \cap \ker O_1^2 = \{0\}.$$

Now, we can focus only on the first subspace to study whether the intersection above depends on k^s or not. By assumption of $\widetilde{A}_0$ invertible and basic algebra (see Appendix A), we have that

$$x_2 \in \widetilde{A}_0^{k^s} [\widetilde{O}_0^2]^{-1}(\operatorname{im} \widetilde{O}_0^1) \Leftrightarrow \widetilde{A}_0^{-k^s} x_2 \in [\widetilde{O}_0^2]^{-1}(\operatorname{im} \widetilde{O}_0^1) \Leftrightarrow \exists \widetilde{x}_2 : \widetilde{O}_0^2 \widetilde{A}_0^{-k^s} x_2 = \widetilde{O}_0^1 \widetilde{x}_2$$

$$\Leftrightarrow \exists \widetilde{x}_2 : \begin{bmatrix} \widetilde{C}_0^2 \widetilde{A}_0^{-k^s} \\ \widetilde{C}_0^2 \widetilde{A}_0^{-k^s+1} \\ \vdots \\ \widetilde{C}_0^2 \widetilde{A}_0^{-k^s+n-1} \end{bmatrix} x_2 = \begin{bmatrix} \widetilde{C}_0^1 \widetilde{x}_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \widetilde{C}_0^2 \widetilde{A}_0^{-k^s} x_2 = \widetilde{C}_0^1 \widetilde{x}_2 \text{ and } \begin{bmatrix} \widetilde{C}_0^2 \widetilde{A}_0^{-k^s+1} \\ \widetilde{C}_0^2 \widetilde{A}_0^{-k^s+2} \\ \vdots \\ \widetilde{C}_0^2 \widetilde{A}_0^{-k_s+n-1} \end{bmatrix} x_2 = 0$$

$$\Leftrightarrow \widetilde{C}_0^2 \widetilde{A}_0^{-k^s} x_2 \in \operatorname{im} \widetilde{C}_0^1 \text{ and } x_2 \in \ker \underbrace{\begin{bmatrix} \widetilde{C}_0^2 \widetilde{A}_0^{-k^s+1} \\ \widetilde{C}_0^2 \widetilde{A}_0^{-k^s+2} \\ \vdots \\ \widetilde{C}_0^2 \widetilde{A}_0^{-k_s+n-1} \end{bmatrix}}_{=:P \in \mathbb{R}^{(p(n-1)) \times (n-n_0)}}. \tag{3.30}$$

Note that the dimension of $\widetilde{A}_0$ is $(n - n_0) \times (n - n_0)$ where $n_0 > 0$ and thus $n - 1 \geq n - n_0$, i.e. the matrix P in the argument (3.30) has sufficient number

of block rows so that by Lemma A.4,

$$\ker P = \ker \begin{bmatrix} \widetilde{C}_0^2 \\ \widetilde{C}_0^2 \widetilde{A}_0 \\ \vdots \\ \widetilde{C}_0^2 \widetilde{A}_0^{n-n_0-1} \end{bmatrix} = \ker \widetilde{O}_0^2.$$

Thus the argument (3.30) is equivalent to

$$\widetilde{C}_0^2 \widetilde{A}_0^{-k^s} x_2 \in \operatorname{im} \widetilde{C}_0^1 \text{ and } x_2 \in \ker \widetilde{O}_0^2. \tag{3.31}$$

Utilizing again Lemma A.4, we have for every $x_2 \in \ker \widetilde{O}_0^2$ that $\widetilde{A}_0^{-k^s} x_2 \in \ker \widetilde{O}_0^2$, in particular, $\widetilde{C}_0^2 \widetilde{A}_0^{-k^s} x_2 = 0 \in \operatorname{im} \widetilde{C}_0^1$. Altogether, we now have

$$x_2 \in \widetilde{A}_0^{k^s} [\widetilde{O}_0^2]^{-1} (\operatorname{im} \widetilde{O}_0^1) \Leftrightarrow x_2 \in \ker \widetilde{O}_0^2$$

which is independent of k^s.

Part 2: Reachability.

The reachable space at $K > k^s + n$ is

$$\mathcal{R}^\sigma_{[0,K]} = A_1^{K-k^s} \mathcal{R}_0 + \mathcal{R}_1$$

with $\mathcal{R}_i = \operatorname{im}[B_i, A_i B_i, \cdots, A_i^{n-1} B_i]$, $i = 0, 1$. Clearly, $\mathcal{R}_1$ is fixed no matter the switching time k^s, and $\mathcal{R}_1$ is A_1-invariant. A nonconstant reachability happens if $\mathcal{R}^\sigma_{[0,K]} = \mathbb{R}^n$ for some k^s and $\mathcal{R}^\sigma_{[0,K]} \subsetneq \mathbb{R}^n$ for some other k^s, or in other words $\dim \mathcal{R}^\sigma_{[0,K]} = n$ for some k^s and $\dim \mathcal{R}^\sigma_{[0,K]} < n$ for some other k^s.

Case 1: A_1 is nonsingular. Assume first for $k^s = K - n$, $\mathcal{R}^\sigma_{[0,K]} = A_1^n \mathcal{R}_0 + \mathcal{R}_1 \subsetneq \mathbb{R}^n$ i.e. unreachable then $A_1^{n+l} \mathcal{R}_0 + \mathcal{R}_1 \subsetneq \mathbb{R}^n$ for any $l \in \mathbb{N}$ i.e. it remains unreachable for any other possible k^s since $\dim A_1^{n+l} \mathcal{R}_0$ cannot increase. Now, if for $k^s = K - n$, $\mathcal{R}^\sigma_{[0,K]} = A_1^n \mathcal{R}_0 + \mathcal{R}_1 = \mathbb{R}^n$ i.e. reachable then from the dimension formula, $\dim A_1^n \mathcal{R}_0 + \dim \mathcal{R}_1 = n$ and $\dim A_1^{n+l} \mathcal{R}_0 + \dim A_1^l \mathcal{R}_1 = n$, and thus $\dim A_1^{n+l} \mathcal{R}_0 + \dim \mathcal{R}_1 = n$ for any $l \in \mathbb{N}$ since $\mathcal{R}_1$ is A_1-invariant. Hence, it stays reachable for any other possible k^s.

Case 2: A_1 is singular. Then, there exists a nonsingular matrix P such that

$$P^{-1} A_1 P = \begin{bmatrix} N & 0 \\ 0 & \widetilde{A}_1 \end{bmatrix}$$

where N is a nilpotent with nilpotency index less than n and $\widetilde{A}_1$ is nonsingular. Applying the state transformation $\widetilde{x} = P^{-1} x$, the system's modes can now be written as

$$(\widetilde{A}_0, \widetilde{B}_0) = (P^{-1} A_0 P, P^{-1} B_0) \text{ and}$$

$$(\widetilde{A}_1, \widetilde{B}_1) = \left(\begin{bmatrix} N & 0 \\ 0 & \widetilde{A}_1 \end{bmatrix} \right), P^{-1} B_1).$$

The reachable space for this new transformed system is

$$\widetilde{\mathcal{R}}^\sigma_{[0,K]} = \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{A}_1^{K-k^s} \end{bmatrix} \widetilde{\mathcal{R}}_0 + \widetilde{\mathcal{R}}_1$$

with $\widetilde{\mathcal{R}}_i = \operatorname{im}[\widetilde{B}_i, \widetilde{A}_i \widetilde{B}_i, \cdots, \widetilde{A}_i^{n-1} \widetilde{B}_i]$. The same arguments as in Case 1 apply

here, and thus the reachability is constant.

Part 3: controllability

The controllable set for the mode sequence $(0,1)$ with the switching time k^s is given by

$$\mathcal{N}_0 = \left[A_0^{k^s}\right]^{-1}\left(\left[A_1^{K-k^s}\right]^{-1}\mathcal{R}_1 + \mathcal{R}_0\right).$$

Now, consider the transformation $\bar{x}(k) = P_{\sigma(k-1)}^{-1}x(k)$ with some invertible $P_0, P_1 \in \mathbb{R}^{n\times n}$ such that $P_0A_0P_0^{-1} = \begin{bmatrix}\widetilde{A}_0 & 0\\ 0 & N_0\end{bmatrix}$ and $P_1A_1P_1^{-1} = \begin{bmatrix}\widetilde{A}_1 & 0\\ 0 & N_1\end{bmatrix}$ with $\widetilde{A}_0$ and $\widetilde{A}_1$ are invertible, $N_0 \in \mathbb{R}^{(n-r_0)\times(n-r_0)}$ and $N_1 \in \mathbb{R}^{(n-r_1)\times(n-r_1)}$ are nilpotent with nilpotency index less than n, r_0 and r_1 are the rank of A_0 and A_1 respectively and not necessarily $r_0 = r_1$. Then, we can rewrite $\mathcal{N}_0$ as

$$\mathcal{N}_0 = \begin{bmatrix}A_0^{k^s} & 0\\ 0 & 0\end{bmatrix}^{-1}\left(\begin{bmatrix}A_1^{K-k^s} & 0\\ 0 & 0\end{bmatrix}^{-1}\mathcal{R}_1 + \mathcal{R}_0\right).$$

For all possible k^s, the dimension of $\begin{bmatrix}A_1^{K-k^s} & 0\\ 0 & 0\end{bmatrix}^{-1}\mathcal{R}_1$ is fixed due to the nonsingularity of $\widetilde{A}_1$, and thus also the dimension of $\mathcal{N}_0$ due to the nonsingularity of $\widetilde{A}_0$. Therefore, it is not possible to have $\mathcal{N}_0 = \mathbb{R}^n$ for some k^s and $\mathcal{N}_0 \subsetneq \mathbb{R}^n$ for some other k^s. □

It is still not clear whether the determinability of single-switch systems is also constant. However, it is conjectured that that property is also constant.

Conjecture 3.35 (Determinability of single-switch LSSs is constant). Consider the InhLSS (3.1) with the number of switches is one, i.e., $J = 1$. Then, its determinability is constant w.r.t. all mode sequences. ◊

Neither complete proof of constant determinability nor counter-example has been derived. One possible attempt to prove constant determinability is by proving that the subspace $\mathcal{Q}^K$ can rewrite

$$\mathcal{Q}^K = \ker C_1 \cap A_1(\ker C_1 \cap A_1(\ldots \cap A_1(\ker C_0 \cap A_0(\ker C_0 \cap A_0 \ldots \\ \cap A_0(\ker C_0 \cap A_0 \ker C_0))))).$$

as

$$\mathcal{Q}^K = \ker C_1 \cap A_1 \ker C_1 \cap A_1^2 \ker C_1 \cap \ldots \cap A_1^{K-k^s-2}\ker C_1 \cap A_1^{K-k^s-1} \\ (\ker C_0 \cap A_0 \ker C_0 \cap A_0^2 \ldots \cap A_0^{k^s-1}\ker C_0). \tag{3.32}$$

Unfortunately, this is not correct since, in general, for any matrix $A \in \mathbb{R}^{n\times n}$ and subspaces $\mathcal{V}, \mathcal{W} \subseteq \mathbb{R}^n$, we only have $A(\mathcal{V}\cap\mathcal{W}) \subseteq A\mathcal{V}\cap A\mathcal{W}$ (with equality if and only if $(\mathcal{V}+\mathcal{W})\cap\ker A = (\mathcal{V}\cap\ker A)+(\mathcal{W}\cap\ker A)$, which holds in particular for any invertible A). However, with a further assumption of A_0 and A_1 being invertible, then (3.32) is true and by the property of $\mathcal{V}\cap A\mathcal{V}\cap\cdots\cap A^n\mathcal{V} =$

$\mathcal{V} \cap A\mathcal{V} \cap \cdots \cap A^{n+\ell}\mathcal{V}$ for all $\ell \in \mathbb{N}$ for any matrix A and any subspace $\mathcal{V}$, we conclude that $\mathcal{Q}^K$ is independent of k^s.

3.5 Concluding Remarks

Observability, determinability, reachability, and controllability for discrete-time linear switched systems have been characterized in this chapter. Necessary and sufficient conditions have been established. Moreover, the notion of constant observability/determinability/reachability/controllability has been introduced to study when those properties do not depend on the switching times. It turned out that one-dimensional systems have constant observability, determinability, reachability, and controllability properties. For single-switch systems only constant observability, reachability, and controllability have been proven. However, it is still not clear whether the determinability of single-switch systems is also constant; this can be considered as one of the future research directions.

Part II

Singular Linear (Switched) Systems

Contents of this part are based on the following papers:

- **Sutrisno** and Stephan Trenn, "Observability of Singular Linear Switched Systems in Discrete Time: Single Switch Case," in *Proc. European Control Conference (ECC)*, 2021. `https://doi.org/10.23919/ECC54610.2021.9654844`
- **Sutrisno** and Stephan Trenn, "Discrete-time Singular Linear (Switched) Systems: Solvability, Reachability, and Controllability Characterizations," in *Proc. 62nd IEEE Conference on Decision and Control (CDC)*, 2023. (*to appear*).
- **Sutrisno** and Stephan Trenn, "Switched linear singular systems in discrete time: solution theory and observability notions," *submitted to journal*.
- **Sutrisno** and Stephan Trenn, "Inhomogeneous singular linear switched systems in discrete time: solvability and reachability under restricted switching signals," *under preparation*.

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Chapter 4 Solvability

"No matter how hard the constraints are, a solution can be found through passion and patience."

Contents

Ordinary systems of the form (3.1) do not possess any solvability issues since for any initial state $x_0 \in \mathbb{R}^n$, any finite time interval $[0, K]$, $K \in \mathbb{N}$, and any switching signal (for switched systems), the system with the initial condition $x(0) = x_0 \in \mathbb{R}^n$ always has a unique solution. All initial states $x_0 \in \mathbb{R}^n$ are then consistent. Furthermore, solutions of ordinary systems are causal in the sense that any solution $x(k)$ at any time instant $k \in \mathbb{N}$ is completely determined by past information (states and inputs), this can also be directly seen from their explicit solution formulas (2.9) for non-switched systems and (3.2) for switched systems.

Meanwhile, singular systems have three solvability issues. To describe the issues, consider the equation $Ex(1) = Ax(0)$ with singular E. The first issue is that not all initial states $x_0 \in \mathbb{R}^n$ are consistent; this is due to x_0 must satisfy $E\xi = Ax_0$ for some $\xi \in \mathbb{R}^n$, i.e., only initial states $x_0 \in A^{-1}(\operatorname{im} E) = \{\, x \in \mathbb{R}^n \mid Ax \in \operatorname{im} E \,\} \subsetneq \mathbb{R}^n$ are consistent. The second solvability issue is that in general, the solution is not unique. Note that a solution $x(1)$ must

satisfy $x(1) \in E^{-1}(Ax_0) = \{E^+Ax_0\} + \ker E$ (see Lemma A.2). The non-uniqueness of $x(1)$ can be seen from this inclusion since all $x(1) \in \ker E$ also solves $Ex(1) = Ax(0)$ (with $x(0) = 0$). This issue does not appear in ordinary systems since if E is nonsingular, then $\ker E = \{0\}$ implies the uniqueness of $x(1)$. The third issue is non-causality in the sense that, in general, a solution at a time instant depends also on future states and inputs. This can be directly seen in singular systems that have the form $\left[\begin{smallmatrix} I & 0 \\ 0 & N \end{smallmatrix}\right] \left[\begin{smallmatrix} \bar{x}_1(k+1) \\ \bar{x}_2(k+1) \end{smallmatrix}\right] = \left[\begin{smallmatrix} J & 0 \\ 0 & I \end{smallmatrix}\right] \left[\begin{smallmatrix} \bar{x}_1(k) \\ \bar{x}_2(k) \end{smallmatrix}\right]$ where N is a nilpotent matrix. From the pure singular subsystem's equation $N\bar{x}_2(k+1) = \bar{x}_2(k)$, notice that $\bar{x}_2(k+1)$ determines $\bar{x}_2(k)$. In practice, this non-causality feature is generally not desired due to the need for future information which is commonly not available yet.

Motivated by those solvability issues, in this chapter, the solution theory for homogeneous and inhomogeneous (both non-switched and switched) singular linear systems is investigated. For homogeneous systems, new solvability notions that require the existence of a unique solution and causality in terms of states will be introduced. For inhomogeneous systems, two types of new solvability notions will be introduced in terms of states and inputs based on whether the current input affects the current state or not. The conditions for those solvability notions will be characterized by utilizing some further notions of index-1 that are developed based on the index-1 notion of a matrix pair presented in the Preliminaries chapter. All solvability notions will require causality because this feature is desired in applications due to future information commonly being not available or uncertain. One may deal with future information by considering it as an uncertain variable and solving the system under uncertainty; however, this is out of the scope of this thesis and can be considered as a future research direction.

4.1 Homogeneous Systems

In this subsection, homogeneous singular linear systems both without switching and with switching are considered. The solvability of this system class will be studied together with introducing its surrogate systems which will be utilized in the observability characterization in the forthcoming Chapter 5. In particular, results in this part are the foundation for studying inhomogeneous systems.

4.1.1 Nonswitched Systems

Consider the system class of discrete-time (unswitched) Homogeneous Singular Linear System (HomSLS) of the form

$$Ex(k+1) = Ax(k), \tag{4.1a}$$
$$y(k) = Cx(k) \tag{4.1b}$$

with $k \in \mathbb{N}$ representing the time instant/step and where $E, A \in \mathbb{R}^{n\times n}, C \in \mathbb{R}^{p\times n}$ are known constant matrices, the matrix E may be singular with rank $E = r \leq n$, $x : \mathbb{N} \to \mathbb{R}^n$ is the state, and $y : \mathbb{N} \to \mathbb{R}^p$ is the output. The state equation (4.1a) can be represented by the matrix pair (E, A), and in the following, they are interchangeable.

Assume the matrix pair (E, A) is regular, see Definition 2.1. Then, by utilizing the QWF (2.2) in Lemma 2.4, the state equation (4.1a) can be transformed into

$$\begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix} \bar{x}(k+1) = \begin{bmatrix} J & 0 \\ 0 & I_{n-r} \end{bmatrix} \bar{x}(k) \tag{4.2}$$

with $T^{-1}x =: \bar{x} = [\bar{x}_1, \bar{x}_2]^\top,\ \bar{x}_1 \in \mathbb{R}^r,\ \bar{x}_2 \in \mathbb{R}^{n-r}$. The subsystem $\bar{x}_1(k+1) = J\bar{x}_1(k)$ is called the ordinary subsystem whereas $N\bar{x}_2(k+1) = J\bar{x}_2(k)$ is called the pure singular subsystem. Using this representation, the explicit solution of (4.1a) on a finite time domain $[0, K],\ K \in \mathbb{N}$ in $\bar{x} = [\bar{x}_1, \bar{x}_2]^\top$ coordinate is given by

$$\bar{x}_1(k) = J^k \bar{x}_1(0), \tag{4.3a}$$
$$\bar{x}_2(k) = N^{K-k} \bar{x}_2(K). \tag{4.3b}$$

This is the classical approach for solving system (4.1) via the state transformation using T in which if system (4.1) is considered on a finite time interval $[0, K]$. For regular (E, A), system (4.1) has always a unique solution for arbitrary $\bar{x}_1(0)$ and $\bar{x}_2(K)$; this can be directly seen from (4.3) in which $\bar{x}_1(0)$ and $\bar{x}_2(K)$ can be freely chosen, and the solution can be derived by forward iteration for $\bar{x}_1$ and backward iteration for $\bar{x}_2$. This means that the regularity of (E, A) is sufficient for system (4.1) to have a unique solution. However, note that the final state $x(K)$ determines the states at $k < K$, i.e., the system is in general not causal in terms of states. A new approach, which utilizes a generalized inverse matrix (see Definition A.1) and projector (see Lemma A.3), will be introduced in the forthcoming Lemma 4.3. This makes it possible to formulate the so-called surrogate system–an ordinary system that has the same solutions–and the explicit solution formula in the original state's coordinate without any state transformation for solvable systems in the sense of the solvability notion that will be introduced in the forthcoming Definition 4.1.

4.1.1.1 Definitions

The following definition introduces a new solvability notion for system (4.1) in which both the existence of a unique solution and causality are taken into

account, see the forthcoming Proposition 4.5 for further discussion.

Definition 4.1 (Solvability of HomSLSs). The HomSLS (4.1) is called **locally uniquely solvable** (for short just **solvable**) if, for all $k_1 \in \mathbb{N}$ and for all $x_0 \in A^{-1}(\operatorname{im} E) =: \mathcal{S}$ there exists a unique solution on $[0, k_1]$ of (4.1) considered on $[0, k_1]$ with $x(0) = x_0$. ◊

Remark 4.2 (Local solvability notion for singular systems). The solvability notion in Definition 4.1 is defined on a finite time interval $[0, k_1]$ where such a solution is a sequence $x(0), x(1), \ldots, x(k_1) \in \mathbb{R}^n$ which satisfies (4.1) for $k = 0, 1, \ldots k_1 - 1$ and furthermore for which another (not necessarily unique) value $x(k_1 + 1) \in \mathbb{R}^n$ exists such that (4.1) also holds for $k = k_1$. Note that this last requirement is different from non-singular systems, where $x(k_1)$ is already uniquely determined by the systems equations considered up to $k_1 - 1$. However, due to the singularity of $E_{\sigma(k_1-1)}$ the value of $x(k_1)$ is not yet fully fixed by the information of (4.1) at $k = k_1 - 1$ and it is necessary to incorporate the additional information which can be concluded for $x(k_1)$ from (4.1) evaluated at $k = k_1$. ◊

4.1.1.2 Characterizations

The characterization for the solvability notion defined in Definition 4.1 is presented in the following lemma, and furthermore, the one-step map and its corresponding surrogate system are also introduced in this lemma.

Lemma 4.3 (Solvability characterization of HomSLSs). The HomSLS (4.1) is solvable in the sense of Definition 4.1 if and only if (E, A) is index-1 (see Definition 2.5). If solvable, its solution satisfies

$$x(k+1) = \Phi_{(E,A)} x(k), \;\; x(0) \in \mathcal{S} \tag{4.4}$$

where $\Phi_{(E,A)} = \Pi_{\mathcal{S}}^{\ker E} E^+ A$ is called the one-step map, $\Pi_{\mathcal{S}}^{\ker E}$ is the canonical projector from $\ker E \oplus \mathcal{S}$ to $\mathcal{S}$, E^+ is a generalized inverse of E (see Definition A.1), and the ordinary system (4.4) is called the surrogate system for (4.1). ◊

Proof. **Part 1: the solvability condition**
Necessity: For any initial value $x_0 \in \mathcal{S}$ there exists a unique solution on $[0, 1]$, and in particular, $x(1)$ is uniquely determined by considering (4.1a) for $k = 0$ and $k = 0$. By Lemma A.2 applied to (4.1a) for $k = 1$, the value $x(1)$ satisfies

$$x(1) \in E^{-1}(Ax_0) = \{E^+ Ax_0\} + \ker E. \tag{4.5}$$

On the other hand, considering (4.1a) at $k = 1$ (not making any assumptions about the unknown $x(2)$), $x(1)$ must satisfy

$$x(1) \in \{A^{-1}(\operatorname{im} E)\} = \mathcal{S}. \tag{4.6}$$

Hence $x(1)$ is uniquely determined for all $x_0 \in \mathcal{S}$ if, and only if,

$$\mathcal{S} \cap (\{E^+Ax_0\} + \ker E) \text{ is a singleton.}$$

Using Lemma A.3 with $\mathbb{U} = \{E^+Ax_0\}$, $\mathbb{Z} = \{0\}$, $\mathcal{V} = \mathcal{S}$ and $\mathcal{W} = \ker E$, $\mathcal{S} \cap (\{E^+Ax_0\} + \ker E)$ is a singleton if and only if $\{E^+Ax_0\} \subseteq \ker E \oplus \mathcal{S}$ which is true if and only if (E, A) is index-1 (by Corollary 2.6).

Sufficiency: This is proved inductively, that if for any $x_0 \in \mathcal{S}$ there exists a unique solution on $[0, k]$, then there also exists a unique solution on $[0, k+1]$. This, together with the trivial observation that $x(0) = x_0$ is the unique solution of (4.1a) with $x(0) = x_0 \in \mathcal{S} = A^{-1}(\operatorname{im} E)$, considered only for $k = 0$ will prove the solvability. Now, given $x(k)$, choose $x(k+1) \in \mathcal{S} \cap (\{E^+Ax(k)\} + \ker E)$ which is possible due to Lemma A.3. Then $x(k+1) \in \{E^+Ax(k)\} + \ker E$ implies that $Ex(k+1) = EE^+Ax(k)$. Since $x(k) \in \mathcal{S}$ (because x is a solution on $[0, k]$), it follows that $Ax(k) \in \operatorname{im} E$, i.e. there exists v such that $Ax(k) = Ev$. Hence $Ex(k+1) = EE^+Ev = Ev = Ax(k)$ which shows that $x(k+1)$ satisfies (4.1a). Furthermore, $x(k+1)$ also satisfies (4.1a) for $k+1$ because $x(k+1) \in \mathcal{S}$. This shows that x is indeed a solution of (4.1a) on $[0, k+1]$. Uniqueness follows from the fact, that by Lemma A.3 the set $\mathcal{S} \cap (\{E^+Ax(k)\} + \ker E)$ is a singleton.

Part 2: the surrogate system (4.4)

For a given solution $x(k)$ at any k of the solvable system (4.1), substituting $\mathbb{U} = \{E^+Ax(k)\}$, $\mathbb{Z} = \{0\}$, $\mathcal{V} = \mathcal{S}$ and $\mathcal{W} = \ker E$ into formula (A.1) provides

$$x(k+1) = (\{E^+Ax(k)\} + \ker E) \cap \mathcal{S} = \Pi_{\mathcal{S}}^{\ker E} E^+Ax(k)$$

i.e. the surrogate system (4.4) is valid. □

If the QWF is utilized, then via the transformed system (4.2) with $N = 0$, the one-step map $\Phi_{(E,A)}$ can be considered with $\Pi_{\mathcal{S}}^{\ker E} = E^+ = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$ i.e.

$$\Phi_{(E,A)} := T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1}. \tag{4.7}$$

This is the one-step map that has been proposed in [15] via QWF, and in addition, $\Phi_{(E,A)}$ in the form of (4.7) is independent of the specific choice of T [15, Lemma 2.2]. If the form $\Phi_{(E,A)} = \Pi_{\mathcal{S}}^{\ker E} E^+A$ is considered, then the one-step map representation $\Phi_{(E,A)}$ or its corresponding surrogate system (4.4) is not unique due to the nonuniqueness of E^+. However, its action to a particular initial state results in a unique solution. This is justified as follows. The generalized inverse E^+ is only applied to vectors from the subspace $A\mathcal{S} = \operatorname{im} E \cap \operatorname{im} A \subseteq \operatorname{im} E$ which implies (cf. the discussion after Definition A.1) that indeed the action of $\Phi_{(E,A)}$ is unique when restricted to the relevant subspace. In particular, if (4.1a) being a non-singular system, then $\Phi_{(E,A)} = E^{-1}A$, and if $E = I$ then $\Phi_{(E,A)} = A$. Consequently, the explicit solution of the solvable

system (4.1) can be written as follows:

$$\begin{aligned} x(k) &= \Phi^k_{(E,A)} x(0),\ x(0) \in \mathcal{S} \\ y(k) &= C\Phi^k_{(E,A)} x(0). \end{aligned} \tag{4.8}$$

The following example illustrates a solvable system and its corresponding one-step map and surrogate system.

Example 4.4. Consider system (4.1a) with

$$(E, A) = \left(\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \right).$$

Geometric computations provide

$$\ker E = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \text{ and } \mathcal{S} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

The matrix pair (E, A) is regular and index-1 since $\ker E \oplus \mathcal{S} = \mathbb{R}^n$, and thus the system is solvable. With

$$\Pi^{\ker E}_{\mathcal{S}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -2 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } E^+ = \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix},$$

its one-step map and surrogate system are given by

$$\Phi_{(E,A)} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ -2 & 0 & 0 & -2 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \text{ and } x(k+1) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ -2 & 0 & 0 & -2 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} x(k),\ x(0) \in \mathcal{S}. \qquad \Diamond$$

4.1.1.3 Discussion on causality

Intuitively, the HomSLS (4.1) is called **causal** (in terms of states) if on any time interval $[0, k_1]$, $k_1 \in \mathbb{N}$, all solutions at any time instant $k \in [0, k_1]$ are determined completely by the initial state $x(0)$. In particular, note that causality does not require that the system has unique solutions.

Note that by requiring the system has a unique solution on every time interval $[0, k]$, $k \in \mathbb{N}$, the solvability notion in Definition 4.1 implicitly requires that the system is causal. A solvable system (4.1) is therefore causal, and furthermore, causality is in fact necessary for solvability; this is shown in the following proposition:

Proposition 4.5 (Solvable HomSLSs are causal). If the HomSLS (4.1) is solvable in the sense of Definition 4.1 if and only if it is causal. $\Diamond$

Proof. This comes from the fact that causality is equivalent to the matrix pair (E, A) being index-1, which can be seen from the explicit solution (4.3b) in which the final state $x(K)$ does not determine $x(K-1), x(K-2), \ldots$ if and

only if $N = 0$ [61]. In particular, solvability implies causality can be directly seen from its surrogate system (4.4) or from its explicit solution formula (4.8) in which only the initial state $x(0)$ determines $x(k)$ for any $k \in \mathbb{N}$. □

Note that only regular singular systems, unless stated otherwise, are considered in this study. In fact, a nonregular singular system may be causal; this is illustrated by the system $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] x(k+1) = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] x(k)$ which is nonregular and is causal.

4.1.2 Switched Systems

Consider now the system class of Homogeneous Singular Linear Switched System (HomSLSS), where each mode is a HomSLS (4.1), of the form

$$E_{\sigma(k)} x(k+1) = A_{\sigma(k)} x(k) \tag{4.9a}$$

$$y(k) = C_{\sigma(k)} x(k) \tag{4.9b}$$

where $\sigma : \mathbb{N} \to \{1, 2, \ldots, \mathrm{p}\}$ is the switching signal determining which mode $\sigma(k)$ is active at time instant k and which has form (2.11); $E_i, A_i \in \mathbb{R}^{n \times n}$, $C_i \in \mathbb{R}^{p \times n}$ are constant matrices for each $i \in \{0, 1, \ldots, \mathrm{p}\}$; E_i may be singular. This system can be represented by the family of matrix pairs $\{(E_0, A_0), (E_1, A_1), \ldots, (E_\mathrm{p}, A_\mathrm{p})\} =: \{(E_i, A_i)\}_{i=0}^{\mathrm{p}}$, and in the following they are interchangeable. For the mode-i, define

$$\mathcal{S}_i := A_i^{-1}(\operatorname{im} E_i) = \{\xi \in \mathbb{R}^n : A_i \xi \in \operatorname{im} E_i\}. \tag{4.10}$$

4.1.2.1 Definitions

The solvability notion in non-switched systems is generalized for switched systems with respect to a given switching signal, this is formally defined as follows:

Definition 4.6 (Solvability notion for HomSLSSs). For a given family of matrix pairs $\{(E_i, A_i)\}_{i=0}^{\mathrm{p}}$ and a given switching signal σ, the HomSLSS (4.9) is said to be **locally uniquely solvable** (for short just **solvable**) if, for all $k_0, k_1 \in \mathbb{N}$, $k_1 > k_0$ and for all $x_{k_0} \in \mathcal{S}_{\sigma(k_0)}$ there exists a unique solution of (4.9) considered on $[k_0, k_1]$ with $x(k_0) = x_{k_0}$. ◊

Note that the solvability notion above requires the ability to consider the SLSS (4.9) starting at an arbitrary initial time and an arbitrary consistent (at this initial time) initial value. Similar to Remark 4.2, the local solvability notion above requires a sequence $x(k_0), x(k_0+1), \ldots, x(k_1) \in \mathbb{R}^n$ which satisfies (4.9) for $k = k_0, k_0+1, \ldots k_1 - 1$ and furthermore for which another (not necessarily unique) value $x(k_1+1) \in \mathbb{R}^n$ exists such that (4.9) also holds for $k = k_1$. Furthermore, it is required to uniquely solve the SLSS on any finite

interval $[k_0, k_1]$, which in particular means that the uniqueness of the final value at k_1 does not depend on the values $x(k)$ for $k > k_1$ i.e. the systems is causal in terms of states, see the forthcoming Corollary 4.26 for further details. This notion is indeed stronger compared to just requiring unique solvability of (4.9) on $[0, \infty)$, but for the latter, a simple characterization for solvability does not exist, see the forthcoming discussion in Remark 4.14.

Three new notions for the family of matrix pairs $\{(E_i, A_i)\}_{i=0}^{\mathrm{p}}$, which are related to the index-1 notion of the individual matrix pair (E_i, A_i), are introduced in the following definition for the switched system (4.9). These notions will be used in the solvability characterizations with respect to different switching signal classes, see the forthcoming Theorem 4.11, Proposition 4.12, and Proposition 4.13.

Definition 4.7 (Jointly index-1, sequentially index-1, and switched index-1 notions). A family of regular matrix pairs $\{(E_i, A_i)\}_{i=0}^{\mathrm{p}}$ is called

- **jointly index-1** if
$$\ker E_j \oplus \mathcal{S}_i = \mathbb{R}^n \ \forall i, j \in \{0, 1, 2, \ldots, \mathrm{p}\} \tag{4.11}$$
- **sequentially index-1** w.r.t. a fixed mode sequence (σ_j) if
$$\ker E_i \oplus \mathcal{S}_i = \mathbb{R}^n \text{ for } i = 0, 1, 2, \ldots, \mathrm{p} \tag{4.12a}$$
$$E_j^+ \left(\operatorname{im} E_j \cap \operatorname{im} A_j\right) \subseteq \ker E_j \oplus \mathcal{S}_{j+1} \text{ for } j = 0, 1, 2, \ldots \tag{4.12b}$$
- **switched index-1** w.r.t. a fixed and known switching signal $\sigma : \mathbb{N} \to \{0, 1, \ldots, \mathrm{p}\}$ if, for $k = 0, 1, 2, \ldots$,
$$E_{\sigma(k)}^+ \left(\operatorname{im} E_{\sigma(k)} \cap \operatorname{im} A_{\sigma(k)}\right) \subseteq \ker E_{\sigma(k)} \oplus \mathcal{S}_{\sigma(k+1)} \tag{4.13}$$

where $\mathcal{S}_i := A_i^{-1}(\operatorname{im} E_i)$ and E_i^+ is a generalized inverse of E_i. ◊

Obviously, jointly index-1 implies sequential index-1 w.r.t. any mode sequence, which in turn implies switched index-1 w.r.t. all switching signals. Meanwhile, sequentially index-1 w.r.t. to a fixed mode sequence implies switched index-1 w.r.t. all switching signals *with* this mode sequence. Both jointly index-1 as well as sequential index-1 imply index-1 of each mode (just choose $i = j$ in the definition and apply Corollary 2.6); however, the converse is not true in general and, furthermore, neither does index-1 for each mode imply switched index-1 nor the other way around. These observations are summarized in the left part of Figure 4.1.

It turns out that a jointly index-1 family of matrix pairs has always a constant rank of E_i; this is proved in the following proposition.

Proposition 4.8 (Jointly index-1 implies E_i have the same rank). If a family of matrix pairs $\{(E_i, A_i)\}_{i=0}^{\mathrm{p}}$ is jointly index-1 then the rank of E_i is constant

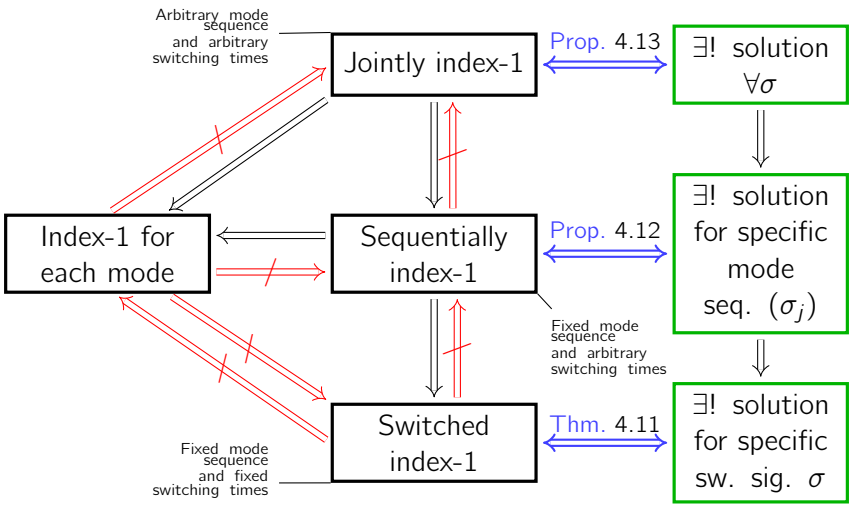


Figure 4.1: Relationship between jointly index-1, sequentially index-1, and switched index-1.

i.e. $r_0 = r_1 = \cdots = r_{\mathrm{p}} =: r$. ◊

Proof. For jointly index-1 families of matrix pairs, the condition (4.11) with $i = j$ implies that $\dim\ker E_i + \dim \mathcal{S}_i = n$ and thus $\dim \mathcal{S}_i = r_i$. This further implies that $\dim\ker E_j + \dim \mathcal{S}_i = n \implies n - r_j + r_i = n \Leftrightarrow r_i = r_j$ for all $i, j \in \{0, 1, \ldots, \mathrm{p}\}$, and thus $\operatorname{rank} E_j$ is constant. □

The converse of the proposition above is not always true i.e. the constant rank of E_i does not imply jointly index-1, see the system in the forthcoming Example 4.9 for a counter-example. In contrast, sequentially or switched index-1 does not imply a constant rank of E_i; see the system in the forthcoming Example 4.10 as an example. As a consequence of E_i having a constant rank, for every jointly index-1 HomSLSS, the nonsingular matrices S_i and T_i can be chosen in such a way that they transform (E_i, A_i) into

$$(S_i E_i T_i, S_i A_i T_i) = \left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} J_i & 0 \\ 0 & I_{n-r} \end{bmatrix} \right), \tag{4.14}$$

for some $J_i \in \mathbb{R}^{r \times r}$.

Before providing the solvability results (already indicated in Figure 4.1), some examples are provided in the following to illustrate the "non-implication" shown in that figure. The fact, that index-1 for each mode is not sufficient for jointly index-1 was already illustrated in [15, Ex. 1.1]. The following example illustrates that sequential index-1 does not imply jointly index-1 in general and

that also index-1 for each mode does not imply sequentially/switched index-1.

Example 4.9. Consider the two matrix pairs

$$(E_0, A_0) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right), \ (E_1, A_1) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \right).$$

Geometric computations provide that

$$\ker E_0 = \operatorname{span}\{(0,0,1)^\top\}, \qquad \mathcal{S}_0 = \operatorname{span}\{(1,0,0)^\top, (0,1,0)^\top\},$$
$$\ker E_1 = \operatorname{span}\{(0,1,0)^\top\}, \qquad \mathcal{S}_1 = \operatorname{span}\{(1,0,-1)^\top, (0,1,-1)^\top\}.$$

Consequently, Each mode is index-1, because $\ker E_i \oplus \mathcal{S}_i = \mathbb{R}^3$, $i = 0, 1$. Furthermore, the family of the two matrix pairs $\{(E_0, A_0), (E_1, A_1)\}$ is:

i) sequentially index-1 with respect to the mode sequence $(0, 1)$, because additionally to each mode being index-1, also $\ker E_0 \oplus \mathcal{S}_1 = \mathbb{R}^3$ which then implies $E_0^+ (\operatorname{im} E_0 \cap \operatorname{im} A_0) \subseteq \ker E_0 \oplus \mathcal{S}_1$.
ii) not jointly index-1, because $\ker E_1 \cap \mathcal{S}_0 = \operatorname{span}\{(0,1,0)^\top\} \neq \{0\}$.
iii) not sequentially index-1 with respect to the mode sequence $(1, 0)$, which also follows from $\ker E_1 \cap \mathcal{S}_0 \neq \{0\}$.
iv) switched index-1 w.r.t. to any switching signal with mode sequence $(0, 1)$ and arbitrary duration times, because they are already sequentially index-1 w.r.t to $(0, 1)$.
v) not switched index-1 w.r.t. to any switching signal with mode sequence $(1, 0)$ with a switch at $k = k_s$, because $\ker E_{\sigma(k_s)} \cap \mathcal{S}_{\sigma(k_s+1)} = \ker E_1 \cap \mathcal{S}_0 \neq \{0\}$.

These observations verify three of the non-implications in Figure 4.1, namely that sequentially index-1 does not imply jointly index-1 and that index-1 for each mode does not imply sequentially and switched index-1. ◊

The next example shows that the property of switched index-1 does not imply in general that each mode is index-1 (and consequently this example can also not be sequentially index-1).

Example 4.10. Consider the matrix pairs

$$(E_0, A_0) = \left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1/4 & 0 \\ 1/2 & 3/4 & 0 \\ 1/2 & 1 & 2 \end{bmatrix} \right),$$
$$(E_1, A_1) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right),$$
$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right).$$

Simple computations provide that

$$\ker E_0 = \operatorname{span}\{(0,1,0)^\top\}, \qquad \mathcal{S}_0 = \operatorname{span}\{(4,0,-1)^\top, (0,2,-1)^\top\},$$
$$\ker E_1 = \operatorname{span}\{(0,0,1)^\top\}, \qquad \mathcal{S}_1 = \operatorname{span}\{(1,0,0)^\top, (0,0,1)^\top\},$$

$$\ker E_2 = \operatorname{span}\{(0,0,1)^\top\}, \qquad \mathcal{S}_2 = \operatorname{span}\{(0,1,0)^\top, (0,0,1)^\top\}.$$

Consequently, the following facts are derived:

i) As an individual system, mode 0 is index-1 ($\ker E_0 \oplus \mathcal{S}_0 = \mathbb{R}^3$) whereas both mode 1 and mode 2 are not index-1 ($\ker E_i \cap \mathcal{S}_i \neq \{0\}, i = 1, 2$).

ii) In view of modes 1 and 2 not being index-1, the family $\{(E_i, A_i) \mid i = 0, 1, 2\}$ cannot be jointly index-1, and also not sequentially index-1 for all mode sequences containing either mode 1 or 2.

iii) It is easily verified that the (sufficient) condition $\ker E_j \oplus \mathcal{S}_i = \mathbb{R}^3$ actually holds for some index pairs, namely all $(j, i) \in \{(0,0), (0,1), (1,0), (2,0)\}$; hence any switching signal which is only composed of these mode transitions (from mode-j to mode-i) leads to the property of switched index-1, an example for such a switching signal is given by

k	0	1	2	3	4	5	6	7	8	...
$\sigma(k)$	2	0	0	0	1	0	0	1	0	...

where for $k > 8$ mode 1 is only active for one time-step each (because the mode sequence cannot contain $(1, 1)$).

Hence, for a specific switching signal, the considered family of regular matrix pairs is switched index-1 while the individual modes are not all index-1 (and hence sequentially index-1 can also not hold). ◊

Based on the observations derived above, the different classes of HomSLSS (4.9) can be categorized along two "dimensions": 1) Properties of the family of matrix pairs concerning the regularity and their index, and 2) The considered class of switching signals (completely arbitrary, mode sequence fixed, switching times and mode sequence fixed). This categorization and the position of the different index-1 notions are illustrated in Figure 4.2.

4.1.2.2 Characterizations

The solvability characterizations (already indicated in Figure 4.1) are presented as follows: the following theorem presents the solvability characterization of the HomSLSS (4.9) with respect to a given switching signal whereas solvability characterizations with respect to mode sequences and arbitrary switching signals are derived from this theorem and will follow later.

Theorem 4.11 (Solvability of HomSLSSs w.r.t. a fixed switching signal). Consider the HomSLSS (4.9) with a corresponding family of matrix pairs $\{(E_i, A_i)\}_{i=0}^{\mathrm{p}}$ and a given switching signal σ of the form (2.11). This system is solvable w.r.t. σ in the sense of Definition 4.6 if, and only if, $\{(E_i, A_i)\}_{i=0}^{\mathrm{p}}$ is switched index-1 w.r.t. σ. Furthermore, if it is solvable, then its solution

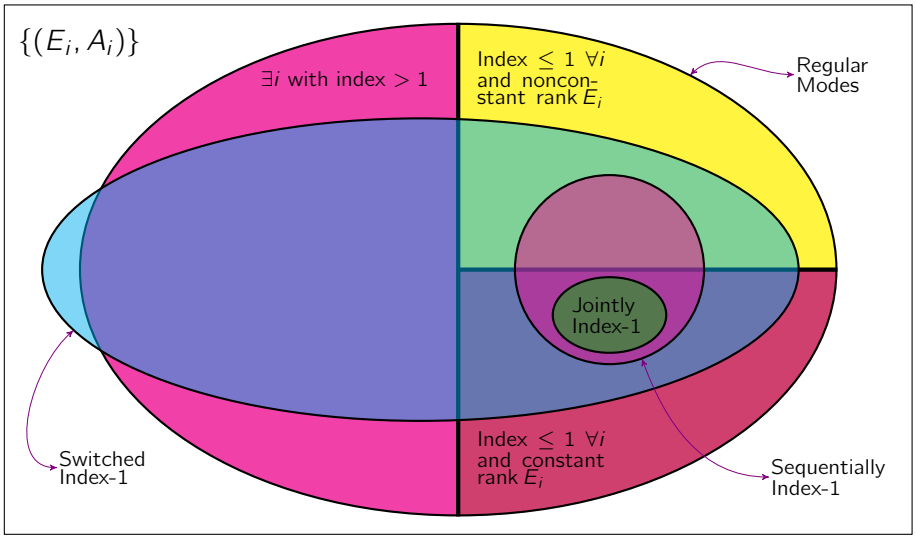


Figure 4.2: Position of jointly, sequential, and switched index-1 with respect to the possible classes of SLSSs.

satisfies

$$x(k+1) = \Phi_{\sigma(k+1),\sigma(k)} x(k) \text{ with } x(0) \in \mathcal{S}_{\sigma(0)}, \ \forall k \in \mathbb{N} \tag{4.15}$$

where $\Phi_{i,j}$ is the so-called one-step map from mode j to mode i given by

$$\Phi_{i,j} := \Pi_{\mathcal{S}_i}^{\ker E_j} E_j^+ A_j \tag{4.16}$$

where $\Pi_{\mathcal{S}_i}^{\ker E_j}$ is the canonical projector from $\ker E_j \oplus \mathcal{S}_i$ to $\mathcal{S}_i$, E_j^+ is a generalized inverse of E_j, and the ordinary system (4.15) is called the surrogate system for (4.9). ◊

Proof. **Necessity:** Let $k_0 \in \mathbb{N}$, $k_1 := k_0 + 1$ and consider the SLSS (4.9) on $[k_0, k_1]$, i.e. $x_0 := x(k_0)$, $x_1 := x(k_0+1)$, $x_2 := x(k_0+2)$ have to satisfy

$$E_{\sigma(k_0)} x_1 = A_{\sigma(k_0)} x_0 \tag{4.17a}$$

$$E_{\sigma(k_1)} x_2 = A_{\sigma(k_1)} x_1. \tag{4.17b}$$

Solvability of (4.9) implies that all elements of the solution set $\{(x_1, x_2)\}$ of the system of linear equations (4.17) for any given $x_0 \in \mathcal{S}_{\sigma(k_0)}$ have a unique first component x_1. Equivalently, $E_{\sigma(k_0)}^{-1}\{A_{\sigma(k_0)} x_0\} \cap A_{\sigma(k_1)}^{-1}(\operatorname{im} E_{\sigma(k_1)})$ must be a singleton. Using Lemma A.2, the latter can be rewritten as $(\{E_{\sigma(k_0)}^+ A_{\sigma(k_0)} x_0\} + \ker E_{\sigma(k_0)}) \cap \mathcal{S}_{\sigma(k_1)}$. Using Lemma A.3 for $\mathbb{Z} = \{0\}$, $\mathbb{U} = E_{\sigma(k_0)}^+ A_{\sigma(k_0)} \mathcal{S}_{\sigma(k_0)}$, $\mathcal{V} = \mathcal{S}_{\sigma(k_1)}$, and $\mathcal{W} = \ker E_{\sigma(k_0)}$, the unique solvability of (4.17) is equivalent to

$$E_{\sigma(k_0)}^+ A_{\sigma(k_0)} \mathcal{S}_{\sigma(k_0)} \subseteq \mathcal{S}_{\sigma(k_1)} \oplus \ker E_{\sigma(k_0)}.$$

From $A_{\sigma(k_0)}\mathcal{S}_{\sigma(k_0)} = A_{\sigma(k_0)}A^{-1}_{\sigma(k_0)}(\operatorname{im} E_{\sigma(k_0)}) = \operatorname{im} E_{\sigma(k_0)} \cap \operatorname{im} A_{\sigma(k_0)}$ the condition (4.13) for $k = k_0$ is indeed necessary for solvability of (4.9) on $[k_0, k_0+1]$.
Sufficiency: It will be shown that for each $k_0, k_1 \in \mathbb{N}$ with $k_1 > k_0$ and for each $x_0 \in \mathcal{S}_{\sigma(k_0)}$ the sequence $x : [k_0, k_1] \to \mathbb{R}^n$ given by $x(k_0) = x_0$ and $x(k+1) = \Phi_{\sigma(k+1),\sigma(k)}x(k)$ is the unique solution of SLSS (4.9) with $x(k_0) = x_0$ on $[k_0, k_1]$. Inductively, assume for $k \geq k_0$ that it is already known that $x(k) \in \mathcal{S}_{\sigma(k)}$ and that x is the unique solution on $[k_0, k]$ with $x(k_0) = x_0$ (which is satisfied for $k = k_0$). Now, it will be shown that x is the unique solution on $[k_0, k+1]$, which by an induction argument then concludes the proof. In order to show the former, it just needs to show that $x(k+1) = \Phi_{\sigma(k+1),\sigma(k)}x(k)$ satisfies

$$\begin{aligned} E_{\sigma(k)}x(k+1) &= A_{\sigma(k)}x(k), \\ E_{\sigma(k+1)}x_2 &= A_{\sigma(k+1)}x(k+1), \end{aligned} \tag{4.18}$$

for some $x_2 \in \mathbb{R}^n$ and that there is no other possible value for $x(k+1)$ satisfying this equation. The same arguments as in the necessity part of this proof conclude that for any $x(k) \in \mathcal{S}_{\sigma(k)}$ there is a unique value for $x(k+1)$ satisfying (4.18) if (4.13) is satisfied at k. Furthermore, by Lemma A.3, this value is uniquely given by

$$x(k+1) = \Pi^{\ker E_{\sigma(k)}}_{\mathcal{S}_{\sigma(k+1)}} E^{+}_{\sigma(k)} A_{\sigma(k)}x(k) = \Phi_{\sigma(k+1),\sigma(k)}x(k).$$

Furthermore, $x(k+1) \in \operatorname{im} \Pi^{\ker E_{\sigma(k)}}_{\mathcal{S}_{\sigma(k+1)}} = \mathcal{S}_{\sigma(k+1)}$, which concludes the proof. □

The theorem above reveals that the property of switched index-1 is necessary and sufficient for the solvability of (4.9) with respect to a given switching signal. The following proposition reveals that sequentially index-1 is necessary and sufficient for the solvability of (4.9) under a fixed mode sequence and with arbitrary switching times.

Proposition 4.12 (Solvability of HomSLSSs under a fixed mode sequence). Consider the HomSLSS (4.9) and a given mode sequence (σ_j). This system is solvable (in the sense of Definition 4.6) w.r.t. any switching signal with this given mode sequence if, and only if, $\{(E_i, A_i)\}_{i=0}^{\mathrm{p}}$ is sequentially index-1 w.r.t. (σ_j). Furthermore, in the case of solvability, the one-step map $\Phi_{i,j}$ in (4.16) and the surrogate system (4.15) are also valid. ◊

Proof. Sufficiency is clear from the fact that (4.12) implies (4.13) for any switching signal with the given mode sequence. For the necessity, first, observe that due to Theorem 4.11 the condition (4.13) needs to hold for all possible switching signals with the given mode sequence. In particular, for all $k, j \in \mathbb{N}$ and all switching signals with $\sigma(k) = \sigma(k+1) = \sigma_j$ the necessary

condition (4.12) implies $\ker E_{\sigma_j} \cap \mathcal{S}_{\sigma_j} = \{0\}$. The latter implies that the matrix pair $(E_{\sigma_j}, A_{\sigma_j})$ must be index-1 (by Corollary 2.6), i.e. (4.12a) must hold. Furthermore, (4.13) must hold at any switch from mode σ_j to σ_{j+1} (in view of Theorem 4.11), which concludes the proof. □

Meanwhile, it can be shown, that the property of jointly index-1 is in fact necessary and sufficient for the solvability of (4.9) under arbitrary switching signals; this is given in the following proposition.

Proposition 4.13 (Solvability of HomSLSSs under arbitrary switching signals, cf. Theorem 3.5 in [15]). The HomSLSS (4.9) is solvable (in the sense of Definition 4.6) w.r.t. all switching signals if, and only if, $\{(E_i, A_i)\}$ is jointly index-1. Furthermore, in the case of solvability, the one-step map $\Phi_{i,j}$ in (4.16) and the surrogate system (4.15) are also valid. ◊

Proof. The proof is similar to the proof of Proposition 4.12. The main idea is highlighted as follows. Sufficiency is the direct consequence of (4.11) implies (4.13) for any arbitrary switching signal. For the necessity, by Theorem 4.11, the condition (4.13) i.e. $\{E^+_{\sigma(k)} A_{\sigma(k)} x(k)\} \subseteq \ker E_{\sigma(k)} \oplus \mathcal{S}_{\sigma(k+1)}$ needs to hold for arbitrary $\sigma(k)$ and $\sigma(k+1)$ which implies $\ker_{\sigma(k)} \cap \mathcal{S}_{\sigma(k+1)}$ for arbitrary $\sigma(k)$ and $\sigma(k+1)$. This implies, by Corollary 2.6 and Proposition 4.8, $\{(E_i, A_i)\}$ is jointly index-1, which concludes the proof. □

Compared to Theorem 3.5 in [15], which utilizes QWF in formulating the surrogate system, the proposition above utilizes a generalized inverse matrix; this makes the proof more straightforward. In particular, based on the results in the proposition above, to ensure the existence and uniqueness of a solution of (4.9) for general switching signals, it is in general not enough to assume that each matrix pair (E_i, A_i) is regular (in contrast to the continuous-time case, see e.g. [62]), even assuming that each matrix pair is index-1 is not sufficient (this is shown by the system in Example 4.9).

Remark 4.14 (Discussion on the solvability notion). One may wonder, why the **local** solvability in Definition 4.6 is considered instead of just requiring that there exists a unique solution on $[0, \infty)$ for every consistent initial value $x(0)$. The following switched system illustrates the fundamental difference between both approaches:

$$\begin{array}{c|c|c} k = 0: & k = 1, 2: & k \geq 3: \\ 0 = x(k) & \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] x(k+1) = x(k) & 0 = x(k) \end{array}$$

It is easily seen that, $x(k) = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, $k \geq 0$, is the only (and hence unique) solution on $[0, \infty)$ with consistency space $\mathcal{S}_0 = \{0\}$. However, if the switched system

is considered only on the time interval $[1,2]$, then any solution $x(1) = \left[\begin{smallmatrix} x_{11} \\ x_{12} \end{smallmatrix}\right]$ and $x(2) = \left[\begin{smallmatrix} x_{21} \\ x_{22} \end{smallmatrix}\right]$ needs to satisfy

$$\begin{aligned} k = 1: &\quad \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} x_{21} \\ x_{22} \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_{11} \\ x_{21} \end{smallmatrix}\right], \\ k = 2: &\quad \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_{21} \\ x_{22} \end{smallmatrix}\right], \end{aligned}$$

for some $\alpha, \beta \in \mathbb{R}$. First observe that any solution must satisfy $x_{11} \overset{k=1}{=} x_{22} \overset{k=2}{=} 0$, however $\mathcal{S}_1 = \operatorname{im} \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, which means that **not** for all $x(1) \in \mathcal{S}_1$ a solution on $[1,2]$ exists. Secondly, $x_{22} = \beta$ is not uniquely determined (without taking the behavior of the switched system at $k = 3$ into account). This shows that a switched system that is globally uniquely solvable for all consistent initial values is not necessarily locally uniquely solvable (while the converse is of course true). In fact, the example illustrates that for an only globally solvable system the consistency of the state value $x(k)$ is in general not only determined by the active mode k ($k = 1$ in the example) but also depends on future modes. Furthermore, the state $x(k+1)$, in general, cannot uniquely be determined from the knowledge of $x(k)$ together with the knowledge of modes k and $k+1$ ($k+1 = 2$ in the example). So in both cases, knowledge of the future behavior of the switched system is necessary to conclude existence and/or uniqueness which in most cases is not desirable. This is in fact related to the concept of causality with respect to the switching signal in the sense of Definition 4.25, see also the forthcoming Corollary 4.26. It should also be noted that the time duration which is needed to look ahead to decide about existence and uniqueness grows with the index of the corresponding matrix pairs involved, in general, if a mode has index ν and this mode is also active for at least ν time steps, then one needs to look ahead $\nu - 1$ steps to conclude existence and uniqueness (in the example the index was two and it was necessary to look one step ahead). ◊

In Fig. 4.1, it is already indicated that in the case of a fixed switching signal, the index-1 condition for the individual modes is also necessary (in addition to not being sufficient). This is discussed in the following remark.

Remark 4.15 (Index-1 of individual modes). From Proposition 4.12 and 4.13, switched systems that are solvable for all switching signals or fixed mode sequences with arbitrary switching times must be composed of index-1 modes. In contrast, from Theorem 4.11, a solvable switched system for a fixed switching signal may contain modes with higher indexes (more than one). However, these higher index modes can only be active for one isolated time instant, because for each mode i which is active for at least two consecutive time-steps, the switched index-1 condition (4.13) implies $\ker E_i \cap \mathcal{S}_i = \{0\}$ which in turn implies index-1 for mode i; see also Example 4.10 and the forthcoming Example 4.24 for more explanations with illustrations. ◊

Remark 4.16 (Well-definedness of the one-step map fir HomSLSSs). Regarding the effect of the nonuniqueness of the pseudo-inverse to the solvability characterization, similar results as in non-switched systems apply here i.e. first, although the pseudo-inverse $E^+_{\sigma(k}$ in (4.13) is not unique, the validity of (4.13) does not depend on the specific choice of the pseudo-inverse (cf. the discussion after Definition A.1). Second, the one-step map matrix $\Phi_{i,j}$, is in general **not unique** (due to this nonuniqueness of E_j^+ chosen in the calculation). However, this pseudo-inverse is only applied to vectors from the subspace $A_i\mathcal{S}_i = \operatorname{im} E_i \cap \operatorname{im} A_i \subseteq \operatorname{im} E_i$ which implies (cf. the discussion after Definition A.1) that indeed the action of $\Phi_{i,j}$ is unique when restricted to the relevant subspace. In particular, for calculations, the well-known Moore-Penrose inverse can also be used, for which efficient algorithms are available in the literature, e.g. by using a singular value decomposition [63]. Furthermore, the restriction to the subspace $E_j^+ A_j \mathcal{S}_j = E_j^+(\ker E_j \cap \operatorname{im} A_j) \subseteq \ker E_j \oplus \mathcal{S}_i$ implies that also the action of $\Pi_{\mathcal{S}_i}^{\ker E_j}$ is well defined. In particular, the projector $\Pi_{\mathcal{S}_i}^{\ker E_j}$ can arbitrarily be extended to a projector defined on the whole of $\mathbb{R}^n$ without changing the effect of the one-step map $\Phi_{i,j}$. Altogether, the above discussion shows that the one-step map $\Phi_{i,j}$ is in fact a well-defined map from $\mathcal{S}_j$ to $\mathcal{S}_i$. ◊

In terms of QWF (4.14), an alternative form of the one-step map for jointly/sequentially index-1 HomSLSSs is presented in the following corollary.

Corollary 4.17 (Simplified one-step map for jointly/sequentially index-1 HomSLSSs). Consider the jointly/sequential index-1 HomSLSS (4.9). One (simple) choice of generalized inverses for E_j is $E_j^+ = T_j \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] S_j$ where T_j and S_j are given as in (4.14). Then

$$\Phi_{i,j} := \Pi_{\mathcal{S}_i}^{\ker E_j} \Phi_j \tag{4.19}$$

where $\Phi_j := \Phi_{(E_j,A_j)} = T_j \left[\begin{smallmatrix} J_j & 0 \\ 0 & 0 \end{smallmatrix}\right] T_j^{-1}$ is the one-step map of the individual mode-j. ◊

Remark 4.18 (One-step map for switched index-1 HomSLSSs). For a switched index-1 HomSLSS (4.9) with a given switching signal, the choice $\Phi_{i,j} = \Pi_{\mathcal{S}_i}^{\ker E_j} \Phi_{(E_j,A_j)}$ derived in Corollary 4.17 is only valid for modes j that have index-1 matrix pairs (E_j, A_j). As shown in the proof of Proposition 4.12, if a mode i appears for at least two consecutive time steps, then the corresponding matrix pair (E_i, A_i) must be index-1. Hence, a possible choice for $\Phi_{i,i}$ is $\Phi_{i,i} = \Pi_{\mathcal{S}_i}^{\ker E_i} \Phi_{(E_i,A_i)}$ which (not surprisingly) simplifies to the (non-switched) one-step map $\Phi_{i,i} = \Phi_{(E_i,A_i)} =: \Phi_i$ because $\operatorname{im} \Phi_i \subseteq \mathcal{S}_i$ and hence $\Pi_{\mathcal{S}_i}^{\ker E_i}$ has no further effect. ◊

Remark 4.19 (Regularity of individual modes). It is well known that for unswitched systems, regularity is necessary for the existence and uniqueness of a solution, see e.g. [11, 64, 61]. Thus, when considering arbitrary switching times, each mode considered on its activation interval can be seen as an unswitched system. From this point of view, the regularity of each mode is then necessary for the existence and uniqueness of solutions. However, when considering a fixed switching signal, regularity is in fact not necessary anymore. This is shown by the system

$$\begin{array}{c|c|c} k=0: & k=1: & k\geq 2 \\ x(1)=x(0) & 0\cdot x(2)=0\cdot x(1) & 0\cdot x(k+1)=x(k) \end{array}$$

where the second mode is not regular, however, the whole switched system has the unique solution $(x_0, x_0, 0, 0, \ldots)$. This also justifies the left part of Fig. 4.2 in which a switched index-1 system can contain nonregular modes. ◊

Inspired by the above remark, it can then be observed that the switched index-1 condition (4.13) (without the additional regularity assumption) is in fact a necessary and sufficient condition for the solvability of general time-varying singular systems of the form $E(k)x(k+1) = A(k)x(k)$. This is formally stated in the following corollary.

Corollary 4.20 (Solvability of time-varying singular linear systems). The general time-varying singular linear system

$$E(k)x(k+1) = A(k)x(k),\ \ k = 0, 1, \ldots$$

is solvable in the sense of Definition 4.6 (with switching signal $\sigma(k) = k$) if, and only if, for $k = 0, 1, \ldots$

$$E(k)^+ \left(\operatorname{im} E(k) \cap \operatorname{im} A(k)\right) \subseteq \ker E(k) \oplus \mathcal{S}(k+1).$$

In the case of solvability, its corresponding (time-varying) one-step map is given by $\Phi(k) := \Pi^{\ker E(k)}_{\mathcal{S}(k+1)} E(k)^+ A(k)$. ◊

Now, by using the one-step map matrix and its corresponding surrogate system (4.15) given in the Theorem 4.11, the explicit solution formula of system (4.9) can then be written as follows:

$$x(k) = \Phi_{\sigma(k),\sigma(k-1)}\Phi_{\sigma(k-1),\sigma(k-2)}\cdots\Phi_{\sigma(1),\sigma(0)}x(0). \tag{4.20}$$

In particular, the explicit solution formula at switching times $k_j^s,\ j = 0, 1, \ldots$ can also be derived; this is presented in the following corollary.

Corollary 4.21 (Solutions at switching times for HomSLSSs). Consider the solvable HomSLSS (4.9) w.r.t. the fixed switching signal σ of the form (2.11). Its solution at the switching time k_j^s is given by

$$x(k_j^s) = \Psi_\sigma(j, 0)x(0) \tag{4.21}$$

where for $j = 0, 1, \ldots$

$$\Psi_\sigma(j,0) = \Phi_{\sigma_j,\sigma_{j-1}}\Phi_{\sigma_{j-1}}^{k_j^s-k_{j-1}^s-1}\Phi_{j-1,j-2}\Phi_{j-2}^{k_{j-1}^s-k_{j-2}^s-1}\cdots\Phi_{\sigma_1,\sigma_0}\Phi_{\sigma_0}^{k_1^s-1}. \quad (4.22)$$

In particular, the matrix $\Psi_\sigma(j,0)$ can be rewritten in a recursive form as

$$\Psi_\sigma(j,0) = \Phi_{\sigma_j,\sigma_{j-1}}\Phi_{\sigma_{j-1}}^{k_j^s-k_{j-1}^s-1}\Psi_\sigma(j-1,0) \quad (4.23)$$

with $\Psi_\sigma(0,0) = I_n$. ◊

The second formula for $\Psi_\sigma(j,0)$ in (4.23) is more computationally friendly since it does not contain repetitive calculations as in (4.22). Matrix $\Psi_\sigma(j,0)$ above maps the initial value to the solution at a switching time that is useful in observability characterization later. Moreover, in the following corollary, a state transition matrix that maps the state at a certain switching time k_j^s to the state at the final time $K > k_J^s$ is defined by utilizing the surrogate system (4.15).

Corollary 4.22 (Final state transition matrix for HomSLSSs). The solution of a switched index-1 SLSS (4.9) w.r.t. switching signal (2.11) satisfies

$$x(K) = \Psi_\sigma^*(K,j)x(k_j^s) \quad (4.24)$$

where for $j = J, J-1, \ldots, 0$ and with $\Phi_{\sigma_0,\sigma_{-1}} = I_n$,

$$\begin{aligned}\Psi_\sigma^*(K,j) =& \Phi_{\sigma_J}^{K-k_J^s}\Phi_{\sigma_J,\sigma_{J-1}}\Phi_{\sigma_{J-1}}^{k_J^s-k_{J-1}^s-1}\Phi_{\sigma_{J-1},\sigma_{J-1}}\Phi_{\sigma_{J-2}}^{k_{J-1}^s-k_{J-2}^s-1}\cdots \\ & \Phi_{\sigma_j,\sigma_{j-1}}\Phi_j^{k_{j+1}^s-k_j^s-1}.\end{aligned}$$ ◊

The matrix $\Psi_\sigma^*(K,j)$ is called the final state transition matrix from switching times and will be used in the determinability characterization later in Chapter 5. Furthermore, it can be rewritten in the following recursive form which is more computationally friendly

$$\Psi_\sigma^*(K,j) = \Psi_\sigma^*(K,j+1)\Phi_{\sigma_j,\sigma_{j-1}}\Phi_j^{k_{j+1}^s-k_j^s-1} \quad (4.25)$$

with $\Psi_\sigma^*(K,J) = \Phi_{\sigma_J}^{K-k_J^s}$.

The following examples illustrate solutions of HomSLSSs which were calculated by using the one-step map formula introduced in Theorem 4.11.

Example 4.23. Consider the HomSLSS (4.9) composed of modes as in Example 4.9, which is sequential index-1 w.r.t. the mode sequence $(\sigma_k) = (0,1)$. Employing the QWF (4.14) and the generalized inverse formula in Remark 4.17 provides $S_0 = T_0 = I$,

$$S_1 = \begin{bmatrix}1&0&0\\0&-1&1\\0&0&1\end{bmatrix},\ T_1 = \begin{bmatrix}1&0&0\\0&1&1\\-1&-1&0\end{bmatrix},\ E_0^+ = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\ E_1^+ = \begin{bmatrix}1&0&0\\0&-1&1\\-1&1&-1\end{bmatrix},$$

and the one-step map formula (4.16) yields the matrices $\Phi_{i,j}$ that map mode j to mode i as follows

$$\Phi_{0,0} = \Phi_0 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\ \Phi_{1,1} = \Phi_1 = \begin{bmatrix}1&0&0\\1&0&1\\-2&0&-1\end{bmatrix}, \Phi_{1,0} = \begin{bmatrix}1&0&0\\0&1&0\\-1&-1&0\end{bmatrix}.$$

Its explicit solution under the switching signal $\sigma(k) = 0$ for $k < k^s$ and 1 for $k \geq k^s$ is then given by

$$x(k) = \begin{cases} \Phi_{0,0}^{k} x(0) & k < k^s \\ \Phi_{1,0}\Phi_0^{k^s-1} x(0) & k = k^s \\ \Phi_1^{k-k^s}\Phi_{1,0}\Phi_0^{k^s-1} x(0) & k > k^s. \end{cases} \qquad \Diamond$$

Example 4.24. Consider the system in Example 4.10 where any switching signal composed of mode transitions from mode-j to mode-i $(j, i) \in \{(0, 0), (0, 1), (1, 0), (2, 0)\}$ leads to the property of switched index-1 as long as mode-1 and mode-2 are active only for one time step as discussed in Example 4.10. By choosing

$$E_0^+ = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; E_1^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; E_2^+ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

the one-step map matrices from mode j to mode i, $\Phi_{i,j}$, are given by

$$\Phi_{0,2} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{2} & -\frac{3}{4} \end{bmatrix}, \Phi_{0,0} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{4} & -\frac{3}{4} & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix}, \Phi_{1,0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix}, \Phi_{0,1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}.$$

Clearly, it is switched index-1 w.r.t. the mode sequence $(\sigma_k) = (2, 0, 1, 0)$ under switching times $k_1^s = 1$, $k_2^s = 4$ and $k_3^s = 5$. Its solution with the initial value $x(0) = (0, 1, 5)^\top$ is shown in Fig. 4.3. $\Diamond$

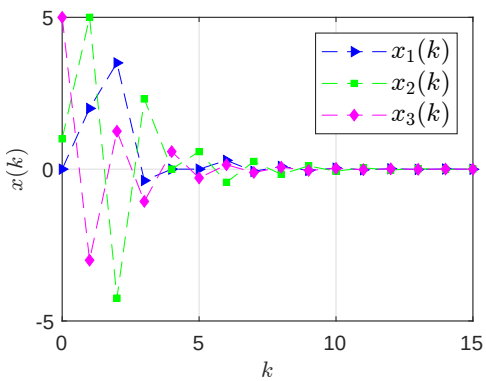


Figure 4.3: A solution trajectory of the system in Example 4.24

4.1.2.3 Discussion on causality

With respect to a given and fixed switching signal, the definition of causality in terms of states for non-switched systems (4.1) is carried over without change. In terms of switching signals, if the switching signal in the future is changed, it is possible to have different solutions in the past with the same switching signal in the past [15]. This is also not desired in the practice of control designs since

the controller cannot determine uniquely the results of the control actions with respect to switching signals. Thus, the following causality notion in terms of switching signals is also desired.

Definition 4.25 (Causality in terms of switching signals, [15]). The HomSLSS (4.9) is called **causal in terms of switching signals** w.r.t. a set of switching signals Ψ if for all $\sigma \in \Psi$ and all corresponding solutions x the implication

$$\sigma(k) = \widetilde{\sigma}(k)\ \forall k \leq \widetilde{k} \implies \exists \text{ sol. } \widetilde{x} \text{ of (4.9) with } \widetilde{\sigma} : \widetilde{x}(k) = x(k)\ \forall k \leq \widetilde{k}.$$

holds for any switching signal $\widetilde{\sigma} \in \Psi$ and any $\widetilde{k} \in \mathbb{N}$. ◊

In other words, system (4.9) is said to be causal in terms of switching signals if changing the switching signal in the future does not make it necessary to change the solution in the past.

As desired, apart from causality in terms of states, solvability in the sense of Definition 4.6 implies causality in terms of switching signals, this is formally stated in the following corollary.

Corollary 4.26 (Solvability implies causality in terms of states and switching signals). Every solvable HomSLSS (4.9) is causal in terms of states. Furthermore, if the HomSLSS (4.9) is solvable w.r.t. all switching signals/a fixed mode sequence, then it is causal in terms of switching signals (in the sense of Definition 4.25) w.r.t. all switching signals/the given mode sequence. ◊

These are direct consequences of the system having a unique solution for all involved switching signals on $[k_0, k_1]$ for arbitrary $k_0, k_1 \in \mathbb{N}$. In particular, the causality in terms of states can also be directly seen from the surrogate system (4.15) or its corresponding explicit solution formula (4.20) in which $x(k)$ is determined completely by $x(0)$.

Remark 4.27 (Solvability provides well-posedness). The jointly/sequentially/switched index-1 condition provides the well-posedness of (4.9) under arbitrary switching signals/fixed mode sequences/fixed switching signals. Possessing this well-posedness means that the system has a unique solution under the considered switching signals, causal in terms of switching signals, and also causal in terms of states; this is valid for systems solved on finite time intervals. This well-posedness is important when the system is observed on a finite time domain, for example for observability and determinability analysis studied in Chapter 5. ◊

4.2 Inhomogeneous Systems

Systems with inputs are studied in this section. The flow of the study is similar to the study for systems without inputs discussed in the previous section, i.e., the study starts with non-switched systems and continues with switched systems. For the solvability analysis, new solvability notions will be introduced with some further features due to the presence of inputs. Causality in terms of states (and switching signals in switched systems) will be preserved, and furthermore, two different causality notions with respect to inputs will be introduced, see the forthcoming Definition 4.36 for non-switched systems and Definition 4.44 for switched systems and the discussion after them.

4.2.1 Nonswitched Systems

Consider the discrete-time Inhomogeneous Singular Linear System (InhSLS) of the form

$$Ex(k+1) = Ax(k) + Bu(k), \ k \in \mathbb{N} \tag{4.26a}$$

$$y(k) = Cx(k) + Du(k) \tag{4.26b}$$

where $E, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$ are known and constant matrices, E may be singular with $\operatorname{rank} E = r \leq n$, $x(k) \in \mathbb{R}^n$ is the state at time instant/step $k \in \mathbb{N}$, $u \in \mathbb{R}^m$ is the input, and $y(k) \in \mathbb{R}^p$ is the output at k. This system can be represented by the matrix triplet (E, A, B). For this system or its corresponding matrix triplet (E, A, B), recall the following set as defined in Lemma 2.9

$$\widehat{\mathcal{S}} := A^{-1}(\operatorname{im}[E, B]) = \{\xi \in \mathbb{R}^n : A\xi \in \operatorname{im}[E, B]\}. \tag{4.27}$$

Under the regularity assumption of the matrix pair (E, A) and by utilizing the QWF (2.6), system (4.26) can be represented by

$$\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \bar{x}(k+1) = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \bar{x}(k) + \begin{bmatrix} B^J \\ B^N \end{bmatrix} u(k). \tag{4.28}$$

where $\bar{x} = T^{-1}x =: \left[\begin{smallmatrix} \bar{x}_1 \\ \bar{x}_2 \end{smallmatrix}\right]$, $\bar{x}_1 \in \mathbb{R}^r$, $\bar{x}_2 \in \mathbb{R}^{n-r}$. The subsystem $\bar{x}_1(k+1) = J\bar{x}_1(k) + B^J u(k)$ is called the pure ordinary subsystem whereas $N\bar{x}_2(k+1) = J\bar{x}_2(k) + B^N u(k)$ is called the pure singular subsystem. By iterating (4.28) over time steps on the finite time domain of interest $[0, K]$, $K \in \mathbb{N}$, the explicit solution of (4.26) on $[0, K]$ in $\bar{x}$ coordinate is given by

$$\bar{x}_1(k) = J^k \bar{x}_1(k) + \sum_{i=0}^{k-1} J^{k-i-1} B^J u(i), \tag{4.29a}$$

$$\bar{x}_2(k) = N^{K-k} \bar{x}_2(K) + \sum_{i=0}^{K-k-1} N^i B^N u(k+i). \tag{4.29b}$$

It can be seen from (4.29b) that in general, on a finite time domain $[0, K]$, the final state $x(K)$ determines the preceding states $x(K-1), x(K-2), \ldots$. In other words, solutions of system (4.26) on $[0, K]$ are determined not only by past (and current) states and inputs but also by future states and inputs. Again, this feature is generally not desired in applications since future values are commonly not available or uncertain, and making decisions under certainty is generally more preferred.

4.2.1.1 Definitions

As a motivation for the new solvability notions considered in this study, consider first system (4.26) only with $k = 0$, i.e., consider the equation $Ex(1) = Ax(0) + Bu(0)$. The first important observation is that if the existence of a solution $x(1)$ is only required with a certain input $u(0)$, then a consistent initial value $x(0)$ must satisfy $x(0) \in A^{-1}(\operatorname{im} E - \{Bu(0)\}) = (A^+ \operatorname{im} E - \{A^+u(0)\}) + \ker A = A^{-1}(\operatorname{im} E) - \{A^+u(0)\} = \mathcal{S} - \{A^+u(0)\}$; this set is indeed the consistency set of the homogeneous system shifted by $u(0)$. Furthermore, the set $\mathcal{S} - \{A^+u(0)\}$ is dependent on $u(0)$, and a solvability notion can then be defined w.r.t. a fixed input $u(0)$, and the existence of a unique solution is required for all initial values in that set. Based on this solvability notion, a further solvability notion can then be defined by requiring only the existence of $u(0)$ such that $Ex(1) = Ax(0) + Bu(0)$ is solvable w.r.t. $u(0)$. Motivated by those observations, two new solvability notions are introduced in the following definition for system (4.26).

Definition 4.28 (Solvability w.r.t. a fixed input sequence for InhSLSs). System (4.26) is called **solvable w.r.t. a fixed input sequence** $(u_k) = (u(0), u(1), \ldots)$ if for all $k_1 \in \mathbb{N}$ and all $x_0 \in \widehat{\mathcal{S}}^{u(0)} = A^{-1}(\operatorname{im} E) - \{A^+Bu(0)\} = \mathcal{S} - \{A^+Bu(0)\}$, there exists a unique solution of (4.26) on $[0, k_1]$ with $x(0) = x_0$ and with the input (u_k). Furthermore, system (4.26) is called **weakly solvable** if there is an input sequence (u_k) such that the system is solvable w.r.t. (u_k). $\Diamond$

Now, if it is required that $Ex(1) = Ax(0) + Bu(0)$ is solvable w.r.t. arbitrary $u(0)$, then the existence of a unique solution for $x(1)$ is required for all initial values $x(0) \in A^{-1}(\operatorname{im} E - \operatorname{im} B) = A^{-1}(\operatorname{im}[E, B])$. This leads to the solvability notion defined in the following definition.

Definition 4.29 (Strong solvability notion for InhSLSs). System (4.26) is called **strongly solvable** if for all $k_1 \in \mathbb{N}$, all $x_0 \in \widehat{\mathcal{S}} = A^{-1}(\operatorname{im}[E, B]$, and all input sequences (u_k), there exists a unique solution of (4.26) on $[0, k_1]$ with $x(0) = x_0$ and with the input (u_k). $\Diamond$

Note that by utilizing the preimage property in Lemma A.2, the set $\widehat{\mathcal{S}}$ can be rewritten as $\widehat{\mathcal{S}} = A^{-1}(\operatorname{im} E - \operatorname{im} B) = A^{+}(\operatorname{im} E - \operatorname{im} B) + \ker A = A^{-1}(\operatorname{im} E) - \operatorname{im} A^{+}B = \mathcal{S} - \operatorname{im} A^{+}B$. Then, the strong solvability notion above is indeed a generalization of the solvability notion for homogeneous systems (with $B = 0$) in Definition 4.1.

Remark 4.30 (Weak solvability vs strong solvability). Weak solvability requires only the existence of an input sequence such that for all $x_0 \in \widehat{\mathcal{S}}^{u(0)}$, the system with $x(0) = x_0$ has a unique solution. Note that for this solvability notion, a given input sequence corresponds to the solutions with the initial conditions $x(0) = x_0 \in \widehat{\mathcal{S}}^{u(0)}$. On the other hand, strong solvability requires that for all input sequences, the system with $x(0) = x_0$ has a unique solution for any $x_0 \in \widehat{\mathcal{S}}$. Thus, strong solvability implies weak solvability. The converse is not always true; this is confirmed by the system

$$\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] x(k+1) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] x(k) + \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] u(k), \;\; u \in \mathbb{R}. \tag{4.30}$$

with $\ker E = \operatorname{span}\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, $\mathcal{S} = \operatorname{span}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\widehat{\mathcal{S}} = \mathbb{R}^2$. Let $x = [x_1, x_2]^\top$. This system can be rewritten as

$$\left[\begin{smallmatrix} x_1(k+1) \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_1(k) \\ x_2(k) \end{smallmatrix}\right] + \left[\begin{smallmatrix} u(k) \\ u(k) \end{smallmatrix}\right]$$

This system is weakly solvable since with $u(k) = 0 \; \forall k \in \mathbb{N}$, the system becomes a homogeneous system, and since $\ker E \cap \mathcal{S} = \{0\}$, it has a unique solution on any $[0, k]$ and any $x(0) \in \mathcal{S}^{u(0)} = \mathcal{S}$. However, it is not strongly solvable since for some $x(0) \in \widehat{\mathcal{S}} = \mathbb{R}^2$ and some input sequences, the system has no solution. For example, take $x(0) = (1, 1)^\top \in \widehat{\mathcal{S}}$ and $u(k) = 0 \; \forall k$ that violates the equation $0 = x_2(0) + u(0)$. ◊

4.2.1.2 Characterizations

The characterization for the solvability notion defined in Definition 4.28 is presented in the following theorem.

Theorem 4.31 (Solvability Characterization w.r.t. a fixed input sequence for InhSLSs). System (4.26) is **solvable** w.r.t. a fixed input sequence (u_k) in the sense of Definition 4.28 if and only if for $k = 0, 1, \ldots$

$$E^{+}A\mathcal{S} + \{p^{u(k+1),u(k)}\} \subseteq \ker E \oplus \mathcal{S} \tag{4.31}$$

where $p^{k,k+1} = (E^{+} - A^{+})Bu(k) + A^{+}Bu(k+1)$. In that case, its solution at any k satisfies the surrogate system

$$x(k+1) = \Phi x(k) + \Theta u(k) + \Gamma u(k+1), \;\; x(0) \in \widehat{\mathcal{S}}^{u(0)} \tag{4.32}$$

where $\Phi = \Pi_{\mathcal{S}}^{\ker E} E^{+}A$, $\Theta = \Pi_{\mathcal{S}}^{\ker E} E^{+}B$, $\Gamma = (\Pi_{\mathcal{S}}^{\ker E} - I)A^{+}B$, E^{+} and A^{+} are a generalized inverse of E and A respectively, and $\Pi_{\mathcal{V}}^{\mathcal{W}}$ is the canonical

projector from $\mathcal{V} \oplus \mathcal{W}$ to $\mathcal{V}$. In particular, $x(k) \in \widehat{\mathcal{S}}^{u(k)}$ for all $k \in \mathbb{N}$. $\Diamond$

Proof. The idea of the proof is based on the proof of the solvability characterization for homogeneous systems in Lemma 4.3. Here, the input term is added.

Step 1: the solvability condition

Necessity: First note that every solution at any k, $x(k)$, satisfies the inclusion $x(k) \in \widehat{\mathcal{S}}^{u(k)} = A^{-1}(\operatorname{im} E - \{Bu(k)\}) = \mathcal{S} - \{A^+Bu(k)\}$ due to $x(k)$ satisfying $Ex(k+1) = Ax(k) + Bu(k)$. Now, for the given input sequence (u_k), take a solution at any k, $x(k)$. By Lemma A.2 the solution $x(k+1)$ of $Ex(k+1) = Ax(k) + Bu(k)$ satisfies

$$x(k+1) \in E^{-1}\{Ax(k) + Bu(k)\} = \{E^+Ax(k) + E^+Bu(k)\} + \ker E.$$

The solution $x(k+1)$ also satisfies $E\xi = Ax(k+1) + Bu(k+1)$ for some $\xi \in \mathbb{R}^n$, i.e.,

$$\begin{aligned} x(k+1) &\in A^{-1}\left(\{-Bu(k+1)\} + \operatorname{im} E\right) \\ &= \{-A^+Bu(k+1)\} + A^+(\operatorname{im} E) + \ker A \qquad (4.33) \\ &= \{-A^+Bu(k+1)\} + A^{-1}(\operatorname{im} E) = \{-A^+Bu(k+1)\} + \mathcal{S}. \end{aligned}$$

Altogether,

$$x(k+1) \in (\{E^+Ax(k)+E^+Bu(k)\}+\ker E)\cap(\{-A^+Bu(k+1)\}+\mathcal{S}) \quad (4.34)$$

By Lemma A.3 with $\mathbb{U} = E^+A\widehat{\mathcal{S}}^{u(k)}+\{E^+Bu(k)\}$, $\mathcal{W} = \ker E$, $\mathbb{Z} = \{-A^+Bu(k+1)\}$, and $\mathcal{V} = \mathcal{S}$, the uniqueness of $x(k+1)$ implies

$$E^+A\widehat{\mathcal{S}}^{u(k)} + \{E^+Bu(k) + A^+Bu(k+1)\} \subseteq \ker E \oplus \mathcal{S}$$

or equivalently (by the definition of $\widehat{\mathcal{S}}^{u(k)}$)

$$E^+A\mathcal{S} + \{(E^+ - A^+)Bu(k) + A^+Bu(k+1)\} \subseteq \ker E \oplus \mathcal{S},$$

and this holds for $k = 0, 1, \ldots$, hence, the inclusion (4.31) holds.

Sufficiency: This is proved inductively, that if for any $x_0 \in \mathcal{S}^{u(0)}$ there exists a unique solution on $[0, k]$ with the given input sequence (u_k), then there also exists a unique solution on $[0, k+1]$ with (u_k). This, together with the trivial observation that $x(0) = x_0$ is the unique solution of (4.26) with $x(0) = x_0 \in \mathcal{S}^{u(0)}$ considered only for $k = 0$ will prove the solvability.

For a given $x(k)$ and with the given (u_k), choose

$$x(k+1) \in \left(\{-A^+Bu(k+1)\} + \mathcal{S}\right) \cap \left(\{E^+Ax(k) + E^+Bu(k)\} + \ker E\right)$$

which is possible due to Lemma A.3. Then, $x(k+1) \in \{E^+Ax(k)+E^+Bu(k)\}+\ker E$ implies that $Ex(k+1) = EE^+Ax(k)+E+E^+Bu(k)$. From $x(k) \in \mathcal{S}^{u(k)}$, it follows that $Ax(k) + Bu(k) \in \operatorname{im} E$, i.e. there exists v such that $Ax(k) + Bu(k) = Ev$. Hence $Ex(k+1) = EE^+Ev = Ev = Ax(k) + Bu(k)$ which shows that $x(k+1)$ satisfies (4.26). Furthermore, $x(k+1)$ also satisfies (4.26) for $k+1$ because $x(k+1) \in \mathcal{S}^{u(k+1)}$. This shows that x is indeed a solution

of (4.26) on $[0, k+1]$. Uniqueness follows from the fact that by Lemma A.3, the set $\big(\{-A^+Bu(k+1)\} + \mathcal{S}\big) \cap \big(\{E^+Ax(k) + E^+Bu(k)\} + \ker E\big)$ is a singleton.

Step 2: the surrogate system (4.32)
Applying formula (A.1) to (4.34) provides

$$\begin{aligned} x(k+1) =& \Pi_{\mathcal{S}}^{\ker E}\left(E^+Ax(k) + E^+Bu(k) + A^+Bu(k+1)\right) - A^+Bu(k+1) \\ =& \Pi_{\mathcal{S}}^{\ker E}E^+Ax(k) + \Pi_{\mathcal{S}}^{\ker E}E^+Bu(k) + \left(\Pi_{\mathcal{S}}^{\ker E} - I\right)A^+Bu(k+1) \end{aligned}$$

i.e. $x(k+1)$ satisfies (4.32). Finally, the inclusion $x(k) \in \widehat{\mathcal{S}}^{u(k)}$ for all $k \in \mathbb{N}$ is a direct consequence of $x(k)$ solves $Ex(k+1) = Ax(k) + Bu(k)$ with the given $u(k)$. □

The characterization for solvability w.r.t. a fixed input sequence above leads to the following proposition which says that system (4.26) is weakly solvable if and only if the matrix pair (E, A) is index-1.

Proposition 4.32 (Weak solvability characterization of InhSLSs). The InhSLS (4.26) is **weakly solvable** in the sense of Definition 4.28 if, and only if, (E, A) is index-1 in the sense of Definition 2.5. ◊

Proof. The matrix pair (E, A) being index-1 implies weak solvability is obvious. Take the zero input $u(k) = 0\ \forall k \in \mathbb{N}$. Then, with this zero input, $\ker E \oplus \mathcal{S} = \mathbb{R}^n$ implies that (4.31) holds. Thus, the system is solvable w.r.t. the zero input which proves weak solvability.
Now, assume the system is weakly solvable, i.e., there is an input sequence $(\bar{u}_k)$ such that (4.31) holds which implies $\ker E \cap \mathcal{S} = \{0\}$. In view of Corollary 2.6, it implies that (E, A) is index-1. □

The condition for strong solvability can be derived by imposing the solvability condition (4.31) to all pairs of initial values and input sequences $(x(0), (u_k))$, $x(0) \in \widehat{\mathcal{S}}$, $u(k) \in \mathbb{R}^m$. However, a simpler characterization can be derived by considering arbitrary initial values and inputs in the proof; the result is presented in the following proposition.

Proposition 4.33 (Strong solvability characterization for InhSLSs). System (4.26) is **strongly solvable** in the sense of Definition 4.29 if, and only if,

$$E^+A\widehat{\mathcal{S}} + \operatorname{im} E^+B \subseteq \ker E \oplus \widehat{\mathcal{S}}. \tag{4.35}$$

where $\widehat{\mathcal{S}} = A^{-1}(\operatorname{im}[E, B])$. In that case, its solution satisfies

$$x(k+1) = \widehat{\Phi}x(k) + \widehat{\Theta}u(k),\ x(0) \in \widehat{\mathcal{S}},\ k = 0, 1, \ldots \tag{4.36}$$

where $\widehat{\Phi} = \Pi_{\widehat{\mathcal{S}}}^{\ker E}E^+A$, $\widehat{\Theta} = \Pi_{\widehat{\mathcal{S}}}^{\ker E}E^+B$, E^+ and A^+ are a generalized inverse of E and A respectively, and $\Pi_{\mathcal{V}}^{\mathcal{W}}$ is the canonical projector from $\mathcal{V} \oplus \mathcal{W}$ to $\mathcal{V}$.

In particular, $x(k) \in \widehat{\mathcal{S}}$ for all $k \in \mathbb{N}$, $\diamondsuit$

Proof. **Step 1: solvability condition**
Necessity: Take a solution at any k, $x(k)$. By the preimage property, the solution $x(k+1)$ of $Ex(k+1) = Ax(k) + Bu(k)$ satisfies

$$x(k+1) \in E^{-1}(\{Ax(k) + Bu(k)\}) = \{E^+Ax(k) + E^+Bu(k)\} + \ker E.$$

The solution $x(k+1)$ also satisfies $E\xi_1 = Ax(k+1) + B\xi_2$, $\xi_1 \in \mathbb{R}^n$, $\xi_2 \in \mathbb{R}^m$ or equivalently

$$x(k+1) \in A^{-1}\operatorname{im}[E, B] = \widehat{\mathcal{S}}. \tag{4.37}$$

Altogether,

$$x(k+1) \in (\{E^+Ax(k) + E^+Bu(k)\} + \ker E) \cap \widehat{\mathcal{S}}.$$

By Lemma A.3 with $\mathbb{U} = E^+A\widehat{\mathcal{S}} + \operatorname{im} E^+B$, $\mathbb{Z} = \{0\}$, $\mathcal{V} = \widehat{\mathcal{S}}$ and $\mathcal{W} = \ker E$, the uniqueness of $x(k+1)$ implies

$$E^+A\widehat{\mathcal{S}} + \operatorname{im} E^+B \subseteq \ker E \oplus \widehat{\mathcal{S}}.$$

Sufficiency: It will be shown that for all $x(0) = x_0 \in \widehat{\mathcal{S}}$ and all $u(0) \in \mathbb{R}^m$ there exists a unique $x(1)$ which satisfies (4.26) at $k = 0$ and $k = 1$, i.e.

$$Ex(1) = Ax(0) + Bu(0)$$
$$E\xi = Ax(1) + B\nu$$

for some $\xi \in \mathbb{R}^n$ and $\nu \in \mathbb{R}^m$. The latter is equivalent to

$$x(1) \in E^{-1}\{Ax(0) + Bu(0)\} = \{E^+Ax_0 + E^+Bu(0)\} + \ker E$$
$$x(1) \in A^{-1}\operatorname{im}[E, B] = \widehat{\mathcal{S}}.$$

By Lemma A.3, the condition $E^+A\widehat{\mathcal{S}} + \operatorname{im} E^+B \subseteq \ker E \oplus \widehat{\mathcal{S}}$ implies that

$$(\{E^+Ax_0 + E^+Bu(0)\} + \ker E) \cap \widehat{\mathcal{S}}$$

is a singleton for all $x_0 \in \widehat{\mathcal{S}}$ and all $u(0) \in \mathbb{R}^m$, hence there is exists a unique $x(1)$ satisfying (4.26). Repeating the argument now inductively, it can be shown that $x(k)$ is uniquely determined by $x(0), x(1), \ldots, x(k-1)$ and $u(0), u(1), \ldots, u(k-1)$ for all $k = 1, 2, \ldots, k_1$.
Step 2: the surrogate system (4.36)
Applying formula (A.1) in Lemma A.3 with $\mathbb{Z} = \{0\}$, $\mathbb{U} = E^+A\widehat{\mathcal{S}} + \operatorname{im} E^+B$, $\mathcal{V} = \widehat{\mathcal{S}}$ and $\mathcal{W} = \ker E$ proves that the solution $x(k+1)$ satisfies (4.36). Finally, the inclusion $x(k) \in \widehat{\mathcal{S}}$ is a direct consequence of $x(k)$ solving (4.26); this can also be seen from (4.37). $\square$

A simpler condition of the solvability condition in the proposition above can even be derived by utilizing the strictly index-1 notion defined in Definition 2.8. This is presented in the following proposition.

Proposition 4.34 (Simplified characterization for strong solvability of InhSLSs). System (4.26) is **strongly solvable** in the sense of Definition 4.29 if, and only if, (E, A, B) is strictly index-1 in the sense of Definition 2.8. ◊

Proof. In view of Lemma 2.9, it suffices to prove that (4.35) holds if and only if $\ker E \cap \widehat{\mathcal{S}} = \{0\}$. The necessity is obvious since the condition (4.35) implies $\ker E \cap \widehat{\mathcal{S}} = \{0\}$. For the sufficiency, by Lemma 2.9, the strictly index-1 condition $\ker E \oplus \widehat{\mathcal{S}} = \mathbb{R}^n$ implies that the strong solvability condition (4.35) holds. □

By utilizing the surrogate system (4.32), the explicit solution of a solvable system (4.26) w.r.t. a given input sequence can be written as

$$\begin{aligned} x(k) &= \Phi^k x(0) + \Sigma_{i=0}^{k-1} \Phi^{k-i-1} \left(\Theta u(i) + \Gamma u(i+1)\right), \ x(0) \in \widehat{\mathcal{S}}^{u(0)} \\ y(k) &= C\Phi^k x(0) + \Sigma_{i=0}^{k-1} C\Phi^{k-i-1} \left(\Theta u(i) + \Gamma u(i+1)\right) + Du(k). \end{aligned} \tag{4.38}$$

Meanwhile, the explicit solution of a strongly solvable system (4.44) can be written as

$$\begin{aligned} x(k) &= \widehat{\Phi}^k x(0) + \sum_{i=0}^{k-1} \widehat{\Phi}^{k-i-1} \widehat{\Theta} u(i), \ x(0) \in \widehat{\mathcal{S}} \\ y(k) &= C\widehat{\Phi}^k x(0) + \sum_{i=0}^{k-1} C\widehat{\Phi}^{k-i-1} \widehat{\Theta} u(i) + Du(k). \end{aligned} \tag{4.39}$$

Remark 4.35 (Solvability w.r.t. a time-dependent set of inputs). If the solvability notion in Definition 4.28 is defined w.r.t. input sequences (u_k) where $u(k)$ is taken from a time-dependent set $\mathbb{U}(k) \subseteq \mathbb{R}^m$, then a further solvability notion w.r.t. $\mathbb{U}(k)$ can be defined as follows: system (4.26) is called **solvable w.r.t.** $\mathbb{U}(k)$ if for all $k_1 \in \mathbb{N}$, all $x_0 \in \widehat{\mathcal{S}}^{\mathbb{U}(0)} = \mathcal{S} - A^+ B\mathbb{U}(0) = \{\xi - \eta \in \mathbb{R}^n \mid \xi \in \mathcal{S}, \eta \in A^+ B\mathbb{U}(0)\}$, and all input sequences $(u_k) \mid_{u(k) \in \mathbb{U}(k)}$, there exists a unique solution of (4.26) on $[0, k_1]$ with $x(0) = x_0$. This solvability notion is suitable for control designs with **constrained inputs**. The characterization for this solvability notion can be derived by imposing the solvability condition (4.31) to all involved input sequences. Therefore, system (4.26) is solvable w.r.t. $\mathbb{U}(k)$ if, and only if, for $k = 0, 1, \ldots$

$$E^+ A(\mathcal{S} - A^+ B\mathbb{U}(k)) + E^+ B\mathbb{U}(k) + A^+ B\mathbb{U}(k+1) \subseteq \ker E \oplus \mathcal{S}$$

which can be rewritten as

$$E^+ A\mathcal{S} + (E^+ AA^+ B + E^+ B)\mathbb{U}(k) + A^+ B\mathbb{U}(k+1) \subseteq \ker E \oplus \mathcal{S} \tag{4.40}$$

Furthermore, the surrogate system (4.32) is also valid, and in particular, $x(k) \in \mathcal{S} - E^+ B\mathbb{U}(k)$ for all $k \in \mathbb{N}$. ◊

4.2.1.3 Discussion on causality

With respect to states, system (4.26) is desired to have the feature in which the current state is determined only by past states. However, with respect to inputs, the current input affecting the current state is still desirable in discrete-time systems for particular classes such as systems in which the time instant represents a period of time. For example, in inventory models, the time instant may represent a day, the current input can be considered as the action at the beginning of the day, and the current state (inventory level) can be considered as the value at the end of the day. Therefore, the systems can be classified into two causality notions: causal and strictly causal.

Intuitively, system (4.44) is said to be **causal** if on any time interval $[0, k_1]$, $k_1 \in \mathbb{N}$, all solutions at any time instant $k \in [0, k_1]$ are determined completely by the initial condition $x(0)$, past inputs $u(0), u(1), \ldots, u(k-1)$ and the current input $u(k)$. Meanwhile, it is said to be **strictly causal** if it is causal and the current input $u(k)$ does not determine the current state $x(k)$. The formal definitions are given in the following. First, let (x, u) be a pair of state and input that solves (4.26) on $[0, \infty)$, $x_{[0,k]} = (x(0), x(1), \ldots, x(k))$ be a sequence of solutions on $[0, k]$, and $u_{[0,k]} = (u(0), u(1), \ldots, u(k))$ be an input sequence on $[0, k]$. Define the sets

$$\Omega := \{ \ (x, u) \ | \ (x, u) \text{ solves } (4.26) \ \}, \tag{4.41}$$

$$\Sigma_k^u := \left\{ \ (x', u') \in \Omega \ \middle| \ u'_{[0,k]} = u_{[0,k]} \ \right\} \tag{4.42}$$

$$\Delta_k^u := \{ \ u' \ | \ \exists x' : (x', u') \in \Sigma_k^u \ \}. \tag{4.43}$$

Intuitively, the set Σ_k^u contains all pairs of state and input sequences that solves (4.26) on $[0, \infty)$ whose input is equal to a given input u on $[0, k]$. Meanwhile, the set Δ_k^u contains all input sequences u' in which there exists a solution x' such that the pair $(x', u') \in \Sigma_k$.

Definition 4.36 (Causality notions for InhSLSs). The InhSLS (4.26) is called

- **causal** if $\forall k \in \mathbb{N}, \ \forall u' \in \Delta_k^u : x_{[0,k]} = x'_{[0,k]}$,
- **strictly causal** if causal and $\forall k \in \mathbb{N}, \ \forall u' \in \Delta_{k-1}^u : x_{[0,k]} = x'_{[0,k]}$.

where $x_{[0,k]}$ and $x'_{[0,k]}$ are solutions of (4.26) on $[0, k]$.

$\Diamond$

From the definition above, in the first (nonstrict) causality notion, the input at k, $u(k)$, also determines $x(k)$. Meanwhile, in the strict causality, the current input $u(k)$ does not determine $x(k)$, i.e., $x(k)$ is determined only by the past information; ordinary systems $x(k+1) = Ax(k) + Bu(k)$ possess this strict causality. Note that causality notions do not make sense to be defined for

systems having no solutions, therefore, the existence of a solution is required; However, the solution is not necessarily unique.

By definition, strict causality implies causality. The converse is not always true, this can be seen from system (4.30) in Remark 4.30 in which the substate x_2 at time instant k depends on $u(k)$, which means that the system is causal and not strictly causal.

The solvability characterizations obtained in the previous section lead to the following proposition which reveals that weak solvability implies (nonstrict) causality and strong solvability implies strict causality.

Proposition 4.37 (Solvability implies causality (InhSLSs)).

(i) Every weakly solvable InhSLS (4.26) is causal

(ii) Every strongly solvable InhSLS (4.26) is strictly causal. ◊

Proof. These are direct consequences of the systems being (weakly or strongly) solvable. The non-strict causality of a weakly solvable system can be directly seen from its surrogate system (4.32) in which $x(k)$ is completely determined by $x(t),\ t < k$ and $u(\ell),\ \ell \leq k$. Meanwhile, the strict causality of a strongly solvable system can be directly seen from its surrogate system (4.36) in which $x(k)$ is completely determined by $x(t)$ and $u(t)$ with $t < k$. □

4.2.2 Switched Systems

Consider now the class of discrete-time Inhomogeneous Singular Linear Switched Systems (InhSLSSs) of the form

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k),\ \ k \in \mathbb{N} \tag{4.44a}$$

$$y(k) = C_{\sigma(k)}x(k) + D_{\sigma(k)}u(k) \tag{4.44b}$$

where $k \in \mathbb{N}$ is the time instant, $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m, m \in \mathbb{N}$ is the input, $\sigma : \mathbb{N} \to \{0, 1, 2, \ldots, \mathrm{p}\}$ is the switching signal of the form (2.11) determining which mode $\sigma(k)$ is active at time instant k, $E_i, A_i \in \mathbb{R}^{n\times n}$, and $B_i \in \mathbb{R}^{n\times m}$ are constant matrices. The matrices E_i may be nonsingular unless stated otherwise, and thus systems given by (4.44) also cover ordinary systems.

For each mode $i \in \{0, 1, \ldots, \mathrm{p}\}$, recall the set $\mathcal{S}_i$ as in (4.10), and define

$$\widehat{\mathcal{S}}_i := A_i^{-1}(\mathrm{im}[E_i, B_i]) = \{\xi \in \mathbb{R}^n : A_i\xi \in \mathrm{im}[E_i, B_i]\}. \tag{4.45}$$

Each mode-i of (4.44) can be represented by the matrix triplet (E_i, A_i, B_i), and the state's equation (6.6) can be represented by the family of matrix triplets $\{(E_i, A_i, B_i)\}_{i=0}^{\mathrm{p}}$. Under regularity assumption for each (E_i, A_i), its corresponding QWF as in (2.6) for mode i is given by

$$(S_iE_iT_i, S_iA_iT_i, S_iB_i) = \left(\begin{bmatrix} I & 0 \\ 0 & N_i \end{bmatrix}, \begin{bmatrix} J_i & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} B_i^J \\ B_i^N \end{bmatrix}\right). \tag{4.46}$$

4.2.2.1 Definitions

As in the solution theory for non-switched inhomogeneous systems in the previous subsection, by considering a fixed input sequence or an arbitrary input sequence, it is also possible to have various solvability notions for switched systems. Furthermore, following the flow of the study for homogeneous systems, with respect to switching signals, the solvability is defined w.r.t. a fixed switching signal; this is presented in the following definition.

Definition 4.38 (Solvability of InhSLSSs w.r.t. a fixed switching signal). The InhSLSS (4.44) is called

- **solvable w.r.t. a fixed switching signal** σ **and a fixed input sequence** (u_k) if for all $k_0, k_1 \in \mathbb{N}$ with $k_1 > k_0$ and all $x_{k_0} \in \widehat{\mathcal{S}}^{u(k_0)}_{\sigma(k_0)} = \mathcal{S}_{\sigma(k_0)} - \{A^+_{\sigma(k_0)} B_{\sigma(k_0)} u(k_0)\}$, there exists a unique solution of (4.44) on $[k_0, k_1]$, $(x(k_0), x(k_0+1), \ldots, x(k_1))$, with $x(k_0) = x_{k_0}$ and with (u_k). In particular, system (4.44) is called **weakly solvable** w.r.t. σ if there is an input sequence (u_k) such that the system is solvable w.r.t. σ and (u_k).

- **strongly solvable w.r.t. a fixed switching signal** σ if, for all $k_0, k_1 \in \mathbb{N}$, $k_1 > k_0$, all $x_{k_0} \in \widehat{\mathcal{S}}_{\sigma(k_0)} = A^{-1}_{\sigma(k_0)}(\operatorname{im}[E_{\sigma(k_0)}, B_{\sigma(k_0)}])$, and all input sequences (u_k), there exists a unique solution of (4.44) on $[k_0, k_1]$ with $x(k_0) = x_{k_0}$. $\Diamond$

Inspired by the index-1 notions for homogeneous systems in Definition 4.7, strictly index-1 notions are introduced in the following definition, which will be used later in the solvability characterizations for inhomogeneous systems, see the forthcoming Theorem 4.40. These notions are based on the strictly index-1 notion in Definition 2.8 for a matrix triplet (E, A, B), and are defined for the family of matrix triplets $\{(E_i, A_i, B_i)\}_{i=0}^{\mathrm{p}}$. Note that they are defined with respect to the subspaces $\widehat{\mathcal{S}}_i = A_i^{-1}(\operatorname{im}[E_i, B_i])$.

Definition 4.39 (Strictly index-1 notions of family of matrix triplets). A family of matrix triplets $\{(E_i, A_i, B_i)\}_{i=0}^{\mathrm{p}}$ is called

- **jointly strictly index-1** if
$$\ker E_i \oplus \widehat{\mathcal{S}}_j = \mathbb{R}^n \ \forall i, j \in \{0, 1, \ldots, \mathrm{p}\} \tag{4.47}$$

- **sequentially strictly index-1** w.r.t. a fixed mode sequence $(\sigma_j)_{j\in\mathbb{N}}$ if
$$\ker E_i \oplus \widehat{\mathcal{S}}_i = \mathbb{R}^n \ \forall i \in \{0, 1, 2, \ldots, \mathrm{p}\} \tag{4.48a}$$
$$E^+_{\sigma_j} A_{\sigma_j} \widehat{\mathcal{S}}_{\sigma_j} + \operatorname{im} E^+_{\sigma_j} B_{\sigma_j} \subseteq \ker E_{\sigma_j} \oplus \widehat{\mathcal{S}}_{\sigma_{j+1}} \quad \text{for } j = 0, 1, 2, \ldots \tag{4.48b}$$

- **switched strictly index-1** w.r.t. a fixed switching signal σ if, for $k = 0, 1, 2, \ldots$,
$$E^+_{\sigma(k)} A_{\sigma(k)} \widehat{\mathcal{S}}_{\sigma(k)} + \operatorname{im} E^+_{\sigma(k)} B_{\sigma_{(k)}} \subseteq \ker E_{\sigma(k)} \oplus \widehat{\mathcal{S}}_{\sigma(k+1)}. \tag{4.49}$$

where $\widehat{\mathcal{S}}_i = A_i^{-1}(\operatorname{im}[E_i, B_i])$ and E_i^+ is a generalized inverse of E_i. ◊

4.2.2.2 Characterizations

Now, the characterizations for the solvability notions defined in Definition 4.38 for the InhSLSS (4.44) are presented as follows:

Theorem 4.40 (Solvability of InhSLSSs w.r.t. a fixed switching signal). Consider the InhSLSS (4.44) with a fixed switching signal σ of the form (2.11). This system is

- **solvable** w.r.t. σ and a fixed input sequence (u_k) if and only if for $k = 0, 1, \ldots$
$$E^+_{\sigma(k)} A_{\sigma(k)} \mathcal{S}_{\sigma(k)} + \left\{ p^{u(k+1),u(k)}_{\sigma(k+1),\sigma(k)} \right\} \subseteq \ker E_{\sigma(k)} \oplus \mathcal{S}_{\sigma(k+1)} \tag{4.50}$$
where
$$p^{u(k+1),u(k)}_{\sigma(k+1),\sigma(k)} = (E^+_{\sigma(k)} - A^+_{\sigma(k)}) B_{\sigma(k)} u(k) + A^+_{\sigma(k+1)} B_{\sigma(k+1)} u(k+1).$$
In this case, its solution satisfies the surrogate system
$$\begin{aligned} x(k+1) =& \Phi_{\sigma(k+1),\sigma(k)} x(k) + \Theta_{\sigma(k+1),\sigma(k)} u(k) \\ &+ \Gamma_{\sigma(k+1),\sigma(k)} u(k+1) \end{aligned} \tag{4.51}$$
with $x(0) \in \widehat{\mathcal{S}}^{u(0)}_{\sigma(0)}$ and where the matrices $\Phi_{i,j} = \Pi^{\ker E_j}_{\mathcal{S}_i} E^+_j A_j$, $\Theta_{i,j} = \Pi^{\ker E_j}_{\mathcal{S}_i} E^+_j B_j$, $\Gamma_{i,j} = (\Pi^{\ker E_j}_{\mathcal{S}_i} - I) A^+_i B_i$ are the one-step maps from mode j to mode i, E_i^+ is a generalized inverse of E_i and $\Pi^{\ker E_j}_{\mathcal{S}_i}$ is the canonical projector from $\ker E_j \oplus \mathcal{S}_i$ to $\mathcal{S}_i$. In particular, $x(k) \in \widehat{\mathcal{S}}^{u(k)}_{\sigma(k)} = \mathcal{S}_{\sigma(k)} - \{E^+_{\sigma(k)} B_{\sigma(k)} u(k)\}$ for all $k \in \mathbb{N}$.

- **strongly solvable** w.r.t. σ if and only if $\{(E_i, A_i, B_i)\}_{i=0}^{\mathfrak{p}}$ is switched strictly index-1 w.r.t. σ. In this case, its solution satisfies the surrogate system
$$x(k+1) = \widehat{\Phi}_{\sigma(k+1),\sigma(k)} x(k) + \widehat{\Theta}_{\sigma(k+1),\sigma(k)} u(k), \; x(0) \in \widehat{\mathcal{S}}_{\sigma(0)} \tag{4.52}$$
where the matrices $\widehat{\Phi}_{i,j} = \Pi^{\ker E_j}_{\widehat{\mathcal{S}}_i} E^+_j A_j$ and $\widehat{\Theta}_{i,j} = \Pi^{\ker E_j}_{\widehat{\mathcal{S}}_i} E^+_j B_j$ are the one-step maps from mode j to mode i. In particular, $x(k) \in \widehat{\mathcal{S}}_{\sigma(k)}$ for all $k \in \mathbb{N}$. ◊

Proof. The idea of the proof is based on the proof of the solvability of Hom-SLSSs in Theorem 4.11 and the proof of the solvability of (non-switched) InhSLSs in Theorem 4.31.

Step 1.a: solvability w.r.t. σ and (u_k)

Necessity: First, let $k_0 \in \mathbb{N}$, $k_1 := k_0 + 1$ and consider the InhSLSS (4.44) on $[k_0, k_1]$ with solutions $x_0 := x(k_0)$, $x_1 := x(k_0 + 1)$, $x_2 := x(k_0 + 2)$ for some $u(k_0) = u_0$ and $u(k_1) = u_1$ satisfying

$$E_{\sigma(k_0)}x_1 = A_{\sigma(k_0)}x_0 + B_{\sigma(k_0)}u_0 \tag{4.53a}$$

$$E_{\sigma(k_1)}x_2 = A_{\sigma(k_1)}x_1 + B_{\sigma(k_1)}u_1. \tag{4.53b}$$

By Lemma A.2, the former equation implies

$$x_1 \in \{E^+_{\sigma(k_0)}A_{\sigma(k_0)}x_0 + E^+_{\sigma(k_0)}B_{\sigma(k_0)}u_0\} + \ker E_{\sigma(k_0)} \tag{4.54}$$

whereas the latter implies

$$\begin{aligned} x_1 &\in \{A^+_{\sigma(k_1)}E_{\sigma(k_1)}x_2 - A^+_{\sigma(k_1)}B_{\sigma(k_1)}u_1\} + \ker A_{\sigma(k_1)} \\ &= \{-A^+_{\sigma(k_1)}B_{\sigma(k_1)}u_1\} + A^{-1}_{\sigma(k_1)}(\operatorname{im} E_{\sigma(k_1)}) \\ &= \{-A^+_{\sigma(k_1)}B_{\sigma(k_1)}u_1\} + \mathcal{S}_{\sigma(k_1)}. \end{aligned} \tag{4.55}$$

Since the system is solvable, i.e., x_1 is unique, the intersection

$$\begin{aligned} &\Big(\{E^+_{\sigma(k_0)}A_{\sigma(k_0)}x_0 + E^+_{\sigma(k_0)}B_{\sigma(k_0)}u_0\} + \ker E_{\sigma(k_0)}\Big) \\ &\qquad \cap \Big(\{-A^+_{\sigma(k_1)}B_{\sigma(k_1)}u_1\} + \mathcal{S}_{\sigma(k_1)}\Big) \end{aligned}$$

is a singleton. By Lemma A.3 with $\mathbb{U} = \{E^+_{\sigma(k_0)}A_{\sigma(k_0)}x_0 + E^+_{\sigma(k_0)}B_{\sigma(k_0)}u_0\}$, $\mathbb{Z} = \{-A^+_{\sigma(k_1)}B_{\sigma(k_1)}u_1\}$, $\mathcal{V} = \mathcal{S}_{\sigma(k_1)}$, and $\mathcal{W} = \ker E_{\sigma(k_0)}$, it implies

$$\begin{aligned} &E^+_{\sigma(k_0)}A_{\sigma(k_0)}\widehat{\mathcal{S}}^{u(k_0)}_{\sigma(k_0)} + \{E^+_{\sigma(k_0)}B_{\sigma(k_0)}u(k_0) + A^+_{\sigma(k_1)}B_{\sigma(k_1)}u(k_1)\} \\ &\qquad \subseteq \ker E_{\sigma(k_0)} \oplus \mathcal{S}_{\sigma(k_1)} \end{aligned}$$

as desired.

Sufficiency: It will be shown that for each $k_0, k_1 \in \mathbb{N}$ with $k_1 > k_0$ and for each $x_0 \in \mathcal{S}_{\sigma(k_0)}$ with the given input sequence (u_k), the sequence $x : [k_0, k_1] \to \mathbb{R}^n$ given by $x(k_0) = x_0$ and $x(k+1) = \Phi_{\sigma(k+1),\sigma(k)}x(k) + \Theta_{\sigma(k+1),\sigma(k)}u(k) + \Gamma_{\sigma(k+1),\sigma(k)}u(k+1)$ is the unique solution of InhSLSS (4.44) with $x(k_0) = x_0$ on $[k_0, k_1]$. Inductively, assume for $k \geq k_0$ that it is already known that $x(k) \in \mathcal{S}_{\sigma(k)}$ and that x is the unique solution on $[k_0, k]$ with $x(k_0) = x_0$ (which is satisfied for $k = k_0$). Now, it will be shown that x is the unique solution on $[k_0, k+1]$, which by an induction argument then concludes the proof. In order to show the former, it just needs to show that $x(k+1) = \Phi_{\sigma(k+1),\sigma(k)}x(k) + \Theta_{\sigma(k+1),\sigma(k)}u(k) + \Gamma_{\sigma(k+1),\sigma(k)}u(k+1)$ satisfies

$$\begin{aligned} E_{\sigma(k)}x(k+1) &= A_{\sigma(k)}x(k), \\ E_{\sigma(k+1)}x_2 &= A_{\sigma(k+1)}x(k+1), \end{aligned} \tag{4.56}$$

for some $x_2 \in \mathbb{R}^n$ and that there is no other possible value for $x(k+1)$ satisfying this equation. For a given $x(k)$ and with the given (u_k), choose

$$x(k+1) = \Big(\{E^+_{\sigma(k)}A_{\sigma(k)}x(k) + E^+_{\sigma(k)}B_{\sigma(k)}u(k)\} + \ker E_{\sigma(k)}\Big) \cap \Big(\{-A^+_{\sigma(k+1)}B_{\sigma(k+1)}u(k+1)\} + \mathcal{S}_{\sigma(k+1)}\Big)$$

which is possible due to Lemma A.3. Then, $x(k+1) \in \{E^+_{\sigma(k)}A_{\sigma(k)}x(k) + E^+_{\sigma(k)}B_{\sigma(k)}u(k)\} + \ker E_{\sigma(k)}$ implies that $E_{\sigma(k)}x(k+1) = E_{\sigma(k)}E^+_{\sigma(k)}A_{\sigma(k)}x(k) + E^+_{\sigma(k)}B_{\sigma(k)}u(k)$. From $x(k) \in \mathcal{S}^{u(k)}_{\sigma(k)}$, it follows that $A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) \in \operatorname{im} E_{\sigma(k)}$, i.e. there exists v such that $A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) = E_{\sigma(k)}v$. Hence $E_{\sigma(k)}x(k+1) = E_{\sigma(k)}E^+_{\sigma(k)}E_{\sigma(k)}v = E_{\sigma(k)}v = A_{\sigma(k)}x(k)+B_{\sigma(k)}u(k)$ which shows that $x(k+1)$ satisfies (4.44). Furthermore, $x(k+1)$ also satisfies (4.44) for $k+1$ because $x(k+1) \in \mathcal{S}^{u(k+1)}_{\sigma(k+1)}$. This shows that x is indeed a solution of (4.26) on $[0, k+1]$. Uniqueness follows from the fact that by Lemma A.3, the set $\big(\{E^+_{\sigma(k)}A_{\sigma(k)}x(k) + E^+_{\sigma(k)}B_{\sigma(k)}u(k)\} + \ker E_{\sigma(k)}\big) \cap \big(\{-A^+_{\sigma(k+1)}B_{\sigma(k+1)}u(k+1)\} + \mathcal{S}_{\sigma(k+1)}\big)$ is a singleton.

Step 1.b: The surrogate system (4.51)

For a solvable system, from the first part of the proof, it is already known that at any time instant k,

$$x(k+1) = \Big(\{E^+_{\sigma(k)}A_{\sigma(k)}x(k) + E^+_{\sigma(k)}B_{\sigma(k)}u(k)\} + \ker E_{\sigma(k)}\Big) \cap \Big(\{-A^+_{\sigma(k+1)}B_{\sigma(k+1)}u(k+1)\} + \mathcal{S}_{\sigma(k+1)}\Big).$$

Applying formula (A.1) with $\mathbb{U} = \{E^+_{\sigma(k)}A_{\sigma(k)}x(k) + E^+_{\sigma(k)}B_{\sigma(k)}u(k)\}$, $\mathbb{Z} = \{-A^+_{\sigma(k+1)}B_{\sigma(k+1)}u(k+1)\}$, $\mathcal{V} = \mathcal{S}_{\sigma(k+1)}$, and $\mathcal{W} = \ker E_{\sigma(k)}$ yields

$$\begin{aligned} x(k+1) =& \Pi^{\ker E_{\sigma(k)}}_{\mathcal{S}_{\sigma(k+1)}}\big(E^+_{\sigma(k)}A_{\sigma(k)}x(k) + E^+_{\sigma(k)}B_{\sigma(k)}u(k) \\ &+ A^+_{\sigma(k+1)}B_{\sigma(k+1)}u(k+1)\big) - A^+_{\sigma(k+1)}B_{\sigma(k+1)}u(k+1) \end{aligned}$$

and thus

$$\begin{aligned} x(k+1) = \Pi^{\ker E_{\sigma(k)}}_{\mathcal{S}_{\sigma(k+1)}}E^+_{\sigma(k)}A_{\sigma(k)}x(k) + \Pi^{\ker E_{\sigma(k)}}_{\mathcal{S}_{\sigma(k+1)}}E^+_{\sigma(k)}B_{\sigma(k)}u(k) \\ + \Big(\Pi^{\ker E_{\sigma(k)}}_{\mathcal{S}_{\sigma(k+1)}} - I\Big)A^+_{\sigma(k)}B_{\sigma(k)}u(k+1) \end{aligned} \tag{4.57}$$

i.e. $x(k+1)$ satisfies (4.51).

Step 2: Strong solvability w.r.t. the fixed switching signal σ

Step 2: The solvability condition

The proof is similar to Step 1 above, however, in strong solvability, it has to be proved that the system has a unique solution for every input sequence. The proof is as follows:

Necessity: Using the same notations for solutions of the InhSLSS (4.44) on

$[k_0, k_1]$, for any arbitrary u_0, there exists a unique solution x_1 satisfying

$$E_{\sigma(k_0)}x_1 = A_{\sigma(k_0)}x_0 + B_{\sigma(k_0)}u_0 \tag{4.58a}$$

$$E_{\sigma(k_1)}\xi = A_{\sigma(k_1)}x_1 + B_{\sigma(k_1)}\eta \tag{4.58b}$$

for some $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^m$. Then, by Lemma A.2

$$x_1 \in \{E^+_{\sigma(k_0)}A_{\sigma(k_0)}x_0 + E^+_{\sigma(k_0)}B_{\sigma(k_0)}u_0\} + \ker E_{\sigma(k_0)} \tag{4.59a}$$

$$x_1 \in A^{-1}_{\sigma(k_1)}\left(\operatorname{im}[E_{\sigma(k_1)}, B_{\sigma(k_1)}]\right) = \widehat{\mathcal{S}}_{\sigma(k_1)}. \tag{4.59b}$$

The uniqueness of x_1 implies that

$$x_1 = \left(\{E^+_{\sigma(k_0)}A_{\sigma(k_0)}x_0 + E^+_{\sigma(k_0)}B_{\sigma(k_0)}u_0\} + \ker E_{\sigma(k_0)}\right) \cap \widehat{\mathcal{S}}_{\sigma(k_1)} \tag{4.60}$$

is a singleton. By Lemma A.3 with $\mathbb{Z} = \{0\}$, $\mathbb{U} = \{E^+_{\sigma(k_0)}A_{\sigma(k_0)}x_0 + E^+_{\sigma(k_0)}B_{\sigma(k_0)}u_0\}$, $\mathcal{V} = \widehat{\mathcal{S}}_{\sigma(k_1)}$, and $\mathcal{W} = \ker E_{\sigma(k_0)}$, it implies

$$E^+_{\sigma(k_0)}A_{\sigma(k_0)}\widehat{\mathcal{S}}_{\sigma(k_0)} + E^+_{\sigma(k_0)}\operatorname{im} B_{\sigma(k_0)} \subseteq \ker E_{\sigma(k_0)} \oplus \widehat{\mathcal{S}}_{\sigma(k_1)}$$

i.e., $\{(E_i, A_i, B_i)\}_{i=0}^{\mathrm{p}}$ is switched strictly index-1 w.r.t. σ, see Definition 4.39.

Sufficiency: Now, it will be shown show that for each $k_0, k_1 \in \mathbb{N}$ with $k_1 > k_0$, for each $x_0 \in \mathcal{S}_{\sigma(k_0)}$, and for each input sequence $(u_k)\,|_{u(k)\in\mathbb{R}^m}$, the sequence $x : [k_0, k_1] \to \mathbb{R}^n$ given by $x(k_0) = x_0$ and $x(k+1) = \widehat{\Phi}_{\sigma(k+1),\sigma(k)}x(k) + \widehat{\Theta}_{\sigma(k+1),\sigma(k)}u(k)$ is the unique solution of InhSLSS (4.44) with $x(k_0) = x_0$ on $[k_0, k_1]$. Again, by inductive arguments, assume first for $k \geq k_0$ that $x(k) \in \widehat{\mathcal{S}}_{\sigma(k)}$ and that x is the unique solution on $[k_0, k]$ with $x(k_0) = x_0$ (which is satisfied for $k = k_0$). Now, it will be shown that x is the unique solution on $[k_0, 1]$, which by an induction argument then concludes the proof. To show the former, it will be shown that $x(k+1) = \widehat{\Phi}_{\sigma(k+1),\sigma(k)}x(k) + \widehat{\Theta}_{\sigma(k+1),\sigma(k)}u(k)$ satisfies

$$\begin{aligned} E_{\sigma(k)}x(k+1) &= A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k), \\ E_{\sigma(k+1)}x_2 &= A_{\sigma(k+1)}x(k+1) + B_{\sigma(k+1)}u(k+1), \end{aligned} \tag{4.61}$$

for some $x_2 \in \mathbb{R}^n$ and is unique. First, note that (4.61) can be rewritten as

$$\begin{aligned} x(k+1) &\in \{E^+_{\sigma(k)}A_{\sigma(k)}x(k) + E^+_{\sigma(k)}B_{\sigma(k)}u(k)\} + \ker E_{\sigma(k)} \\ x(k+1) &\in A^{-1}_{\sigma(k+1)}\operatorname{im}[E_{\sigma(k+1)}, B_{\sigma(k+1)}] = \widehat{\mathcal{S}}_{\sigma(k+1)}. \end{aligned} \tag{4.62}$$

For an arbitrary pair $(x(k), u(k))$,

$$x(k+1) = \left(\{E^+_{\sigma(k)}A_{\sigma(k)}x(k) + E^+_{\sigma(k)}B_{\sigma(k)}u(k)\} + \ker E_{\sigma(k)}\right) \cap \widehat{\mathcal{S}}_{\sigma(k+1)}$$

which is possible due to Lemma A.3. Then, $x(k+1) \in \{E^+_{\sigma(k)}A_{\sigma(k)}x(k) + E^+_{\sigma(k)}B_{\sigma(k)}u(k)\} + \ker E_{\sigma(k)}$ implies that $E_{\sigma(k)}x(k+1) = E_{\sigma(k)}E^+_{\sigma(k)}A_{\sigma(k)}x(k) + E^+_{\sigma(k)}B_{\sigma(k)}u(k)$. From $x(k) \in \widehat{\mathcal{S}}_{\sigma(k)}$, it follows that $A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) \in \operatorname{im} E_{\sigma(k)}$, i.e. there exists v such that $A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) = E_{\sigma(k)}v$. Hence $E_{\sigma(k)}x(k+1) = E_{\sigma(k)}E^+_{\sigma(k)}E_{\sigma(k)}v = E_{\sigma(k)}v =$

$A_{\sigma(k)}x(k)+B_{\sigma(k)}u(k)$ which shows that $x(k+1)$ satisfies (4.44). Furthermore, $x(k+1)$ also satisfies (4.44) for $k+1$ because $x(k+1) \in \widehat{\mathcal{S}}_{\sigma(k+1)}$. This shows that x is indeed a solution of (4.26) on $[0, k+1]$. Uniqueness follows from the fact that by Lemma A.3, the set $\big(\{E^+_{\sigma(k)}A_{\sigma(k)}x(k) + E^+_{\sigma(k)}B_{\sigma(k)}u(k)\} + \ker E_{\sigma(k)}\big) \cap \widehat{\mathcal{S}}_{\sigma(k+1)}$ is a singleton.

Step 2.b: The surrogate system (4.52)

Applying formula (A.1) to (4.60) with $\mathbb{U} = E^+_{\sigma(k)}A_{\sigma(k)}\widehat{\mathcal{S}}_{\sigma(k)} + \operatorname{im} E^+_{\sigma(k)}B_{\sigma(k)}$, $\mathcal{V} = \widehat{\mathcal{S}}_{\sigma(k+1)}$ and $\mathcal{W} = \ker E_{\sigma(k)}$ proves that the solution $x(k+1)$ at any k satisfies (4.52). □

Based on the theorem above, the characterizations for weak solvability can then be derived. Furthermore, the characterization for strong solvability w.r.t. a fixed mode sequence or all switching signals can also then be derived. Those characterizations are derived by imposing the solvability condition w.r.t. a fixed switching signal to every switching signal involved in the characterization. These are presented in the following proposition.

Proposition 4.41 (Solvability characterizations for InhSLSSs w.r.t. a mode sequence or all switching signals). The InhSLSS (4.44) is

- **weakly solvable** w.r.t. the fixed switching signal σ/fixed mode sequence (σ_j)/all switching signals if and only if $\{(E_i, A_i)\}_{i=0}^{\mathrm{p}}$ is switched index-1 w.r.t. σ/sequentially index-1 w.r.t. (σ_j)/jointly index-1.
- **strongly solvable** w.r.t. the fixed mode sequence $(\sigma_j)_{j\in\mathbb{N}}$/all switching signals if and only if $\{(E_i, A_i, B_i)\}_{i=0}^{\mathrm{p}}$ is sequentially strictly index-1 w.r.t. $(\sigma_j)_{j\in\mathbb{N}}$/jointly strictly index-1. ◊

Proof. **Part 1: Weak solvability**

Step 1.a: weak solvability w.r.t. the fixed switching signal σ

Necessity: It will be shown that if there is an input sequence $(\bar{u}_k)$ such that (4.50) holds with the given switching signal σ, then the switched index-1 condition (4.13) holds w.r.t. σ. For any k with the input sequence $(\bar{u}_k)$, the condition (4.50) means that $E^+_{\sigma(k)}A_{\sigma(k)}\mathcal{S}_{\sigma(k)} + p_{\sigma(k+1),\sigma(k)}$ is an affine subset of $\ker E_{\sigma(k)} \oplus \mathcal{S}_{\sigma(k+1)}$. This implies that $E^+_{\sigma(k)}A_{\sigma(k)}\mathcal{S}_{\sigma(k)} \subseteq \ker E_{\sigma(k)} \oplus \mathcal{S}_{\sigma(k+1)}$, i.e., $\{(E_i, A_i)\}_{i=0}^{\mathrm{p}}$ is switched index-1 w.r.t. σ.

Sufficiency: For any switched index-1 InhSLSS w.r.t. σ, the switched index-1 condition (4.13) implies (4.50) with $u \equiv 0$, which shows that there is the input sequence $u \equiv 0$ such that (4.50) holds and thus the system is weakly solvable.

Step 1.a: weak solvability w.r.t. the fixed mode sequence $(\sigma_j)_{j\in\mathbb{N}}$

Sufficiency: This is obvious by taking $u(k) = 0\ \forall k \in \mathbb{N}$. With these zero inputs, the system is now a homogeneous SLSS, and sequentially index-1 of

$\{(E_i, A_i)\}_{i=0}^{\mathrm{p}}$ implies the existence of a unique solution on $[k_0, k_1]$ for arbitrary $k_0, k_1 \in \mathbb{N},\ k_1 > k_0$ for any switching signal with the given mode sequence.
Necessity: The existence of an input sequence (u_k) such that the system is solvable w.r.t. (u_k) and all possible switching signals with the given mode sequence implies that (4.50) holds for each switching signal with the given mode sequence. Then, for all k, j, and all switching signals with $\sigma(k) = \sigma(k+1) = \sigma_j$, $\ker E_{\sigma_j} \cap \mathcal{S}_{\sigma_j} = \{0\}$, which in the view of Corollary 2.6, condition (4.12a) must hold. Furthermore, (4.50) must hold at any switch from mode σ_j to σ_{j+1}, which, in view of Step 1 of this proof, (4.12b) must hold. Altogether, (4.12) holds, i.e., $\{(E_i, A_i)\}_{i=0}^{\mathrm{p}}$ is sequentially index-1.
Step 1.b: weak solvability w.r.t. all switching signals
The sufficiency is obvious since jointly index-1 implies sequentially index-1 w.r.t. all mode sequences. For the necessity, the condition (4.50) being true for all switching signals implies that $\ker E_{\sigma(k)} \cap \mathcal{S}_{\sigma(k+1)} = \{0\}$ with arbitrary $\sigma(k)$ and $\sigma(k+1)$. Thus, by Corollary 2.6 and Proposition 4.8, $\{(E_i, A_i)\}$ is jointly index-1.
Part 2: Strong solvability
The proof is similar to the proof of weak solvability and is done by imposing the switched strict index-1 condition to all switching signals with the given mode sequence/all switching signals. □

Remark 4.42 (Solvability w.r.t. a time-dependent set of inputs). As in non-switched systems, if the solvability notion in Definition 4.38 is defined w.r.t. input sequences (u_k) where $u(k)$ is taken from a time-dependent set $\mathbb{U}(k) \subseteq \mathbb{R}^m$, the notion of solvability w.r.t. $\mathbb{U}(k)$ can then be defined as follows: the InhSLSS (4.44) is called **solvable w.r.t. a fixed switching signal σ and a time-dependent set of inputs** $\mathbb{U}(k)$ if for all $k_0, k_1 \in \mathbb{N}$ with $k_1 > k_0$, all $x_{k_0} \in \widehat{\mathcal{S}}_{\sigma(k_0)}^{\mathbb{U}(k_0)} = \mathcal{S}_{\sigma(k_0)} - A_{\sigma(k_0)}^{+} B_{\sigma(k_0)} \mathbb{U}(k_0)$, and all input sequences $(u_k)\,|_{u(k)\in\mathbb{U}(k)}$, there exists a unique solution of (4.44) on $[k_0, k_1]$ with $x(k_0) = x_{k_0}$. By imposing the solvability condition (4.50) to all involved input sequences, system (4.44) is solvable w.r.t. σ and $\mathbb{U}(k)$ if, and only if, for $k = 0, 1, \ldots$

$$\begin{aligned} E_{\sigma(k)}^{+} A_{\sigma(k)} \left(\mathcal{S}_{\sigma(k)} - A_{\sigma(k)}^{+} B_{\sigma(k)} \mathbb{U}(k)\right) + E_{\sigma(k)}^{+} B_{\sigma(k)} \mathbb{U}(k) \\ + A_{\sigma(k)}^{+} B_{\sigma(k)} \mathbb{U}(k+1) \subseteq \ker E_{\sigma(k)} \oplus \mathcal{S}_{\sigma(k+1)} \end{aligned} \tag{4.63}$$

which can be rewritten as

$$\begin{aligned} E_{\sigma(k)}^{+} A_{\sigma(k)} \mathcal{S}_{\sigma(k)} + (E_{\sigma(k)}^{+} A_{\sigma(k)} A_{\sigma(k)}^{+} B_{\sigma(k)} + E_{\sigma(k)}^{+} B_{\sigma(k)}) \mathbb{U}(k) + \\ A_{\sigma(k)}^{+} B_{\sigma(k)} \mathbb{U}(k+1) \subseteq \ker E_{\sigma(k)} \oplus \mathcal{S}_{\sigma(k+1)}. \end{aligned} \tag{4.64}$$

Furthermore, the surrogate system (4.51) is also valid, and in particular, $x(k) \in \widehat{\mathcal{S}}_{\sigma(k)}^{\mathbb{U}(k)} = \mathcal{S}_{\sigma(k)} - E_{\sigma(k)}^{+} B_{\sigma(k)} \mathbb{U}(k)$ for all $k \in \mathbb{N}$. ◊

From the establishment of the surrogate systems (4.51) for weakly solvable systems and (4.52) for strongly solvable systems, it is now possible to write its explicit formulas in terms of its one-step maps. Furthermore, the explicit solution formulas can then be written in the system's original coordinate without any state transformation; this is due to those surrogate systems, which are provided by Theorem 4.40.

First, by utilizing (4.51), the explicit solution of the weakly solvable InhSLSS (4.44) can be written as

$$\begin{aligned} x(k) = &\prod_{j=0}^{k-1} \Phi_{\sigma(k-j),\sigma(k-j-1)} x(0) \\ &+ \prod_{j=1}^{k-1} \Phi_{\sigma(k-j),\sigma(k-j-1)} \left(\Theta_{\sigma(1),\sigma(0)} u(0) + \Gamma_{\sigma(1),\sigma(0)} u(1)\right) \\ &+ \prod_{j=2}^{k-1} \Phi_{\sigma(k-j),\sigma(k-j-1)} \left(\Theta_{\sigma(2),\sigma(1)} u(1) + \Gamma_{\sigma(2),\sigma(1)} u(2)\right) + \cdots \\ &+ \Phi_{\sigma(k),\sigma(k-1)} \left(\Theta_{\sigma(k-1),\sigma(k-2)} u(k-2) \right. \\ &+ \left. \Gamma_{\sigma(k-1),\sigma(k-2)} u(k-1)\right) \\ &+ \left(\Theta_{\sigma(k),\sigma(k-1)} u(k-1) + \Gamma_{\sigma(k),\sigma(k-1)} u(k)\right) \end{aligned} \tag{4.65}$$

with $x(0) \in \mathcal{S}_{\sigma(0)}^{u(0)}$. Second, by utilizing (4.52), the explicit solution of the strongly solvable InhSLSS (4.44) can be written as

$$\begin{aligned} x(k) = &\prod_{j=0}^{k-1} \widehat{\Phi}_{\sigma(k-j),\sigma(k-j-1)} x(0) \\ &+ \prod_{j=1}^{k-1} \left(\widehat{\Phi}_{\sigma(k-j),\sigma(k-j-1)}\right) \widehat{\Theta}_{\sigma(1),\sigma(0)} u(0) \\ &+ \prod_{j=2}^{k-1} \left(\widehat{\Phi}_{\sigma(k-j),\sigma(k-j-1)}\right) \widehat{\Theta}_{\sigma(2),\sigma(1)} u(2) + \cdots \\ &+ \widehat{\Phi}_{\sigma(k),\sigma(k-1)} \widehat{\Theta}_{\sigma(k-1),\sigma(k-2)} u(k-2) + \widehat{\Theta}_{\sigma(k),\sigma(k-1)} u(k-1). \end{aligned} \tag{4.66}$$

with $x(0) \in \widehat{\mathcal{S}}_{\sigma(0)}$. In particular, solutions at switching times can also be derived from the solution formula above. This is given in the following corollary and will be used in the reachability characterization later; only the strongly solvable systems are considered here.

Corollary 4.43 (Solutions of strongly solvable InhSLSSs at switching times). Under a fixed switching signal (2.11), the solution of the strongly

solvable InhSLSS (4.44) at any switching time k_j^s is given by

$$\begin{aligned} x(k_j^s) =& \widehat{\Psi}_\sigma(j,0)x(0) + \widehat{\Psi}_\sigma(j,1)\widehat{\Phi}_{\sigma_1,\sigma_0}R_{\sigma_0}(k_1^s-1)\begin{bmatrix} u(k_1^s-2) \\ \vdots \\ u(0) \end{bmatrix} \\ &+ \widehat{\Psi}_\sigma(j,1)\widehat{\Theta}_{\sigma_1,\sigma_0}u(k_1^s-1) \\ &+ \widehat{\Psi}_\sigma(j,2)\widehat{\Phi}_{\sigma_2,\sigma_1}R_{\sigma_1}(k_2^s-k_1^s-1)\begin{bmatrix} u(k_2^s-2) \\ \vdots \\ u(k_1^s) \end{bmatrix} \\ &+ \widehat{\Psi}_\sigma(j,2)\widehat{\Theta}_{\sigma_2,\sigma_1}u(k_2^s-1) + \cdots \\ &+ \widehat{\Psi}_\sigma(j,j)\widehat{\Phi}_{\sigma_j,\sigma_{j-1}}R_{\sigma_{j-1}}(k_j^s-k_{j-1}^s-1)\begin{bmatrix} u(k_j^s-2) \\ \vdots \\ u(k_{j-1}^s) \end{bmatrix} \\ &+ \widehat{\Psi}_\sigma(j,j)\widehat{\Theta}_{\sigma_j,\sigma_{j-1}}u(k_j^s-1) \end{aligned} \tag{4.67}$$

where for $i,j,h \in \mathbb{N},\ j \geq h > 0$,

$$R_i(k) = [\widehat{\Theta}_i, \widehat{\Phi}_i\widehat{\Theta}_i, \cdots, \widehat{\Phi}_i^{k-1}\widehat{\Theta}_i]$$

$$\widehat{\Psi}_\sigma(j,h) = \widehat{\Phi}_{\sigma_j,\sigma_{j-1}}\widehat{\Phi}_{\sigma_{j-1}}^{k_j^s-k_{j-1}^s-1}\widehat{\Phi}_{\sigma_{j-1},\sigma_{j-1}}\widehat{\Phi}_{\sigma_{j-2}}^{k_{j-1}^s-k_{j-2}^s-1}\cdots\widehat{\Phi}_{\sigma_{h+1},\sigma_h}\widehat{\Phi}_{\sigma_h}^{k_{h+1}^s-k_h^s-1}$$

with $\widehat{\Psi}_\sigma(j,j) = I_n$, and $\widehat{\Psi}_\sigma(j,h)$ is called the state transition matrix for system (4.44) from the switching time k_h^s to the switching time k_j^s, and the matrices $\widehat{\Phi}_{i,j}$ and $\widehat{\Theta}_{i,j}$ are one-step maps given in (4.52). In particular,

$$\begin{aligned} x(k_j^s) =& \widehat{\Phi}_{\sigma_j,\sigma_{j-1}}\widehat{\Phi}_{\sigma_{j-1}}^{k_j^s-k_{j-1}^s-1}x(k_{j-1}^s) \\ &+ \widehat{\Phi}_{\sigma_j,\sigma_{j-1}}R_{\sigma_{j-1}}(k_j^s-k_{j-1}^s-1)\begin{bmatrix} u(k_j^s-2) \\ \vdots \\ u(k_{j-1}^s) \end{bmatrix} \\ &+ \widehat{\Theta}_{\sigma_j,\sigma_{j-1}}u(k_j^s-1). \end{aligned} \tag{4.68}$$

Moreover, the matrix $\widehat{\Psi}_\sigma(j,h)$ can also be rewritten in a recursive form as

$$\widehat{\Psi}_\sigma(j,h) = \widehat{\Phi}_{\sigma_j,\sigma_{j-1}}\widehat{\Phi}_{\sigma_{j-1}}^{k_j^s-k_{j-1}^s-1}\widehat{\Psi}_\sigma(j-1,h) \tag{4.69}$$

with $\widehat{\Psi}_\sigma(h,h) = I_n$. ◊

This formula is derived by extracting the solution from (4.66) at the switching time k_j^s and will be used in the reachability and controllability characterizations in the forthcoming Chapter 6 (Reachability and Controllability). Furthermore, only the solution formula (at switching times) for strongly solvable systems is exposed here since the study for reachability and controllability in this thesis is carried out only for strongly solvable systems.

4.2.2.3 Discussion on causality

For switched systems, the causality notions in Definition 4.36 for non-switched systems are defined with respect to the class of the switching signal: fixed switching signals, fixed mode sequences, and arbitrary switching signals. If a fixed switching signal is considered, then the causality notions in Definition 4.36 can be carried over without change since the solutions observed in the analysis are only those that correspond to the given switching signal only. Meanwhile, in the causality with respect to a fixed mode sequence or arbitrary switching signals, the solutions observed in the analysis are those that correspond to all involved switching signals. The formal definition is then needed to be stated only for a fixed switching signal; this is defined as follows:

Definition 4.44 (Causality Notions for InhSLSSs w.r.t. a fixed switching signal). The InhSLSS (4.44) is called (strictly) causal w.r.t. a fixed switching signal σ if the system with the given switching signal is (strictly) causal in the sense of Definition 4.36. ◊

Note that the causality notion above is defined in terms of states and inputs. For switched systems (4.44), similar to the HomSLSSs studied in the previous section, the causality property in terms of switching signal in the sense of Definition 4.25 is also desired here, and is indeed achieved by the solvability notions considered in this study, see the forthcoming Corollary 4.46.

As desired, by observing the solvability characterizations above, every weakly solvable system is (nonstrict) causal and every strongly solvable system is strictly causal. This is shown in the following proposition.

Proposition 4.45 (Solvability implies causality (InhSLSSs)).

(i) Every weakly solvable InhSLSS (4.44) is causal w.r.t. the corresponding switching signals

(ii) Every strongly solvable InhSLSS (4.44) is strictly causal w.r.t. the corresponding switching signals. ◊

Proof. As in Proposition 4.37, this proposition is a direct consequence of the system being (weakly or strongly) solvable. The non-strict causality of a weakly solvable system can be directly seen from its surrogate system (4.51) or its explicit solution formula (4.65) in which $x(k)$ is completely determined by $x(t),\ t < k$ and $u(\ell),\ \ell \leq k$. The strict causality of a strongly solvable system can be directly seen from its surrogate system (4.52) or its explicit solution formula (4.66) in which $x(k)$ is completely determined by $x(t)$ and $u(t)$ with $t < k$. □

As the solvability notion for HomSLSSs, the solvability notions for InhSLSSs also imply causality in terms of switching signals in the sense of Definition 4.25.

Corollary 4.46 (Weak/strong solvability implies causality in terms of switching signals). If the InhSLSS (4.44) is weakly or strongly solvable w.r.t. a fixed mode sequence/all switching signals, then it is causal in terms of switching signals (in the sense of Definition 4.25) w.r.t. the given mode sequence/all switching signals. ◊

This corollary is a direct consequence of the system having a unique solution for every involved switching signal on $[k_0, k_1]$ for any arbitrary $k_0, k_1 \in \mathbb{N}$. This causality in terms of switching signals can also be directly seen from the explicit solution formula (4.65) for weakly solvable systems or (4.66) for strongly solvable systems in which for every consistent initial value x_0 and every switching signal, the system with $x(0) = x_0$ on every time interval $[0, k_1]$ has a unique solution. This means that with the same switching signal on $[0, k_1]$, the solution on $[0, k_1]$ is the same no matter the switching signal on $[k_1 + 1, \infty)$.

4.2.2.4 Illustrative Examples

This section is closed with the following examples which illustrate non-solvable and solvable inhomogeneous switched systems.

Example 4.47. Consider system (4.44) composed of two modes given by the matrix triplets

$$(E_0, A_0, B_0) = \left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\right),$$

$$(E_1, A_1, B_1) = \left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\right).$$

Geometric computations provide that $\ker E_0 = \operatorname{span}\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \ker E_1 = \operatorname{span}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \widehat{\mathcal{S}}_0 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \widehat{\mathcal{S}}_1 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Each mode is strictly index-1 since $\ker E_i \oplus \widehat{\mathcal{S}}_i = \mathbb{R}^2$, $i = 0, 1$, and thus each mode as an individual system is strongly solvable. However, $\ker E_0 \cap \widehat{\mathcal{S}}_1 \neq \{0\}$ and $\ker E_1 \cap \widehat{\mathcal{S}}_0 \neq \{0\}$. Thus, it is not possible to have a strongly solvable switched system w.r.t. switching signals that contain mode transitions between those modes. This in particular shows that having solvable modes is not sufficient to have a solvable switched system. ◊

Example 4.48. Consider system (4.44) composed of modes represented by the matrix triplets

$$(E_0, A_0, B_0) = \left(\left[\begin{smallmatrix} -1 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 \\ 0 \\ 0 \end{smallmatrix}\right]\right)$$

$$(E_1, A_1, B_1) = \left(\left[\begin{smallmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ -1 \\ 0 \end{smallmatrix}\right]\right)$$

with

$$\begin{aligned}\ker E_0 &= \operatorname{span}\{(1,1,0)^\top\}\\ \ker E_1 &= \operatorname{span}\{(1,1,1)^\top\}\\ \widehat{\mathcal{S}}_0 &= \operatorname{span}\{(1,0,0)^\top,(0,0,1)^\top\}\\ \widehat{\mathcal{S}}_1 &= \operatorname{span}\{(1,0,1)^\top,(0,1,1)^\top\}.\end{aligned}$$

The family of those matrix triplets is jointly strictly index-1. Thus, switched systems composed of those modes are strongly solvable. With

$$E_0^+ = \begin{bmatrix} -1/2 & 0 & 0\\ 1/2 & 0 & 0\\ -1 & -1 & 0\end{bmatrix}, \qquad E_1^+ = \begin{bmatrix} -1/3 & 1/3 & 0\\ -1/3 & -2/3 & 0\\ 2/3 & 1/3 & 0\end{bmatrix},$$

$$\Pi_{\widehat{\mathcal{S}}_0}^{\ker E_0} = \begin{bmatrix} 1 & -1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}, \qquad \Pi_{\widehat{\mathcal{S}}_1}^{\ker E_1} = \begin{bmatrix} 0 & -1 & 1\\ -1 & 0 & 1\\ -1 & -1 & 2\end{bmatrix},$$

$$\Pi_{\widehat{\mathcal{S}}_1}^{\ker E_0} = \begin{bmatrix} 1/2 & -1/2 & 1/2\\ -1/2 & 1/2 & 1/2\\ 0 & 0 & 1\end{bmatrix}, \qquad \Pi_{\widehat{\mathcal{S}}_0}^{\ker E_1} = \begin{bmatrix} 1 & -1 & 0\\ 0 & 0 & 0\\ 0 & -1 & 1\end{bmatrix},$$

we have the surrogate system (4.52) with the one-step maps

$$\widehat{\Phi}_{0,0} = \begin{bmatrix} -1 & 1 & -1\\ 0 & 0 & 0\\ -1 & 0 & -2\end{bmatrix}, \qquad \widehat{\Phi}_{1,0} = \begin{bmatrix} -1 & 1/2 & -3/2\\ 0 & -1/2 & -1/2\\ -1 & 0 & -2\end{bmatrix},$$

$$\widehat{\Phi}_{0,1} = \begin{bmatrix} 0 & -1 & 0\\ 0 & 0 & 0\\ 0 & -1 & -1\end{bmatrix}, \qquad \widehat{\Phi}_{1,1} = \begin{bmatrix} 0 & -1 & -1\\ 0 & 0 & -1\\ 0 & -1 & -2\end{bmatrix},$$

$$\widehat{\Theta}_{0,0} = \begin{bmatrix} 1\\ 0\\ 1\end{bmatrix}, \qquad \widehat{\Theta}_{1,0} = \begin{bmatrix} 1\\ 0\\ 1\end{bmatrix},$$

$$\widehat{\Theta}_{0,1} = \begin{bmatrix} -1\\ 0\\ 0\end{bmatrix}, \qquad \widehat{\Theta}_{1,1} = \begin{bmatrix} 0\\ 1\\ 1\end{bmatrix}. \qquad \Diamond$$

Example 4.49. Consider system (6.6) composed of the following two modes

$$(E_0, A_0, B_0) = \left(\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}, \begin{bmatrix} 0\\ 1\\ 0\end{bmatrix}\right),$$

$$(E_1, A_1, B_1) = \left(\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}, \begin{bmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}, \begin{bmatrix} 0\\ 0\\ 0\end{bmatrix}\right).$$

Geometric computations provide

$$\ker E_0 = \{0\}, \qquad \widehat{\mathcal{S}}_0 = \mathbb{R}^3,$$

$$\ker E_1 = \operatorname{span}\begin{pmatrix} 0\\ 0\\ 1\end{pmatrix}, \qquad \widehat{\mathcal{S}}_1 = \operatorname{span}\left\{\begin{pmatrix} 1\\ 0\\ 0\end{pmatrix}, \begin{pmatrix} 0\\ 1\\ 0\end{pmatrix}\right\}.$$

Since $\ker E_i \oplus \widehat{\mathcal{S}}_i = \mathbb{R}^3$, both matrix triplets (E_0, A_0, B_0) and (E_1, A_1, B_1) are strictly index-1, and hence each individual mode is strongly solvable. Furthermore, $E_0^+ A_0 \widehat{\mathcal{S}}_0 + \operatorname{im} E_0^+ B_0 \subseteq \ker E_0 \oplus \widehat{\mathcal{S}}_1$, i.e., the family of matrix triplets $\{(E_i, A_i, B_i)\}_{i=0}^2$ is sequentially strictly index-1 with respect to the mode sequence $(\sigma) = (0,1)$ and thus switched systems composed of those modes are strongly solvable w.r.t. $(0,1)$. With

$$E_0^+ = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \qquad E_1^+ = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}, \qquad \Pi_{\widehat{\mathcal{S}}_0}^{\ker E_0} = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},$$

$$\Pi_{\widehat{\mathcal{S}}_1}^{\ker E_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \Pi_{\widehat{\mathcal{S}}_1}^{\ker E_0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

the corresponding surrogate system (4.52) with the mode sequence $(0, 1)$ is given with the one-step maps

$$\widehat{\Phi}_{0,0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \widehat{\Phi}_{1,0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \widehat{\Phi}_{1,1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\widehat{\Theta}_{0,0} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \widehat{\Theta}_{1,0} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \widehat{\Theta}_{1,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Note that $\{(E_i, A_i, B_i)\}_{i=0}^2$ is not sequentially strictly index-1 with respect to the mode sequence $(\sigma) = (1, 0)$ since $\ker E_1 \cap \widehat{\mathcal{S}}_0 = \operatorname{span}(0, 0, 1)^\top \neq \{0\}$, and thus switched systems composed of those modes are not solvable w.r.t. all switching signals with the mode sequence $(1, 0)$. In particular, this shows that the solvability is dependent on mode sequences, and sequentially strictly index-1 does not imply jointly strictly index-1. $\Diamond$

Example 4.50. Consider system (6.6) composed of the following three modes

$$(E_0, A_0, B_0) = \left(\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right),$$

$$(E_1, A_1, B_1) = \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right).$$

$$(E_2, A_2, B_2) = \left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right).$$

Geometric computations provide

$$\ker E_0 = \operatorname{span} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \widehat{\mathcal{S}}_0 = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\},$$

$$\ker E_1 = \operatorname{span} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \widehat{\mathcal{S}}_1 = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\},$$

$$\ker E_2 = \operatorname{span} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \widehat{\mathcal{S}}_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

For all pairs (i, j), $\ker E_i \oplus \widehat{\mathcal{S}}_j = \mathbb{R}^3$ holds, thus the family of those matrix triplets is jointly strictly index-1, and hence any switched system composed of those modes is strongly solvable w.r.t. all switching signals. With $\Pi_{\widehat{\mathcal{S}}_i}^{\ker E_j} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ for all i, j and

$$E_0^+ = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 0 \end{bmatrix}, \qquad E_1^+ = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 1 & 1 \\ 0 & -\frac{1}{2} & 0 \end{bmatrix}, \qquad E_2^+ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

the corresponding surrogate system (4.52) is given with the one-step maps

$$\widehat{\Phi}_{0,0} = \widehat{\Phi}_{1,0} = \widehat{\Phi}_{2,0} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\widehat{\Phi}_{0,1} = \widehat{\Phi}_{1,1} = \widehat{\Phi}_{2,1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\widehat{\Phi}_{0,2} = \widehat{\Phi}_{1,2} = \widehat{\Phi}_{2,2} = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\widehat{\Theta}_{0,0} = \widehat{\Theta}_{1,0} = \widehat{\Theta}_{2,0} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix},$$
$$\widehat{\Theta}_{0,1} = \widehat{\Theta}_{1,1} = \widehat{\Theta}_{2,1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$
$$\widehat{\Theta}_{0,2} = \widehat{\Theta}_{1,2} = \widehat{\Theta}_{2,2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \qquad \Diamond$$

4.3 Concluding Remarks

In this chapter, solvability notions for the well-posedness of singular linear systems have been proposed both for systems without switching and with switching and also both for systems without inputs and with inputs. Those solvability notions have been characterized with the help of the index-1 notion and its variations. For solvability under arbitrary switching times, index-1 of each mode is necessary. A somewhat surprising result is that each mode being index-1 is neither sufficient nor necessary for the solvability with a fixed switching signal. Furthermore, for solvable systems, one-step maps that map a solution at a certain time instant to the solution at the next time instant have been derived, which then produce surrogate systems–ordinary (switched) systems that have the same behavior.

Besides guaranteeing the existence and uniqueness of a solution, the proposed solvability notions also guarantee causality in terms of states in which future states do not determine the past and current states and, for switched systems, causality in terms of switching signals in which changing the switching signal in the future does not necessarily change the solution in the past. Furthermore, for systems with inputs, two solvability notions are offered: weak solvability and strong solvability. For strongly solvable systems, the current input does not determine the current state (ordinary systems have this feature) whereas, for weakly solvable systems, the current input determines the current state, which is not the case for ordinary systems.

The system studied in this chapter is considered under a finite number of modes. In practice, a switched system could have an infinite number of modes, and the solution theory derived from this study can still be applied. If the switched system has an infinite number of modes, then the jointly index-1 notion is not practical. However, the switched and sequentially index-1 notions are still practical by checking the condition online.

Therefore, Theorem 4.11 is still valid for a fixed switching signal σ with infinitely many modes, and the system is solvable w.r.t. σ if and only if the switched index-1 condition (4.13) is satisfied for $k = 0, 1, \ldots$.

Meanwhile, the solvability for a fixed mode sequence with arbitrary switching times can be utilized by checking the solvability condition online for

$k = 0, 1, \ldots,$ and if at some time instant, the system stays in the mode, then it suffices to check the first condition for the currently active mode.

In contrast, the condition for the solvability w.r.t. all switching signals for systems with infinitely many modes is in general not practical since the condition (4.11) needs to hold for all pairs of modes. Nevertheless, it is still practical for some particular switched systems with infinitely many modes under some restrictive situations such as the modes that can be parameterized in which the solvability condition can be checked for all possible values of the parameter.

Chapter 5 | Observability and Determinability

"Not all problems are solvable but all can be handled."

Contents

In this chapter, observability, which has been defined for ordinary systems in Chapter 3 as a notion for investigating whether the information from the output is enough or not to provide complete information about the past state, is studied for singular systems in discrete time both for systems without switching and with switching. Furthermore, determinability is also studied in this chapter for singular systems as an alternative notion besides observability. As an additional insight, instead of past states, determinability investigates whether the information about the output is enough to provide complete information about the future states.

Instead of using the original singular systems' equations, those notions are studied via their surrogate systems introduced in Chapter 4, which makes the study more straightforward. Geometric approaches are used in the characterizations in which the role of the consistency space can be seen directly from the characterization results. As in the study for ordinary systems in Chapter 3, the study for singular systems is also complemented with constant observability and determinability.

The studies for those notions will be considered first only for homogeneous systems of the form (4.1) or (4.9). After that, by some arguments from linearity, it will be shown that the results are also valid for inhomogeneous systems of the form (4.26) or (4.44); this will be discussed in Proposition 5.9 for nonswitched systems and Proposition 5.20 for switched systems.

5.1 Definitions

Recall the HomSLSS (4.9) under switching signals which have form (2.11). The basic intuition for the observability notion considered in this study for singular systems is similar to the intuition for linear systems discussed in Chapter 3. As an additional insight, intuitively, system (4.9) is called observable on a finite time interval w.r.t. a fixed and known switching signal if its state on that time interval is uniquely determined by its output on that time interval under the given switching signal. This is formally defined as follows:

Definition 5.1 (Observability of SLSSs). The HomSLSS (4.9) is called *observable* on $[0, K]$ w.r.t. a fixed switching signal σ of the form (2.11) if the following implication holds:

$$y'_{[0,K]} \equiv y''_{[0,K]} \implies x'_{[0,K]} = x''_{[0,K]} \tag{5.1}$$

where $(x'_{[0,K]}, y'_{[0,K]})$ and $(x''_{[0,K]}, y''_{[0,K]})$ are two arbitrary pairs of state-output solutions of (4.9) under σ on $[0, K]$ with $x'(0), x''(0) \in \mathcal{S}_{\sigma(0)}$. ◊

By linearity, the observability condition in the definition above can be simplified into the condition in the following proposition:

Proposition 5.2 (Zero observability of SLSSs). The HomSLSS (4.9) is observable w.r.t. a fixed switching signal σ of the form (2.11) on $[0, K]$ if and only if the following implication holds:

$$y_{[0,K]} \equiv 0 \implies x_{[0,K]} \equiv 0. \tag{5.2}$$

where $y_{[0,K]}$ and $x_{[0,K]}$ are arbitrary output and state respectively of (4.9) on $[0, K]$ under σ. ◊

Proof. The necessity is obvious by considering the trivial solution

$(x_{[0,K]}, y_{[0,K]}) \equiv (0,0)$. For the sufficiency, consider two arbitrary pairs of state-output solutions $(x'_{[0,K]}, y'_{[0,K]})$ and $(x''_{[0,K]}, y''_{[0,K]})$ of (4.9) on $[0,K]$ under σ with $y'_{[0,K]} \equiv y''_{[0,K]}$. By linearity, $x := x' - x''$ is also a solution of (4.9) under σ on $[0,K]$ with output $y = y' - y'' = 0$. Hence, the implication (5.2) implies $x_{[0,K]} \equiv 0$, i.e., $x'_{[0,K]} \equiv x''_{[0,K]}$. □

For a solvable system (4.9) (in the sense of Definition 4.6), it follows that $x_{[0,K]} \equiv 0$ if, and only if, $x(0) = 0$. Thus, the observability condition (5.2) for solvable systems (4.9) can be reduced into the condition in the following corollary.

Corollary 5.3 (Zero-output-zero-initial-state observability of SLSSs). A solvable HomSLSS (4.9) is observable w.r.t. a fixed switching signal σ of the form (2.11) on $[0,K]$ if and only if the following implication holds:

$$y_{[0,K]} \equiv 0 \implies x(0) = 0. \tag{5.3}$$

where $y_{[0,K]}$ is the output of (4.9) on $[0,K]$ under σ. ◊

The observability characterization above also explains that observability concerns recovering the state in the *past*, or equivalently the initial value, from the measured output values. Meanwhile, for some applications (e.g. designing feedback rules based on observers) it may however be more relevant to recover the present state from the already measured output. This ability, which has been introduced in Definition 3.13 for ordinary linear switched systems, is called determinability and is formally defined for singular linear systems of the form (4.9) as follows:

Definition 5.4 (Determinability of SLSSs). The HomSLSS (4.9) is called *determinable* on $[0,K]$ w.r.t. a fixed switching signal of the form (2.11) if, and only if, the following implication holds:

$$y'_{[0,K]} \equiv y''_{[0,K]} \implies x'(K) = x''(K) \tag{5.4}$$

where $(x'_{[0,K]}, y'_{[0,K]}), (x''_{[0,K]}, y''_{[0,K]})$ are two arbitrary pairs of solutions of (4.9) on $[0,K]$ under σ. ◊

Similar to observability, under the solvability assumption in the sense of Definition 4.6, the determinability condition in the definition above can be simplified as the zero determinability condition presented in the following proposition.

Proposition 5.5 (Zero determinability of SLSSs). The solvable HomSLSS (4.9) is determinable on $[0,K]$ w.r.t. a fixed switching signal σ of the form (2.11) if, and only if, the following implication holds:

$$y_{[0,K]} \equiv 0 \implies x(K) = 0. \tag{5.5}$$

where $y_{[0,K]}$ is the output of (4.9) on $[0,K]$ under σ. ◊

Proof. The proof is similar to the proof of Proposition 5.2. The necessity is obvious by considering $x(K) = 0$ and $y_{[0,K]} \equiv 0$. For the sufficiency, consider two arbitrary pairs of final-state and output $(x'(K), y'_{[0,K]})$ and $(x''(K), y''_{[0,K]})$ of (4.9) on $[0, K]$ under σ with $y'_{[0,K]} \equiv y''_{[0,K]}$. By linearity, $x(K) := x'(K) - x''(K)$ is also a solution of (4.9) at $k = K$ under σ with output $y = y' - y'' = 0$. Hence, the implication (5.5) implies $x(K) = 0$, i.e., $x'(K) = x''(K)$. □

5.2 Characterizations: Nonswitched Case

The characterizations start with the nonswitched case, i.e., (nonswitched) singular linear systems of the form (4.1) (without inputs) or (4.26) (with inputs) are being studied in this subsection. Consider first the homogeneous systems (4.1), and later, it will be shown that the characterization results for homogeneous systems are also valid for inhomogeneous systems.

Note that the nonswitched system class (4.1) is indeed a particular case of the switched system class (4.9) under constant switching signals. Therefore, the observability and determinability for nonswitched systems can be studied in the sense of Definitions 5.1 and 5.4 under constant switching signals, and particular definitions for nonswitced systems are then not needed. Furthermore, notations in the solvability study in Section 4.1.1 are also used here. Recall the output explicit solution of (4.1) as follows:

$$y(k) = C\Phi^k_{(E,A)}x(0), \quad k \in [0, K] \tag{5.6}$$

as well as $x(0) \in \mathcal{S} := A^{-1}(\operatorname{im} E)$. By using this explicit solution, the observability characterization for solvable HomSLSs can then be derived, this is presented in the following lemma.

Lemma 5.6 (Observability characterization of SLSs). The solvable HomSLS (4.1) is observable on $[0, K]$ if, and only if,

$$\mathcal{S} \cap \mathcal{O}^K = \{0\}, \tag{5.7}$$

where $\mathcal{O}^K := \ker[C^\top, (C\Phi_{(E,A)})^\top, \ldots, (C\Phi^K_{(E,A)})^\top]^\top$. ◊

Proof. Taking the explicit output solution over the time domain $[0, K]$ provides

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(K) \end{bmatrix} = \begin{bmatrix} C \\ C\Phi_{(E,A)} \\ C\Phi^2_{(E,A)} \\ \vdots \\ C\Phi^K_{(E,A)} \end{bmatrix} x(0) = O^K x(0).$$

with $O^K := [C^\top, (C\Phi_{(E,A)})^\top, \ldots, (C\Phi^K_{(E,A)})^\top]^\top$. The implication (5.3) is true

if and only if $\mathcal{O}^K := \ker O^K = 0$. From the fact that $x(0) \in \mathcal{S}$, the system is observable if and only if $\mathcal{S} \cap \mathcal{O}^K = \{0\}$. □

Note that due to Cayley-Hamilton, $\mathcal{O}^K = \mathcal{O}^{n-1}$ if $K \geq n-1$, but $\mathcal{O}^K \supsetneq \mathcal{O}^{n-1}$ is possible; in particular, the unobservable space (i.e. the subspace of all initial values $x(0)$ which produce a zero output) depends on the length K of the considered interval when K is small compared to the system dimension n. This is a major difference to the continuous time case, where the unobservable space (given by $\mathcal{O}^{n-1}$) is independent of the length of the observation interval. This makes the observability analysis for switched systems more challenging in the discrete-time case compared to the continuous-time case.

For the determinability characterization of the HomSLS (4.1), define the following sequence of subspaces:

$$\begin{aligned} \mathcal{D}^0 &= \ker C \cap \mathcal{S} \\ \mathcal{D}^k &= \ker C \cap \Phi \mathcal{D}^{k-1}, \;\; k = 1, 2, \ldots \end{aligned} \tag{5.8}$$

with $\Phi := \Phi_{(E,A)}$ is given as in Lemma 4.3. The determinability characterization is presented in the forthcoming Lemma 5.8, where the preceding lemma will be used in its proof.

Lemma 5.7 (Solutions of (4.1) **belong to the subspace** (5.8)**).** Consider a solvable HomSLS with corresponding subspace sequence (5.8). For every $k \in [0, K]$, $x_k \in \mathcal{D}^k$, $x_k \in \mathbb{R}^n$ if, and only if, there exists a solution of (4.1) with $x(k) = x_k$ and $y(i) = 0$ for all $i \in [0, k]$. ◊

Proof. Sufficiency: For $k = 0$, $x_0 \in \mathcal{S}$ implies existence of a solution with $x(0) = x_0$, and $x_0 \in \ker C$ implies that $y(0) = Cx_0 = 0$. The proof for $k > 0$ proceeds inductively. Assume the claim holds for $k-1$. From $x_k \in \mathcal{D}^k \subseteq \Phi\mathcal{D}^{k-1}$, it follows the existence of a $x_{k-1} \in \mathcal{D}^{k-1}$ with $x_k = \Phi x_{k-1}$. By inductive assumption, there exists a solution x of (4.1) on $[0, k-1]$ with $x(k-1) = x_{k-1}$ and $y(i) = 0$ for all $i \in [0, k-1]$. By Theorem 4.3, setting $x(k) = x_k = \Phi x_{k-1}$ yields a solution on $[0, k]$ and from $x_k \in \mathcal{D}^k \subseteq \ker C$ it follows that also $y(k) = Cx_k = 0$ which concludes the sufficiency part of the proof.

Necessity: For $k = 0$, it is obvious since every solution x of SLS (4.1) needs to satisfy $x(0) \in \mathcal{S}$, and $y(0) = 0$ implies $x(0) \in \ker C$. Again, the proof for $k > 0$ proceeds inductively. Consider a solution x of SLS (4.1) with $y(i) = 0$ for all $i \in [0, k]$. This implies $x(k) \in \ker C$ and $x(k) = \Phi x(k-1)$. Using the inductivity assumption, $x(k-1) \in \mathcal{D}^{k-1}$, because $y(i) = 0$ for all $i \in [0, k-1]$. Hence $x(k) \in \ker C \cap \Phi\mathcal{D}^{k-1} = \mathcal{D}^k$ as desired. □

Lemma 5.8 (Determinability characterization for SLSs). A solvable SLS (4.1) is determinable on $[0, K]$ if, and only if,

$$\mathcal{D}^K = \{0\}, \tag{5.9}$$

where $\mathcal{D}^k$ is given by (5.8). ◊

Proof. The necessity is obvious from the fact that the implication (5.5) implies $\mathcal{D}^K = \{0\}$ (by Lemma 5.7). For the sufficiency, $\mathcal{D}^K = \{0\}$ implies that for all $i \in [0, K]$, $y(i) = 0$ and $x(K) = 0$ (also by Lemma 5.7), i.e., the implication (5.5) holds. □

Discussion on Inhomogeneous Systems

Now, for systems with inputs, consider the observability notion in the sense that the InhSLS (4.26) is said to be observable if it is observable for all (unbounded) input sequences. Similarly, the InhSLS (4.26) is said to be determinable if it is determinable for all (unbounded) input sequences. The following proposition states that the observability and determinability of InhSLSs can be investigated via their homogeneous forms.

Proposition 5.9 (Observability/determinability of InhSLSs). The weakly or strongly solvable InhSLS (4.26) is observable (determinable) on the finite time domain $[0, K]$, $K \in \mathbb{N}$ if and only if its homogeneous form (4.1) is observable (determinable) on $[0, K]$. ◊

Proof. At any time instant k, any two pairs of state and output measurement $(x'_u(k), y'_u(k))$ and $(x''_u(k), y''_u(k))$ of (4.26) with the same input $u(k)$ satisfy

$$Ex'_u(k+1) = Ax'_u(k) + Bu(k) \text{ and } Ex''_u(k+1) = Ax''_u(k) + Bu(k),$$
$$y'_u(k) = Cx'_u(k) + Du(k) \text{ and } y''_u(k) = Cx''_u(k) + Du(k).$$

The difference between those two states and outputs satisfy $E(x'_u(k+1) - x''_u(k+1)) = A(x'_u(k) - x''_u(k)) + Bu(k) - Bu(k)$, i.e., $x'_u(k) - x''_u(k) =: x'''_u(k)$ satisfies $Ex'''_u(k+1) = Ax'''_u(k)$, and $y'_u(k) - y''_u(k) = C(x'_u(k) - x'''_u(k)) + Du(k) - Du(k)$, i.e., $y'_u(k) - y''_u(k) =: y'''_u(k)$ satisfies $y'''_u(k) = Cx'''_u(k)$. Hence, the difference between those solutions of inhomogeneous systems is also the solution of the homogeneous system (4.1). This concludes that the observability/determinability of inhomogeneous systems and homogeneous systems imply each other. □

It can also be seen intuitively from the output equation (4.39) in which the matrices $\widehat{\Phi}$, $\widehat{\Theta}$, C, and D and inputs $u(k)$ for all k are known, and the last two terms in (4.39) can be subtracted from the output measurement $y(k)$. Therefore, the state $x_{[0,K]}$ is determined only by the observation of the output

$y_{[0,K]}$, which means that the observability and determinability are independent of inputs.

5.3 Characterizations: Single Switch Case

In this section, observability and determinability are studied for singular linear switched systems under fixed switching signals. However, to show precise insights into the observability and determinability characterizations for switched systems, first, the characterizations are discussed for single switch switching signals. Based on these results, characterizations for general switching signals are then developed. The study also starts with homogeneous systems, and later, it will be shown that the characterization results are also valid for inhomogeneous systems. Thus, in this section, the study is restricted to only single switch switching signals, i.e., consider the HomSLSS (4.9) with the switching signal given by

$$\sigma(k) = \begin{cases} 0, & 0 \leq k < k^s, \\ 1, & k^s \leq k \leq K, \end{cases} \tag{5.10}$$

with k^s the switching time, see also Figure 5.1 for illustration.

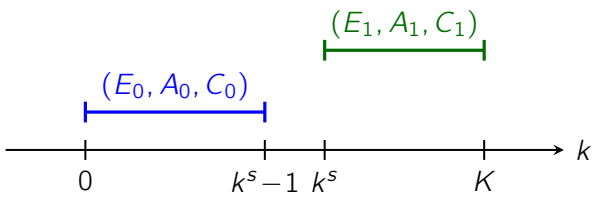


Figure 5.1: Single switch switching signal

5.3.1 Arbitrary Switching Time and Observation Time

First, the characterization is formulated with arbitrary switching time and observation final time, i.e., each mode is active for an arbitrary duration. Later, the characterization is formulated under the dwell-time condition in which each mode is active for at least n time steps. This will end up in simpler characterizations. The observability characterization for the solvable HomSLSS (4.9) under the single switch switching signal (5.10) is presented in the following theorem.

Theorem 5.10 (Observability of SLSSs: single switch case). Consider the HomSLSS (4.9) with the single switch switching signal (5.10) and assume it

is solvable in the sense of Definition 4.6 with corresponding one-step maps Φ_0, Φ_1 given by (4.7) and $\Phi_{1,0}$ as in (4.16). Then, system (4.9) is observable on $[0, K]$ w.r.t. (5.10) if, and only if,

$$\mathcal{S}_0 \cap \mathcal{O}_0^{k^s-1} \cap \left[\Phi_{1,0}\Phi_0^{k^s-1}\right]^{-1}\left(\mathcal{O}_1^{K-k^s}\right) = \{0\}, \tag{5.11}$$

where, for $i = 0, 1$ and $k \in \mathbb{N}$, $\mathcal{O}_i^k := \ker[C_i^\top, (C_i\Phi_i)^\top, \dots, (C_i\Phi_i^k)^\top]^\top$. $\Diamond$

Proof. The output of the system can be expressed in terms of the initial value $x(0) = x_0$ as follows:

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(k^s-1) \\ y(k^s) \\ y(k^s+1) \\ \vdots \\ y(K) \end{bmatrix} = \begin{bmatrix} C_0 \\ C_0\Phi_0 \\ \vdots \\ C_0\Phi_0^{k^s-1} \\ C_1\Phi_{1,0}\Phi_0^{k^s-1} \\ C_1\Phi_1\Phi_{1,0}\Phi_0^{k^s-1} \\ \vdots \\ C_1\Phi_1^{K-k^s}\Phi_{1,0}\Phi_0^{k^s-1} \end{bmatrix} x_0 = O^{k^s,K}x_0$$

with $O^{k^s,K} := [C_0^\top, (C_0\Phi_0)^\top, \dots, (C_0\Phi_0^{k^s-1})^\top, (C_1\Phi_{1,0}\ \Phi_0^{k^s-1})^\top, (C_1\Phi_1\Phi_{1,0}\ \Phi_0^{k^s-1})^\top, \dots, (C_1\Phi_1^{K-k^s}\Phi_{1,0}\Phi_0^{k^s-1})^\top]^\top$. Then,

$$\ker O^{k^s,K} =: \mathcal{O}^{k^s,K} = \mathcal{O}_0^{k^s-1} \cap \left[\Phi_{1,0}\Phi_0^{k^s-1}\right]^{-1}\left(\mathcal{O}_1^{K-k^s}\right),$$

(by the fact that $\ker(O\Phi) = \Phi^{-1}(\ker O)$ for any matrices O and Φ of appropriate size). Note that here Φ^{-1} denotes, in general, the preimage and not the inverse matrix (see the discussion after Definition 2.1).

Sufficiency: Assume $0 \neq x_0 \in \mathcal{S}_0 \cap \mathcal{O}^{k^s,K}$. Then there exists a unique, non-trivial solution x with $x(0) = x_0$. Since $x(0) \in \mathcal{O}^{k^s,K}$ then $y(k) = 0$, $0 \leq k \leq K$. This means that there exists a non-trivial solution of x with zero output. Hence, (4.1) is not observable.

Necessity: Consider a solution of (4.9) then $x(0) \in \mathcal{S}_0$. Furthermore, if $y(k) = 0$ for all $k \in [0, K]$, then $x(0) \in \mathcal{O}^{k^s,K}$. Hence $x(0) \in \mathcal{S}_0 \cap \mathcal{O}^{k^s,K} = \{0\}$, which shows the desired implication (5.3), i.e., $y_{[0,K]} \equiv 0 \implies x(0) = 0$ holds. $\square$

In general, the second and the third subspaces in the observability condition (5.11) depend on the switching time since k^s explicitly appears on them. This means that in discrete time, changing the switching time can change the observability property (see the forthcoming Example 5.27 for illustration). In contrast, the observability condition in continuous time does not depend on the switching time in the single switch case (see [59, Theorem 9]).

For the determinability characterization, define the following sequence of

subspaces:

$$\begin{aligned}
\mathcal{P}^0 &= \ker C_0 \cap \mathcal{S}_0 \\
\mathcal{P}^k &= \ker C_0 \cap \Phi_0 \mathcal{P}^{k-1}, \ k = 1, 2, \dots k^s - 1 \\
\mathcal{P}^{k^s} &= \ker C_1 \cap \Phi_{1,0} \mathcal{P}^{k^s - 1} \\
\mathcal{P}^k &= \ker C_1 \cap \Phi_1 \mathcal{P}^{k-1}, \ k = k^s + 1, k^s + 2, \dots
\end{aligned} \tag{5.12}$$

where $\Phi_{i,j}$ are given by (4.16) in Theorem 4.11. The determinability characterization is presented in the forthcoming Theorem 5.12, where the preceding lemma, which is the counterpart of Lemma 5.7 for nonswitched systems, will be used in its proof. Even though it is similar to the nonswitched case, the complete proof is provided for completeness.

Lemma 5.11 (Solutions $x(k)$ of (4.9) belongs to $\mathcal{P}^k$ in (5.12)). Consider a solvable HomSLSS under the single switching signal (6.7) and the subspace sequence (5.12). For every $k \in [0, K]$, a vector $x_k \in \mathbb{R}^n$ satisfies $x_k \in \mathcal{P}^k$ if, and only if, there exists a solution of (4.9) with $x(k) = x_k$ and $y(i) = 0$ for all $i \in [0, k]$. ◊

Proof. For all k with $k < k^s$, the statement is true by Lemma 5.7 for nonswitched systems. Then, it suffices to prove for k with $k \geq k^s$.
Step 1: $k = k^s$
($\Longrightarrow$): The inclusion $x_{k^s} \in \mathcal{P}^{k^s} \subseteq \Phi_{1,0}\mathcal{P}^{k^s-1}$ implies the existence of a $x_{k^s-1} \in \mathcal{P}^{k^s-1}$ with $x_{k^s} = \Phi_{1,0} x_{k^s-1}$. By Theorem 4.11, setting $x(k^s) = x_{k^s} = \Phi_{1,0} x(k^s - 1)$ yields a solution at $k = k^s$ and from $x_{k^s} \in \mathcal{P}^{k^s} \subseteq \ker C_1$ it follows that also $y(k) = C_1 x_{k^s} = 0$ which concludes this part of the proof.
($\Longleftarrow$): Consider a solution $x(k^s)$ of (4.9) with $y(k^s) = 0$. This implies $x(k^s) \in \ker C_1$ and $x(k^s) = \Phi_{1,0} x(k^s - 1)$. From the knowledge $x(k^s - 1) \in \mathcal{P}^{k^s-1}$, $x(k^s) \in \ker C_1 \cap \Phi_{1,0}\mathcal{P}^{k^s-1} = \mathcal{P}^{k^s}$ as desired.
Step 2: $k > k^s$
($\Longrightarrow$): This proceeds inductively. Assume the claim holds for $k - 1$ with $k > k^s + 2$. From $x_k \in \mathcal{P}^k \subseteq \Phi_1 \mathcal{P}^{k-1}$, it follows the existence of a $x_{k-1} \in \mathcal{P}^{k-1}$ with $x_k = \Phi_2 x_{k-1}$. By the inductive assumption, there exists a solution x of (4.9) on $[0, k-1]$ with $x(k-1) = x_{k-1}$ and $y(i) = 0$ for all $i \in [0, k-1]$. By Theorem 4.11, setting $x(k) = x_k = \Phi_1 x_{k-1}$ yields a solution on $[0, k]$ and from $x_k \in \mathcal{P}^k \subseteq \ker C_1$ it follows that also $y(k) = C_1 x_k = 0$ which concludes this part of the proof.
($\Longleftarrow$): This part also proceeds inductively. Consider a solution x of (4.9) with $y(i) = 0$ for all $i \in [0, k]$. This implies $x(k) \in \ker C_1$ and $x(k) = \Phi_1 x(k-1)$. Using the inductivity assumption, $x(k-1) \in \mathcal{P}^{k-1}$, because $y(i) = 0$ for all $i \in [0, k-1]$. Hence $x(k) \in \ker C_1 \cap \Phi_1 \mathcal{P}^{k-1} = \mathcal{P}^k$, which completes the proof. □

Theorem 5.12 (Determinability characterization: single switch case). A solvable HomSLSS (4.9) is determinable on $[0, K]$ w.r.t. the single switch switching signal (5.10) if, and only if,

$$\mathcal{P}^K = \{0\}, \tag{5.13}$$

where $\mathcal{P}^k$ is given by (5.12). ◊

Proof. The necessity is obvious from the fact that the implication (5.5) implies $\mathcal{P}^K = \{0\}$ (by Lemma 5.11). For the sufficiency, $\mathcal{P}^K = \{0\}$ implies that for all $i \in [0, K]$, $y(i) = 0$ and $x(K) = 0$ (also by Lemma 5.11), i.e., the implication (5.5) holds. □

5.3.2 Large Enough Observation Time

If the observation time (or activation time) for each mode is long enough, i.e., $k^s \geq n$ and $K - k^s \geq n$, the dependence on the switching time k^s in the observability characterization (5.11) can be reduced by exploiting the Cayley-Hamilton-Theorem as follows:

Corollary 5.13 (Observability with long enough activation time). Consider the solvable HomSLSS (4.9) with the single switch switching signal (5.10) and assume $n \leq k^s \leq K - n$. Then, system (4.9) is observable on $[0, K]$ w.r.t. (5.10) if, and only if,

$$\mathcal{S}_0 \cap \mathcal{O}_0 \cap \left[\Phi_{1,0}\Phi_0^{k^s-1}\right]^{-1}(\mathcal{O}_1) = \{0\} \tag{5.14}$$

where for $i = 1, 2$, $\mathcal{O}_i := \ker[C_i^\top, (C_i\Phi_i)^\top, \ldots, (C_i\Phi_i^{n-1})^\top]^\top$. Furthermore, if there is some $\nu \in \mathbb{N}$ with $0 < \nu < n$ such that $\Phi_0^{\nu+1} = \Phi_0^\nu$, then (4.9) is observable on $[0, K]$ if, and only if,

$$\mathcal{S}_0 \cap \mathcal{O}_0 \cap [\Phi_{1,0}\Phi_0^\nu]^{-1}(\mathcal{O}_1) = \{0\}. \qquad ◊$$

Remark 5.14 (Simpler conditions for determinability). It is still not clear whether a similar simplification for determinability exists. This is due to that in general, for noninvertible matrix $M \in \mathbb{R}^{n\times n}$ and subspaces $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n$, only the inclusion $M(\mathcal{A} \cap \mathcal{B}) \subseteq M\mathcal{A} \cap M\mathcal{B}$ is always true, and thus the subspace $\mathcal{P}^k$ in (5.12) cannot be rewritten explicitly in terms of powers of certain matrices as in the observability condition (5.11). ◊

5.4 Characterizations: Multiple Switches Case

Using the intuition behind the characterizations under single switch switching signals, the characterizations under general switching signals are developed in

the forthcoming Theorem 5.15 for observability and Theorem 5.19 for determinability.

5.4.1 Observability

Using notations from Corollary 4.21, the following theorem presents the observability characterization of system (4.9) under general switching signals of the form (2.11).

Theorem 5.15 (Observability of SLSSs under general switching signals). Consider the HomSLSS (4.9) on $[0, K]$ and assume it is solvable (in the sense of Definition 4.6) w.r.t. the fixed switching signal σ of the form (2.11). Then, this system is observable on $[0, K]$ w.r.t. σ if, and only if,

$$\mathcal{S}_{\sigma_0} \cap \bigcap_{j=0}^{J} \Psi_\sigma(j,0)^{-1}(\mathcal{O}_{\sigma_j}^{k_{j+1}^s - k_j^s - 1}) = \{0\} \tag{5.15}$$

where $\Psi_\sigma(j,0)$ is given by (4.23) (which is not assumed to be invertible, in particular, $\Psi_\sigma(j,0)^{-1}$ stands for the preimage) and, for $k \in \mathbb{N}$,

$$\mathcal{O}_{\sigma_j}^k = \ker[C_{\sigma_j}^\top, (C_{\sigma_j}\Phi_{\sigma_j})^\top, \dots, (C_{\sigma_j}\Phi_{\sigma_j}^k)^\top]^\top. \tag{5.16}$$

$\Diamond$

Proof. The vector of the outputs over the time interval $[0, K]$, $y_{[0,K]} = [y_{[0,k_1^s)}^\top, y_{[k_1^s,k_2^s)}^\top, \dots, y_{[k_J^s,K]}^\top]^\top$ with $y_{[k_j^s,k_{j+1}^s)}^\top = [y(k_j^s)^\top, y(k_j^s+1)^\top, \dots, y(k_{j+1}^s - 1)^\top]^\top$, can be written as:

$$\begin{bmatrix} y_{[0,k_1^s)}^\top \\ y_{[k_1^s,k_2^s)}^\top \\ \vdots \\ y_{[k_J^s,K]}^\top \end{bmatrix} = \underbrace{\begin{bmatrix} \mathcal{O}_{\sigma_0}^{k_1^s-1} \\ \mathcal{O}_{\sigma_1}^{k_2^s-k_1^s-1}\Psi_1(1,0) \\ \mathcal{O}_{\sigma_2}^{k_3^s-k_2^s-1}\Psi_2(2,0) \\ \dots \\ \mathcal{O}_{\sigma_J}^{K-k_J^s-1}\Psi_\sigma(J,0) \end{bmatrix}}_{O_{[0,K]}} x_0 = O_{[0,K]}x_0 \tag{5.17}$$

with $O_{\sigma_j}^k = [C_{\sigma_j}^\top, (C_{\sigma_j}\Phi_{\sigma_j})^\top, \dots, (C_{\sigma_j}\Phi_{\sigma_j}^k)^\top]^\top$. By using the fact that $\ker(O\Phi) = \Phi^{-1}(\ker O)$ for any matrices O and Φ of appropriate size,

$$\begin{aligned} \ker O_{[0,K]} = \mathcal{O}_{\sigma_0}^{k_1^s-1} \cap \Psi_1(1,0)^{-1}(\mathcal{O}_{\sigma_1}^{k_2^s-k_1^s-1}) \cap \dots \\ \cap \Psi_\sigma(J,0)^{-1}(\mathcal{O}_{\sigma_J}^{K-k_J^s-1}) =: \mathcal{O}_{[0,K]}. \end{aligned}$$

Sufficiency: Assume $0 \neq x_0 \in \mathcal{S}_{\sigma_0} \cap \mathcal{O}_{[0,K]}$. Then there exists a unique solution x of the system (4.9) with $x(0) = x_0 \in \mathcal{S}_{\sigma_0}$. Since $x(0) \in \mathcal{O}_{[0,K]}$ it follows from above that $y(k) = 0$, $0 \leq k \leq K$. This means that there exists a nontrivial solution of x with zero output. Hence, system (4.9) is not observable.

Necessity: Consider a solution of (4.9) then $x(0) \in \mathcal{S}_{\sigma_0}$. Furthermore, if $y(k) = 0$ for all $k \in [0, K]$, then $x(0) \in \mathcal{O}_{[0,K]}$. Hence $x(0) \in \mathcal{S}_{\sigma_0} \cap \mathcal{O}_{[0,K]} = \{0\}$. □

The subspace on the left-hand side in (5.15) is called the unobservable space of system (4.9), and this system is observable if, and only if, the unobservable space is a singleton set with the zero vector.

By exploiting the Cayley-Hamilton Theorem, it is possible to simplify (5.16) when the observation time is long enough and each mode is active for at least n time instants. This is summarized by the following corollary.

Corollary 5.16 (Observability of SLSSs under dwell-time assumption). Consider a solvable HomSLSS (4.9) w.r.t. the fixed switching signal σ of the form (2.11) and assume each mode is long enough active, i.e., $k^s_{j+1} - k^s_j \geq n-1$, for all j. Then, this system is observable on $[0, K]$ w.r.t. σ if, and only if,

$$\mathcal{S}_{\sigma_0} \cap \bigcap_{j=0}^{J} \Psi_\sigma(j, 0)^{-1}(\ker O_{\sigma_j}) = \{0\} \tag{5.18}$$

where $O_{\sigma_j} := O^{n-1}_{\sigma_j}$ for $j = 0, 1, ..., J$. In particular, if $K \geq k^s_J + n - 1$, then observability does not depend on the total length of the observation interval $[0, K]$. ◊

Remark 5.17. In (5.15), the switching times explicitly occur in $\mathcal{O}^{k^s_{j+1}-k^s_j-1}_{\sigma_j}$ and implicitly in $\Psi_\sigma(j, 0)$, which indicate that, in general, observability depends on the switching times (and not only on the mode sequence), and changing the switching times may produce different observability characterization results. Furthermore, assuming that each mode is active long enough, the dependence of observability on the switching times is only partially removed in (5.18) because $\Psi_\sigma(j, 0)$ still depends on them. ◊

5.4.2 Determinability

Based on the sequence of subspaces for the single switch case defined in (5.12), define the following sequence of subspaces for the solvable HomSLSS (4.9) w.r.t. a fixed switching signal σ of the form (2.11), which will play a crucial role for characterizing determinability:

$$\begin{aligned} \mathcal{Q}^0 &= \ker C_{\sigma(0)} \cap \mathcal{S}_{\sigma(0)} \\ \mathcal{Q}^k &= \ker C_{\sigma(k)} \cap \Phi_{\sigma(k),\sigma(k-1)} \mathcal{Q}^{k-1}, k = 1, 2, \ldots, K \end{aligned} \tag{5.19}$$

Lemma 5.18 (Solutions $x(k)$ belongs to $\mathcal{Q}^k$). Consider a solvable HomSLSS w.r.t. a fixed switching signal σ of the form 2.11 with corresponding subspace

sequence (5.19). For every $k \in [0, K]$, $x_k \in \mathcal{Q}^k$, $x_k \in \mathbb{R}^n$ if, and only if, there exists a solution of (4.9) with $x(k) = x_k$ and $y(i) = 0$ for all $i \in [0, k]$. ◊

Proof. The proof is the generalization of the proof of Lemma 5.11 for single switch case.
Sufficiency: For $k = 0$ the claim is clear, because $x_0 \in \mathcal{S}_{\sigma_0}$ implies existence of a solution with $x(0) = x_0$ and $x_0 \in \ker C_{\sigma_0}$ implies that $y(0) = C_{\sigma_0} x_0 = 0$. For $k > 0$, it proceeds inductively, i.e., first, assume the claim holds for $k - 1$. From $x_k \in \mathcal{Q}^k \subseteq \Phi_{\sigma(k),\sigma(k-1)} \mathcal{Q}^{k-1}$ it follows the existence of a $x_{k-1} \in \mathcal{Q}^{k-1}$ with $x_k = \Phi_{\sigma(k),\sigma(k-1)} x_{k-1}$. By inductive assumption, there exists a solution x of (4.9) on $[0, k-1]$ with $x(k-1) = x_{k-1}$ and $y(i) = 0$ for all $i \in [0, k-1]$. By Theorem 4.11, setting $x(k) = x_k = \Phi_{\sigma(k),\sigma(k-1)} x_{k-1}$ yields a solution on $[0, k]$ and from $x_k \in \mathcal{Q}^k \subseteq C_{\sigma(k}$ it follows that also $y(k) = C_{\sigma_k} x_k = 0$ which concludes the sufficiency part of the proof.
Necessity: For $k = 0$ the claim is clearly true, because every solution x of SLSS (4.9) needs to satisfy $x(0) \in \mathcal{S}_{\sigma_0}$ and $y(0) = 0$ implies $x(0) \in \ker C_{\sigma_0}$. For $k > 0$, it again proceeds inductively. Therefore, consider a solution x of SLSS with $y(i) = 0$ for all $i \in [0, k]$. This implies $x(k) \in \ker C_{\sigma(k)}$ and $x(k) = \Phi_{\sigma(k),\sigma(k-1)} x(k-1)$. Using the inductivity assumption, $x(k-1) \in \mathcal{Q}^{k-1}$, because $y(i) = 0$ for all $i \in [0, k-1]$. Hence $x(k) \in \ker C_{\sigma(k)} \cap \Phi_{\sigma(k),\sigma(k-1)} \mathcal{Q}^{k-1} = \mathcal{Q}^k$ as desired. □

The determinability characterization is then can be directly obtained from Lemma 5.18.

Theorem 5.19 (Determinability characterization for SLSSs under general switching signals). Consider the solvable SLSS (4.9) w.r.t. a switching signal σ of the form (2.11). Then, system (4.9) is determinable on $[0, K]$ w.r.t. σ if, and only if,

$$\mathcal{Q}^K = \{0\} \tag{5.20}$$

where $\mathcal{Q}^K$ is given by (5.19). ◊

Proof. The necessity is obvious from the fact that the implication (5.5) implies $\mathcal{Q}^K = \{0\}$ (by Lemma 5.18). For the sufficiency, $\mathcal{Q}^K = \{0\}$ implies that for all $i \in [0, K]$, $y(i) = 0$ and $x(K) = 0$ (also by Lemma 5.18), i.e., the implication (5.5) holds. □

In contrast to the characterization of observability given in Theorem 5.15, the dependence of determinability on the switching times is not so apparent. However, by introducing the family of maps $\Omega_{i,j}$ which map a subspace $\mathcal{T}$ to

$$\Omega_{i,j} \mathcal{T} := \ker C_i \cap \Phi_{i,j} \mathcal{T},$$

it can be seen that $\mathcal{Q}^k = \Omega_{\sigma(k),\sigma(k-1)}\mathcal{Q}^{k-1}$. In particular, for a switching signal given by (2.11), it can be concluded that the undeterminable space $\mathcal{Q}^{k^s_{j+1}-1}$ can be expressed in terms of corresponding powers of the operator $\Omega_{\sigma_j,\sigma_j}$ applied to $\mathcal{Q}^{k^s_j}$, i.e.

$$\mathcal{Q}^{k^s_{j+1}-1} = \Omega^{k^s_{j+1}-k^s_j-1}_{\sigma_j,\sigma_j}\mathcal{Q}^{k^s_j} = \Omega^{k^s_{j+1}-k^s_j-1}_{\sigma_j,\sigma_j}\Omega_{\sigma_j,\sigma_{j-1}}\mathcal{Q}^{k^s_j-1}.$$

Consequently, $\mathcal{Q}^K$ can be calculated by the following nested formula:

$$\mathcal{Q}^K = \Omega^{K-k^s_J-1}_{\sigma_J,\sigma_J}\Omega_{\sigma_J,\sigma_{J-1}}(\Omega^{k^s_J-k^s_{J-1}-1}_{\sigma_{J-1},\sigma_{J-1}}\Omega_{\sigma_J,\sigma_{J-1}}(\cdots(\Omega^{k^s_1-1}_{\sigma_0,\sigma_0}\mathcal{Q}^0)),$$

which clearly shows the dependence on the switching times. However, it seems not possible to easily simplify this expression in case the mode durations are sufficiently large (as in Corollary 5.16).

5.4.3 Inhomogeneous Systems

Now, for switched systems with inputs, consider the observability notion in the sense that the InhSLSS (4.44) is said to be observable if it is observable in the sense of Definition 5.1 for all (unbounded) input sequences. Similarly, the InhSLSS (4.44) is said to be determinable if it is determinable in the sense of Definition 5.4 for all (unbounded) input sequences. As in nonwsitched systems, it is true that under a fixed and known switching signal, the observability and determinability of InhSLSSs can be investigated via their homogeneous forms. This is stated in following proposition, which is the switched version of Proposition 5.9.

Proposition 5.20 (Observability/determinability of InhSLSSs). The weakly or strongly solvable InhSLSS (4.44) is observable (determinable) on the finite time domain $[0, K]$, $K \in \mathbb{N}$ w.r.t. a fixed and known switching signal σ of the form (2.11) if and only if its homogeneous form (4.9) is observable (determinable) on $[0, K]$ w.r.t. σ. ◊

Proof. Since fixed switching signals are being considered, the arguments are similar to the proof of Proposition 5.9; for completeness, the complete proof is provided as follows. Consider a fixed switching signal σ of the form (2.11). At any time instant k, Let $(x'_u(k), y'_u(k))$ and $(x''_u(k), y''_u(k))$ be any two pairs of state and output measurement of (4.44) under σ with the same input $u(k)$. Then, they satisfy

$$\begin{aligned} E_{\sigma(k)}x'_u(k+1) &= A_{\sigma(k)}x'_u(k) + B_{\sigma(k)}u(k) \\ y'_u(k) &= C_{\sigma(k)}x'_u(k) + D_{\sigma(k)}u(k) \end{aligned}$$

and

$$E_{\sigma(k)}x''_u(k+1) = A_{\sigma(k)}x''_u(k) + B_{\sigma(k)}u(k)$$

$$y''_u(k) = C_{\sigma(k)}x''_u(k) + D_{\sigma(k)}u(k)$$

respectively. Take the difference between the two states, then it satisfies

$$E_{\sigma(k)}(x'_u(k+1) - x''_u(k+1)) = A_{\sigma(k)}(x'_u(k) - x''_u(k)) + B_{\sigma(k)}u(k) - B_{\sigma(k)}u(k)$$

i.e., $x'_u(k) - x''_u(k) =: x'''_u(k)$ satisfies $E_{\sigma(k)}x'''_u(k+1) = A_{\sigma(k)}x'''_u(k)$. Now, take the difference between the two outputs, then it satisfies

$$y'_u(k) - y''_u(k) = C_{\sigma(k)}(x'_u(k) - x'''_u(k)) + D_{\sigma(k)}u(k) - D_{\sigma(k)}u(k)$$

i.e., $y'_u(k) - y''_u(k) =: y'''_u(k)$ satisfies $y'''_u(k) = C_{\sigma(k)}x'''_u(k)$. Altogether, the difference between the solutions of inhomogeneous systems is also the solution of their homogeneous forms. Hence, the observability/determinability of inhomogeneous systems and homogeneous systems imply each other. □

The intuition of this bi-implication between observability/determinability of inhomogeneous switched systems and homogeneous systems can also be explained as follows. Under a fixed switching signal, the matrices Φ_i, Θ_i, Γ_i, $\widehat{\Phi}_i, \widehat{\Theta}_i$, $\widehat{\Gamma}_i$, and D_i for all i and inputs $u(k)$ for all k in the output equation (4.65) and (4.66) are known. Thus, the last two terms in (4.65) and (4.66) can be subtracted from the output measurement $y(k)$. Therefore, the state $x_{[0,K]}$ is determined only by the observation of the output $y_{[0,K]}$. This means that the observability and determinability, under fixed switching signals, are independent of inputs, which then concludes that the InhSLSS (4.44) is observable (determinable) on a given time interval $[0, K]$ and switching signal σ if and only if its homogeneous form (4.9) is observable (determinable) on $[0, K]$ w.r.t. σ.

5.5 Constant Observability and Determinability

As already discussed in Section 5.2, observability and determinability for non-switched linear systems depend on the length of the interval of consideration, which is intuitively clear because in general, it takes some time for the initial value to propagate through the different states until it reaches the output. However, it can be seen that a system is observable/determinable for some time interval if, and only if, it is observable/determinable for any interval of length n or larger, where n is the state dimension. For switched linear systems, there are therefore two independent reasons why observability/determinability depends on the mode durations: On the one hand, for short mode durations, observability/determinability depends on the mode duration because the local observability/determinability properties of the individual modes depend on the mode durations; on the other hand, the switches themselves introduce a dependency of observability/determinability from the mode duration. Here

the second phenomenon is of more relevance as it only occurs for switched systems. Assuming that each mode is active sufficiently long indeed simplifies the observability characterization (Corollary 5.16), however, the mode duration dependency is not eliminated, see Example 5.27. While a general characterization of system classes for which observability/determinability is independent of the mode duration (under the assumption that each mode is active long enough) seems difficult to find, two sufficient conditions have been derived: for the single switch case, observability and determinability do not depend on the mode duration if each mode is at least n times steps active, which is in line with the results from the continuous time case and also rather intuitive. The other case is the two-dimensional strictly singular case. Both cases are considered in the following in detail.

To be precise, recall the definition of constant observability/determinability for ordinary systems in Definition 3.32 and adjust it for system (4.9) as follows:

Definition 5.21 (Constant observability/determinability of SLSSs). Consider the HomSLSS (4.9). Its observability (determinability) is called **constant** (under slow switching) w.r.t. a mode sequence (σ_j) of the form (2.11) if it is either observable (determinable) on $[0, K]$ for all $\sigma \in \mathbb{S}^n_{(\sigma_j)}$ and all $K > (J+1)n+1$ or unobservable (undeterminable) on $[0, K]$ for all $\sigma \in \mathbb{S}^n_{(\sigma_j)}$ and all $K > (J+1)n+1$. ◊

Note that the notion of constant observability/determinability only makes sense if $k^s_{J+1} = K+1 > (J+1)n$, otherwise $\mathbb{S}^n_{[0,K]}$ contains exactly one switching signal ($K+1 = (J+1)n$) or is empty ($K+1 < (J+1)n$).

5.5.1 The two-dimension case

It has been proved in Section 3.4 that for one-dimensional (non-singular) linear switched systems, the observability is constant for any mode sequence, whereas non-constant observabilities occur in systems with two- or more-dimensional states. In singular systems with two-dimensional states and singular E_is, the one-step map matrices are always singular, and thus the behavior of the system lies in zero- or one-dimensional space. This leads to the following proposition.

Proposition 5.22 (Constant observability/determinability of two-dimensional SLSSs). Consider the HomSLSS (4.9) with two-dimensional states and singular E_is, and assume it is solvable w.r.t. a given mode sequence (σ_j). Then, its observability and determinability are constant w.r.t. (σ_j). In this case, the matrix (4.23) can be simplified as

$$\Psi_\sigma(j, 0) = \Phi_{\sigma_j,\sigma_{j-1}} \Phi_{\sigma_{j-1}} \Psi_\sigma(j-1, 0) \tag{5.21}$$

with $\Psi_\sigma(0, 0) = I_n$. ◊

Proof. Note that $\Phi_{\sigma_i,\sigma_j} \in \mathbb{R}^{n\times n}$ $\forall i,j$ is singular. Now, consider two-dimensional matrices $\Phi_{\sigma_i,\sigma_j} \in \mathbb{R}^2$. From basic algebra, there exists invertible matrix Q_{σ_i,σ_j} such that

$$Q_{\sigma_i,\sigma_j}^{-1}\Phi_{\sigma_i,\sigma_j}Q_{\sigma_i,\sigma_j} = \begin{bmatrix} \kappa_{\sigma_i,\sigma_j} & 0 \\ 0 & 0 \end{bmatrix} \tag{5.22}$$

where $\kappa_{\sigma_i,\sigma_j} \in \mathbb{R}$ and for simplification, denote $\kappa_{\sigma_i,\sigma_i} = \kappa_{\sigma_i}$ and $Q_{\sigma_i,\sigma_i} = Q_{\sigma_i}$.
Part I: Observability
The observability condition is $\mathcal{S}_{\sigma_0} \cap \ker O_{[0,K]} = \{0\}$ where $O_{[0,K]}$ is as in (5.17). With the invertible matrices $Q_{i,j}$ given in (5.22), the subspace $\ker O_{[0,K]}$ depends on the switching times k_j^s if, and only if, $\ker \widetilde{O}_{[0,K]}$ depends on the switching times k_j^s where

$$\widetilde{O}_{[0,K]} = \begin{bmatrix} \widetilde{O}_{[0,k_1^s)} \\ \widetilde{O}_{[k_1^s,k_2^s)} \\ \vdots \\ \widetilde{O}_{[k_J^s,k_{J+1}^s)}, \end{bmatrix}, \quad \widetilde{O}_{[0,k_1^s)} = \begin{bmatrix} \widetilde{C}_0 \\ \widetilde{C}_0\widetilde{\Phi}_0 \\ \vdots \\ \widetilde{C}_1\widetilde{\Phi}_0^{k_1^s-1} \end{bmatrix}$$

and for $j = 1, 2, \ldots, J$,

$$\widetilde{O}_{[k_j^s,k_{j+1}^s)} = \begin{bmatrix} \widetilde{C}_j\widetilde{\Phi}_{j,j-1}\widetilde{N}_{j-1}\cdots\widetilde{\Phi}_0^{k_1^s-1} \\ \widetilde{C}_j\widetilde{\Phi}_j\widetilde{N}_j\widetilde{\Phi}_{j,j-1}\widetilde{N}_{j-1}\cdots\widetilde{\Phi}_0^{k_1^s-1} \\ \vdots \\ \widetilde{C}_j\widetilde{\Phi}_J^{k_{j+1}^s-k_j^s-1}\widetilde{N}_J\widetilde{\Phi}_{j,j-1}\widetilde{N}_{j-1}\cdots\widetilde{\Phi}_0^{k_1^s-1} \end{bmatrix}$$

and where $\widetilde{C}_j := C_{\sigma_j}Q_{\sigma_j}$, $\widetilde{N}_j = Q_{\sigma_j}^{-1}Q_{\sigma_j,\sigma_{j-1}}$ are invertible matrices for all j, and $\widetilde{\phi}_{i,j} := Q_{\sigma_i,\sigma_j}^{-1}\Phi_{\sigma_i,\sigma_j}Q_{\sigma_i,\sigma_j}$, $\widetilde{\phi}_i := \widetilde{\phi}_{\sigma_i,\sigma_i}$ $\forall i,j$ are in the form of (5.22). The kernel of the matrix $\widetilde{O}_{[0,K]}$ can be rewritten as

$$\begin{aligned} \ker\begin{bmatrix} \widetilde{C}_0 \\ \widetilde{C}_0\widetilde{\Phi}_0 \end{bmatrix} \cap \ker\begin{bmatrix} s_{1,1}\kappa_0^{k_1^s-1} & 0 \\ \vdots & \\ s_{1,p}\kappa_0^{k_1^s-1} & 0 \end{bmatrix} \cap \ker\begin{bmatrix} s_{2,1}\kappa_1^{k_2^s-k_1^s-1}\kappa_0^{k_1^s-1} & 0 \\ \vdots & \\ s_{2,p}\kappa_1^{k_2^s-k_1^s-1}\kappa_0^{k_1^s-1} & 0 \end{bmatrix} \cap \cdots \\ \cap \ker\begin{bmatrix} s_{J+1,1}\kappa_J^{k_{J+1}^s-k_J^s-1}\cdots\kappa_1^{k_2^s-k_1^s-1}\kappa_0^{k_1^s-1} & 0 \\ \vdots & \\ s_{J+1,p}\kappa_J^{k_{J+1}^s-k_J^s-1}\cdots\kappa_1^{k_2^s-k_1^s-1}\kappa_0^{k_1^s-1} & 0 \end{bmatrix} \end{aligned} \tag{5.23}$$

where $s_{i,j}$ are some scalars. The subspaces from the second to the last in (5.23) have special forms of $\operatorname{span}\binom{0}{*}$ that leads to a situation in which it is not possible to have $\mathcal{S}_{\sigma_0} \cap \ker O_{[0,K]}$ for some switching time(s) and not a singleton for some other switching times. This yields the conclusion that observability is constant.
Part II: Determinability

The subspace $\mathcal{Q}^K$ can be rewritten as the following nested intersections of subspaces

$$\begin{aligned}\mathcal{Q}^K &= \ker C_{\sigma(K)} \cap \Phi_{\sigma(K),\sigma(K-1)}(\ker C_{\sigma(K-1)} \cap \cdots \\ &\cap \Phi_{\sigma(k),\sigma(k-1)}(\ker C_{\sigma(k-1)} \cap \cdots \cap \Phi_{\sigma(0)}(\ker C_{\sigma(0)} \cap \mathcal{S}_{\sigma(0)}))).\end{aligned} \tag{5.24}$$

First, $\Phi_{i,j} = 0$ for some i, j or $\ker C_i = \{0\}$ for some i or $\ker C_{\sigma(0)} \cap \mathcal{S}_{\sigma(0)} = \{0\}$, or $\mathcal{S}_{\sigma(0)} = \{0\}$ yields constant determinability (the system is always determinable). Thus, it suffices to prove constant determinability with $\Phi_{i,j} \neq 0$ (but singular) for all i, j, $\ker C_i \neq \{0\}$ for all i, $\ker C_{\sigma(0)} \cap \mathcal{S}_{\sigma(0)} \neq \{0\}$, and $\dim \mathcal{S}_{\sigma(0)} = 1$. By utilizing the invertible matrices $Q_{i,j}$ given in (5.22), the subspace $\mathcal{Q}^K$ depends on the switching times k_j^s if and only if $\widetilde{\mathcal{Q}}^K$ depends on the switching times where

$$\begin{aligned}\widetilde{\mathcal{Q}}^K &= \ker \widetilde{C}_{\sigma(K)} \cap \widetilde{\Phi}_{\sigma(K),\sigma(K-1)}(\ker \widetilde{C}_{\sigma(K-1)} \cap \cdots \\ &\cap \widetilde{\Phi}_{\sigma(k),\sigma(k-1)}(\ker \widetilde{C}_{\sigma(k-1)} \cap \cdots \cap \widetilde{\Phi}_{\sigma(0)} \widetilde{\mathcal{Q}}^0))\end{aligned}$$

with $\widetilde{\mathcal{Q}}^0 = \mathcal{Q}^0$. Let for $j = 0, 1, \ldots, J$, $\widetilde{\mathcal{Q}}^{k_j^s} = \operatorname{span}\left(\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}\right)$ where a_j and b_j some scalars, and let for $i \in \{0, 1, \ldots, \mathrm{p}\}$, $\widetilde{\Phi}_{\sigma(i)} = Q_{\sigma(i)} \Phi_{\sigma(i)} Q_{\sigma(i)}^{-1} = \left[\begin{smallmatrix} p_i & 0 \\ q_i & 0 \end{smallmatrix}\right]$ with $p_i \neq 0$ and q_i some scalars. It will be shown that for any $j = 0, 1, \ldots, J$,

$$\mathcal{Q}^{k_j^s+1} = \mathcal{Q}^{k_j^s+2} = \cdots = \mathcal{Q}^{k_{j+1}^s-1}. \tag{5.25}$$

First, note that $\left[\begin{smallmatrix} p_i & 0 \\ q_i & 0 \end{smallmatrix}\right]^2 \operatorname{span}\left(\begin{smallmatrix} a_i \\ b_i \end{smallmatrix}\right) = \left[\begin{smallmatrix} p_i & 0 \\ q_i & 0 \end{smallmatrix}\right] \operatorname{span}\left(\begin{smallmatrix} p_i a_i \\ q_i a_i \end{smallmatrix}\right) = \operatorname{span}\left(\begin{smallmatrix} p_i p_i a_i \\ q_i p_i a_i \end{smallmatrix}\right) = \operatorname{span}\left(\begin{smallmatrix} p_i a_i \\ q_i a_i \end{smallmatrix}\right) = \left[\begin{smallmatrix} p_i & 0 \\ q_i & 0 \end{smallmatrix}\right] \operatorname{span}\left(\begin{smallmatrix} a_i \\ * \end{smallmatrix}\right) = \left[\begin{smallmatrix} p_i & 0 \\ q_i & 0 \end{smallmatrix}\right] \operatorname{span}\left(\begin{smallmatrix} a_i \\ b_i \end{smallmatrix}\right)$. Now, assume first $\ker C_{\sigma_j} = \mathbb{R}^2$, then (5.25) holds since

$$\begin{aligned}
\widetilde{\mathcal{Q}}^{k_j^s+1} &= \widetilde{\Phi}_{\sigma_j} \widetilde{\mathcal{Q}}^{k_j^s} = \left[\begin{smallmatrix} p_j & 0 \\ q_j & 0 \end{smallmatrix}\right] \operatorname{span}\left(\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}\right) \\
\widetilde{\mathcal{Q}}^{k_j^s+2} &= \widetilde{\Phi}_{\sigma_j} \widetilde{\mathcal{Q}}^{k_j^s+1} = \left[\begin{smallmatrix} p_j & 0 \\ q_j & 0 \end{smallmatrix}\right]^2 \operatorname{span}\left(\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}\right) = \widetilde{\mathcal{Q}}^{k_j^s+1} \\
\widetilde{\mathcal{Q}}^{k_j^s+3} &= \widetilde{\Phi}_{\sigma_j} \widetilde{\mathcal{Q}}^{k_j^s+2} = \left[\begin{smallmatrix} p_j & 0 \\ q_j & 0 \end{smallmatrix}\right]^3 \operatorname{span}\left(\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}\right) = \widetilde{\mathcal{Q}}^{k_j^s+1} \\
&\vdots \\
\widetilde{\mathcal{Q}}^{k_{j+1}^s-1} &= \widetilde{\Phi}_{\sigma_j} \widetilde{\mathcal{Q}}^{k_{j+1}^s-2} = \left[\begin{smallmatrix} p_j & 0 \\ q_j & 0 \end{smallmatrix}\right]^{k_{j+1}^s-k_j^s-1} \operatorname{span}\left(\begin{smallmatrix} a_j \\ b_j \end{smallmatrix}\right) = \widetilde{\mathcal{Q}}^{k_j^s+1}.
\end{aligned}$$

Now, for the case $\{0\} \neq \ker C_{\sigma_j} \subsetneq \mathbb{R}^2$, let $\ker C_{\sigma_j} = \operatorname{span}\left(\begin{smallmatrix} c_j \\ d_j \end{smallmatrix}\right)$ where c_j and

d_j some scalars. If $\widetilde{\mathcal{Q}}^{k_j^s+1} = \{0\}$, then clearly (5.25) holds, otherwise,

$$\begin{aligned}
\widetilde{\mathcal{Q}}^{k_j^s+1} &= \operatorname{span}\begin{pmatrix} c_j \\ d_j \end{pmatrix} \cap \begin{bmatrix} p_j & 0 \\ q_j & 0 \end{bmatrix} \operatorname{span}\begin{pmatrix} a_j \\ b_j \end{pmatrix} = \begin{bmatrix} p_j & 0 \\ q_j & 0 \end{bmatrix} \operatorname{span}\begin{pmatrix} a_j \\ b_j \end{pmatrix} \\
\widetilde{\mathcal{Q}}^{k_j^s+2} &= \operatorname{span}\begin{pmatrix} c_j \\ d_j \end{pmatrix} \cap \begin{bmatrix} p_j & 0 \\ q_j & 0 \end{bmatrix}^2 \operatorname{span}\begin{pmatrix} a_j \\ b_j \end{pmatrix} = \mathcal{Q}^{k_j^s+1} \\
\widetilde{\mathcal{Q}}^{k_j^s+3} &= \operatorname{span}\begin{pmatrix} c_j \\ d_j \end{pmatrix} \cap \begin{bmatrix} p_j & 0 \\ q_j & 0 \end{bmatrix}^3 \operatorname{span}\begin{pmatrix} a_j \\ b_j \end{pmatrix} = \mathcal{Q}^{k_j^s+1} \\
&\vdots \\
\widetilde{\mathcal{Q}}^{k_{j+1}^s-1} &= \operatorname{span}\begin{pmatrix} c_j \\ d_j \end{pmatrix} \cap \begin{bmatrix} p_j & 0 \\ q_j & 0 \end{bmatrix}^{k_{j+1}^s-k_j^s-1} \operatorname{span}\begin{pmatrix} a_j \\ b_j \end{pmatrix} = \widetilde{\mathcal{Q}}^{k_j^s+1}.
\end{aligned}$$

Altogether, $\mathcal{Q}^K$ under slow switching (each mode is active for at least two time steps) does not depend on switching times. This concludes that the determinability of two-dimensional systems is constant. □

5.5.2 The single switch case

There are switched systems that have a non-constant observability property with respect to any dwell time, for example, see the system in the forthcoming Example 5.27. On the other hand, there are switched systems that have a constant observability property for a sufficiently large dwell time, see the system in Example 5.28 for an example. The latter example had only two modes and it will be shown that in fact, all switched systems with only two modes have the constant observability property. Therefore, the SLSS (4.9) with only one switch is considered in the following discussion, and the attention is now restricted to the specific mode sequence $(0, 1)$, i.e., consider the switched system

$$\begin{aligned}
E_0 x(k+1) &= A_0 x(k), \quad y(k) = C_0 x(k), \quad k \in [0, k^s) \\
E_1 x(k+1) &= A_1 x(k), \quad y(k) = C_1 x(k), \quad k \in [k^s, K]
\end{aligned} \tag{5.26}$$

and assume it is sequentially index-1 w.r.t. to the mode sequence $(0, 1)$ to ensure the existence and uniqueness of solutions for an arbitrary switching time $k^s \in (0, K)$.

The following result presents the constant observability for system (4.9) under single switch switching signals.

Proposition 5.23 (Constant observability of SLSSs under single switch switching signals). The HomSLSS (5.26) with exactly one switch has the property of constant observability. ◊

Proof.
Case 1: The matrix Φ_0 is nonsingular.

In this case, the consistency space is $\mathcal{S}_0 = \mathbb{R}^n$. According to Corollary 5.16, the switched system is observable if and only if

$$\ker O_0 \cap \ker[O_1 \Phi_{1,0} \Phi_0^{k^s}] = \{0\}$$

or, equivalently (by invertibility of $\Phi_0^{k^s}$),

$$\Phi_0^{k^s} \ker O_0 \cap \ker[O_1 \Phi_{1,0}] = \{0\}.$$

Since $\Phi_0^{k^s} \ker O_0 = \ker O_0 \Phi_0^{-k^s} = \ker O_0$ according to Lemma A.4 in the Appendix it follows that observability does not depend on k^s.

Case 2: The matrix Φ_0 is singular.

Consider the (real) Jordan canonical form of Φ_0, i.e.

$$\Phi_0 = U_0 \begin{bmatrix} N_0 & 0 \\ 0 & \widetilde{\Phi}_0 \end{bmatrix} U_0^{-1}$$

where U_0 is invertible, $N_0 \in \mathbb{R}^{n_0 \times n_0}$ is a nilpotent with nilpotency index at most $n_0 < n$, and $\widetilde{\Phi}_0 \in \mathbb{R}^{n-n_0 \times n-n_0}$ is of full rank. By the coordinate transformation $\widetilde{x}(k) = U_0^{-1} x(k)$, for $k < k^s$,

$$\widetilde{x}(k+1) = U_0^{-1} x(k+1) = \begin{bmatrix} N_0 & 0 \\ 0 & \widetilde{\Phi}_0 \end{bmatrix} \widetilde{x}(k)$$

and the corresponding output vector can be rewritten as

$$y(k) = C_0 U_0 \widetilde{x}(k) = \widetilde{C}_0 \widetilde{x}(k) = [\widetilde{C}_0^1, \widetilde{C}_0^2] \widetilde{x}(k).$$

Now, the observability condition becomes

$$\widetilde{\mathcal{S}}_0 \cap \ker \widetilde{O}_0 \cap \ker \left(\widetilde{O}_1 \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{\Phi}_0^{k^s} \end{bmatrix} \right) \tag{5.27}$$

where $\widetilde{\mathcal{S}}_0 = \left[U_0^{-1} A U_0 \right]^{-1} \operatorname{im} \left[U_0^{-1} E U_0 \right]$,

$$\widetilde{O}_0 = \begin{bmatrix} C_0 U_0 \\ C_0 U_0 \begin{bmatrix} N_0 & 0 \\ 0 & \widetilde{\Phi}_0 \end{bmatrix} \\ \vdots \\ C_0 U_0 \begin{bmatrix} N_0 & 0 \\ 0 & \widetilde{\Phi}_0 \end{bmatrix}^{n-1} \end{bmatrix} = \begin{bmatrix} \widetilde{C}_0^1 & \widetilde{C}_0^2 \\ \widetilde{C}_0^1 N_0 & \widetilde{C}_0^2 \widetilde{\Phi}_0 \\ \vdots & \vdots \\ \widetilde{C}_0^1 N_0^{n-1} & \widetilde{C}_0^2 \widetilde{\Phi}_0^{n-1} \end{bmatrix} =: [\widetilde{O}_0^1, \widetilde{O}_0^2]$$

and $\widetilde{O}_1 = O_1 \Phi_{1,0} =: [\widetilde{O}_1^1, \widetilde{O}_1^2]$. Note that $\widetilde{\mathcal{S}}_0$ in (5.27) is independent of k^s, hence it suffices to show that

$$\ker \widetilde{O}_0 \cap \ker \left(\widetilde{O}_1 \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{\Phi}_0^{k^s} \end{bmatrix} \right) \tag{5.28}$$

is independent of k^s.

Now, for an arbitrary vector $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \mathbb{R}^n$, $x_1 \in \mathbb{R}^{n_0}$, $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \ker \left(\widetilde{O}_1 \left[\begin{smallmatrix} 0 & 0 \\ 0 & \widetilde{\Phi}_0^{k^s} \end{smallmatrix}\right] \right)$ if, and only if, x_1 is arbitrary and $x_2 \in [\widetilde{\Phi}_0^{k^s}]^{-1} \ker \widetilde{O}_1^2$. Furthermore, $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \ker \widetilde{O}_0 = \ker[\widetilde{O}_0^1, \widetilde{O}_0^2]$ if, and only if, $\widetilde{O}_0^1 x_1 = -\widetilde{O}_0^2 x_2$. In other words, any $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$

is in the intersection (5.28) if, and only if,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \begin{bmatrix} -[\widetilde{O}_0^1]^{-1}\widetilde{O}_0^2 \\ I \end{bmatrix} \left([\widetilde{O}_0^2]^{-1}(\operatorname{im} \widetilde{O}_0^1) \cap \widetilde{\Phi}_0^{-k^s} \ker \widetilde{O}_1^2 \right)$$

where $[\widetilde{O}_0^1]^{-1}$ and $[\widetilde{O}_0^2]^{-1}$ stand for the (set-valued) preimage operator. In particular, (5.28) is independent of k^s if $[\widetilde{O}_0^2]^{-1}(\operatorname{im} \widetilde{O}_0^1) \cap \widetilde{\Phi}_0^{-k^s} \ker O_1^2$ is independent of k^s. Since $\widetilde{\Phi}_0$ is invertible, the latter is independent of k^s if, and only if, $\widetilde{\Phi}_0^{k^s}[\widetilde{O}_0^2]^{-1}(\operatorname{im} \widetilde{O}_0^1)$ is independent of k^s. Now, observe that

$$x_2 \in \widetilde{\Phi}_0^{k^s}[\widetilde{O}_0^2]^{-1}(\operatorname{im} \widetilde{O}_0^1) \Leftrightarrow \exists x_1 : \widetilde{O}_0^2 \widetilde{\Phi}_0^{-k^s} x_2 = \widetilde{O}_0^1 x_1.$$

From the definition of $\widetilde{O}_0^1$ and $\widetilde{O}_0^2$, the latter is equivalent to

$$\exists x_1 : \begin{bmatrix} \widetilde{C}_0^2 \widetilde{\Phi}_0^{-k^s} \\ \widetilde{C}_0^2 \widetilde{\Phi}_0^{-k^s+1} \\ \vdots \\ \widetilde{C}_0^2 \widetilde{\Phi}_0^{-k^s+n_0-1} \end{bmatrix} x_2 = \begin{bmatrix} \widetilde{C}_0^1 \\ \widetilde{C}_0^1 N_0 \\ \vdots \\ \widetilde{C}_0^1 N_0^{n_0-1} \end{bmatrix} x_1 \text{ and } \begin{bmatrix} \widetilde{C}_0^2 \widetilde{\Phi}_0^{-k^s+n_0} \\ \widetilde{C}_0^2 \widetilde{\Phi}_0^{-k^s+n_0+1} \\ \vdots \\ \widetilde{C}_0^2 \widetilde{\Phi}_0^{-k^s+n-1} \end{bmatrix} x_2 = 0.$$

Lemma A.4 implies that

$$\ker \begin{bmatrix} \widetilde{C}_0^2 \widetilde{\Phi}_0^{-k^s+n_0} \\ \widetilde{C}_0^2 \widetilde{\Phi}_0^{-k^s+n_0+1} \\ \vdots \\ \widetilde{C}_0^2 \widetilde{\Phi}_0^{-k^s+n-1} \end{bmatrix} = \ker \begin{bmatrix} \widetilde{C}_0^2 \\ \widetilde{C}_0^2 \widetilde{\Phi}_0 \\ \vdots \\ \widetilde{C}_0^2 \widetilde{\Phi}_0^{n-n_0-1} \end{bmatrix} = \ker \widetilde{O}_0^2$$

hence $x_2 \in \widetilde{\Phi}_0^{k^s}[\widetilde{O}_0^2]^{-1}(\operatorname{im} \widetilde{O}_0^1)$ if, and only if

$$\begin{bmatrix} \widetilde{C}_0^2 \widetilde{\Phi}_0^{-k^s} \\ \widetilde{C}_0^2 \widetilde{\Phi}_0^{-k^s+1} \\ \vdots \\ \widetilde{C}_0^2 \widetilde{\Phi}_0^{-k^s+n_0-1} \end{bmatrix} x_2 \in \operatorname{im} \begin{bmatrix} \widetilde{C}_0^1 \\ \widetilde{C}_0^1 N_0 \\ \vdots \\ \widetilde{C}_0^1 N_0^{n_0-1} \end{bmatrix} \text{ and } x_2 \in \ker \widetilde{O}_0^2.$$

Utilizing again Lemma A.4, every $x_2 \in \ker \widetilde{O}_0^2$ satisfies $\widetilde{\Phi}_0^i x_2 \in \ker \widetilde{O}_0^2$ for all $i \in \mathbb{Z}$, and hence $\widetilde{C}_0^2 \widetilde{\Phi}_0^i x_2 = 0$ and, in particular,

$$\begin{bmatrix} \widetilde{C}_0^2 \widetilde{\Phi}_0^{-k^s} \\ \widetilde{C}_0^2 \widetilde{\Phi}_0^{-k^s+1} \\ \vdots \\ \widetilde{C}_0^2 \widetilde{\Phi}_0^{-k^s+n_0-1} \end{bmatrix} x_2 = 0 \in \operatorname{im} \begin{bmatrix} \widetilde{C}_0^1 \\ \widetilde{C}_0^1 N_0 \\ \vdots \\ \widetilde{C}_0^1 N_0^{n_0-1} \end{bmatrix}.$$

In other words,

$$x_2 \in \widetilde{\Phi}_0^{k^s}[\widetilde{O}_0^2]^{-1}(\operatorname{im} \widetilde{O}_0^1) \Leftrightarrow x_2 \in \ker \widetilde{O}_0^2$$

which is independent of k^s as desired. □

Similar to the determinability of single-switch ordinary systems, it is still not clear whether the determinability of single-switch singular linear systems is also constant. Neither a counter-example nor the proof for constant determinability has been derived. Nevertheless, by similar observation as in the discussion

after Conjecture 3.35 for ordinary systems, it is also conjectured that the determinability of SLSSs under single switch switching signals is constant w.r.t. all mode sequences.

Conjecture 5.24 (Determinability of single-switch SLSSs is constant). Consider the solvable SLSS (4.9) with the number of switches is one. Then, its determinability is constant w.r.t. all mode sequences. ◊

5.6 Illustrative Examples

Examples in this section illustrate the solution theory derived in this chapter. Some systems also play as counter-examples for some results in the theoretical parts. First, the following example illustrates an unobservable switched system w.r.t. a fixed switching signal.

Example 5.25. Recall Example 4.24. With $C_0 = (1/4, 2/4, 1)$, $C_1 = (0, 1, -1)$ and $C_2 = (0, 1, 0)$, the switched system under the same switching signal is not observable on $[0, 7]$ since the unobservable space in (5.15) is $\operatorname{span}\{(0, 0, 1)^\top\} \neq \{0\}$. ◊

The second example below shows that determinability is indeed a weaker property of observability.

Example 5.26. Consider the SLSS (4.9) with the following system's matrices

$$(E_0, A_0, C_0) = \left(\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}^\top \right),$$

$$(E_1, A_1, C_1) = \left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}^\top \right)$$

and with the mode sequence $(0, 1)$. It is sequentially index-1 w.r.t. this mode sequence, and thus it is switched index-1 w.r.t. any (single switch) switching signal with the same mode sequence. Using Theorem 5.15 it can be shown that this SLSS is unobservable on $[0, 12]$ for any switching time k^s with $1 \leq k^s \leq 12$. On the other hand, only for $k^s = 1$, the SLSS is not determinable on $[0, 12]$, for all other switching times $k_s \geq 2$ the system is determinable. This shows that the final state can be recovered although the initial state cannot be recovered from the same output measurement. ◊

Next, the following example demonstrates the dependence of observability on switching times.

Example 5.27. Consider the HomSLSS (4.9) composed of two modes given

by

$$(E_0, A_0, C_0) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \right),$$
$$(E_1, A_1, C_1) = \left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \right)$$

with

$$\ker E_0 = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \qquad \mathcal{S}_0 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\},$$
$$\ker E_1 = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \qquad \mathcal{S}_1 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

It can be seen that this system is sequentially index-1 w.r.t. the mode sequence $(\sigma_k) = (0, 1, 0)$, and thus it is switched index-1 w.r.t. all switching signals with the same mode sequence with any arbitrary switching times k_1^s (from the first mode to the second mode) and k_2^s (from the second mode to the third mode), and any final time K with $k_1^s < k_2^s < K$.

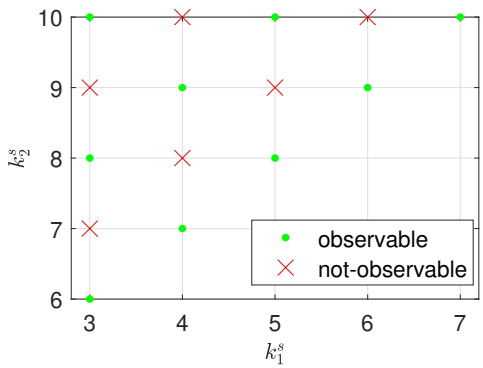
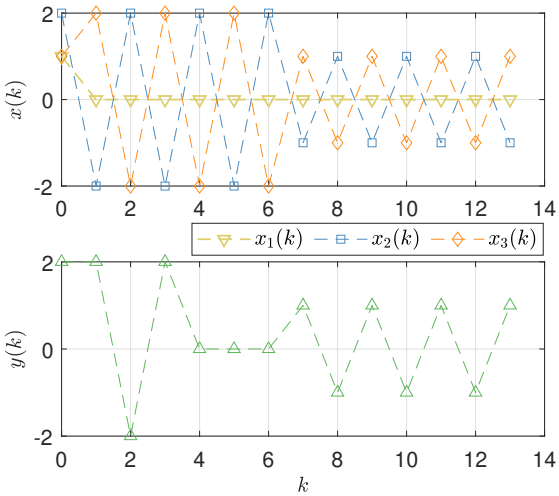


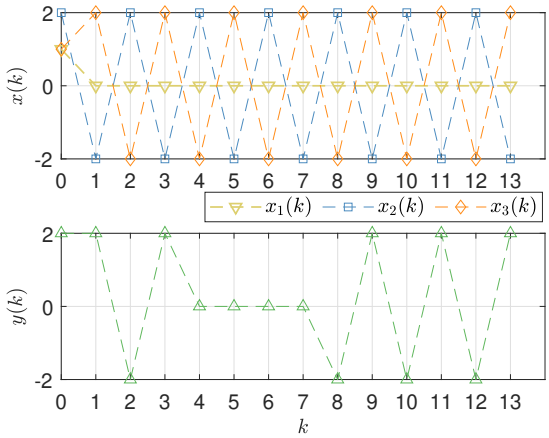
Figure 5.2: Switching times vs observability of Example 5.27

With respect to the mode sequence $(\sigma_k) = (0, 1, 0)$, with switching times from the range $3 \leq k_1^s$, $k_1^s + 2 < k_2^s \leq 10$, $K = k_2^2 + 3$ (i.e. satisfying the dwell time condition from Corollary 5.16) the observability property is shown in Figure 5.2. It can be seen that, indeed, the observability of this system depends on the switching times; in fact, the SLSS is observable if, and only if, the mode duration $k_2^s - k_2^s$ of mode 1 is odd. The state and output trajectories for the specific switching times $k_1^s = 3$ and $k_2^s = 6$ (odd mode duration of mode 1) as well as for $k_1^s = 3$ and $k_2^s = 7$ (even mode duration of mode 1) are shown in Figures 5.3a and 5.3b, respectively. ◊

The dependence of observability on the switching times also occurs in continuous time (see [56, Th. 12]). However, in contrast to the continuous time case, this dependence on the switching time also occurs for the single-switch case as the following example shows.



(a) $k_1^s = 3, k_2^s = 6$, observable



(b) $k_1^s = 3, k_2^s = 7$, not-observable

Figure 5.3: State and output for Example 5.27

Example 5.28. Consider the HomSLSS (4.9) observed on the time interval $[0, K]$ with $K = 10$, and composed by two modes with matrices

$$(E_0, A_0, C_0) = \left(\begin{bmatrix} 0&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&0&0\\1&0&0&0\\1&1&0&1\\0&0&1&0 \end{bmatrix}, \begin{bmatrix} 0\\1\\1\\1 \end{bmatrix}^\top \right)$$

$$(E_1, A_1, C_1) = \left(\begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&1&1&1\\0&0&0&1\\1&0&0&1\\1&0&1&1 \end{bmatrix}, \begin{bmatrix} 1\\1\\0\\1 \end{bmatrix}^\top \right).$$

Geometric computations provide

$$\ker E_0 = \operatorname{span}\left\{ \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix} \right\}, \qquad \mathcal{S}_0 = \operatorname{span}\left\{ \begin{pmatrix} 1\\0\\0\\-1 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\-1 \end{pmatrix} \right\},$$

$$\ker E_1 = \operatorname{span}\left\{ \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix} \right\}, \qquad \mathcal{S}_1 = \operatorname{span}\left\{ \begin{pmatrix} 1\\0\\0\\-1 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix} \right\}.$$

Consequently, the switched system is jointly index-1, because $\mathcal{S}_i \cap \ker E_j = \{0\}$, $i, j = 0, 1$. The corresponding one-step-map matrices are

$$\Phi_0 = \begin{bmatrix} 0&0&0&0\\1&0&0&0\\0&0&0&0\\-1&0&0&0 \end{bmatrix}, \ \Phi_1 = \begin{bmatrix} -1&0&0&0\\0&1&0&0\\0&0&0&0\\1&0&0&0 \end{bmatrix}, \ \Phi_{1,0} = \begin{bmatrix} 0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \ \Phi_{0,1} = \begin{bmatrix} -1&0&0&0\\0&1&0&0\\0&0&0&0\\1&-1&0&0 \end{bmatrix}.$$

Both mode 0 and mode 1 as individual systems are not-observable on $[0, K]$ since

$$\mathcal{S}_0 \cap \mathcal{O}_0 = \operatorname{span}\begin{pmatrix} 0\\1\\0\\-1 \end{pmatrix} \neq \{0\}, \text{ and}$$

$$\mathcal{S}_1 \cap \mathcal{O}_1 = \operatorname{span}\begin{pmatrix} 1\\0\\0\\-1 \end{pmatrix} \neq \{0\}.$$

Consider now switched systems with single switch switching signals with mode sequences $(\sigma_k) = (0, 1)$ and $(1, 0)$. With respect to both those mode sequences, the system is sequentially index-1, and thus it is switched index-1 w.r.t. all switching signals with that mode sequence (with any arbitrary switching time k^s).

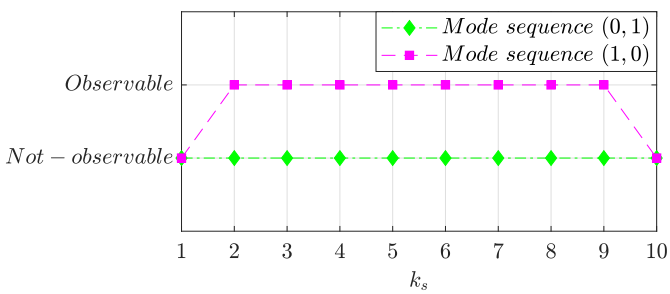


Figure 5.4: Switching time vs observability of Example 5.28

The dependence of observability on various switching times $k^s \in [1, K]$ is illustrated in Fig. 5.4. While for the mode sequence $(0, 1)$ the switched system is unobservable for all possible switching times, the observability for mode sequence $(1, 0)$ depends on the switching time (for $k_s = 1$ or $k_s = 10$ the switched system is not observable, while it is observable for all other k_s). It should be noted however that when restricting to the case of a minimal dwell-time as in Corollary 5.16, observability becomes independent from the switching times for this example. $\Diamond$

The system in Example 5.28 above shows that even if all modes of the switched systems are unobservable on a given interval $[0, K]$, the switched system considered on the same interval can be observable.

It is also possible to have unobservable switched systems from observable modes. This is rather obvious since it can happen only under fast switchings that can cause a situation in which the information from the output measurements is not sufficient to recover the states. Under slow switchings, it is not possible to have that case, this can be directly seen, w.l.o.g., from the observability condition (5.18) for single switch systems under slow switchings where if the initial mode is observable (i.e. $\mathcal{O}_0 = \{0\}$) then the switched system is also observable.

5.7 Concluding Remarks

Necessary and sufficient conditions for observability and determinability characterizations of singular linear systems in discrete time, both for nonswitched and switched cases, have been presented in this chapter. For switched systems, the characterizations were first discussed with single switch switching signals, which have clearer intuition. Characterizations for systems under general switching signals were then developed based on the understanding of the results under single-switch systems.

Switching from one mode (or subsystem) to another mode could "improve" or "worsen" the property of observability and determinability carried by the individual modes. This is drawn from the fact that switching may produce an observable switched system from unobservable individual modes, and, on the other hand, switching could also result in an unobservable switched system composed of observable modes. Switching signals play their roles here. In particular, changing the mode sequence may also change observability/determinability.

The switching time variable appears explicitly in the conditions, which means that in general, the observability and determinability properties depend on the switching time. However, the observability of systems with two-

dimensional states and systems with single switch switching signals does not depend on the switching time. Meanwhile, determinability does not depend on the switching time for two-dimensional systems. Unfortunately, it is still not clear whether it depends on switching time or not for single-switch systems.

Nevertheless, the results in this chapter still leave many open problems which could be considered as future research directions. First, the dependence of observability/determinability on the switching time could be further studied for systems with higher dimensional states. Second, study whether determinability is also constant for single switch systems. And, third, observer designs are also interesting to study by utilizing results from this chapter.

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Chapter 6 | Reachability and Controllability

"With the right switching action, bad subsystems can form good systems."

Contents

Reachability and controllability notions, which have been introduced for the ordinary linear switched system (3.1) in Chapter 3, are studied in this chapter for the singular linear switched system (4.44). Recall that reachability deals with investigating whether a (final) state can be reached from any (consistent) initial state within a certain finite time interval with an input sequence. If a state variable is independent of the input, then it is not possible to transfer this state variable to some final state, and therefore the whole system is said to be unreachable. This unreachable feature is not desired in practice since a solution to a control problem may not exist.

Meanwhile, controllability deals with investigating whether all (consistent) initial states can be brought to zero within some finite time interval with some input sequence. This notion is indeed a particular case of reachability and is also

called null controllability or controllability to zero. In some control purposes such as regularization, having only controllability property is sufficient and the reachability property is not necessary.

In this chapter, those two notions are studied for singular systems both without switching and with switching. Similar to the study of observability and determinability in Chapter 5, the characterizations are formulated by utilizing surrogate systems introduced in Chapter 4 instead of using the original singular systems' equations. Again, this approach makes the study more straightforward, and the role of the consistency space can be seen directly from the characterization results. Moreover, constant reachability and controllability in which those properties are independent of switching times will also be discussed.

6.1 Nonswitched Systems

Recall the state's equation of the InhSLS (4.26) as follows:

$$Ex(k+1) = Ax(k) + Bu(k), \ k \in \mathbb{N} \tag{6.1}$$

and assume that this system is **strongly solvable** in the sense of Definition 4.29. Recall also its corresponding consistency set

$$\widehat{\mathcal{S}} := A^{-1}(\mathrm{im}[E, B]) = \{\xi \in \mathbb{R}^n : A\xi \in \mathrm{im}[E, B]\}.$$

Under the strong solvability assumption, only strongly solvable systems, which then are also strictly causal, are considered here. This means that the analysis of reachability and controllability deals only with past inputs as in ordinary systems. Analysis for weakly solvable systems, which then also takes the current input into account, can be considered as a future research direction.

6.1.1 Definitions

Recall that the basic intuition for reachability is to find the set of all final states reachable within finite time steps starting from a given initial state. Meanwhile, controllability (to zero) deals with finding initial values that can be brought to zero within some finite time steps. Those two notions are in fact equivalent when considering continuous-time non-switched systems, see e.g. [65, Lem. 2.3]. However, they are not equivalent in discrete time; this is already well-known in ordinary systems, see e.g. [34]. For singular systems, it will be shown that this nonequivalence between reachability and controllability is also true, see the forthcoming Remark 6.8.

Consider first the notion of reachability from zero in which the system starts from the origin, and a (final) state is said to be reachable from zero if the zero initial state can be brought to that final state within a finite time

interval. It is formally defined as follows:

Definition 6.1 (Reachable states from zero). A state $x_f \in \widehat{\mathcal{S}}$ of (6.1) is called **reachable from zero** on $[0, K]$, $K \in \mathbb{N}$ if with $x(0) = 0$, there exists an input sequence $u(0), u(1), .., u(K-1)$ such that $x(K) = x_f$. $\Diamond$

Note that only states in the consistency set $\widehat{\mathcal{S}}$ are considered in the definition above. This is due to the fact that all solution states at any time instant belong to $\widehat{\mathcal{S}}$ (see Proposition 4.33), which also means that any state that does not belong to $\widehat{\mathcal{S}}$ is always unreachable from zero.

A final state that is reachable from zero is in fact also reachable from any arbitrary initial state, see the forthcoming Remark 6.6. However, in the characterization, the definition of reachability from zero above is used to have a simpler proof. Next, the set of states that are reachable from zero is introduced in the following definition together with the reachability notion of system (6.1).

Definition 6.2 (Reachable set from zero and reachability). The reachable set (from zero) of system (6.1) on $[0, K]$ is the set of all states $x_f \in \widehat{\mathcal{S}}$ that are reachable from zero on $[0, K]$ and denoted by $\mathcal{R}_{[0,K]}$. Furthermore, the InhSLS (6.1) is called **reachable (from zero)** on $[0, K]$ if $\mathcal{R}_{[0,K]} = \widehat{\mathcal{S}}$. $\Diamond$

The controllability to zero, or null-controllability, on a certain time interval $[0, K]$ is formally defined for system (6.1) in the following definition. The intuition of this notion is that a consistent initial state is said to be controllable to zero if this initial state can be brought to zero within a certain time interval with some input sequence.

Definition 6.3 (Controllability to zero). A consistent initial state $x_0 \in \widehat{\mathcal{S}}$ of (6.1) is called **controllable to zero** on $[0, K]$, $K \in \mathbb{N}$ if with $x(0) = x_0$, there exists an input sequence $(u(0), u(1), .., u(K-1))$ such that $x(K) = 0$. $\Diamond$

Again, only initial states in the consistency set $\widehat{\mathcal{S}}$ are considered due to the inclusion $x(0) \in \widehat{\mathcal{S}}$ from Proposition 4.33. This also means that any non-consistent initial state is always uncontrollable to zero; this is rather obvious since the system with inconsistent initial states does not have solutions. Next, the set of controllable (to zero) initial states and controllability (to zero) are introduced in the following definition.

Definition 6.4 (Controllable set to zero and controllability). The controllable set (to zero) of system (6.1) on $[0, K]$ is the set of all initial states $x_0 \in \widehat{\mathcal{S}}$ which are controllable to zero on $[0, K]$ and denoted by $\mathcal{C}_{[0,K]}$. Furthermore, the InhSLS (6.1) is called **controllable (to zero)** on $[0, K]$ if $\mathcal{C}_{[0,K]} = \widehat{\mathcal{S}}$. $\Diamond$

6.1.2 Characterizations

The reachability characterization of system (6.1) is presented in the following lemma whereas the controllability characterization will follow afterwards.

Lemma 6.5 (Reachability characterization of SLSs). Consider the strongly solvable InhSLS (6.1), and let $\mathcal{R}_{[0,K]}$ be its reachable set from zero on $[0, K]$. Then

$$\mathcal{R}_{[0,K]} = \widehat{\mathcal{S}} \cap \operatorname{im} R(K) \tag{6.2}$$

where $R(k) = [\widehat{\Theta}, \widehat{\Phi}\widehat{\Theta}, \cdots, \widehat{\Phi}^{k-1}\widehat{\Theta}]$ and the matrices $\widehat{\Phi}$ and $\widehat{\Theta}$ are as in (4.36). In particular, the InhSLS (6.1) is reachable from zero if, and only if, $\widehat{\mathcal{S}} \cap \operatorname{im} R(K) = \widehat{\mathcal{S}}$, or equivalently, $\widehat{\mathcal{S}} \subseteq \operatorname{im} R(K)$. ◊

Proof. From the explicit solution formula (4.39), the solution of (6.1) at $k = K > 0$ with $x(0) = 0$ can be written as

$$\begin{aligned} x(K) =& \widehat{\Theta}u(K-1) + \widehat{\Phi}\widehat{\Theta}u(K-2) + \cdots + \widehat{\Phi}^{K-2}\widehat{\Theta}u(1) + \widehat{\Phi}^{K-1}\widehat{\Theta}u(0) \\ =& [\widehat{\Theta}, \widehat{\Phi}\widehat{\Theta}, \cdots, \widehat{\Phi}^{K-2}\widehat{\Theta}, \widehat{\Phi}^{K-1}\widehat{\Theta}] \begin{bmatrix} u(K-1) \\ u(K-2) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix}. \end{aligned} \tag{6.3}$$

Proof of $\mathcal{R}_{[0,K]} \subseteq \widehat{\mathcal{S}} \cap \operatorname{im} R(K)$: pick any reachable state $x(K) \in \mathcal{R}_{[0,K]}$. Then, there exists an input sequence $(u(0), u(1), ..., u(K-1))$ such that (6.3) holds i.e.

$$x(K) \in \operatorname{im}[\widehat{\Theta}, \widehat{\Phi}\widehat{\Theta}, \cdots, \widehat{\Phi}^{K-2}\widehat{\Theta}, \widehat{\Phi}^{K-1}\widehat{\Theta}] =: \operatorname{im} R(K).$$

On the other hand, from Proposition 4.33, $x(k) \in \widehat{\mathcal{S}}$ for all $k \geq 0$. Thus, $x(K) \in \widehat{\mathcal{S}} \cap \operatorname{im} R(K)$, and hence $\mathcal{R}_{[0,K]} \subseteq \widehat{\mathcal{S}} \cap \operatorname{im} R(K)$.
Proof of $\mathcal{R}_{[0,K]} \supseteq \widehat{\mathcal{S}} \cap \operatorname{im} R(K)$: pick any $x_f \in \widehat{\mathcal{S}} \cap \operatorname{im} R(K)$. Then, $x_f \in \operatorname{im} R(K)$ implies that there exists a vector $\bar{u} \in \mathbb{R}^{(K \times m) \times 1}$ such that

$$[\widehat{\Theta}, \widehat{\Phi}\widehat{\Theta}, \cdots, \widehat{\Phi}^{K-2}\widehat{\Theta}, \widehat{\Phi}^{K-1}\widehat{\Theta}]\bar{u} = x_f$$

i.e. x_f is reachable (from zero) by considering $\bar{u}$ as the input. Thus, $x_f \in \mathcal{R}_{[0,K]}$, and hence $\widehat{\mathcal{S}} \cap \operatorname{im} R(K) \subseteq \mathcal{R}_{[0,K]}$. □

Remark 6.6 (Reachability from zero ⇔ reachability from arbitrary initial state). Note that reachability from zero on $[0, K]$ is equivalent to **reachability** on $[0, K]$ in the sense that all final states $x_f \in \mathcal{R}_{[0,K]}$ are reachable from any consistent initial state $x(0) \in \widehat{\mathcal{S}}$. This is due to the fact that putting the term containing nonzero initial values x_0 in (4.39) into (6.3) also yields (6.2). In particular, the reachable set is in fact a (linear) subspace in $\mathbb{R}^n$ since $\widehat{\mathcal{S}}$ and $\operatorname{im} R(k)$ are subspaces in $\mathbb{R}^n$. ◊

Lemma 6.7 (Controllability characterization of SLSs). Consider the solvable InhSLS (6.1), and let $\mathcal{C}_{[0,K]}$ be its controllable set to zero on $[0, K]$. Then

$$\mathcal{C}_{[0,K]} = \widehat{\mathcal{S}} \cap \left[\widehat{\Phi}^K\right]^{-1} (\operatorname{im} R(K)) \,. \tag{6.4}$$

In particular, the InhSLS is controllable to zero if, and only if, $\mathcal{C}_{[0,K]} = \widehat{\mathcal{S}}$, or equivalently $\widehat{\mathcal{S}} \subseteq \left[\widehat{\Phi}^K\right]^{-1} (\operatorname{im} R(K))$. ◊

Proof. Consider the solution of (6.1) at $k = K > 0$ with $x(0) = x_0 \in \widehat{\mathcal{S}}$ via (4.39). By setting the solution at $k = K$ as zero we have

$$0 = x(K) = \widehat{\Phi}^K x_0 + [\widehat{\Theta}, \widehat{\Phi}\widehat{\Theta}, \cdots, \widehat{\Phi}^{K-2}\widehat{\Theta}, \widehat{\Phi}^{K-1}\widehat{\Theta}] \begin{bmatrix} u(K-1) \\ u(K-2) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix} . \tag{6.5}$$

Proof of $\mathcal{C}_{[0,K]} \subseteq \widehat{\mathcal{S}} \cap \left[\widehat{\Phi}^K\right]^{-1} (\operatorname{im} R(K))$: pick any controllable to zero state $x_0 \in \mathcal{C}_{[0,K]}$. Then, there exists an input sequence $(u(0), u(1), ..., u(K-1))$ such that (6.5) holds, thus $x_0 \in \left[\widehat{\Phi}^K\right]^{-1} (\operatorname{im} R(K))$. Since $x_0 \in \widehat{\mathcal{S}}$, we have $x_0 \in \widehat{\mathcal{S}} \cap \left[\widehat{\Phi}^K\right]^{-1} (\operatorname{im} R(K))$. Hence $\mathcal{C}_{[0,K]} \subseteq \widehat{\mathcal{S}} \cap \left[\widehat{\Phi}^K\right]^{-1} (\operatorname{im} R(K))$.
Proof of $\mathcal{C}_{[0,K]} \supseteq \widehat{\mathcal{S}} \cap \left[\widehat{\Phi}^K\right]^{-1} (\operatorname{im} R(K))$: pick any $\xi \in \widehat{\mathcal{S}} \cap \left[\widehat{\Phi}^K\right]^{-1} (\operatorname{im} R(K))$. Then, $\widehat{\Phi}^K \xi \in \operatorname{im} R(K)$, which implies that there exists a vector $\bar{u} \in \mathbb{R}^{(K \times m) \times 1}$ such that $R(K)\bar{u} = \widehat{\Phi}^K \xi$ or $0 = \widehat{\Phi}^K \xi - R(K)\bar{u}$ i.e. $x(0) = \xi$ is controllable to zero by considering $-\bar{u}$ as the input. Thus, $\xi \in \mathcal{C}_{[0,K]}$, and hence $\widehat{\mathcal{S}} \cap \left[\widehat{\Phi}^K\right]^{-1} (\operatorname{im} R(K)) \subseteq \mathcal{C}_{[0,K]}$. □

Remark 6.8 (Reachability vs Controllability). In ordinary systems, there are three important observations regarding the relationship between reachability and controllability i.e. (1) reachability implies controllability to zero, (2) controllability to zero does not always imply reachability, and (3) they are equivalent when the state's coefficient matrix is nonsingular [66]. For solvable singular systems, with singular matrix E, the first two statements are still true, however, in contrast, the equivalency between reachability and controllability to zero never happens as the matrix $\widehat{\Phi}$ in (4.36) is always singular. The proof for the first statement is obvious since, in reachability, the zero (final) state is also reachable from any consistent initial value i.e. it is controllable to zero. The second statement is illustrated by the forthcoming Example 6.9 as a counter-example. ◊

Example 6.9. Consider system (6.1) with

$$(E, A, B) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right).$$

Its consistency space is $\widehat{\mathcal{S}} = \operatorname{span}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. It is strongly solvable since, with e.g. $E^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\operatorname{im}[E^+A, E^+B] = \operatorname{im}\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \operatorname{span}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \subset \ker E \oplus \widehat{\mathcal{S}} = \mathbb{R}^3$. With $\Pi_{\widehat{\mathcal{S}}}^{\ker E} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, its corresponding surrogate system (4.36) is given with $\widehat{\Phi} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ and $\widehat{\Theta} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Since for all $K > 0$, $\operatorname{im} R(K) = \operatorname{span}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\mathcal{R}_{[0,K]} = \widehat{\mathcal{S}} \cap \operatorname{im} R(K) = \{0\}$ i.e. the system is not reachable on $[0, K]$ for any $K \geq 0$. However, it is controllable to zero on $[0, K]$ for any $K > 0$; this can be seen from the fact that $\mathcal{C}_{[0,K]} = \widehat{\mathcal{S}} \cap \left[\widehat{\Phi}^K\right]^{-1} (\operatorname{im} R(K)) = \operatorname{span}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \widehat{\mathcal{S}}$. $\Diamond$

6.2 Switched Systems

Recall the state's equation of the InhSLSS (4.44) as follows:

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k), \;\; k \in \mathbb{N} \tag{6.6}$$

and a fixed and known switching signal of interest σ of the form (2.11). Assume it is **strongly solvable** w.r.t. σ in the sense of Definition 4.38. Recall also the individual mode's consistency set

$$\widehat{\mathcal{S}}_i := A_i^{-1}(\operatorname{im}[E_i, B_i]) = \{\xi \in \mathbb{R}^n : A_i\xi \in \operatorname{im}[E_i, B_i]\}.$$

Under the strong solvability assumption w.r.t. the given switching signal, similar to the study for non-switched systems in the previous subsection, the analysis of reachability and controllability for system (6.6) focuses only on solvable systems that are strictly causal, and analysis for the weakly solvable switched system (6.6) can be considered as a future research direction.

6.2.1 Definitions

The reachability and controllability are studied for system (6.6) with respect to a fixed time domain $[0, K]$ in which the system is composed of modes from the set $\{0, 1, ..., \mathrm{p}\}$ and with a mode sequence $\sigma_0, \sigma_1, ..., \sigma_J$, see again Fig. 2.1. The reachability and controllability notions for non-switched systems are generalized for switched systems in the following definitions, which are indeed similar to the notions defined for ordinary switched systems, however, note that the notions are defined with respect to the consistency set $\widehat{\mathcal{S}}_i$ instead of $\mathbb{R}^n$.

Definition 6.10 (Reachability from zero InhSLSSs). A state $x_f \in \widehat{\mathcal{S}}_{\sigma_J}$ of the InhSLSS (6.6) is called **reachable from zero** on $[0, K]$, $K \in (k_J^s, k_{J+1}^s)$ w.r.t. a fixed and known switching signal σ of the form (2.11) if with $x(0) = 0$, there exists an input sequence $(u(0), u(1), .., u(K-1))$ such that the solution $x(K)$ of (6.6) under σ satisfies $x(K) = x_f$. ◊

Definition 6.11 (Reachable set and reachability of InhSLSSs). The reachable set from zero of the InhSLSS (6.6) on $[0, K]$, $K \in (k_J^s, k_{J+1}^s)$ w.r.t. σ of the form (2.11) is the set of all final states $x_f \in \widehat{\mathcal{S}}_{\sigma_J}$ that are reachable from zero on $[0, K]$ and denoted by $\mathcal{R}^{\sigma}_{[0,K]}$. In particular, system (6.6) is called **reachable from zero** on $[0, K]$ w.r.t. σ if $\mathcal{R}^{\sigma}_{[0,K]} = \widehat{\mathcal{S}}_{\sigma_J}$. ◊

The reachability notion above is defined w.r.t. a fixed switching signal and can be further defined w.r.t. a fixed mode sequence or all switching signals by requiring reachability w.r.t. every involved switching signal.

Definition 6.12 (Controllability to zero of InhSLSSs). A consistent initial state $x_0 \in \widehat{\mathcal{S}}_{\sigma_0}$ of (6.6) is called **controllable to zero** on $[0, K]$, $K \in (k_J^s, k_{J+1}^s)$ w.r.t. a fixed and known switching signal σ of the form (2.11) if with $x(0) = x_0$, there exists an input sequence $(u(0), u(1), .., u(K-1))$ such that $x(K)$ of (6.6) under σ satisfies $x(K) = 0$. ◊

Definition 6.13 (Controllable set and controllability of InhSLSSs). The controllable set to zero of the InhSLSS (4.44) on $[0, K]$, $K \in (k_J^s, k_{J+1}^s)$ is the set of all consistent initial states $x_0 \in \widehat{\mathcal{S}}_{\sigma_0}$ which are controllable to zero on $[0, K]$ w.r.t. σ and denoted by $\mathcal{C}^{\sigma}_{[0,K]}$. In particular, system (6.6) is called **controllable to zero** on $[0, K]$ w.r.t. σ if $\mathcal{C}^{\sigma}_{[0,K]} = \widehat{\mathcal{S}}_{\sigma_0}$. ◊

Similar to reachability, controllability can also be defined with respect to a fixed mode sequence or all switching signals by requiring controllability w.r.t. every switching signal with the given mode sequence or all switching signals.

6.2.2 Characterizations: single switch case

The study in this subsection is restricted to only single switch switching signals considered on the finite time domain $[0, K]$, $K \in \mathbb{N}$ of the form (see also Fig. 6.1 for illustration)

$$\sigma(k) = \begin{cases} 0, & 0 \leq k < k^s, \\ 1, & k^s \leq k \leq K. \end{cases} \tag{6.7}$$

Thus, switched systems that are composed of two modes are considered; it starts from mode (E_0, A_0, B_0) with the corresponding consistency space $\widehat{\mathcal{S}}_0$ and switches at the switching time k^s to mode (E_1, A_1, B_1) with the corresponding

consistency space $\widehat{\mathcal{S}}_1$.

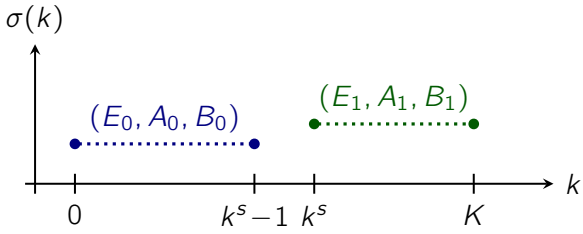

Figure 6.1: Single switch switching signal illustration

This study under single switch switching signals is presented to show detailed characterizations, and the understanding from the single switch case will be used as a foundation to study switched systems under general switching signals.

Let $\mathcal{R}_i(k) = \operatorname{im} R_i(k) = \operatorname{im}\left[\widehat{\Theta}_i, \widehat{\Phi}_i\widehat{\Theta}_i, \cdots, \widehat{\Phi}_i^{k-1}\widehat{\Theta}_i\right]$ for mode $i = 0, 1$, and define the following subspaces

$$\begin{aligned} \mathcal{P}_0 =& \widehat{\mathcal{S}}_0 \cap \mathcal{R}_0(k^s - 1), \\ \mathcal{P}_1 =& \widehat{\mathcal{S}}_1 \cap \left(\widehat{\Phi}_1^{K-k^s}\widehat{\Phi}_{1,0}\mathcal{P}_0 + \operatorname{im}\widehat{\Phi}_1^{K-k^s}\widehat{\Theta}_{1,0} + \mathcal{R}_1(K - k^s)\right). \end{aligned} \tag{6.8}$$

The reachability characterization for InhSLSSs under single switch switching signals is presented in the following theorem in which the subspace $\mathcal{P}_1$ defined above is in fact the reachable set of system (6.6) under the switching signal (6.7).

Theorem 6.14 (Reachability characterization for single switch SLSSs). Consider the solvable InhSLSS (6.6) and let $\mathcal{R}^{\sigma}_{[0,K]}$ be its reachable set on $[0, K]$ w.r.t. the single switch switching signal (6.7). Then $\mathcal{P}_1 = \mathcal{R}^{\sigma}_{[0,K]}$ where $\mathcal{P}_1$ is given by (6.8). In particular, the InhSLSS (6.6) is reachable if, and only if, $\mathcal{P}_1 = \widehat{\mathcal{S}}_1$. $\Diamond$

Proof. From the explicit solution formula (4.66), the solution of (6.6) with $x(0) = 0$ at $k = K > k^s$ can be written as

$$\begin{aligned} x(K) = R_1(K - k^s)\begin{bmatrix} u(K-1) \\ u(K-2) \\ \vdots \\ u(k^s) \end{bmatrix} + \widehat{\Phi}_1^{K-k^s}\widehat{\Theta}_{1,0}u(k^s - 1) \\ + \widehat{\Phi}_1^{K-k^s}\widehat{\Phi}_{1,0}R_0(k^s - 1)\begin{bmatrix} u(k^s-2) \\ u(k^s-3) \\ \vdots \\ u(0) \end{bmatrix}. \end{aligned} \tag{6.9}$$

Proof of $\mathcal{P}_1 \supseteq \mathcal{R}^{\sigma}_{[0,K]}$: Pick any reachable state $x(K) \in \mathcal{R}^{\sigma}_{[0,K]}$. Then, there exists an input sequence $(u(0), u(1), ..., u(K-1))$ and a corresponding solution $x(k)$, $k = 0, 1, \ldots, K-1$ such that (6.9) is satisfied i.e. $x(K) \in \widehat{\Phi}_1^{K-k^s}\widehat{\Phi}_{1,0}\mathcal{R}_0(k^s-1) + \operatorname{im}\widehat{\Phi}_1^{K-k^s}\widehat{\Theta}_{1,0} + \mathcal{R}_1(K-k^s)$. On the other hand, from Theorem 4.40, $x(k)$–the solution of (6.6) at time instant k–satisfies the inclusion $x(k) \in \widehat{\mathcal{S}}_0$ for all $k \in [0, k^s)$ and $x(k) \in \widehat{\mathcal{S}}_1$ for all $k \in [k^s, K]$. Thus, $x(K) \in \widehat{\mathcal{S}}_1 \cap \left(\widehat{\Phi}_1^{K-k^s}\widehat{\Phi}_{1,0}\mathcal{P}_0 + \operatorname{im}\widehat{\Phi}_1^{K-k^s}\widehat{\Theta}_{1,0} + \mathcal{R}_1(K-k^s)\right) = \mathcal{P}_1$, and hence $\mathcal{R}^{\sigma}_{[0,K]} \subseteq \mathcal{P}_1$.
Proof of $\mathcal{P}_1 \subseteq \mathcal{R}^{\sigma}_{[0,K]}$: Pick any $x_f \in \mathcal{P}_1$. Then, there exists a vector $\bar{u} \in \mathbb{R}^{(K\times m)\times 1}$ with the structure $\bar{u} = \begin{bmatrix}\bar{u}_1\\ \bar{u}_2\\ \bar{u}_3\end{bmatrix}$ with $\bar{u}_1 \in \mathbb{R}^{(k^s-1\times m)\times 1}$, $\bar{u}_2 \in \mathbb{R}^{m\times 1}$, and $\bar{u}_3 \in \mathbb{R}^{(K-k^s\times m)\times 1}$ such that

$$R_1(K-k^s)\bar{u}_1 + \widehat{\Phi}_1^{K-k^s}\widehat{\Theta}_{1,0}\bar{u}_2 + \widehat{\Phi}_1^{K-k^s}\widehat{\Phi}_{1,0}R_0(k^s-1)\bar{u}_3 = x_f$$

i.e. x_f is reachable (from zero) by considering $\bar{u}$ as the input. Thus, $x_f \in \mathcal{R}^{\sigma}_{[0,K]}$, and hence $\mathcal{P}_1 \subseteq \mathcal{R}^{\sigma}_{[0,K]}$. Altogether, we get $\mathcal{P}_1 = \mathcal{R}^{\sigma}_{[0,K]}$. □

Remark 6.15 (Reachability from zero ⇔ reachability from any initial consistent state). A similar result as in Remark 6.6 is also derived here in which reachable from zero on $[0, K]$ is equivalent to **reachable** on $[0, K]$ i.e. every $x_f \in \mathcal{R}^{\sigma}_{[0,K]}$ is reachable from any consistent initial value $x_0 \in \widehat{\mathcal{S}}_0$. This can be seen from the fact that putting the term of the solution that contains the nonzero initial value, $\widehat{\Phi}_1^{K-k^s}\widehat{\Phi}_{1,0}\widehat{\Phi}_0^{k^s-1}x_0$, into (6.9) yields the same result. ◊

For the controllability characterization, define the subspaces

$$\begin{aligned} \mathcal{Q}_1 &= \widehat{\mathcal{S}}_1 \cap \left[\widehat{\Phi}_1^{K-k^s}\right]^{-1}\mathcal{R}_1(K-k^s) \quad \text{and} \\ \mathcal{Q}_0 &= \widehat{\mathcal{S}}_0 \cap \left[\widehat{\Phi}_{1,0}\widehat{\Phi}_0^{k^s-1}\right]^{-1}\left[\mathcal{Q}_1 + \widehat{\Phi}_{1,0}\mathcal{R}_0(k^s-1) + \operatorname{im}\widehat{\Theta}_{1,0}\right]. \end{aligned} \tag{6.10}$$

The following theorem reveals that the subspace $\mathcal{Q}_0$ defined above is in fact the controllable set of system (6.6) under the single switch switching signal (6.7).

Theorem 6.16 (Controllability characterization for single switch SLSSs). Consider the strongly solvable InhSLSS (6.6). Let $\mathcal{C}^{\sigma}_{[0,K]}$ be its **controllable set to zero** on $[0, K]$ w.r.t. the single switch switching signal given by (6.7). Then $\mathcal{C}^{\sigma}_{[0,K]} = \mathcal{Q}_0$ where $\mathcal{Q}_0$ is defined in (6.10). In particular, the InhSLSS (6.6) is controllable to zero if, and only if, $\mathcal{Q}_0 = \widehat{\mathcal{S}}_0$. ◊

Proof. Setting the solution at $k = K > k^s$ of (6.6) under the single switch

switching signal (6.7) with $x(0) = x_0 \in \widehat{\mathcal{S}}_0$ as zero provides

$$0 = x(K) = \widehat{\Phi}_1^{K-k^s} x(k^s) + \left[\widehat{\Theta}_1, \widehat{\Phi}_1\widehat{\Theta}_1, \cdots, \widehat{\Phi}_1^{K-k^s-1}\widehat{\Theta}_1\right] \begin{bmatrix} u(K-1) \\ u(K-2) \\ \vdots \\ u(k^s) \end{bmatrix} \quad (6.11)$$

i.e. $x(k^s) \in \left[\widehat{\Phi}_1^{K-k^s}\right]^{-1} \mathcal{R}_1(K-k^s)$. The solution at $k = k^s$ can be written as

$$x(k^s) = \widehat{\Phi}_{1,0}\widehat{\Phi}_0^{k^s-1} x_0 + \widehat{\Phi}_{1,0} R_0(k^s-1) \begin{bmatrix} u(k^s-2) \\ u(k^s-3) \\ \vdots \\ u(0) \end{bmatrix} + \widehat{\Theta}_{1,0} u(k^s-1) \quad (6.12)$$

i.e. $x_0 \in \left[\widehat{\Phi}_{1,0}\widehat{\Phi}_0^{k^s-1}\right]^{-1} \left[\{x(k^s)\} + \widehat{\Phi}_{1,0}\mathcal{R}_0(k^s-1) \ + \operatorname{im}\widehat{\Theta}_{1,0}\right]$.

Proof of $\mathcal{C}^{\sigma}_{[0,K]} \subseteq \mathcal{Q}_0$: pick any controllable to zero state $x_0 \in \mathcal{C}^{\sigma}_{[0,K]}$. Then, there exists an input sequence $(u(0), u(1), ..., u(K-1))$ such that (6.11) holds. Together with the knowledge of $x(k^s) \in \widehat{\mathcal{S}}_1$ from Theorem 4.40, it implies that

$$x(k^s) \in \mathcal{S}_1 \cap \left[\widehat{\Phi}_1^{K-k^s}\right]^{-1} \mathcal{R}_1(K-k^s) = \mathcal{Q}_1$$

and, by the knowledge of $x_0 \in \widehat{\mathcal{S}}_0$, it further implies that

$$x_0 \in \widehat{\mathcal{S}}_1 \cap \left[\widehat{\Phi}_{1,0}\widehat{\Phi}_0^{k^s-1}\right]^{-1} \left[\mathcal{Q}_1 + \widehat{\Phi}_{1,0}\mathcal{R}_0(k^s-1) + \operatorname{im}\widehat{\Theta}_{1,0}\right] = \mathcal{Q}_0.$$

Hence $\mathcal{C}^{\sigma}_{[0,K]} \subseteq \mathcal{Q}_0$.

Proof of $\mathcal{Q}_0 \subseteq \mathcal{C}^{\sigma}_{[0,K]}$: pick any $\xi \in \mathcal{Q}_0$. Then, $\widehat{\Phi}_{1,0}\widehat{\Phi}_0^{k^s-1}\xi = \left[\varsigma + \widehat{\Phi}_{1,0}\mathcal{R}_0(k^s-1)\bar{u}_1 + \widehat{\Theta}_{1,0}\bar{u}_2\right]$, for some $\varsigma \in \mathcal{Q}_1$, $\bar{u}_1 \in \mathbb{R}^{(k^s\times m)\times 1}$ and $\bar{u}_2 \in \mathbb{R}^{m\times 1}$. The inclusion $\varsigma \in \mathcal{Q}_1$ implies that there exists a vector $\bar{u}_3 \in \mathbb{R}^{(K-k^s)\times m\times 1}$ such that $\widehat{\Phi}_1^{K-k^s}\varsigma = \mathcal{R}_1(K-k^s)\bar{u}_3$. Now, take $\bar{u} \in \mathbb{R}^{(K\times m)\times n}$ of the form $\bar{u} = \left[\begin{smallmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{smallmatrix}\right]$. Then with this input, $x(0) = \xi$ is brought to zero on $[0, K]$ i.e. $x(0) = \xi$ is controllable to zero. Thus, $\xi \in \mathcal{C}^{\sigma}_{[0,K]}$, and hence $\mathcal{Q}_0 \subseteq \mathcal{C}^{\sigma}_{[0,K]}$. Altogether, $\mathcal{Q}_0 = \mathcal{C}^{\sigma}_{[0,K]}$. □

6.2.3 Characterizations: general switch case

General switching signals of the form (2.11) are now considered in this subsection. Results from the single switch case are generalized as follows. First, for the reachability characterization, let $\mathcal{R}_i(k) = \operatorname{im} R_i(k) = \operatorname{im}\left[\widehat{\Theta}_i, \widehat{\Phi}_i\widehat{\Theta}_i, \cdots, \widehat{\Phi}_i^{k-1}\widehat{\Theta}_i\right]$ for mode $i \in \{0, 1, ..., \mathrm{p}\}$, and for a fixed switching signal σ of the form (2.11), define the following subspaces:

$$
\begin{aligned}
\mathcal{P}_0 =& \widehat{\mathcal{S}}_0 \cap \mathcal{R}_0(k_1^s - 1),\\
\mathcal{P}_j =& \widehat{\mathcal{S}}_{\sigma_j} \cap \Big(\widehat{\Phi}_{\sigma_j}^{k_{j+1}^s - k_j^s - 1} \widehat{\Phi}_{\sigma_j,\sigma_{j-1}} \mathcal{P}_{j-1} + \operatorname{im} \widehat{\Phi}_{\sigma_j}^{k_{j+1}^s - k_j^s - 1} \widehat{\Theta}_{\sigma_j,\sigma_{j-1}}\\
&\qquad + \mathcal{R}_{\sigma_j}(k_{j+1}^s - k_j^s - 1)\Big) \text{ for } j = 1, 2, \ldots, J,\\
\mathcal{P}_K =& \widehat{\mathcal{S}}_{\sigma_J} \cap \Big(\widehat{\Phi}_{\sigma_J}^{K - k_J^s} \mathcal{P}_J + \mathcal{R}_{\sigma_J}(K - k_J^s)\Big) \text{ for } K \in (k_J^s, k_{J+1}^s).
\end{aligned}
\tag{6.13}
$$

Furthermore, consider the following lemma which says that every state that is reachable from zero at a switching time belongs to $\mathcal{P}_j$ and vice versa. This lemma will be used later to provide the reachability characterization for system (3.1a) under the general switching signal (2.11).

Lemma 6.17 (Reachable states at switching times of InhSLSSs). Consider a strongly solvable InhSLSS (6.6) with the zero initial condition $x(0) = 0$ and the corresponding subspace sequence (6.13). For $j = 1, 2, \ldots, J$, $x_j \in \mathcal{P}_j$ if, and only if, there exists a solution $x(k_j^s)$ of (6.6) and an input sequence $(u(0), u(1), \ldots, u(k_j^s - 1))$ such that $x(k_j^s) = x_j$. ◊

Proof. ($\Longrightarrow$): For $j = 1$, the claim is clear from the proof of the reachability characterization for the single switch case in Theorem 6.14. For $j > 1$, it proceeds inductively as follows. Assume the claim holds for $j - 1$. From

$$x_j \in \mathcal{P}_j \subseteq \Big(\widehat{\Phi}_{\sigma_j}^{k_j^s - k_{j-1}^s - 1} \widehat{\Phi}_{\sigma_j,\sigma_{j-1}} \mathcal{P}_{j-1} + \operatorname{im} \widehat{\Phi}_{\sigma_j}^{k_{j+1}^s - k_j^s - 1} \widehat{\Theta}_{\sigma_j,\sigma_{j-1}} + \mathcal{R}_{\sigma_j}(k_{j+1}^s - k_j^s - 1)\Big),$$

it follows the existence of a vector $x_{j-1} \in \mathcal{P}_{j-1}$ and an input sequence $(u(k_{j-1}^s), u(k_{j-1}^s + 1), \ldots, u(k_j^s - 1))$ with

$$
\begin{aligned}
x_j =& \Phi_{\sigma_j}^{k_{j+1}^s - k_j^s - 1} \widehat{\Phi}_{\sigma_j,\sigma_{j-1}} x_{j-1} + R_{\sigma_j}(k_j^s - k_{j-1}^s - 1) \begin{bmatrix} u(k_j^s - 2) \\ \vdots \\ u(k_{j-1}^s) \end{bmatrix}\\
&+ \widehat{\Phi}_{\sigma_j}^{k_{j+1}^s - k_j^s - 1} \widehat{\Theta}_{\sigma_j,\sigma_{j-1}} u(k_j^s - 1).
\end{aligned}
\tag{6.14}
$$

By inductive assumption, there exists a solution $x(k_{j-1}^s)$ of (4.44) and an input sequence $(u(0), u(1), \ldots, u(k_j^s - 1)$ with $x(k_{j-1}^s) = x_{j-1}$. By Theorem 4.40, setting $x(k_j^s) = x_j$ with x_j given by (6.14) concludes this part of the proof.
($\Longleftarrow$): For $j = 1$ the claim is also clear (by the proof of the single switch case). For $j > 1$, it proceeds again inductively as follows. Consider a solution x of (6.6) with the input sequence $(u(0), u(1), \ldots)$. This implies $x(k_j^s) \in \mathcal{S}_{\sigma_j}$ for $j = 1, 2, \ldots$ (by Theorem 4.40) and, by Corollary 4.43,

$$
\begin{aligned}
x(k_j^s) =& \Phi_{\sigma_j}^{k_{j+1}^s - k_j^s - 1} \widehat{\Phi}_{\sigma_j,\sigma_{j-1}} x(k_{j-1}^s) + \mathcal{R}_{\sigma_j}(k_j^s - k_{j-1}^s - 1) \begin{bmatrix} u(k_j^s - 2) \\ \vdots \\ u(k_{j-1}^s) \end{bmatrix}\\
&+ \widehat{\Phi}_{\sigma_j}^{k_{j+1}^s - k_j^s - 1} \widehat{\Theta}_{\sigma_j,\sigma_{j-1}} u(k_j^s - 1).
\end{aligned}
$$

Using the inductivity assumption, it is known that $x(k_{j-1}^s) \in \mathcal{P}_{j-1}$. Hence

$$x(k_j^s) \in \mathcal{S}_{\sigma_j} \cap \Big(\widehat{\Phi}_{\sigma_j}^{k_{j+1}^s - k_j^s - 1} \widehat{\Phi}_{\sigma_j,\sigma_{j-1}} \mathcal{P}_{j-1} + \operatorname{im} \widehat{\Phi}_{\sigma_j}^{k_{j+1}^s - k_j^s - 1} \widehat{\Theta}_{\sigma_j,\sigma_{j-1}} + \mathcal{R}_{\sigma_j}(k_{j+1}^s - k_j^s - 1)\Big) = \mathcal{P}_j$$

as desired. □

The lemma above tells us that a state at a switching time $k = k_j^s$ is reachable (from zero) if and only if it belongs to the set $\mathcal{P}_j$. Now, to investigate whether a state at time instant $k = K$ with $K \in (k_J^s, k_{J+1}^s)$ is reachable (from zero), it suffices to add the reachable set (from zero) at the switching time $k = k_J^s$ with the reachable states from $x(k_J^s)$ at $k = K$. This is presented in the following theorem, which then provides the reachability characterization for strongly solvable InhSLSSs under general switching signals.

Theorem 6.18 (Reachability characterization of InhSLSSs under general switching signals). Consider the InhSLSS (6.6) and assume it is strongly solvable w.r.t. a fixed and known switching signal σ of the form (2.11). Let $\mathcal{R}_{[0,K]}^{\sigma}$ be its reachable set on $[0, K]$ w.r.t. σ. Then $\mathcal{P}_K = \mathcal{R}_{[0,K]}^{\sigma}$ where $\mathcal{P}_K$ is given by (6.13). In particular, the InhSLSS (6.6) is reachable w.r.t. σ if, and only if, $\mathcal{P}_K = \widehat{\mathcal{S}}_J$. ◊

Proof. From the explicit solution formula (4.67), the solution of (4.44) with $x(0) = 0$ at the time instant $k = K \in (k_J^s, k_{J+1}^s)$ can be written as

$$x(K) = \widehat{\Phi}_{\sigma_J}^{K-k_J^s} x(k_J^s) + R_J(K - k_J^s) \begin{bmatrix} u(K-1) \\ \vdots \\ u(k_J^s) \end{bmatrix} \tag{6.15}$$

Proof of $\mathcal{P}_J \supseteq \mathcal{R}_{[0,K]}^{\sigma}$: Pick any reachable state $x(K) \in \mathcal{R}_{[0,K]}^{\sigma}$. Then, there exists an input sequence $(u(0), u(1), ..., u(K-1))$ and a solution $x(k_J^s)$ such that (6.15) is satisfied, i.e., $x(K) \in \widehat{\Phi}_1^{K-k_J^s} \mathcal{P}_J + \mathcal{R}_J(K - k_J^s)$. On the other hand, from the proof of Theorem 4.40, note that $x(K) \in \widehat{\mathcal{S}}_J$. Thus, $x(K) \in \widehat{\mathcal{S}}_J \cap \Big(\widehat{\Phi}_J^{K-k_J^s} \mathcal{P}_J + \mathcal{R}_J(K - k_J^s)\Big) = \mathcal{P}_K$, and hence $\mathcal{R}_{[0,K]}^{\sigma} \subseteq \mathcal{P}_K$.

Proof of $\mathcal{P}_K \subseteq \mathcal{R}_{[0,K]}^{\sigma}$: From Lemma 6.17, it is already known that any vector $x_J \in \mathcal{P}_J$ is reachable from zero, i.e., there exists an input sequence $\bar{u}_1 = (u(0), u(1), \ldots, u(k_J^s - 1))$ such that $x(k_J^s) = x_J$. Now, pick any $x_K \in \mathcal{P}_K$. Then, there exists a vector $\bar{u}_K \in \mathbb{R}^{((K-k_J^s)\times m)\times 1}$ such that

$$\widehat{\Phi}_1^{K-k^s} x(k_J^s) + R_J(K - k_J^s)\bar{u}_K = x_K$$

i.e. x_K is reachable from zero by considering $\left(\begin{smallmatrix} \bar{u} \\ \bar{u}_K \end{smallmatrix}\right)$ as the input. Thus, $x_K \in \mathcal{R}_{[0,K]}^{\sigma}$, and hence $\mathcal{P}_K \subseteq \mathcal{R}_{[0,K]}^{\sigma}$. Altogether, $\mathcal{P}_K = \mathcal{R}_{[0,K]}^{\sigma}$. □

By the same arguments as in Remark 6.15, reachable from zero on $[0, K]$ w.r.t. σ is equivalent to **reachable** on $[0, K]$ w.r.t. σ, i.e., every $x_f \in \mathcal{R}^{\sigma}_{[0,K]}$ is reachable from any consistent initial value $x_0 \in \widehat{\mathcal{S}}_0$. Now, for the controllability characterization under general switching signals, with a fixed switching signal σ of the form (2.11), define the sequence of subspaces for $j = J-1, J-2, ..., 0$:

$$\begin{aligned}\mathcal{Q}_J =& \widehat{\mathcal{S}}_{\sigma_J} \cap \left[\widehat{\Phi}^{K-k^s_J}_{\sigma_J}\right]^{-1} \mathcal{R}_{\sigma_J}(K - k^s_J),\\ \mathcal{Q}_j =& \widehat{\mathcal{S}}_{\sigma_j} \cap \left[\widehat{\Phi}_{\sigma_{j+1},\sigma_j}\widehat{\Phi}^{k^s_{j+1}-k^s_j-1}_{\sigma_j}\right]^{-1}\\ &\left[\mathcal{Q}_{j+1} + \widehat{\Phi}_{\sigma_{j+1},\sigma_j}\mathcal{R}_{\sigma_j}(k^s_{j+1} - k^s_j - 1) + \operatorname{im}\widehat{\Theta}_{\sigma_{j+1},\sigma_j}\right].\end{aligned} \tag{6.16}$$

By utilizing the sequence of subspaces $\mathcal{Q}_j$ defined above, the controllability characterization for InhSLSSs under general switching signals is presented in the forthcoming Corollary 6.20 with the help of the following lemma.

Lemma 6.19 (Controllable states at switching times of InhSLSSs). Consider a strongly solvable InhSLSS (6.6) on a finite time interval $[0, K]$ with a fixed switching signal σ of the form (2.11) and the corresponding subspace sequence (6.16). For $j = J, J-1, ..., 0$, $x_j \in \mathcal{Q}_j$ if, and only if, there exists a solution $x(k^s_j)$ of (6.6) and an input sequence $(u(k^s_j), u(k^s_j + 1), ..., u(K))$ such that $x(K) = 0$. ◊

Proof. ($\Longrightarrow$): For $j = J$, the claim is clear from the proof of the controllability characterization for the nonswitched case in Lemma 6.7. For $j = J-1$, the claim is also clear from the proof of the controllability characterization for the single switch case in Theorem 6.16. For $j < J-1$, it proceeds inductively as follows. Assume the claim holds for $j+1$. From

$$\begin{aligned}x_j \in \mathcal{Q}_j \subseteq& \left[\widehat{\Phi}_{\sigma_{j+1},\sigma_j}\widehat{\Phi}^{k^s_{j+1}-k^s_j-1}_{\sigma_j}\right]^{-1}\\ &\left[\mathcal{Q}_{j+1} + \widehat{\Phi}_{\sigma_{j+1},\sigma_j}\mathcal{R}_{\sigma_j}(k^s_{j+1} - k^s_j - 1) + \operatorname{im}\widehat{\Theta}_{\sigma_{j+1},\sigma_j}\right],\end{aligned}$$

it follows the existence of a vector $x_{j+1} \in \mathcal{Q}_{j+1}$ and an input sequence $(u(k^s_{j+1}-1), u(k^s_{j+1} - 2), \ldots, u(k^s_j))$ with

$$\begin{aligned}x_j =& \left[\widehat{\Phi}_{\sigma_{j+1},\sigma_j}\widehat{\Phi}^{k^s_{j+1}-k^s_j-1}_{\sigma_j}\right]^{-1}\left[\left\{x_{j+1} + \widehat{\Phi}_{\sigma_{j+1},\sigma_j}R_{\sigma_j}(k^s_{j+1} - k^s_j - 1)\right.\right.\\ &\left.\left.\begin{bmatrix} u(k^s_{j+1}-2)\\ \vdots\\ u(k^s_j)\end{bmatrix} + \widehat{\Theta}_{\sigma_{j+1},\sigma_j}u(k^s_{j+1} - 1)\right\}\right].\end{aligned} \tag{6.17}$$

By inductive assumption, there exists a solution $x(k^s_{j+1})$ of (4.44) and an input sequence $(u(k^s_{j+1}), u(k^s_{j+1} + 1), \ldots, u(K-1))$ with $x(k^s_{j+1}) = x_{j+1}$ such that $x(K) = 0$. By Theorem 4.40, setting $x(k^s_j) = x_j$ with x_j given by (6.17)

concludes this part of the proof.
($\Longleftarrow$): The claim for $j = J$ and $j = J-1$ is clear also from the proof of the nonswitched case in Lemma 6.7 and the single switch case in Theorem 6.16 respectively. For $j < J-1$, it again proceeds inductively as follows. Assume the claim holds for $j+1$. Consider a solution x of (6.6) with the input sequence $(u(k_j^s), u(k_j^s+1), \ldots, u(K-1))$ such that $x(K) = 0$. This implies $x(k_j^s) \in \mathcal{S}_{\sigma_j}$ (by Theorem 4.40) and

$$x(k_j^s) = \Big[\widehat{\Phi}_{\sigma_{j+1},\sigma_j}\widehat{\Phi}_{\sigma_j}^{k_{j+1}^s-k_j^s-1}\Big]^{-1}\Bigg[\Bigg\{x(k_{j+1}^s) + \widehat{\Phi}_{\sigma_{j+1},\sigma_j}R_{\sigma_j}(k_{j+1}^s - k_j^s - 1)$$
$$\begin{bmatrix} u(k_{j+1}^s-2) \\ \vdots \\ u(k_j^s) \end{bmatrix} + \widehat{\Theta}_{\sigma_{j+1},\sigma_j}u(k_{j+1}^s - 1)\Bigg\}\Bigg].$$

(by Corollary 4.43). Using the inductivity assumption, it is known that $x(k_{j+1}^s) \in \mathcal{Q}_{j+1}$. Hence

$$x(k_j^s) \in \mathcal{S}_{\sigma_j} \cap \big[\widehat{\Phi}_{\sigma_{j+1},\sigma_j}\widehat{\Phi}_{\sigma_j}^{k_{j+1}^s-k_j^s-1}\big]^{-1}\big[\mathcal{Q}_{j+1} + \widehat{\Phi}_{\sigma_{j+1},\sigma_j}\mathcal{R}_{\sigma_j}(k_{j+1}^s - k_j^s - 1)$$
$$+ \operatorname{im}\widehat{\Theta}_{\sigma_{j+1},\sigma_j}u(k_{j+1}^s - 1)\big] = \mathcal{Q}_j$$

which completes the proof. □

Corollary 6.20 (Controllability characterization of InhSLSSs under general switching signals). Consider the InhSLSS (6.6) and assume it is strongly solvable w.r.t. a fixed and known switching signal σ of the form (2.11). Let $\mathcal{C}_{[0,K]}^{\sigma}$ be its **controllable set to zero** on $[0, K]$ w.r.t. σ. Then

$$\mathcal{C}_{[0,K]}^{\sigma} = \mathcal{Q}_0 \tag{6.18}$$

where $\mathcal{Q}_0$ is defined in (6.16). In particular, the InhSLSS (6.6) is controllable to zero if, and only if, $\mathcal{Q}_0 = \widehat{\mathcal{S}}_0$. ◊

6.2.4 Discussion on Constant Reachability and Controllability

For the completeness of the study, constant reachability and controllability are also discussed here as the counterpart of the study for observability and determinability. Consider the following definition for constant reachability and controllability, which is the analog of the constant observability and determinability in Definition 5.21.

Definition 6.21 (Constant Reachability/Controllability of InhSLSSs). The reachability (controllability) of the InhSLSS (6.6) is called **constant** (under slow switching) w.r.t. a mode sequence (σ_j) of the form (2.11) if it is either reachable (controllable) on $[0, K]$ for all $\sigma \in \mathbb{S}_{(\sigma_j)}^{[n]}$ and all $K \geq (J+1)n+1$ or

unreachable (uncontrollable) on $[0, K]$ for all $\sigma \in \mathbb{S}^{[n]}_{(\sigma_j)}$ and all $K \geq (J+1)n+1$. ◊

Whereas for two-dimensional systems, it has been proved that observability and determinability are constant, it is still not clear whether reachability and controllability are also constant. However, based on the observation of the solution trajectories in which any solution vector of two-dimensional system (3.1a) with singular E_i has only one nonzero component at the most, it leads to the hypothesis of constant reachability and controllability.

Meanwhile, for single-switch systems, since the characterizations of reachability and controllability contain nested intersections, similar challenges as in the determinability of ordinary switched systems and singular-switched systems also happen in the attempt of proving constant reachability and controllability. However, based on similar observations as in ordinary systems, it is also conjectured that the reachability and controllability of single-switch systems are also constant.

Conjecture 6.22 (Two-dimensional and single-switch InhSLSSs have constant reachability and controllability). The reachability and controllability of two-dimensional and single-switch InhSLSS (6.6) with singular E_i are constant w.r.t. all mode sequences. ◊

To illustrate the reachability and controllability analysis, some academic examples are discussed in the following. Some systems in the examples also illustrate some important observations related to reachability and controllability, for example, the reachability or controllability of individual modes does not guarantee the reachability or controllability of switched systems composed of those modes.

Example 6.23. Recall the switched system in Example 4.49 which is strongly solvable w.r.t. every switching signal with the mode sequence $(0, 1)$. The corresponding surrogate system w.r.t. the mode sequence $(0, 1)$ with the (single) switching time k^s, $\widehat{\mathcal{S}}_0 = \mathbb{R}^3$, and $\widehat{\mathcal{S}}_1 = \operatorname{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$ is given by

$$\begin{aligned}
k < k^s: &\quad x(k+1) = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} x(k) + \begin{bmatrix}0\\1\\0\end{bmatrix} u(k)\\
k = k^s: &\quad x(k+1) = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} x(k) + \begin{bmatrix}0\\1\\0\end{bmatrix} u(k)\\
k > k^s: &\quad x(k+1) = \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix} x(k) + \begin{bmatrix}0\\0\\0\end{bmatrix} u(k).
\end{aligned}$$

Each mode as an individual system is unreachable and uncontrollable. Under the mode sequence $(0, 1)$ with the switching time $n < k^s < K - n$, the switched system is always unreachable and uncontrollable on $[0, K]$ with $K = 11$ with any switching time k^s. ◊

Example 6.24. Recall the switched system in Example (4.50) which is strongly solvable w.r.t. all switching signals. Consider the switched system that starts with mode-0 and switches at k_1^s to mode-1 and switches to mode-2 at k_2^s. All modes as individual systems are unreachable. Modes-0 and 1 are controllable, however, mode-2 is uncontrollable. With the mode sequence $(0, 1, 2)$ and with the switching times k_1^s and k_2^s, the switched system is unreachable but controllable on $[0, K]$ with $K = 20$ w.r.t. switching signals with any possible k_1^s and k_2^s. $\Diamond$

6.3 Concluding Remarks

Reachability and controllability notions for strongly solvable inhomogeneous singular linear switched systems in discrete time have been introduced in this chapter. Necessary and sufficient conditions have also been established to characterize those two notions. The characterizations utilized the surrogate systems established in Chapter 4.

Regarding the independence of those properties on switching times, it is still not clear whether the reachability and controllability for some particular systems are independent of switching times. Nevertheless, based on similar observations as in ordinary systems, it is conjectured that those properties for two-dimensional and single-switch systems do not depend on switching times.

Part III

Singular Nonlinear (Switched) Systems

Contents of this part are based on the following papers:

- **Sutrisno** and Stephan Trenn, "Nonlinear Switched Singular Systems in Discrete Time: The One-step Map and Stability Under Arbitrary Switching Signals," *European Journal of Control*, in press. `https://doi.org/10.1016/j.ejcon.2023.100852`
- **Sutrisno**, Hao Yin, Stephan Trenn and Bayu Jayawardhana, "Nonlinear singular switched systems in discrete-time: solution theory and incremental stability under restricted switching signals," in *Proc. 62nd IEEE Conference on Decision and Control (CDC)*, 2023.

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Chapter 7 | Singular Nonlinear (Switched) Systems

"In life, switching may result in an unstable situation but it is always worth a try."

Contents

Extensions to singular nonlinear (switched) systems are presented in this chapter based on results from studies for singular linear (switched) systems discussed in the previous chapters. The key idea in establishing the solvability characterization, one-step map, and surrogate systems in linear systems is utilized again to study the solvability and formulate surrogate systems for singular nonlinear (switched) systems.

7.1 Solvability

Singular nonlinear systems also have the three solvability issues as in singular linear systems discussed in the introduction part of Chapter 4. To deal with this, solution theory is studied in this section. New solvability notions will be

introduced based on solvability notions for linear systems with some changes to the consistency set.

7.1.1 Nonswitched Systems

Consider first non-switched Singular Nonlinear Systems (SNSs) of the form

$$Ex(k+1) = F(x(k)),\ \ k = 0, 1, \ldots \tag{7.1}$$

where $x \in \mathbb{R}^n$ is the state, $E \in \mathbb{R}^{n\times n}$ is singular with $\operatorname{rank} E = r < n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous nonlinear function. For system (7.1), define the set $\widetilde{\mathcal{S}} := \{\ x \in \mathbb{R}^n \mid F(x) \in \operatorname{im} E\ \}$.

The solvability notion for singular linear systems in Definition 4.1 is carried over with a change in the consistency set from $\widetilde{\mathcal{S}}$ to $\widetilde{\mathcal{S}}$. Note that for linear systems, $\widetilde{\mathcal{S}}$ is a (sub)space of $\mathbb{R}^n$, meanwhile, for nonlinear systems, $\widetilde{\mathcal{S}}$ is in general a manifold. However, the basic idea for the solvability is still the same i.e. the system is said to be solvable if a unique solution exists for any arbitrary finite time domain and for any arbitrary consistent initial value $x(0) \in \widetilde{\mathcal{S}}$, and in particular, the solution is desired to be causal in terms of states. This is formally defined as follows:

Definition 7.1 (Solvability of SNSs). The SNS (7.1) is called **locally uniquely solvable** (for short just **solvable**) if, for all $k \in \mathbb{N}$ and for all $x_0 \in \widetilde{\mathcal{S}}$ there exists a unique solution on $[0, k]$ of (7.1) considered on $[0, k]$ with $x(0) = x_0$. $\Diamond$

A similar fact as in singular linear systems regarding the local solvability notion above also happens here, i.e. this solvability notion is stronger compared to the common solvability notion for ordinary systems where the unique solution is required on $[0, \infty)$ for all (consistent) initial values. However, having the former solvability notion will guarantee the existence of the one-step map and its corresponding surrogate system for system (7.1), and it is not always possible to have a one-step map for the latter solvability notion (see the forthcoming Remark 7.8). Furthermore, note that every non-singular system (i.e. E is non-singular) is locally solvable, in fact, solutions are already uniquely determined on $[0, k]$ by only considering (7.1) on $[0, k-1]$. This is in contrast to the singular case, where the algebraic constraints at k are usually needed to determine uniquely the value of $x(k)$.

From basic algebra, for any singular matrix E, there exist invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that $SET = \left[\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right]$. By using the state transformation $T^{-1}x(k) = \left(\begin{smallmatrix} v(k) \\ w(k) \end{smallmatrix}\right)$, $v \in \mathbb{R}^r$, $w \in \mathbb{R}^{n-r}$, system (7.1) can be rewritten as

$$\left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} v(k+1) \\ w(k+1) \end{smallmatrix}\right] = SF\left(T \left[\begin{smallmatrix} v(k) \\ w(k) \end{smallmatrix}\right]\right) =: \left[\begin{smallmatrix} G(v(k),w(k)) \\ H(v(k),w(k)) \end{smallmatrix}\right]. \tag{7.2}$$

The representation above decouples (7.1) into pure ordinary subsystem in v and pure singular subsystem or algebraic constraint in w. In this decoupled representation, system (7.1) can be illustrated by Fig. 7.1.

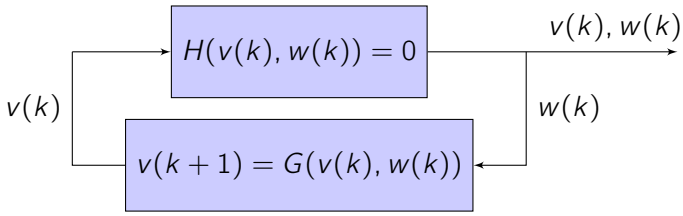

Figure 7.1: Block diagram of (7.2)

To be able to ensure existence and uniqueness of a solution of the switched system (7.1), the following assumption is considered.

Assumption 7.2 (The consistency set of SNSs is assumed to be a linear subspace). The set $\widetilde{\mathcal{S}} := \{ x \in \mathbb{R}^n \mid F(x) \in \operatorname{im} E \}$ of (7.1) is a linear subspace in $\mathbb{R}^n$. ◊

At first glance, this assumption looks rather restrictive, however, in general, the set $\widetilde{\mathcal{S}}$ is a differentiable manifold at least locally and then a (local) nonlinear coordinate transformation can be applied to obtain a linear subspace $\widetilde{\mathcal{S}}$.

Remark 7.3. The algebraic constraint $H(v, w) = 0$ in (7.2) is, in general, nonlinear even if $\widetilde{\mathcal{S}}$ is a subspace. However, if $\widetilde{\mathcal{S}}$ is a subspace in $\mathbb{R}^n$, then there exists a matrix K such that $\widetilde{\mathcal{S}} = \ker K$; in particular, for $[P, Q] := KT$ where T is the coordinate transformation as in (7.2), $H(v, w) = 0$ if, and only if, $Pv + Qw = 0$. Thus, for every $k \in \mathbb{N}$, the nonlinear algebraic constraint $H(v(k), w(k)) = 0$ can be replaced by the linear algebraic constraint

$$0 = Pv(k) + Qw(k). \tag{7.3}$$

As a consequence, the nonlinearity appears now only on $G(v, w)$. To find such matrices P and Q, take a matrix K such that $\widetilde{\mathcal{S}} = \ker K$, and by using the coordinate transformation T as in (7.2) we have for $[P, Q] := KT$ that $H(v, w) = 0$ if, and only if, $Pv + Qw = 0$. ◊

The following lemma provides two characterizations for the solvability of system (7.1) under Assumption 7.2.

Lemma 7.4 (Solvability Characterization for SNSs). The following statements are equivalent:

(i) System (7.1) under Assumption 7.2 is solvable in the sense of the Definition 7.1

(ii) The matrix Q in (7.3) is nonsingular

(iii) $\mathcal{T} \subseteq \ker E \oplus \widetilde{\mathcal{S}}$ where $\mathcal{T} = \{E^+F(\varsigma) \mid \varsigma \in \widetilde{\mathcal{S}}\}$, i.e. $\mathcal{T}$ is the range of $\tau : \widetilde{\mathcal{S}} \to \mathbb{R}^n$ with $\tau(\varsigma) = E^+F(\varsigma)$. ◊

Proof. (i) ⇒ (ii): The set $\widetilde{\mathcal{S}}$ being a subspace implies the existence of the equivalent linear algebraic constraint of the form (7.3), hence system (7.1) can equivalently be rewritten as

$$\begin{cases} v(k+1) = G(v(k), w(k)), \ k = 0, 1, \dots \\ \qquad\quad 0 = Pv(k) + Qw(k) \end{cases}$$

Consider this system on $[0, 1]$, then it reads

$$\begin{array}{r|l} v(1) = G(v(0), w(0)) & v(2) = G(v(1), w(1)) \\ 0 = Pv(0) + Qw(0) & 0 = Pv(1) + Qw(1) \end{array}$$

where $(v(0), w(0))$ is given, and thus $v(1)$ is also given. Seeking a contradiction assume that the square matrix Q is singular. Then it is first of all not guaranteed anymore that for the specific $v(1)$ a solution $w(1)$ exists with $0 = Pv(1) + Qw(1)$. If $w(1)$ exists at all it is not unique because Q has a nontrivial kernel. Hence we have non-existence or non-uniqueness of solutions of (7.1) considered on the interval $[0, 1]$, contradicting (i).
(ii) ⇒ (i): Nonsingularity of Q implies that the algebraic constraints are equivalent to $w(k) = Q^{-1}Pv(k)$, which then leads to the uniquely solvable nonsingular system $v(k+1) = \bar{G}(v(k))$ with $\bar{G}(v) = G(v, Q^{-1}Pv)$. Transforming this unique solution back to its original coordinates provides a unique solution x on any interval $[0, k]$.
(i) ⇒ (iii): By assumption for any initial value x_0 there exists a unique solution on $[0, 1]$, in particular $x(1)$ is uniquely determined by considering (7.1) for $k = 0$ and $k = 1$. By Lemma A.2 applied to (7.1) for $k = 0$ the value $x(1)$ satisfies

$$x(1) \in E^{-1}(F(x_0)) = \{E^+F(x_0)\} + \ker E. \tag{7.4}$$

On the other hand, considering (7.1) at $k = 1$ (not making any assumptions about the unknown $x(2)$, the state $x(1)$ must satisfy

$$x(1) \in \{x \in \mathbb{R}^n | F(x) \in \operatorname{im} E\} = \widetilde{\mathcal{S}}. \tag{7.5}$$

Hence $x(1)$ is uniquely determined for all $x_0 \in \widetilde{\mathcal{S}}$ if, and only if, $\widetilde{\mathcal{S}} \cap (\{E^+F(x_0)\} + \ker E)$ is a singleton. Lemma A.3 with $\mathbb{Z} = \{0\}$, $\mathbb{U} = \mathcal{T}$, $\mathcal{V} = \widetilde{\mathcal{S}}$, and $\mathcal{W} = \ker E$ concludes (iii).
(iii) ⇒ (i): This is proved inductively, that if for any $x_0 \in \widetilde{\mathcal{S}}$ there exists a unique solution on $[0, k]$, then there also exists a unique solution on $[0, k+1]$. This together with the trivial observation that $x(0) = x_0$ is the unique solution of (7.1), $x(0) = x_0$, considered only for $k = 0$ will prove (i). Now,

given $x(k)$, we choose $x(k+1) \in \widetilde{\mathcal{S}} \cap (\{E^+F(x(k))\} + \ker E)$ which is possible due to Lemma A.3. Then $x(k+1) \in \{E^+F(x(k))\} + \ker E$ implies that $Ex(k+1) = EE^+F(x(k))$. Since $x(k) \in \widetilde{\mathcal{S}}$ (because x is a solution on $[0,k]$), it follows that $F(x(k)) \in \operatorname{im} E$, i.e. there exists v such that $F(x(k)) = Ev$. Hence $Ex(k+1) = EE^+Ev = Ev = F(x(k)$ which shows that $x(k+1)$ satisfies (7.1). Furthermore, $x(k+1)$ also satisfies (7.1) for $k+1$ because $x(k+1) \in \widetilde{\mathcal{S}}$. This shows that x is indeed a solution of (7.1) on $[0,k+1]$. Uniqueness follows from the fact, that by Lemma A.3 the set $\widetilde{\mathcal{S}} \cap (\{E^+F(x(k))\} + \ker E)$ is a singleton. □

Lemma 7.4 provides two alternatives for checking whether system (7.1) is solvable. The condition (ii) requires, first, transforming the original system into (7.2)'s form, and then finding Q by using Remark 7.3. Meanwhile, the condition (iii) uses data from the original system directly, which requires fewer computation steps. In particular, using formula (A.1) in Lemma A.3 and the same arguments as in the proof for Lemma 7.4 we arrive at the following one-step map that allows one to obtain an equivalently surrogate ordinary system for (7.1):

Corollary 7.5 (The one-step map and surrogate systems of SNSs). Consider the SNS (7.1) under Assumption 7.2. If solvable, its solution satisfies the following surrogate system for $k = 0, 1, \ldots$

$$x(k+1) = \Phi(x(k)) = \Pi_{\widetilde{\mathcal{S}}}^{\ker E} E^+F(x(k)),\ x(0) = x_0 \in \widetilde{\mathcal{S}} \tag{7.6}$$

where E^+ is a generalized inverse of E and $\Pi_{\widetilde{\mathcal{S}}}^{\ker E}$ is the canonical projector from $\ker E \oplus \widetilde{\mathcal{S}}$ to $\widetilde{\mathcal{S}}$. Furthermore, any solution of (7.6) with $x(0) \in \widetilde{\mathcal{S}}$ also solves (7.1). In particular, $x(k) \in \widetilde{\mathcal{S}}$ for all $k \in \mathbb{N}$. ◊

The function $\Phi(x(k))$ is the one-step map for the SNS (7.1), and the ordinary nonlinear system (7.6) is the surrogate system for (7.1). Since the original singular system (7.1) is solvable only with the initial condition $x(0) = x_0 \in \widetilde{\mathcal{S}}$, the surrogate system (7.6) is then considered only with this initial condition although, as an ordinary system, it might also be solvable with some initial condition $x(0) = \widetilde{x}_0 \notin \widetilde{\mathcal{S}}$.

Remark 7.6 (The nonuniqueness of generalized inverses). Note that the generalized inverse matrix E^+, in general, is not unique, and thus applying different E^+ could provide different $\mathcal{T}$ in Lemma 7.4 and different one-step maps in the surrogate system (7.6). However, condition (iii) in Lemma 7.4 as well as the restriction of Φ on $\widetilde{\mathcal{S}}$ will give the same results regardless of the choice of E^+ used in the calculation, i.e., the nonuniqueness of E^+ has no effect on the solution characterization/formula; the justification for this

statement is similar to the arguments for linear systems in Chapter 4; however, for completeness, the proof for the nonlinear system (7.1) is provided as follows. On the one hand, $\{F(\varsigma) \mid \varsigma \in \widetilde{\mathcal{S}}\} = \{F(\zeta) \mid \zeta \in \mathbb{R}^n\} \cap \operatorname{im} E \subseteq \operatorname{im} E$. On the other hand, for any two different generalized inverses E_1^+ and E_2^+ of E, $(E_1^+ - E_2^+)y \in \ker E$ for all $y \in \operatorname{im} E$. Altogether, the difference between two different $\mathcal{T}_1$ and $\mathcal{T}_2$ which corresponds to two different generalized inverses E_1^+ and E_2^+ respectively is contained in $\ker E$, i.e., the action of $\Phi(x)$ in (7.6) is unique when restricted to the relevant subspace. Thus, choosing different generalized inverse matrices results in the same solution. The well-known Moore-Penrose inverse, which can be easily computed using the singular value decomposition, can also be utilized here to calculate the generalized inverse matrix. ◊

By utilizing the one-step map Φ and its corresponding surrogate system (7.6), now it is possible to write the explicit solution of (7.1) with the initial condition $x(0) = x_0 \in \widetilde{\mathcal{S}}$, i.e.,

$$x(k) = \underbrace{(\Phi \circ \Phi \circ \cdots \circ \Phi)}_{k \text{ times}}(x_0)$$

where $\Phi(\cdot)$ is as given in (7.6). The following example illustrates the solution theory above.

Example 7.7. Consider the SNS (7.1) with

$$(E, F(x)) = \left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} x_1^{\frac{1}{3}} + x_2^{\frac{1}{3}} \\ x_1^{\frac{1}{3}} - x_2^{\frac{1}{3}} \end{bmatrix} \right)$$

with

$$\ker E = \operatorname{span}\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \text{ and } \widetilde{\mathcal{S}} = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{pmatrix} x_1^{\frac{1}{3}} + x_2^{\frac{1}{3}} \\ x_1^{\frac{1}{3}} - x_2^{\frac{1}{3}} \end{pmatrix} \in \operatorname{span}\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \right\} = \operatorname{span}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right).$$

Since $\ker E \oplus \widetilde{\mathcal{S}} = \mathbb{R}^n$, the condition (iii) in Lemma 7.4 is satisfied (independently of what $\mathcal{T}$ is), and thus this system is solvable and has a unique solution for every initial value $x_0 \in \widetilde{\mathcal{S}} = \operatorname{span}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Furthermore, with $\Pi_{\widetilde{\mathcal{S}}}^{\ker E} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $E^+ = \left[\begin{smallmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{smallmatrix}\right]$, the one-step map is given by $\Phi(x) = \left(\begin{smallmatrix} x_1^{\frac{1}{3}} \\ 0 \end{smallmatrix}\right)$ and each solution satisfies the surrogate system

$$x(k+1) = \left[\begin{smallmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{smallmatrix}\right] F(x(k)) = \left(\begin{smallmatrix} x_1(k)^{\frac{1}{3}} \\ 0 \end{smallmatrix}\right). \qquad ◊$$

Remark 7.8 (Discussion on the local solvability notion for SNSs). It is not always possible to establish a one-step map for system (7.1) if only global solvability on $[0, \infty)$ is assumed instead of the local solvability in the sense of

Definition 7.1. This is illustrated by the following "counter-example":

$$\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] x(k+1) = \begin{pmatrix} x_1(k)^{\frac{1}{3}} \\ x_2(k)^{\frac{1}{3}} \end{pmatrix}, \ k = 0, 1, \ldots \tag{7.7}$$

with $\widetilde{\mathcal{S}} = \operatorname{im}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. For this system, considered on $[0, \infty)$, the unique solution is given by $x(k) = 0$ for all $k > 0$, because $x_2(k) = 0$ for all k and $x_1(k) = x_2(k+1) = 0$ for all k. However, if we consider the system on $[0, 1]$, the system has a non-unique local solution because $x_1(1)$ can be arbitrary. This, in particular, shows that the solvability on $[0, \infty)$ does not imply the solvability in the sense of Definition 7.1, however, the converse is indeed always true. Now, since $x_1(1)$ is free, we cannot determine it only from the current and past information, and thus the one-step map, which depends only on the current and past information, cannot exist. Therefore, the solvability notion given in Definition 7.1 is necessary for the existence of the one-step map, which in turn is needed to study switched systems (where at a given time k it may not be clear yet what the mode at $k+1$ will be. ◊

7.1.2 Switched Systems

Consider now the following Singular Nonlinear Switched System (SNSS) where each mode is the SNS (7.1):

$$E_{\sigma(k)} x(k+1) = F_{\sigma(k)}(x(k)) \tag{7.8}$$

where σ is the switching signal of the form (2.11), $E_i \in \mathbb{R}^{n \times n}$ are singular and $F_i : \mathbb{R}^n \to \mathbb{R}^n$ are continuous nonlinear functions. We refer to the pair (E_i, F_i) as the mode-i. Define for each mode $i \in \{0, 1, \ldots, \mathrm{p}\}$ the set $\widetilde{\mathcal{S}}_i := \{\ x \in \mathbb{R}^n \mid F_i(x) \in \operatorname{im} E_i\ \}$. The solvability notion for non-switched systems in Definition 7.1 is generalized for switched systems w.r.t. a switching signal as follows:

Definition 7.9 (Solvability notion for SNSSs). The SNSS (7.8) is called **locally uniquely solvable** (for short just **solvable**) w.r.t. a fixed and known switching signal σ of the form (2.11) if, for all $k_0, k_1 \in \mathbb{N}$, $k_1 > k_0$ and all $x_{k_0} \in \widetilde{\mathcal{S}}_{\sigma(k_0)}$ there exists a unique solution of (7.8) under σ considered on $[k_0, k_1]$ with $x(k_0) = x_{k_0}$. ◊

For the given switching signal, this solvability notion requires the existence of a unique solution considered on any time interval with any arbitrary initial time instant and, furthermore, for any consistent initial value at that initial time instant. In particular, the SNSS (7.8) is solvable w.r.t. the mode sequence $(\sigma_0, \sigma_1, \ldots)$ if it is solvable w.r.t. all switching signals with the mode sequence $(\sigma_0, \sigma_1, \ldots)$ and with arbitrary switching times. As for non-switched systems, the subspace assumption in Assumption 7.2 is also considered here for all modes

composing the switched system. The reason for considering this assumption for switched system (2.14) is similar to the reason for having Assumption 7.2 for nonswitched systems.

Assumption 7.10 (The consistency sets of all modes are subspaces). For every $i \in \{0, 1, ..., \mathrm{p}\}$, $\widetilde{\mathcal{S}}_i$ is a linear subspace in $\mathbb{R}^n$. ◊

A similar observation in linear systems also happens here where solvability for individual modes is, in general, not sufficient for switched systems composed of those modes to be solvable, see the system in Example 7.14 for justification for this. The solvability characterization under the given assumption above is presented in the following theorem.

Theorem 7.11 (Solvability characterization of SNSSs w.r.t. a fixed switching signal). The SNSS (7.8) under Assumption 7.10 is solvable w.r.t. a fixed and known switching signal $\sigma : \mathbb{N} \to \{0, 1, ..., \mathrm{p}\}$ of the form (2.11) in the sense of Definition 7.9 if, and only if,

$$\mathcal{T}_{\sigma(k)} \subseteq \ker E_{\sigma(k)} \oplus \widetilde{\mathcal{S}}_{\sigma(k+1)} \quad \text{for } k = 0, 1, 2, \ldots \tag{7.9}$$

where $\mathcal{T}_i = \left\{ E_i^+ F_i(\varsigma) \;\middle|\; \varsigma \in \widetilde{\mathcal{S}}_i \right\}$. Furthermore, if it is solvable, its solution at any time instant $k \in \mathbb{N}$ satisfies the following surrogate ordinary system

$$x(k+1) = \Phi_{\sigma(k+1),\sigma(k)}(x(k)), \; x(0) \in \widetilde{\mathcal{S}}_{\sigma(0)} \tag{7.10}$$

where $\Phi_{i,j}$ is called the one-step map from mode-j to mode-i given by

$$\Phi_{i,j}(x(k)) := \Pi_{\widetilde{\mathcal{S}}_i}^{\ker E_j} E_j^+ F_j(x(k)), \tag{7.11}$$

the matrix E_j^+ is a generalized inverse of E_j and $\Pi_{\widetilde{\mathcal{S}}_i}^{\ker E_j}$ is the canonical projector from $\widetilde{\mathcal{S}}_i \oplus \ker E_j$ to $\widetilde{\mathcal{S}}_i$. In particular, $x(k) \in \widetilde{\mathcal{S}}_{\sigma(k)}$ for all $k \in \mathbb{N}$. ◊

Proof. Due to Assumption 7.10, the proof is similar to the proof of the solvability for linear singular switched systems in Chapter 4 where the solvability condition (7.9) in nonlinear systems corresponds to the switched index-1 condition in linear systems. For completeness, the complete proof is briefly provided as follows.

Step 1: The solvability characterization

Necessity: Consider a solution on an arbitrary interval $[k, k+1]$ and, w.l.o.g., let $\sigma(k) = j$ and $\sigma(k+1) = i$. For a given $x(k) \in \widetilde{\mathcal{S}}_i$, in order to have a unique $x(k+1)$ w.r.t. the given switching signal, the following system of equations must have a unique solution for $x(k+1)$:

$$E_j x(k+1) = F_j(x(k)), \tag{7.12a}$$

$$E_i x(k+2) = F_i(x(k+1)), \tag{7.12b}$$

Equation (7.12a) is equivalent to $x(k+1) \in E_j^{-1} F_j(x(k))$ which by Lemma A.2 is equivalent to

$$x(k+1) \in \{E_j^+ F_j(x(k))\} + \ker E_j. \tag{7.13}$$

Since the solution is considered only on $[k, k+1]$, the value $x(k+2)$ in (7.12b) is arbitrary, hence Equation (7.12b) is equivalent to

$$x(k+1) \in \{x \in \mathbb{R}^n : F_i(x) \in \operatorname{im} E_i\} = \widetilde{\mathcal{S}}_i. \tag{7.14}$$

Altogether,

$$x(k+1) \in \left(\{E_j^+ F_j(x(k))\} + \ker E_j\right) \cap \widetilde{\mathcal{S}}_i. \tag{7.15}$$

By applying $\mathbb{Z} = \{0\}$, $\mathbb{U} = \mathcal{T}_j$, $\mathcal{V} = \widetilde{\mathcal{S}}_i$ and $\mathcal{W} = \ker E_j$ to Lemma A.3, the uniqueness of $x(k+1)$ implies $\mathcal{T}_j \subseteq \ker E_j \oplus \widetilde{\mathcal{S}}_i$. Since $[k, k+1]$ is arbitrary, this condition must hold for all k and thus (7.9) must hold.
Sufficiency: Identical arguments as for the non-switched case allow one to inductively extend any solution x on $[0, k]$ uniquely to a solution on $[0, k+1]$ if (7.9) holds.
Step 2: One-step map and surrogate system (7.10)
By applying formula (A.1) in Lemma A.3 to (7.15) with $\mathbb{Z} = \{0\}$, $\mathbb{U} = \{E_{\sigma(k)}^+ F_{\sigma(k)}(x(k))\}$, $\mathcal{V} = \widetilde{\mathcal{S}}_{\sigma(k+1)}$ and $\mathcal{W} = \ker E_{\sigma(k)}$, the solution $x(k+1)$ satisfies (7.10). Finally, the inclusion $x(k) \in \widetilde{\mathcal{S}}_{\sigma(k)}$ for all $k \in \mathbb{N}$ is a direct consequence of the solution $x(k)$ satisfying $E_{\sigma(k)}\xi = F_{\sigma(k)}(x(k))$ with some $\xi \in \mathbb{R}^n$. □

By utilizing the theorem above, a necessary and sufficient condition for solvability under a fixed mode sequence can then be derived, i.e., by imposing the solvability condition (7.9) to all switching signals that belong to the given mode sequence. This is presented in the following proposition.

Proposition 7.12 (Solvability characterization for SNSSs w.r.t. a fixed mode sequence). The SNSS (7.8) under Assumption 7.10 is solvable w.r.t. all switching signals $\sigma : \mathbb{N} \to \{0, 1, ..., \mathrm{p}\}$ of the form (2.11) with the fixed and known mode sequence $(\sigma_0, \sigma_1, \ldots)$ with arbitrary switching times if, and only if,

$$\mathcal{T}_i \subseteq \ker E_i \oplus \widetilde{\mathcal{S}}_i \quad \text{for all } i \in \{0, 1, \ldots, \mathrm{p}\} \tag{7.16a}$$

$$\mathcal{T}_{\sigma_j} \subseteq \ker E_{\sigma_j} \oplus \widetilde{\mathcal{S}}_{\sigma_{j+1}} \quad \text{for } j = 0, 1, 2, \ldots. \tag{7.16b}$$

Furthermore, if solvable, the surrogate ordinary system (7.10) is valid for every switching signal with the given mode sequence. ◊

Proof. The sufficiency is obvious since (7.16a)-(7.16b) implies that (7.9) is satisfied by all switching signals with the given mode sequence. For the necessity, solvability w.r.t. the given mode sequence $(\sigma_0, \sigma_1, \ldots)$ implies solvability

w.r.t. any arbitrary switching signal with the given mode sequence. Thus, for all $k \in \mathbb{N}$, and all switching signals with $\sigma(k) = \sigma(k+1) = i$, $\mathcal{T}_i \subseteq \ker E_i \oplus \widetilde{\mathcal{S}}_i$ for all $i \in \{0, 1, ..., \mathrm{p}\}$. Furthermore, at all switches from mode σ_j to σ_{j+1}, the condition (7.9) is also satisfied, which implies $\mathcal{T}_{\sigma_j} \subseteq \ker E_{\sigma_j} \oplus \widetilde{\mathcal{S}}_{\sigma_{j+1}}$ for $j = 0, 1, ...$ The validity of the surrogate system (7.10) is a direct consequence of the system being solvable w.r.t. all switching signals within the given mode sequence. □

Now, by imposing the solvability condition (7.9) to all possible switching signals with the given set of modes $\{0, 1, \ldots, \mathrm{p}\}$, a necessary and sufficient condition for solvability w.r.t. all switching signals can then be derived, this is presented in the following proposition.

Proposition 7.13 (Solvability characterization for SNSSs w.r.t. all switching signals). System (7.8) under Assumption 7.10 is solvable w.r.t. all switching signals (in the sense of Definition 7.9) if, and only if,

$$\mathcal{T}_j \subseteq \ker E_j \oplus \widetilde{\mathcal{S}}_i \ \forall i, j \in \{0, 1, ..., \mathrm{p}\}. \tag{7.17}$$

Furthermore, if solvable, the surrogate system (7.10) is valid for any arbitrary switching signal. Again, the validity of the surrogate system (7.10) is also a direct consequence of the system being solvable w.r.t. all switching signals. ◊

Proof. The sufficiency is clear since (7.17) implies that (7.9) holds for any arbitrary switching signal. For the necessity, take any mode sequence $(\sigma_0, \sigma_1, ...)$. The system is solvable w.r.t. this mode sequence, and thus (7.16a)-(7.16b) holds for this mode sequence. Since σ_j and σ_{j+1} are arbitrary, then (7.16b) is satisfied by any pair of modes $(j, j+1)$. Altogether, this implies $\mathcal{T}_j \subseteq \widetilde{\mathcal{S}}_i \oplus \ker E_j \ \forall i, j \in \{0, 1, ..., \mathrm{p}\}$. □

Regarding the nonuniqueness of the generalized inverse matrix E_j^+, the same phenomenon discussed in Remark 7.6 also applies i.e. the nonuniqueness of E_j^+ has no effect on the solution or the formula (7.10). This section is closed with the following examples.

Example 7.14. This example is provided to illustrate solvability characterizations of nonswitched singular nonlinear systems and in particular, to show that having solvable individual modes is in general not sufficient to have solvable switched systems composed of those modes. Consider system (2.14) with

$$(E_0, F_0(x)) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} x_1^{\frac{1}{3}} \\ x_2^{\frac{1}{3}} \end{bmatrix} \right), (E_1, F_1(x)) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} x_1^2 \\ x_1^2 + x_2^2 \end{bmatrix} \right),$$

and with

$$\ker E_0 = \operatorname{span}\{(0, 1)^\top\}, \qquad \widetilde{\mathcal{S}}_0 = \operatorname{span}\{(1, 0)^\top\},$$

$$\ker E_1 = \operatorname{span}\{(1,0)^\top\}, \qquad \widetilde{\mathcal{S}}_1 = \operatorname{span}\{(0,1)^\top\}.$$

For each pair, as an individual system, we have that $\ker E_i \oplus \widetilde{\mathcal{S}}_i = \mathbb{R}^n,\ i = 0,1$ i.e. individual system is solvable. Their solutions are $\begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} = \begin{pmatrix} x_{10}^{\frac{1}{3^k}} \\ 0 \end{pmatrix},\ k = 1,2,\ldots$ and $\begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} = \begin{pmatrix} 0 \\ x_{20}^{2k} \end{pmatrix},\ k = 1,2,\ldots$, respectively. When considering the switching signal $\sigma(k) = 0$ for $k < k^s$ and $\sigma(k) = 1$ for $k \geq k^s$ the switched system reads:

$$\begin{array}{r|r} k < k^s: & k \geq k^s: \\ x_1(k+1) = x_1(k)^{1/3}, & 0 = x_1(k)^2, \\ 0 = x_2(k)^{1/3}, & x_2(k+1) = x_2(k)^2. \end{array}$$

From this, it is clear that once the switch occurs at $k = k^s$, the only solution for x_1 is $x_1(k) = 0$ also before the switch, although x_1 was not restricted for $k < k^s$. Furthermore, $x_2(k^s)$ is not restricted by the above equations and hence uniqueness of solutions with respect to $x(0)$ is not satisfied. ◊

The following example illustrates a system that is solvable w.r.t. all switching signals.

Example 7.15. Consider system (2.14) with

$$(E_0, F_0(x)) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} x_1^{\frac{1}{3}} \\ x_2^{\frac{1}{3}} \end{bmatrix} \right),\ (E_1, F_1(x)) = \left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} x_1^2 + x_2^2 \\ x_2^2 \end{bmatrix} \right).$$

and with

$$\ker E_0 = \operatorname{span}\{(0,1)^\top\}, \qquad \widetilde{\mathcal{S}}_0 = \operatorname{span}\{(1,0)^\top\},$$
$$\ker E_1 = \operatorname{span}\{(0,1)^\top\}, \qquad \widetilde{\mathcal{S}}_1 = \operatorname{span}\{(1,0)^\top\}.$$

A few observations are discussed as follows: since $\ker E_i \oplus \widetilde{\mathcal{S}}_j = \mathbb{R}^n,\ \forall i,j \in \{0,1\}$, then the condition (7.17) holds regardless of the sets $\mathcal{T}_i$, and thus the system is solvable w.r.t. all switching signals. Choosing $E_0^+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $E_1^+ = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$ provides the following one-step maps from mode-j to mode-i:

$$\Phi_{0,0}(x(k)) = \Phi_{1,0}(x(k)) = \begin{pmatrix} x_1^{\frac{1}{3}}(k) \\ 0 \end{pmatrix}$$
$$\Phi_{1,1}(x(k)) = \Phi_{0,1}(x(k)) = \begin{pmatrix} \frac{1}{2}x_1^2(k) + \frac{3}{2}x_2^2(k) \\ 0 \end{pmatrix}.$$

Under the periodic switching signal $\sigma(k) = 1$ for $k \in [0,5) \cup [10,15) \cup \ldots$ and $\sigma(k) = 0$ for $k \in [6,10) \cup [15,20) \cup \ldots$, and with $x(0) = (-\frac{1}{2}, 0)^\top$, the solution is shown in Fig. 7.2. ◊

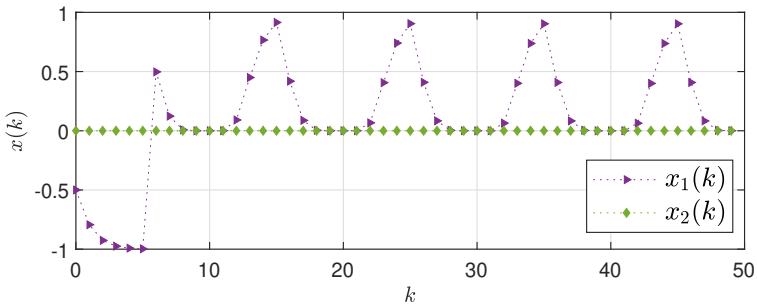

Figure 7.2: Solution of Example 7.15

7.2 Lyapunov Stability

The establishment of surrogate systems for singular systems makes it possible to study further properties such as observability and reachability in Chapters 5 and 6 for linear systems. For nonlinear systems, surrogate systems can also be utilized to study some properties of the system. In this section, Lyapunov stability is studied for the original singular system via its surrogate system. The study focuses on the stability of the zero equilibrium via Lyapunov stability analysis !analysis.

7.2.1 Nonswitched Systems

Recall the SNS (7.1) via its surrogate system (7.6). The stability of an equilibrium point $x_e \in \mathbb{R}^n$ of (7.1), i.e. $Ex_e = F(x_e)$ is studied in this part. Without restriction (by considering the linear coordinate transformation $x \mapsto x - x_e$), assume that $x_e = 0$, consequently $F(0) = 0$ and hence $\Phi(0) = 0$ i.e. $x_e = 0$ is also an equilibrium point of (7.6). Suppose $\Phi(0) = 0$ i.e. $x = 0$ is an equilibrium point for (7.6). This can also be generalized for a nonzero equilibrium: when $x = x_e \neq 0$ is the equilibrium point we are investigating, the new state $\widehat{x} = x - x_e$ provides 0 as an equilibrium point in $\widehat{x}$ coordinate. However, this coordinate transformation is not needed if $F(0) = 0$ since it directly implies that $\Phi(0) = 0$.

Definition 7.16 (Stability of zero equilibrium). The equilibrium $x = 0$ of the SNS (7.1) is

- **stable** if for each $\epsilon > 0$ there is $\delta = \delta(\epsilon)$ such that for all solutions x of (7.1)
$$||x(0)|| < \delta \Longrightarrow ||x(k)|| < \epsilon \quad \forall k \geq 0$$

- **asymptotically stable** if it is stable and δ can be chosen such that for all solutions x of (7.1)
$$||x(0)|| < \delta \Longrightarrow \lim_{k\to\infty} x(k) = 0$$
- **unstable** if it is not stable. $\Diamond$

Since the surrogate system (7.6) can be seen as an ordinary system, the stability theory for ordinary systems can be utilized here. The following corollary for the stability of $x = 0$ of (7.6) is a simple consequence from the classical stability theorem for ordinary systems, see e.g. [67, 68, 69].

Corollary 7.17 (Lyapunov stability characterization of SNSs). Consider the solvable singular system (7.1) via its surrogate ordinary system (7.6). Assume $\Phi : \widetilde{\mathcal{S}} \to \mathbb{R}^n$ is continuous on $\widetilde{\mathcal{S}} \subset \mathbb{R}^n$. If there exists a continuous function $V : \widetilde{\mathcal{S}} \to \mathbb{R}$ such that
$$V(0) = 0, V(x) > 0 \ \forall x \in \widetilde{\mathcal{S}} - \{0\}, \text{ and} \tag{7.18}$$
$$V(\Phi(x)) - V(x) \leq 0 \ \forall x \in \widetilde{\mathcal{S}} \tag{7.19}$$
then $x = 0$ is stable for (7.1). Furthermore, if
$$V(\Phi(x)) - V(x) < 0 \ \ \forall x \in \widetilde{\mathcal{S}} - \{0\} \tag{7.20}$$
then $x = 0$ is asymptotically stable for (7.1). $\Diamond$

7.2.2 Switched Systems

Recall the SNSS (7.8) under Assumption 7.10. Assume $x = 0$ is its equilibrium point. Definition 7.16 is carried over for the stability of the zero equilibrium of the SNSS (7.8) by considering the solution under the given fixed and known switching signal, or equivalently by considering the solution of its surrogate system (7.10).

The first approach that can be used to study the stability of $x = 0$, even though it is conservative, is the common Lyapunov function approach. The following corollary is derived from the common Lyapunov stability theorem for the general time-varying nonlinear systems of the form $x(k+1) = f_k(x(k))$ in [70].

Corollary 7.18 (Common Lyapunov function approach for stability of SNSSs). Consider the SNSS (7.8) under Assumption 7.10 with a fixed and known switching signal σ. Assume further that it is solvable w.r.t. σ and $x = 0$ is an equilibrium. Then, $x = 0$ is asymptotically stable w.r.t. σ if there is a function $V : \mathbb{R}^n \to \mathbb{R}$ such that

- V is positive–definite and radially unbounded;

- $V(x(k+1)) - V(x(k)) < 0$ for all solutions of (7.10) under σ. $\Diamond$

Note that in order to check the condition $V(x(k+1)) - V(x(k)) < 0$, one could require that

$$V(\Phi_{\sigma(k+1),\sigma(k)}(x)) - V(x) < 0 \quad \forall x \in \mathbb{R}^n, \text{ for } k = 0, 1, \ldots.$$

The stability can also be checked for all switching signals by imposing the second condition in Corollary 7.18 on all switching signals. Thus, one could require that

$$V(\Phi_{i,j}(x)) - V(x) < 0 \quad \forall i, j \in \{0, 1, \ldots, \mathrm{p}\} \; \forall x \in \mathbb{R}^n.$$

However, this means that system (7.10) is considered as a switched system with p^2 independent different modes (one for each **pair** (i, j)). However, this viewpoint is too conservative because the mode sequences in (7.10) are restricted to those where at time $k+1$ the mode pair (i_{k+1}, j_{k+1}) is related to the mode pair (i_k, j_k) at time k via $i_k = j_{k+1}$. Based on this motivation, the switched Lyapunov function approach is introduced in the following theorem where the "local" Lyapunov functions, which correspond to individual modes, are utilized. However, in this approach, each individual mode is required to be (asymptotically) stable.

Theorem 7.19 (Switched Lyapunov function approach for stability w.r.t. a fixed switching signal for SNSSs). Consider the SNSS (7.8) under Assumption 7.10 with a fixed and known switching signal σ of the form (2.11) via its surrogate ordinary switched system (7.10). Assume for all $i \in \{0, 1, ..., \mathrm{p}\}$, $\Phi_i : \widetilde{\mathcal{S}}_i \to \mathbb{R}^n$ is continuous on $\widetilde{\mathcal{S}}_i \subsetneq \mathbb{R}^n$ and each mode is (asymptotically) stable with corresponding Lyapunov function V_i satisfying Corollary 7.17. If the following conditions hold:

$$V_i(x) = V_j(x) \; \forall x \in \widetilde{\mathcal{S}}_i \cap \widetilde{\mathcal{S}}_j, \; \forall i, j \in \{0, 1, ..., \mathrm{p}\} \text{ and} \tag{7.21a}$$

$$\begin{gathered} V_{\sigma(k+1)}(\Phi_{\sigma(k+1),\sigma(k)}(x)) - V_{\sigma(k)}(x) \\ (<) \leq 0 \; \forall x \in \widetilde{\mathcal{S}}_{\sigma(k)} - \{0\} \text{ for } k = 0, 1, ... \end{gathered} \tag{7.21b}$$

then $x = 0$ is (asymptotically) stable for system (7.8) w.r.t. σ. $\Diamond$

Proof. For the given switching signal, the following Lyapunov function for (7.8) is constructed from the Lyapunov functions of the individual modes V_i:

$$V : \mathbb{R}^n \to \mathbb{R}, \; V(x) = \begin{cases} V_i(x) & \text{if } x \in \widetilde{\mathcal{S}}_i \\ 0 & \text{otherwise.} \end{cases}$$

The first condition in (7.21) ensures that V is well defined for all $x \in \cup_{i=0}^{\mathrm{p}} \widetilde{\mathcal{S}}_i$ i.e. it guarantees that $V(x)$ is unique for every $x \in \cup_{i=0}^{\mathrm{p}} \widetilde{\mathcal{S}}_i$. For the solution $x(k)$ of (7.10) under σ at any $k \in \mathbb{N}$,

$$V(x(k+1)) - V(x(k)) = V_{\sigma(k+1)}(x(k+1)) - V_{\sigma(k)}(x(k))$$

$$= V_{\sigma(k+1)}(\Phi_{\sigma(k+1),\sigma(k)}(x(k)) - V_{\sigma(k)}(x(k)).$$

Now, the second condition in (7.21) yields, by Corollary 7.18, the (asymptotic) stability of the equilibrium $x = 0$ under the given switching signal. □

Consider now the SNSS (7.8) with a fixed mode sequence $(\sigma_0, \sigma_1, ...)$. By requiring the second condition in 7.19 to be satisfied by all switching signals with the given fixed mode sequence, a sufficient condition for the stability w.r.t. the given fixed mode sequence–i.e. stability w.r.t. all switching signals with the given mode sequence–can be derived; this is presented in the following proposition.

Proposition 7.20 (Switched Lyapunov function approach for stability w.r.t. a fixed mode sequence for SNSSs). Consider the SNSS (7.8) under Assumption 7.10 with a fixed and known mode sequence $(\sigma) = (\sigma_0, \sigma_1, \ldots)$ via its surrogate ordinary switched system (7.10). Assume for all $i \in \{0, 1, ..., \mathrm{p}\}$, $\Phi_i : \widetilde{\mathcal{S}}_i \to \mathbb{R}^n$ is continuous on $\widetilde{\mathcal{S}}_i \subsetneq \mathbb{R}^n$ and each mode is (asymptotically) stable with corresponding Lyapunov function V_i satisfying Corollary 7.17. If the following conditions hold

$$V_i(x) = V_j(x) \ \forall x \in \widetilde{\mathcal{S}}_i \cap \widetilde{\mathcal{S}}_j \ \forall i, j \in \{0, 1, ..., \mathrm{p}\}, \tag{7.22a}$$

$$V_i(\Phi_i(x)) - V_i(x)(<) \leq 0 \ \forall i \in \{0, 1, ..., \mathrm{p}\} \ \forall x \in \widetilde{\mathcal{S}}_i - \{0\}, \tag{7.22b}$$

$$V_{j+1}(\Phi_{j+1,j}(x)) - V_j(x)(<) \leq 0 \text{ for all } x \in \widetilde{\mathcal{S}}_j - \{0\} \text{ for } j = 0, 1, ... \tag{7.22c}$$

then $x = 0$ is (asymptotically) stable for system (7.8) w.r.t. the given mode sequence (σ). ◊

Proof. This is a direct consequence of Theorem 7.19 where the second condition in Theorem 7.19 is applied to every individual mode and to every two consecutive modes that appear in the given mode sequence. □

Finally, based on the proposition above, a necessary sufficient for the stability of the origin for system (7.8) for all switching signals can be derived.

Proposition 7.21 (Switched Lyapunov function approach for stability w.r.t. all switching signals for SNSSs). Consider the solvable SNSS (7.8) w.r.t. all switching signals via its surrogate system (7.10). Assume for all $i \in \{0, 1, ..., \mathrm{p}\}$, $\Phi_i : \widetilde{\mathcal{S}}_i \to \mathbb{R}^n$ is continuous on $\widetilde{\mathcal{S}}_i \subset \mathbb{R}^n$ and each mode is (asymptotically) stable with corresponding Lyapunov function V_i. If for all $i, j \in \{0, 1, ..., \mathrm{p}\}$ the following conditions hold:

$$V_i(x) = V_j(x) \ \forall x \in \widetilde{\mathcal{S}}_i \cap \widetilde{\mathcal{S}}_j \tag{7.23a}$$

$$V_i(\Phi_{i,j}(x)) - V_j(x)(<) \leq 0 \ \forall x \in \widetilde{\mathcal{S}}_j - \{0\} \tag{7.23b}$$

then $x = 0$ of (7.8) is (asymptotically) stable for all switching signals. ◊

Proof. This is a direct consequence of Proposition 7.20 where the third condition in Proposition 7.19 must be satisfied by every pair of modes. □

Note that the second condition in the proposition above is necessary only for certain switches i.e. after $\Phi_{i,j}$, and the condition is checked only for switches to $\Phi_{i,j}$ and not for all switches to any other one-step map matrix. This makes the stability theorem above more relaxed compared to the common Lyapunov approach. The following example illustrates the stability analysis by using the condition provided in the proposition above.

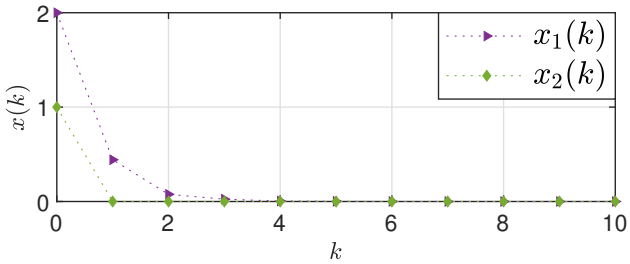

Figure 7.3: A solution of the switched system in Example 7.22

Example 7.22. Consider system (7.8) composed of the following two modes:

$$(E_0, F_0(x)) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} (x_1+1)^{\frac{1}{3}}-1 \\ x_2^{\frac{1}{3}} \end{bmatrix} \right),$$

$$(E_1, F_1(x)) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} (x_1+1)^{\frac{1}{5}}-1 \\ x_2^{\frac{1}{5}} \end{bmatrix} \right),$$

and with

$$\ker E_0 = \operatorname{span}\{(0,1)^\top\}, \qquad \widetilde{\mathcal{S}}_0 = \operatorname{span}\{(1,0)^\top\},$$

$$\ker E_1 = \operatorname{span}\{(0,1)^\top\}, \qquad \widetilde{\mathcal{S}}_1 = \operatorname{span}\{(1,0)^\top\}.$$

Since $\ker E_i \oplus \widetilde{\mathcal{S}}_j = \mathbb{R}^n$, $\forall i,j \in \{0,1\}$, clearly the condition (7.17) holds i.e. the system is solvable for arbitrary switching signals. Choosing $E_0^+ = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $E_1^+ = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and with $\Pi_{\widetilde{\mathcal{S}}_1}^{\ker E_0} = \Pi_{\widetilde{\mathcal{S}}_0}^{\ker E_1} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ provide

$$\Phi_0(x(k)) = \Phi_{0,0}(x(k)) = \Phi_{1,0}(x(k)) = \begin{bmatrix} (x_1+1)^{\frac{1}{3}}-1 \\ 0 \end{bmatrix}$$

and

$$\Phi_1(x(k)) = \Phi_{1,1}(x(k)) = \Phi_{0,1}(x(k)) = \begin{bmatrix} (x_1+1)^{\frac{1}{5}}-1 \\ 0 \end{bmatrix}.$$

As an individual system, $x = 0$ of each mode is stable with the Lyapunov function e.g. $V_i(x) = x_1^2 + x_2^2$, $i = 0, 1$. Clearly, the conditions (i) and (ii) in Proposition 7.21 with strict inequality are satisfied, and moreover $V_0(x) =$

$V_1(x)$). Hence, $x = 0$ of the switched system is asymptotically stable for arbitrary switching signals. With $\sigma(k) = 0$ if $k = 0, 2, 4, \ldots$ and $= 1$ if $k = 1, 3, 5, \ldots$, the trajectory of the solution is illustrated in Fig. 7.3. ◊

7.3 Incremental Stability

While the Lyapunov stability analyzes the stability of an equilibrium point, generally speaking, the incremental stability notion is related to the asymptotic convergence behavior of the solutions to each other or to a particular steady-state trajectory [71]. The surrogate system (7.10) is utilized in this section to study the incremental stability analysis for the original SNSS (7.8).

7.3.1 Nonswitched Systems

Let $x(k; x_0)$ be the solution of (7.1) via (7.6) at time instant $k \in \mathbb{N}$ with the initial condition $x(0) = x_0 \in \widetilde{\mathcal{S}}$. Throughout the rest of the study, following [72], the standard notations for function classes[1] $\mathcal{K}, \mathcal{L}, \mathcal{K}_\infty$, and $\mathcal{KL}$ are used. Moreover, the norm $||\cdot||$ stands for the standard Euclidean norm, $\mathbb{R}_{\geq 0}$ denotes the set of all nonnegative real numbers, and $||x||_{\mathcal{X}}$ denotes the distance of vector x to set $\mathcal{X}$ defined by $||x||_{\mathcal{X}} = \inf_{\xi \in \mathcal{X}} ||x - \xi||$.

The incremental stability notion used in this study for nonswitched singular nonlinear systems of the form (7.1) is formally defined as follows:

Definition 7.23 (Incremental Stability Notion for SNSs). The SNS (7.1) is called *asymptotically incrementally stable* on a positively invariant $\mathcal{X} \subseteq \widetilde{\mathcal{S}} \subsetneq \mathbb{R}^n$ if there exists $\beta \in \mathcal{KL}$ such that

$$||x(k; x_0') - x(k; x_0'')|| \leq \beta(||x_0' - x_0''||, k) \tag{7.24}$$

for all $x_0', x_0'' \in \mathcal{X}$, $k \in \mathbb{Z}_{\geq 0}$ and where $x(k; x_0)$ denotes the solution of (7.1) with initial values x_0. ◊

Compared to other incremental stability notions such as [71, Definition 1], which is also defined globally on $\mathbb{R}^n$, the notion in the definition above is only considered on the subspace $\widetilde{\mathcal{S}}$ which is a strict subspace of $\mathbb{R}^n$ for any solvable SNS (7.1) with singular E (because $\widetilde{\mathcal{S}} \cap \ker E = \{0\}$ is necessary for solvability). The following proposition provides a sufficient and necessary condition for the

[1] A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing. If it is also unbounded, then α belongs to class-$\mathcal{K}_\infty$. Meanwhile, a function $\beta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{L}$ if it is continuous, strictly decreasing, and $\lim_{t \to \infty} \beta(t) = 0$. A function $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{KL}$ if it belongs to class-$\mathcal{K}$ in its first argument and class-$\mathcal{L}$ in its second argument.

incremental stability notion defined above, which is inspired by the condition for ordinary nonlinear systems in [71, Theorem 5].

Proposition 7.24 (Incremental stability of SNSs). Consider the SNS (7.1) under Assumption 7.2 via its surrogate system (7.6). Then, this system is *asymptotically incrementally stable* on $\widetilde{\mathcal{S}}$ if, and only if, there exist a continuous function $V : \widetilde{\mathcal{S}} \times \widetilde{\mathcal{S}} \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\alpha_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ positive definite such that

$$\alpha_1(||x' - x''||) \leq V(x', x'') \leq \alpha_2(||x' - x''||), \tag{7.25a}$$

$$V(\Phi(x'), \Phi(x'')) - V(x', x'') \leq -\alpha_3(||x' - x''||) \tag{7.25b}$$

hold for all $x', x'' \in \widetilde{\mathcal{S}}$. ◊

Proof. The proof is similar to the proof of incremental stability analysis for ordinary systems, however, for completeness, the complete proof is briefly provided as follows: the proof relies on the so-called *augmented system* defined as

$$z(k+1) = \widetilde{\Phi}(z(k)) \tag{7.26}$$

with $z = \left[\begin{smallmatrix} x' \\ x'' \end{smallmatrix}\right] \in \widetilde{\mathcal{S}} \times \widetilde{\mathcal{S}}$ and $\widetilde{\Phi}(z) = \left[\begin{smallmatrix} \Phi(x') \\ \Phi(x'') \end{smallmatrix}\right]$. The diagonal set of this augmented system is defined as the set $\Delta := \left\{ \left[\begin{smallmatrix} x \\ x \end{smallmatrix}\right] \in \widetilde{\mathcal{S}} \times \widetilde{\mathcal{S}} \;\middle|\; x \in \widetilde{\mathcal{S}} \right\}$. The distance from z to Δ is [73, Lemma 2.3]

$$||z||_\Delta := \inf_{w \in \Delta} ||w - z|| = \frac{1}{2}\sqrt{2}||x' - x''|| \tag{7.27}$$

i.e., it is proportional to $||x' - x''||$. The augmented system (7.26) is said to be asymptotically stable with respect to Δ if there exists a class-$\mathcal{KL}$ function β such that

$$||z(k; z(0))||_\Delta \leq \beta(||z(0)||_\Delta, k), \;\; z(0) \in \widetilde{\mathcal{S}} \times \widetilde{\mathcal{S}}. \tag{7.28}$$

for all $k \in \mathbb{N}$. From the relationship (7.27), the condition (7.28) is equivalent to (7.24), and thus system (7.6) is asymptotically incrementally stable on $\widetilde{\mathcal{S}}$ if and only if the augmented system (7.26) is asymptotically stable w.r.t. Δ. By [74, Proposition 2.2 and Theorem 1], system (7.26) is asymptotically stable w.r.t. Δ if, and only if, it admits a continuous Lyapunov function $W : \mathbb{R}^{2n} \to \mathbb{R}_{\geq 0}$ for which there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ positive definite such that

$$\alpha_1(||z||_\Delta) \leq W(z) \leq \alpha_2(||z||_\Delta), \tag{7.29a}$$

$$W(\widetilde{\Phi}(z)) - W(z) \leq -\alpha_3(||z||_\Delta) \tag{7.29b}$$

hold for all $z \in \widetilde{\mathcal{S}} \times \widetilde{\mathcal{S}}$ and all $k \in \mathbb{N}$. The equivalence of (7.29) and (7.25) completes the proof. □

Such function V satisfying (7.25) is called an incremental Lyapunov function for (7.1). A similar idea will also be used in the forthcoming incremental stability analysis for switched systems.

7.3.2 Switched Systems

Consider now the solvable SNSS (7.8) via its surrogate ordinary switched system (7.10). In this case, the surrogate ordinary switched system (7.10) can be seen as a time-varying system, where incremental stability characterization and contraction analysis for time-varying (ordinary) systems can be applicable [75]. However, the existing conditions are required to be checked for every time step, which is not necessary for (7.10) since for some time intervals, the system stays at a certain mode, and thus it can be characterized by mode-wise approach. Furthermore, in the existing studies for time-varying systems, the characterizations for incremental stability were considered in a positively invariant set $\mathbb{X}$, which also serves as the consistency set of the system that is defined globally i.e. on $[k_0, \infty)$. Note in general that the consistency set of SNSS (7.8) which corresponds to each mode may be different i.e. it is not necessary to have $\widetilde{\mathcal{S}}_i = \widetilde{\mathcal{S}}_j,\ i \neq j$. Therefore, in the following, new incremental stability notions with respect to a time-dependent set are defined.

Consider first the time-dependent set $\widehat{\mathcal{S}}(k)$ defined by $\widehat{\mathcal{S}} : \mathbb{N} \to \{\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_1, ..., \widetilde{\mathcal{S}}_{\mathrm{p}}\}$ with $\widehat{\mathcal{S}}(k) = \widetilde{\mathcal{S}}_{\sigma(k)}$. Following Definition 1 in [76] for a time-dependent positively invariant set w.r.t. a dynamical system, by Theorem 4.40, the time-dependent set $\widehat{\mathcal{S}}(k)$ is a time-dependent positively invariant set w.r.t. system (7.8). The incremental stability notion for singular nonlinear switched systems considered in this study is formally defined in the following definition, which is inspired by the notion for ordinary systems [75, Def. 1].

Definition 7.25 (Incremental Stability Notion for SNSSs). The SNSS (7.8) is called **asymptotically incrementally stable** w.r.t. a fixed switching signal σ on a time-dependent positively invariant set $\mathcal{X}(k)$ if there exists $\beta \in \mathcal{KL}$ such that

$$||x'(k; x_0') - x''(k; x_0'')|| \leq \beta(||x_0' - x_0''||, k) \tag{7.30}$$

for all $x_0', x_0'' \in \mathcal{X}(0)$ and all $k \in \mathbb{N}$. ◊

Compared to the notion in [75, Definition 1] which is defined on a constant positive invariant set and is defined also globally on $\mathbb{R}^n$, the notion in Definition 7.25 above is defined on a time-dependent positive invariant set, and furthermore, it cannot be defined globally on $\mathbb{R}^n$ since $\widetilde{\mathcal{S}}_i \subsetneq \mathbb{R}^n$ for all i. Moreover, the incremental stability notion above is defined nonuniformly w.r.t. time since we are interested only with initial conditions $x(0) = x_0 \in \widetilde{\mathcal{S}}_{\sigma(0)}$.

7.3.2.1 Single Lyapunov Function Approach

Under a fixed and known switching signal, the surrogate switched system (7.10) can be seen as a time-varying system without containing the time variable explicitly. Thus, the existing condition for incremental stability for time-varying systems, such as [75, Theorem 9], can be adopted. However, it is only a sufficient condition and not necessary since the set $\widetilde{\mathcal{S}}_{\sigma(k)}$, in general, is a positively invariant set only for the current mode (the mode that is active at time instant k). Now, inspired by the existing characterization for ordinary systems in [75, Theorem 9] and by utilizing the surrogate system (7.10), the following necessary and sufficient condition for incremental stability of system (7.8) on the time-dependent positively invariant set $\widehat{\mathcal{S}}(k)$ is established.

Lemma 7.26 (Single Lyapunov function approach for incremental stability w.r.t. a fixed switching signal for SNSSs). Consider the SNSS (7.8) under Assumption 7.10 via its surrogate switched system (7.10). This system is **asymptotically incrementally stable** w.r.t. a fixed and known switching signal σ on the time-dependent positively invariant set $\widehat{\mathcal{S}}(k)$ if, and only if, there exists a continuous function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and $\alpha_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ positive definite such that for $k = 0, 1, \ldots$

$$\alpha_1(||x'(k) - x''(k)||) \leq V(x'(k), x''(k)) \leq \alpha_2(||x'(k) - x''(k)||), \quad (7.31a)$$

$$V(x'(k+1), x''(k+1)) - V(x'(k), x''(k)) \leq -\alpha_3(||x'(k) - x''(k)||) \quad (7.31b)$$

hold for all solutions x' and x'' of (2.14) with the given switching signal σ. $\Diamond$

Proof. The proof is similar to the proof for time-varying systems in [75, Theorem 9] by considering the switched *augmented system*

$$z(k+1) = \widetilde{\Phi}_k(z(k)) \quad (7.32)$$

with $z(k) = \begin{bmatrix} x'(k) \\ x''(k) \end{bmatrix} \in \widetilde{\mathcal{S}}_{\sigma(k)} \times \widetilde{\mathcal{S}}_{\sigma(k)}$ and $\widetilde{\Phi}(z(k)) = \begin{bmatrix} \Phi_{\sigma(k+1),\sigma(k)}(x'(k)) \\ \Phi_{\sigma(k+1),\sigma(k)}(x''(k)) \end{bmatrix}$ where $\Phi_{i,j}$ is the one-step map as in (7.10). The claim then follows from showing that the switched diagonal set $\Delta(k) := \left\{ \begin{bmatrix} x \\ x \end{bmatrix} \in \widetilde{\mathcal{S}}_{\sigma(k)} \times \widetilde{\mathcal{S}}_{\sigma(k)} \;\middle|\; x \in \widetilde{\mathcal{S}}_{\sigma(k)} \right\}$ is asymptotically stable for (7.32). Similar as in [74, Theorem 1] and [77, Chapter 5], this stability is shown via the existence of a Lyapunov function $W : \mathbb{R}^{2n} \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\alpha_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ positive definite such that for $k = 0, 1, \ldots$

$$\alpha_1(||z(k; x_0)||_{\Delta(k)}) \leq W(z(k; x_0)) \leq \alpha_2(||x(z; x_0)||_{\Delta(k)}), \quad (7.33a)$$

$$W(\Phi_{\sigma(k+1),\sigma(k)}(z(k; x_0))) - W(z(k; x_0)) \leq -\alpha_3(||z(k; x_0)||_{\Delta(k)}), \quad (7.33b)$$

for all $x_0 \in \widetilde{\mathcal{S}}_{\sigma(0)}$. The details are as follows: first, for any k, the distance

formula (7.27) is also valid here, i.e., the distance from $z(k)$ to $\Delta_{\sigma(k)}$ satisfies

$$||z(k)||_{\Delta_{\sigma(k)}} = \inf_{w \in \Delta_{\sigma(k)}} ||w - z(k)|| = \frac{\sqrt{2}}{2}||x'(k) - x''(k)||. \tag{7.34}$$

This means that at any time instant k, $||z(k)||_{\Delta_{\sigma(k)}}$ is proportional to $||x'(k) - x''(k)||$. By (7.34), it can be seen that the SNSS (7.8) via its surrogate system (7.10) is asymptotically incrementally stable on $\widehat{\mathcal{S}}(k)$ if and only if the switched augmented system (7.32) is asymptotically stable w.r.t. $\Delta(k)$ if and only if there exist a continuous function $V : \mathbb{R}^{2n} \to \mathbb{R}_{\geq 0}$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and $\alpha_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ positive definite such that for $k = 0, 1, ...$, (7.33) holds for all $x_0 \in \widehat{\mathcal{S}}_{\sigma(0)}$ (by Lemma 7.29). The equivalence of (7.33) and (7.31) completes the proof. □

A function V that satisfies Lemma 7.26 is called an incremental Lyapunov function. Note that while Lemma 7.26 provided a *characterization* (i.e. necessary *and* sufficient) of incremental stability in terms of a Lyapunov function, one may argue that the condition (7.31) is not practical since it needs to be checked for all explicit solutions. Therefore, we provide a sufficient condition in the following corollary, which is more convenient to check by utilizing the one-step map introduced in Theorem 4.40, in particular, it doesn't require knowledge of the solutions.

Corollary 7.27 (Sufficient Condition for Incremental Stability of SNSSs via Single Lyapunov Function Approach). Consider the SNSS (7.8) under Assumption 7.10 with a fixed and known switching signal σ of the form (2.11) via its surrogate switched system (7.10) and a time-dependent positively invariant set $\widehat{\mathcal{S}}(k)$. If there exist a continuous function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and $\alpha_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ positive definite such that for $k = 0, 1, ...$

$$\alpha_1(||x' - x''||) \leq V(x', x'') \leq \alpha_2(||x' - x''||), \tag{7.35a}$$

$$V(\Phi_{\sigma(k+1),\sigma(k)}(x'), \Phi_{\sigma(k+1),\sigma(k)}(x'')) - V(x', x'') \leq -\alpha_3(||x' - x''||) \tag{7.35b}$$

hold for all $x', x'' \in \widehat{\mathcal{S}}(k)$ then this system is *asymptotically incrementally stable* w.r.t. σ on $\widehat{\mathcal{S}}(k)$. ◊

7.3.2.2 Switched Lyapunov Function Approach

The conditions in Lemma 7.26 and Corollary 7.27 require a single incremental Lyapunov function. If every mode as an individual (non-switched) system is asymptotically incrementally stable on its consistency space, then the corresponding incremental Lyapunov functions of all modes can be utilized to formulate a switched incremental Lyapunov function for the switched system

composed of those modes. This is presented in the forthcoming Theorem 7.30; however, the following notion of stability w.r.t. a time-dependent positively invariant set followed by its characterization is settled prior to it and will be used in the proof of the theorem.

Definition 7.28 (Stability w.r.t. a time-dependent positively invariant set). Consider the SNSS (7.8) under Assumption 7.10 with the fixed and known switching signal σ via its surrogate ordinary system (7.10). This system is called asymptotically stable w.r.t. a time-dependent positively invariant set $\mathcal{X}(k) \subseteq \widetilde{\mathcal{S}}_{\sigma(k)},\ k = 0, 1, \ldots$ if there exists a class-$\mathcal{KL}$ function β such that

$$||x(k; x_0)||_{\mathcal{X}(k)} \leq \beta(||x_0||_{\mathcal{X}(0)}, k) \tag{7.36}$$

holds for all $x_0 \in \widetilde{\mathcal{S}}_{\sigma(0)}$ and all $k \in \mathbb{N}$. ◊

This definition is motivated by the occurrence of the time-dependent positively invariant set $\widehat{\mathcal{S}}(k)$ for the SNSS (7.8), and therefore it is specifically made for this system. It can also be defined for any nonlinear system and any time-dependent positively invariant set $\mathcal{X}(k)$. However, finding such a set is not obvious. One may consider $\mathcal{X}(k) = \mathbb{R}^n$ for ordinary nonlinear systems (or $\mathcal{X}(k) = \widehat{\mathcal{S}}(k)$ for SNSSs), however, this definition becomes meaningless since $||x(k)||_{\mathcal{X}(k)} = 0$ for all $x(k) \in \mathcal{X}(k)$, i.e., the inequality (7.36) is always satisfied by all solutions of the system.

In the following lemma, inspired by [74, Proposition 2.2] for time-varying systems, the characterization for the stability notion defined above is provided for singular switched systems (7.8). Compared to the existing characterization for time-varying ordinary systems, a time-dependent invariant set is considered in this lemma whereas the invariant set considered in [74, Proposition 2.2] for ordinary systems is constant over time, see also [74, Theorem 1] and [77, Chapter 5].

Lemma 7.29 (Asymptotic stability w.r.t. a time-dependent positively invariant set). Consider the SNSS (7.8) under Assumption 7.10 with the fixed and known switching signal σ via its surrogate ordinary system (7.10). This system is asymptotically stable w.r.t. the time-dependent positively invariant set $\mathcal{X}(k) \subsetneq \widetilde{\mathcal{S}}_{\sigma(k)},\ k = 0, 1, \ldots$ if, and only if, there exist a continuous function $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\alpha_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ positive definite such that for $k = 0, 1, \ldots$

$$\alpha_1(||x(k; x_0)||_{\mathcal{X}(k)}) \leq W(x(k; x_0)) \leq \alpha_2(||x(k; x_0)||_{\mathcal{X}(k)}) \tag{7.37a}$$

$$W(\Phi_{\sigma(k+1),\sigma(k)}(x(k; x_0))) - W(x(k; x_0)) \leq -\alpha_3(||x_0||_{\mathcal{X}(0)}) \tag{7.37b}$$

hold for all $x_0 \in \widetilde{\mathcal{S}}_{\sigma(0)}$. ◊

Proof. The proof is similar to the proof for time-varying systems with a con-

stant invariant set. Here, the proof is provided for the case with a time-dependent positively invariant set. For the sufficiency, consider $\alpha = \alpha_3 \circ \alpha_2^{-1}$. From condition (7.37):

$$W(\Phi_{\sigma(k+1),\sigma(k)}(x(k;x_0))) - W(x(k;x_0)) \leq -\alpha(W(x(k;x_0)))$$

for all $x_0 \in \widetilde{\mathcal{S}}_{\sigma(0)}$ and all $k \in \mathbb{N}$. Pick a function $\beta_\alpha \in \mathcal{KL}$ satisfying Lemma 4.3 in [74]; then, by this lemma,

$$W(x(k;x_0)) \leq \beta_\alpha(W(x_0), k)$$

for all $x_0 \in \widetilde{\mathcal{S}}_{\sigma(0)}$ and all $k \in \mathbb{N}$. Now, define $\beta(s,r) = \alpha_1^{-1} \circ \beta_\alpha(\alpha_2(s), r)$, then $\beta \in \mathcal{KL}$ as $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and thus we have

$$||x(k;x_0)||_{\mathcal{X}(k)} \leq \beta(||x_0||_{\mathcal{X}(0)}, k)$$

for all $x_0 \in \widetilde{\mathcal{S}}_{\sigma(0)}$ and all $k \in \mathbb{N}$. Hence, the system is asymptotically stable w.r.t. $\mathcal{X}(k)$. For the necessity, from [78, Proposition 7], for any function $\beta \in \mathcal{KL}$, there exists $\rho_1, \rho_2 \in \mathcal{K}_\infty$ such that

$$\beta(s,r) \leq \rho_1(\rho_2(s)e^{-r}) \ \forall s \geq 0 \ \forall r \geq 0.$$

Then, for any SNSS that is asymptotic stable w.r.t. $\mathcal{X}(k)$, there exist $\rho_1, \rho_2 \in \mathcal{K}_\infty$ such that

$$||x(k;x_0)||_{\mathcal{X}(k)} \leq \rho_1(\rho_2(||x_0||_{\mathcal{X}(0)})e^{-k})$$

for all $x_0 \in \widetilde{\mathcal{S}}_{\sigma(0)}$ and all $k \in \mathbb{N}$. Take $\omega(s) = \rho_1^{-1}(s) \in \mathcal{K}_\infty$, then for $k = 0, 1, ...$

$$\omega(||x(k;x_0)||_{\mathcal{X}(k)}) \leq \rho_2(||x_0||_{\mathcal{X}(0)})e^{-k}.$$

Now, define the continuous function $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ by $W(x_0) = \sum_{k=0}^{\infty} \omega(||x(k;x_0)||_{\mathcal{X}(k)})$. Furthermore, from the last inequality,

$$\omega(||x_0||_{\mathcal{X}(0)}) \leq W(x_0) \leq \sum_{k=0}^{\infty} \rho_2(||x_0||_{\mathcal{X}(0)})e^{-k} \leq \frac{e}{e-1}\rho_2(||x_0||_{\mathcal{X}(0)}),$$

which means that $\sum_{k=0}^{\infty} \omega(||x(k;x_0)||_{\mathcal{X}(k)})$ is convergent; this completes the proof. □

Now, the main theorem for the incremental stability analysis using switched Lyapunov function approach is presented in the following theorem.

Theorem 7.30 (Switched Lyapunov function approach). Consider the solvable SNSS (2.14) under Assumption 7.10. Assume each mode i is asymptotically incrementally stable on $\widetilde{\mathcal{S}}_i$ with the corresponding incremental Lyapunov function $V_i : \widetilde{\mathcal{S}}_i \times \widetilde{\mathcal{S}}_i \to \mathbb{R}_{\geq 0}$ and class-$\mathcal{K}_\infty$ functions α_{1i}, α_{2i} and α_{3i} satisfying Proposition 7.24. If the following three conditions hold:

1. For all $x', x'' \in \widetilde{\mathcal{S}}_i \cap \widetilde{\mathcal{S}}_j$ and all $i, j \in \{0, 1, ..., \mathrm{p}\}$:
$$V_i(x', x'') = V_j(x', x''), \tag{7.38}$$

2. For all $x', x'' \in \widetilde{\mathcal{S}}_i \cup \widetilde{\mathcal{S}}_j$ with $||x'|| = ||x''||$ and all $i, j \in \{0, 1, ..., \mathrm{p}\}$:

$$\alpha_{1i}(||x||) = \alpha_{1j}(||x||),\ \alpha_{2i}(||x||) = \alpha_{2j}(||x||),\ \alpha_{3i}(||x||) = \alpha_{3j}(||x||), \quad (7.39)$$

3. For $k = 0, 1, ...$ and for all $x', x'' \in \widetilde{\mathcal{S}}_{\sigma(k)}$:

$$\begin{aligned} V_{\sigma(k+1)}(\Phi_{\sigma(k+1),\sigma(k)}(x'), \Phi_{\sigma(k+1),\sigma(k)}(x'')) - V_{\sigma(k)}(x', x'') \\ \leq -\alpha_3(||x' - x''||) \end{aligned} \quad (7.40)$$

with $\alpha_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0},\ \alpha_3(||x||) = \alpha_{3i}(||x||)$ if $x \in \widetilde{\mathcal{S}}_i$ and 0 otherwise, then system (2.14) is *asymptotically incrementally stable* w.r.t. the given fixed and known switching signal σ on $\widehat{\mathcal{S}}(k)$. ◊

Proof. For the given switching signal, we construct the following incremental (switched) Lyapunov function from the incremental Lyapunov functions of individual modes as follows:

$$V : \mathbb{R}^{2n} \to \mathbb{R},\ V(x_1, x_2) = \begin{cases} V_i(x_1, x_2) & \text{if } x_1, x_2 \in \widetilde{\mathcal{S}}_i \\ 0 & \text{otherwise.} \end{cases}$$

Condition (7.38) ensures that V is well defined for all $x_1, x_2 \in \mathbb{R}^n$, i.e. it guarantees that $V(x)$ is unique for every $x_1, x_2 \in \mathbb{R}^n$. From the functions $\alpha_{1i}, \alpha_{2i}, \alpha_{3i}$ of all individual modes, we also construct for the switched system the corresponding functions $\alpha_\ell : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0},\ \ell = 1, 2, 3$ defined by

$$\alpha_\ell(||x||) = \begin{cases} \alpha_{\ell i}(||x||) & \text{if } x \in \widetilde{\mathcal{S}}_i \\ 0 & \text{otherwise.} \end{cases}$$

The functions α_ℓ are well defined due to the condition (2). Now, since $\widetilde{\mathcal{S}}_i$ are subspaces, $\{0\} \in \widetilde{\mathcal{S}}_i$ and $\widetilde{\mathcal{S}}_i \cap \widetilde{\mathcal{S}}_j \supseteq \{0\}$. Thus, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ due to conditions in (7.39). Since each V_i and $\alpha_{\ell i},\ \ell = 1, 2, 3$ satisfy (7.25), the functions V and α_ℓ defined above satisfy

$$\alpha_1(||x' - x''||) \leq V(x', x'') \leq \alpha_2(||x' - x''||).$$

and together with condition (7.40) implies the incremental stability on $\widehat{\mathcal{S}}(k)$ w.r.t. the given switching signal σ. □

Such a piecewise function V in the proof above is called a switched Lyapunov function; the term comes from the fact that V switches over the Lyapunov functions of individual modes depending on in which consistency space the solution is at the corresponding time instant. Compared to the single Lyapunov function approach presented in Lemma 7.26, the switched Lyapunov function approach is simpler in terms of finding the Lyapunov function since it is formulated from the Lyapunov functions of the individual modes. However, stability for each mode is required here; this assumption is not required

in the single Lyapunov function approach, i.e. the switched system may contain unstable modes. Nevertheless, the single Lyapunov approach requires a Lyapunov function that fits the whole switched system, which is intuitively more difficult to find. In particular, Lyapunov function construction methods in ordinary systems can be utilized, such as Yoshizawa method [79], least square optimization approach [80], collocation approaches [81, 82] and linear programming approach [83].

Theorem 7.30 can be easily extended to characterize the incremental stability of the SNSS (2.14) with respect to a fixed and known mode sequence in which the switching times are arbitrary, or with respect to arbitrary switching signals (both mode sequence and switching times are arbitrary) by considering the condition in Theorem 7.30 to be satisfied by the involved switching signals. This is discussed in the following remark.

Remark 7.31 (Conditions for Incremental Stability w.r.t. Mode Sequences or All Switching Signals). The SNSS (7.8) is asymptotically incrementally stable on $\widehat{\mathcal{S}}(k)$ w.r.t. all switching signals with the fixed and known mode sequence $(\sigma) = (\sigma_0, \sigma_1, \ldots)$ if Theorem 7.30 is satisfied with the condition (7.40) is replaced by

$$V_i(\Phi_i(x'), \Phi_i(x'')) - V_i(x', x'') \leq \alpha_3(||x' - x''||),\ i = 0, 1, \ldots \mathrm{p},$$
$$V_{j+1}(\Phi_{j+1,j}(x'), \Phi_{j+1,j}(x'')) - V_j(x', x'') \leq \alpha_3(||x' - x''||),\ j = 0, 1, \ldots.$$

Furthermore, the SNSS (7.8) is asymptotically incrementally stable on $\widehat{\mathcal{S}}(k)$ w.r.t. any arbitrary switching signal if Theorem 7.30 is satisfied with the condition (7.40) is replaced by

$$V_i(\Phi_{i,j}(x'), \Phi_{i,j}(x'')) - V_i(x', x'') \leq \alpha_3(||x' - x''||),\ \forall (i, j) \in \{0, 1, ..., \mathrm{p}\}.$$

The conditions above are derived by imposing the condition (7.40) to all involved switching signals and then simplifying the condition at nonswitching times as in the proof of solvability w.r.t. a fixed mode sequence in Proposition 7.12 or the proof of solvability w.r.t. all switching signals in Proposition 7.13. ◊

7.3.2.3 Illustrative Examples

We close this part by providing an example illustrating the derived theoretical results.

Example 7.32. Consider system (7.8) composed of the following two modes:

$$(E_0, F_0(x)) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \sin(x_1)+\cos(x_2) \\ \sin(x_1)-\cos(x_2)x_2+x_3^{\frac{1}{3}} \\ x_3^{\frac{1}{3}} \end{bmatrix} \right),$$

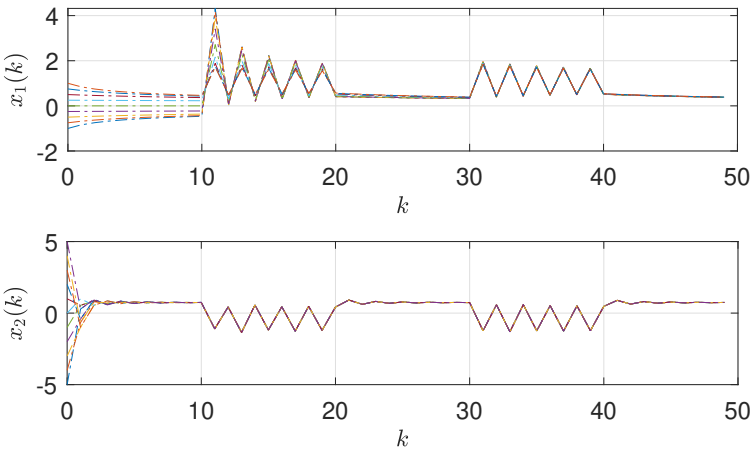

Figure 7.4: Solution trajectories of the switched system in Example 7.32

$$(E_1, F_1(x)) = \left(\begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} e^{1-x_1} + x_3^{\frac{1}{3}} \sin(x_2) \\ e^{1-x_1} - x_3^{\frac{1}{3}} \sin(x_2) x_2 + x_3^{\frac{1}{3}} e^{1-x_1} \\ x_3^{\frac{1}{3}} e^{1-x_1} \end{bmatrix} \right).$$

Geometric computations provide $\ker E_0 = \ker E_1 = \operatorname{span}\{(0,0,1)^\top\}$, $\widetilde{\mathcal{S}}_0 = \widetilde{\mathcal{S}}_1 = \operatorname{span}\{(1,0,0)^\top, (0,1,0)^\top\}$. Since $\ker E_i \oplus \widetilde{\mathcal{S}}_j = \mathbb{R}^n$, $\forall i, j \in \{0,1\}$, the condition (7.17) holds independently of $\mathcal{T}_{\sigma(k)}$ i.e. the switched system is solvable w.r.t. any arbitrary switching signal (each mode as an individual system is also solvable). Choosing $E_0^+ = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $E_1^+ = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and with $\Pi_{\widetilde{\mathcal{S}}_1}^{\ker E_0} = \Pi_{\widetilde{\mathcal{S}}_0}^{\ker E_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ provide the surrogate system (7.10) with

$$\Phi_0(x(k)) = \Phi_{0,0}(x(k)) = \Phi_{1,0}(x(k)) = \begin{bmatrix} \sin(x_1) \\ \cos(x_2) \\ 0 \end{bmatrix}$$

and

$$\Phi_1(x(k)) = \Phi_{1,1}(x(k)) = \Phi_{0,1}(x(k)) = \begin{bmatrix} (1+\frac{1}{2}x_3^{\frac{1}{3}})e^{1-x_1} \\ (-1-x_3^{\frac{1}{3}})e^{1-x_1} + x_3^{\frac{1}{3}} \sin(x_2) x_2 \\ 0 \end{bmatrix}.$$

As an individual system, each mode is incrementally stable by considering the functions $\alpha_1(\xi) = \xi^2$, $\alpha_2(\xi) = \xi^2$, $\alpha_3(\xi) = 0$ for both modes and the incremental Lyapunov function $V_i(x) = x_1^2 + x_2^2 + x_3^2$, $i = 0, 1$. Now, by considering the switched incremental Lyapunov function as in the proof of Theorem 7.30, the switched system is incrementally stable w.r.t. any switching signal. The trajectories of the solutions for x_1 under the periodic switching signal $\sigma(k) = 0$ if $k \in [0, 10) \cup [20, 30) \cup \cdots$ and $\sigma(k) = 1$ if $k \in [10, 20) \cup [30, 40) \cup \cdots$ is

shown in Fig. 7.4, which illustrates incrementally stable trajectories (trajectories of x_3 are not shown since its solution is $x_3(k) = 0$ for $k = 1, 2, \ldots$ and therefore not exciting). $\Diamond$

7.4 Concluding Remarks

The solvability of discrete-time nonlinear singular switched systems with fixed switching signals has been addressed in this chapter. A necessary and sufficient condition has been derived for the solvability under a fixed switching signal. Solvability characterizations for fixed mode sequences (with arbitrary switching times) and for arbitrary switching signals have also been derived by imposing the solvability condition on switching signals involved in the characterization.

The solvability characterization for fixed switching signals presented in Theorem 7.11 is considered with finitely many modes. However, in practice, it is still valid for systems with infinitely many modes by checking the corresponding condition online. Thus, the system is solvable w.r.t. σ (with infinitely many modes) if and only if the condition (7.9) holds for $k = 0, 1, \ldots$. Meanwhile, the solvability for a fixed mode sequence with arbitrary switching times is also still valid for systems with infinitely many modes. The solvability condition in Proposition 7.12 can be checked online for $k = 0, 1, \ldots$, and if at some time instant, the system stays in the mode, then it suffices to check the first condition for the currently active mode. In contrast, as for linear switched systems, the condition for the solvability w.r.t. all switching signals with infinitely many modes is in general not practical since the condition (7.17) needs to hold for all pairs of modes. It is, however, also still practical for some particular switched nonlinear systems with infinitely many modes including systems with modes that can be expressed as a function of a parameter.

Moreover, for solvable systems, the corresponding surrogate nonlinear ordinary systems have been established by utilizing the one-step map from the current mode to the successive mode. Via surrogate systems, sufficient (and necessary) conditions for Lyapunov and incremental stability analyses for the original singular systems have been proposed using single and switched Lyapunov function approaches.

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Chapter 8 Conclusions and Outlooks

"Being singular is completely fine, it just needs
a special approach to treat it."

Contents

8.1 Concluding Remarks

Fundamental properties including observability, determinability, reachability, and controllability of (nonsingular) linear switched systems have been characterized in the first part of this thesis. Necessary and sufficient conditions for their characterizations have been proposed via geometric approaches. For one-dimensional and single-switch systems with slow switching, those properties except the determinability are independent of switching times. It is not proven yet that the determinability is also independent of switching times nor has a counter-example been found, however, it is conjectured that it does not depend on switching times.

Solution theory for singular linear switched systems has been investigated in the second part of the thesis. Solvability notions for the well-posedness of this system class have been defined both for systems without inputs and systems with inputs. Besides the well-posedness, the defined solvability notions guarantee causality and that a surrogate system–an ordinary switched system that has equivalent behavior–can be found. The corresponding necessary and sufficient conditions have also been investigated based on the class of the switching signals. In particular for systems with inputs, the weak solvability

implies (constrict) causality whereas the strong solvability implies strict causality. By utilizing the surrogate systems, the fundamental properties as in the study of ordinary systems have also been characterized under fixed and known switching signals. It has been proven that the observability of two-dimensional and single-switch singular linear switched systems under slow switching is independent of switching times as well as the determinability of two-dimensional systems. Meanwhile, the determinability of single-switch systems is conjectured to be independent of switching times. Furthermore, it is also conjectured that two-dimensional and single-switch systems have constant reachability and controllability.

Via generalization, solution theory for nonlinear singular systems has also been investigated both for systems without switching and with switching, nevertheless, without inputs. Necessary and sufficient conditions as well as the corresponding surrogate systems have also been established. Further results related to the Lyapunov and incremental stability characterizations for this nonlinear system class have also been derived by employing the surrogate systems. In principle, the classic Lyapunov stability theorem can still be used, however, it is not practical for the systems with switching since the consistency set is not the whole space. Switched Lyapunov function is then proposed for this; this is rather more practical which is concluded from theoretical observations and illustrative examples.

8.2 Future Research Directions

Even though solution theory has been extensively studied in this thesis, it still leaves many open problems that can be considered as future research directions, such as further solvability analysis, characterizations for some other fundamental properties, observer designs, control designs, and many more.

For solvability, in the linear case, one may study further singular systems that are not regular, and the term singular singular systems may be used to call this system class. In the nonlinear case, it remains open to characterize solvability for systems with inputs. More general nonlinear singular (switched) systems could also be studied, for example, by replacing the (constant) matrices E_i with some matrix-valued nonlinear functions. Further features such as disturbances, uncertainties, and time delays, among others, can also be further included.

The observability and reachability characterizations as well as the determinability and controllability characterizations derived in this thesis can be further used to study duality notions. Moreover, by utilizing surrogate systems obtained in this thesis, further analysis such as Lyapunov stability (for the linear

case) and other stability notions such as convergent analysis and contraction analysis (for the nonlinear case) can also be studied. For the linear case, it is possible to rewrite the singular switched system (with fixed switching signal) as an ordinary switched system whose stability can be checked with available methods for linear time-varying systems, however, simpler conditions could be derived since the system is not fully time-varying and may stay in a certain mode for a certain mode duration. Furthermore, observer and control designs remain open to study.

The switching signals considered in this thesis are triggered only by time, and thus the corresponding switched systems are a particular class of hybrid systems. Further studies may investigate other classes of hybrid singular systems such as singular switched systems in which the switching is triggered by the state.

In particular, the results with a fixed mode sequence but arbitrary switching times can also be used for switched systems with the switching rule triggered also by events as long as the mode sequence is known. Therefore, one possible extension for future work is designing state-feedback control algorithms for switched systems including ones with event-triggered switching rules. Other possible extensions could be observer designs and further studies for systems with inputs.

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Appendix A | Some Algebraic Properties

Let $\mathcal{V}_1, \mathcal{V}_2$ be any linear subspaces in $\mathbb{R}^n$, $A, M \in \mathbb{R}^{n\times n}$, $n \in \mathbb{N}$ be matrices (not necessarily invertible), B and C are matrices with suitable dimensions, $\operatorname{im} A$ be the image or range of linear transformation A, $\ker A$ be the kernel of A, and $A^{-1}(\mathcal{V}) = \{ \xi \in \mathbb{R}^n \mid A\xi \in \mathcal{V} \}$ be the preimage of A over a set $\mathcal{V} \subseteq \mathbb{R}^n$. Then [84]

1. $A \operatorname{im} B = \operatorname{im} AB$
2. $A^{-1}(\ker B) = \ker BA$
3. $\operatorname{im} A \subseteq \operatorname{im}[A, B]$
4. $\ker \left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right] = \ker A \cap \ker B$
5. $A^{-1}(\mathcal{V}_1 \cap \mathcal{V}_2) = A^{-1}\mathcal{V}_1 \cap A^{-1}\mathcal{V}_2$
6. $A(\mathcal{V}_1 + \mathcal{V}_2) = A\mathcal{V}_1 + A\mathcal{V}_2$
7. $A(\mathcal{V}_1 \cap \mathcal{V}_2) \subseteq A\mathcal{V}_1 \cap A\mathcal{V}_2$ (with equality if and only if $(\mathcal{V}_1 + \mathcal{V}_2) \cap \ker A = (\mathcal{V}_1 \cap \ker A) + (\mathcal{V}_2 \cap \ker A)$, which holds, in particular, for any invertible A)
8. $A^{-1}(\mathcal{V}_1 + \mathcal{V}_2) \supseteq A^{-1}\mathcal{V}_1 + A^{-1}\mathcal{V}_2$ (with equality if and only if $(\mathcal{V}_1 + \mathcal{V}_2) \cap \operatorname{im} A = (\mathcal{V}_1 \cap \operatorname{im} A) + (\mathcal{V}_2 \cap \operatorname{im} A)$, which holds, in particular, for any invertible A).

Definition A.1 (Generalized inverse). For a matrix $M \in \mathbb{R}^{m\times n}$, a generalized inverse of M is defined as a matrix $M^+ \in \mathbb{R}^{n\times m}$ that satisfies $MM^+M = M$. $\Diamond$

A generalized matrix always exists, but is not necessarily unique, one possible choice is the well-known Moore-Penrose pseudoinverse [85]. Furthermore,

for two generalized inverses M_1 and M_2 of M, we have that $(M_1 - M_2)y \in \ker M$ for all $y \in \operatorname{im} M$. In particular, for calculations, the well-known Moore-Penrose inverse can be used, for which efficient algorithms are available in the literature, e.g. by using a singular value decomposition [63].

Lemma A.2 (Preimage property). For any matrix $M \in \mathbb{R}^{n \times n}$ and $y \in \operatorname{im} M$, we have that

$$M^{-1}\{y\} = \{M^+ y\} + \ker M$$

where M^+ is any generalized inverse of M. ◊

Proof. Take any arbitrary $x \in M^{-1}\{y\}$, then by definition $My = x$. Let $x_0 := x - M^+ y$, multiplying both sides with M yields

$$Mx_0 = Mx - MM^+ y = y - MM^+ y.$$

Since $y \in \operatorname{im} M$, we can represent it as $y = Mz$ for some $z \in \mathbb{R}^n$, and consequently

$$Mx_0 = Mz - MM^+ Mz = 0,$$

i.e. $x_0 \in \ker M$ and hence $x = M^+ y + x_0 \in \{M^+ y\} + \ker M$. This concludes $M^{-1}\{y\} \subseteq \{M^+ y\} + \ker M$. To show the converse subspace relation, let $x \in \{M^+ y\} + \ker M$, i.e. $x = M^+ y + x_0$ for some $x_0 \in \ker M$. Multiplying both sides with M gives

$$Mx = MM^+ y + Mx_0 = MM^+ y + 0.$$

Writing $y = Mz$ for some $z \in \mathbb{R}^n$ we have

$$Mx = MM^+ Mz = Mz = y$$

i.e. $x \in M^{-1}\{y\}$. This concludes $M^{-1}\{y\} \supseteq \{M^+ y\} + \ker M$. □

The following lemma provides a property of an intersection of two affine sets and the representation of the intersection via a projector.

Lemma A.3 (Projector of affine spaces). Consider sets $\mathbb{Z}, \mathbb{U} \subseteq \mathbb{R}^n$ and subspaces $\mathcal{V}, \mathcal{W} \subseteq \mathbb{R}^n$. Then, for all pairs $(z, u) \in \mathbb{Z} \times \mathbb{U}$, $(\{z\} + \mathcal{V}) \cap (\{u\} + \mathcal{W})$ is a singleton if, and only if, $\mathbb{U} - \mathbb{Z} \subseteq \mathcal{V} \oplus \mathcal{W}$ where $\mathbb{U} - \mathbb{Z} = \{\ u - z \mid z \in \mathbb{Z}, u \in \mathbb{U}\ \}$. In that case,

$$(\{z\} + \mathcal{V}) \cap (\{u\} + \mathcal{W}) = \Pi_{\mathcal{V}}^{\mathcal{W}}(u - z) + z = \Pi_{\mathcal{W}}^{\mathcal{V}}(z - u) + u, \tag{A.1}$$

where $\Pi_{\mathcal{V}}^{\mathcal{W}} : \mathcal{V} \oplus \mathcal{W} \to \mathcal{V}$ is the canonical projector from $\mathcal{V} \oplus \mathcal{W}$ to $\mathcal{V}$. ◊

Proof. Step 1: We show that for arbitrary pair $(z, u) \in \mathbb{Z} \times \mathbb{U}$, the intersection $(\{z\} + \mathcal{V}) \cap (\{u\} + \mathcal{W})$ is nonempty for all pairs $(z, u) \in \mathbb{Z} \times \mathbb{U}$ if, and only if, $\mathbb{U} - \mathbb{Z} \subseteq \mathcal{V} + \mathcal{W}$.
Step 1a: Necessity.
Seeking a contradiction, assume $\mathbb{U} - \mathbb{Z} \not\subseteq \mathcal{V} + \mathcal{W}$, i.e. there exists $(z, u) \in \mathbb{Z} \times \mathbb{U}$

with $u - z \in \mathbb{U} - \mathbb{Z}$ which is not in $\mathcal{V} + \mathcal{W}$. Choose $x \in (\mathbb{Z} + \mathcal{V}) \cap (\mathbb{U} + \mathcal{W})$, then there is $z \in \mathbb{Z}$, $u \in \mathbb{U}$, $v \in \mathcal{V}$, and $w \in \mathcal{W}$ with $x = z + v = u + w$ such that $u - z = v - w \in \mathcal{V} + \mathcal{W}$, which contradicts the choice of u.
Step 1b: Sufficiency.
Let $u - z \in \mathbb{U} - \mathbb{Z} \subseteq \mathcal{V} + \mathcal{W}$. Choose $v \in \mathcal{V}$ and $w \in \mathcal{W}$ such that $u - z = v + w$. Then $z + v = u - w \in \mathbb{U} + \mathcal{W}$. Hence, $z + v \in (\mathbb{Z} + \mathcal{V}) \cap (\mathbb{U} + \mathcal{W})$, i.e., the latter intersection is not empty.

Step 2: We will prove that if $(\mathbb{Z} + \mathcal{V}) \cap (\mathbb{U} + \mathcal{W})$ is non-empty for at least one $z \in \mathbb{Z}$ and $u \in \mathbb{U}$ then $(\mathbb{Z} + \mathcal{V}) \cap (\mathbb{U} + \mathcal{W})$ is a singleton for each $z \in \mathbb{Z}$ and each $u \in \mathbb{U}$, if, and only if, $\mathcal{V} \cap \mathcal{W}$ is a singleton.
Step 2a: Necessity.
Seeking a contradiction, assume that $\mathcal{V} \cap \mathcal{W} \neq \{0\}$ and choose $0 \neq p \in \mathcal{V} \cap \mathcal{W}$. Choose some $z \in \mathbb{Z}$ and $u \in \mathbb{U}$ for which $(\{z\} + \mathcal{V}) \cap (\{u\} + \mathcal{W})$ is non-empty and choose $x \in (\{z\} + \mathcal{V}) \cap (\{u\} + \mathcal{W})$. Then there is $w \in \mathcal{W}$ with $x = z + v = u + w$. Since $z + v + p = u + w + p$ and $v + p \in \mathcal{V}$ as well as $w + p \in \mathcal{W}$ we arrive at $z + v + p \in (\{z\} + \mathcal{V}) \cap (\{u\} + \mathcal{W})$, and since $z + v + p \neq z + v$, the set $(\{z\} + \mathcal{V}) \cap (\{u\} + \mathcal{W})$ is not a singleton (and also not empty).
Step 2b: Sufficiency.
For some $z \in \mathbb{Z}$ and $u \in \mathcal{U}$ for which $(\{z\} + \mathcal{V}) \cap (\{u\} + \mathcal{W})$ is non-empty, let $x_1, x_2 \in (\{z\} + \mathcal{V}) \cap (\{u\} + \mathcal{W})$. Then, there exists $v_1, v_2 \in \mathcal{V}$ and $w_1, w_2 \in \mathcal{W}$ with $x_1 = z + v_1 = u + w_1$ and $x_2 = z + v_2 = u + w_2$. Consequently, $x_1 - x_2 = z + v_1 - z - v_2 = u + w_1 - u - w_2 = v_1 - v_2 = w_1 - w_2$. Since $v_1 - v_2 \in \mathcal{V} \cap \mathcal{W} = \{0\}$, $v_1 = v_2$, which implies $x_1 = x_2$, i.e., $(\{z\} + \mathcal{V}) \cap (\{u\} + \mathcal{W})$ is a singleton.

Step 3: We show (A.1).
Let $u - z \in \mathbb{U} - \mathbb{Z} \subseteq \mathcal{V} + \mathcal{W}$ and choose (unique) $v \in \mathcal{V}$ and $w \in \mathcal{W}$ such that $u - z = v + w$. Then, $u - z + z = z + v + w$, $\Pi_{\mathcal{V}}^{\mathcal{W}}(u - z) + z = z + v \in \{z\} + \mathcal{V}$, and $u + \Pi_{\mathcal{V}}^{\mathcal{W}}(u - z) + z - u = u + z + v - u = u + z + (u - z - w) - u = u - w \in \{u\} + \mathcal{W}$. Hence $\Pi_{\mathcal{V}}^{\mathcal{W}}(u - z) + z \in (\{z\} + \mathcal{V}) \cap (\{u\} + \mathcal{W})$ which together with Step 2 completes the proof. The second formula is another representation of the first and is derived from $\Pi_{\mathcal{V}}^{\mathcal{W}}(u - z) + z = (I - \Pi_{\mathcal{W}}^{\mathcal{V}})(u - z) + z = (u - z) + \Pi_{\mathcal{W}}^{\mathcal{V}}(z - u) + z = \Pi_{\mathcal{W}}^{\mathcal{V}}(z - u) + u$. □

In other words, for any $z \in \mathbb{Z}$ and any $u \in \mathbb{U}$ satisfying $(u - z) \in \mathbb{U} - \mathbb{Z} \subseteq \mathcal{V} \oplus \mathcal{W}$, there exists $v \in \mathcal{V}$ for which there exists $w \in \mathcal{W}$ with $z + v = u + w =: p$, and this vector is given by $p = \Pi_{\mathcal{V}}^{\mathcal{W}}(u - z) + z = \Pi_{\mathcal{W}}^{\mathcal{V}}(z - u) + u$. In particular, note that the condition $\mathbb{U} - \mathbb{Z} \subseteq \mathcal{V} \oplus \mathcal{W}$ is equivalent to $\mathbb{Z} - \mathbb{U} \subseteq \mathcal{V} \oplus \mathcal{W}$ since $\mathcal{V} \oplus \mathcal{W}$ is a (sub)space.

Lemma A.4 (Kernel equality). For any invertible matrix $A \in \mathbb{R}^{n\times n}$, and any $C \in \mathbb{R}^{m\times n}$, the equality

$$\ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \ker \begin{bmatrix} CA^k \\ CA^{1+k} \\ \vdots \\ CA^{\widehat{n}-1+k} \end{bmatrix} \tag{A.2}$$

holds for every $k \in \mathbb{Z}$ and every $\widehat{n} \geq n$. $\Diamond$

Proof. Consider first $k = 1$. By Cayley-Hamilton theorem, each of the rows of CA^k for $k > n-1$ are linearly dependent on the rows of the observability matrix $[C^\top, (CA)^\top, \dots, (CA^{n-1})^\top]^\top$, hence

$$\ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \\ CA^{n} \\ \vdots \\ CA^{\widehat{n}} \end{bmatrix} \subseteq \ker \begin{bmatrix} CA \\ \vdots \\ CA^{\widehat{n}-1} \\ CA^{\widehat{n}} \end{bmatrix}.$$

To show the converse subspace relation, let $x \in \ker \begin{bmatrix} CA \\ \vdots \\ CA^{\widehat{n}-1} \\ CA^{\widehat{n}} \end{bmatrix}$. Then the Cayley-Hamilton theorem ensures existence of $a_0, a_1, \dots, a_{n-1} \in \mathbb{R}$ such that

$$0 = CA^n x = C \sum_{i=0}^{n} a_i A^i x.$$

By assumption $CA^i x = 0$ for $i = 1, 2, \dots, n-1$, hence we can conclude that

$$0 = \sum_{i=0}^{n} a_i CA^i x = a_0 Cx.$$

Since A is invertible, the characteristic polynomial $\det(sI - A) = \sum_{i=0}^{n} a_i s$ cannot have the root $\lambda = 0$, i.e. $a_0 \neq 0$ and we can conclude that $Cx = 0$. Therefore, $x \in \ker C$ and thus $x \in \ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\widehat{n}-1} \end{bmatrix}$.

Now our claim is that for any $k \in \mathbb{Z}$,

$$\ker \begin{bmatrix} CA^k \\ CA^{1+k} \\ \vdots \\ CA^{\widehat{n}-1+k} \end{bmatrix} = \ker \begin{bmatrix} CA^{k+1} \\ CA^{1+k+1} \\ \vdots \\ CA^{\widehat{n}-1+k+1} \end{bmatrix}, \tag{A.3}$$

from which (A.2) follows inductively. Let $\widehat{C} := CA^k$ (this is also applicable for $k < 0$ since A is invertible) then using the same arguments as in the case of $k = 1$, we have that

$$\ker \begin{bmatrix} \widehat{C} \\ \widehat{C}A \\ \vdots \\ \widehat{C}A^{\widehat{n}-1} \end{bmatrix} = \ker \begin{bmatrix} \widehat{C}A \\ \widehat{C}A^2 \\ \vdots \\ \widehat{C}A^{\widehat{n}} \end{bmatrix} \tag{A.4}$$

i.e. (A.3) holds. □

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Summary

The study in this thesis starts with characterizing four fundamental properties considered for ordinary linear switched systems: observability, determinability, reachability, and controllability. Necessary and sufficient conditions for those properties have been established via geometric approaches under fixed switching signals. Even though the analysis of those features for ordinary linear switched systems has been addressed in existing studies under arbitrary switching signals, characterizations for fixed switching signals are significant due to their less restrictive conditions. Furthermore, studies to investigate whether those properties are independent of switching times are also carried out; this is the main novel aspect of the study for ordinary linear switched systems. It turns out that all of those properties of one-dimensional switched systems under slow switching are independent of switching times. Meanwhile, three properties, observability, reachability, and controllability of single switch systems under slow switching are also independent of switching times. Even though it has not been (completely) proven, from observations on the characterization, it is conjectured that the determinability of single-switch systems is also independent of switching times, and a counter-example has not been found that also supports the conjecture.

Unlike ordinary dynamical systems which are naturally well-posed, singular dynamical systems, in general, are not well-posed due to the presence of three possible solvability issues: inconsistent initial values, nonuniqueness of solutions, and noncausality. The first issue may occur due to the fact that in general, not all initial values in the whole space have a solution whereas the second issue happens when the algebraic constraint does not have a solution or does have many solutions. Meanwhile, the third issue deals with the situation of future states determining past states.

This thesis mainly concerns the well-posedness of singular systems which deals with those solvability issues, both for systems without switching and with switching. The solvability study covers the system classes of homogeneous singular linear (switched) systems, inhomogeneous singular linear (switched) systems, and homogeneous singular nonlinear (switched) systems.

The second part of the thesis focuses on singular linear systems. Solvability notions for the well-posedness of singular linear systems are defined; those notions guarantee the existence and uniqueness of a solution for all consistent initial values and causality. For systems with inputs, unlike ordinary systems which are always solvable for arbitrary input sequences, singular systems are not always solvable with arbitrary input sequences, i.e., the system has solutions only with inputs satisfying the algebraic constraints. Therefore, for nonswitched systems, three types of solvability notions are proposed with respect to the input sequences: first, solvability is defined w.r.t. a fixed input sequence, second, the notion of weak solvability in which only the existence of an input sequence such that the system is solvable w.r.t. that input sequence, and third, the notion of strong solvability in which solvability for all input sequences is required. For switched systems, those notions are first defined with respect to a fixed switching signal, and further notions are then defined with respect to a fixed mode sequence (with arbitrary switching times) and arbitrary switching signals.

Necessary and sufficient conditions for the well-posedness of each singular system class have been derived. All solvability characterizations are done with the help of the so-called index-1 notion and its variations. For nonswitched systems, the characterization is done with the help of the index-1 notion for homogeneous systems, and the strict index-1 for inhomogeneous systems. Those index-1 notions are then generalized to the so-called switched (strict) index-1 for a fixed switching signal, sequential (strict) index-1 for a fixed mode sequence with arbitrary switching times, and joint (strict) index-1 for arbitrary switching signals. The solvability is then characterized with the help of those further index-1 notions, and a solvability notion corresponds to a specific index-1 notion.

Furthermore, alongside resolving the solvability issues described above, all solvability notions also guarantee the existence of surrogate systems–ordinary systems that have equivalent behavior with the corresponding singular systems–for each corresponding singular system class. Surrogate systems are successfully established for solvable systems. By utilizing surrogate systems, observability, determinability, reachability, and controllability are then characterized for singular linear systems, both with and without switching. Necessary and sufficient conditions for those features are derived via geometric approaches. Furthermore, as in the study part for ordinary systems, studies to investigate whether those fundamental properties are independent of switching times are also carried out for singular linear switched systems under slow switching. Results showed that the observability of two-dimensional and single-switch systems and the determinability of two-dimensional systems are independent

of switching times. Proofs for the other properties have not been derived as well as counter-examples have not been found, nevertheless, it is conjectured that the determinability of single-switch systems, reachability of both two-dimensional and single-switch systems, and controllability of both two-dimensional and single-switch systems are also independent of switching times.

The last part of the thesis deals with singular nonlinear systems, both without and with switching. Necessary and sufficient conditions for solvability are provided, and furthermore, the corresponding surrogate systems are also established. By utilizing surrogate systems, Lyapunov and incremental stability properties are analyzed. Sufficient (and necessary) conditions have also been derived via single Lyapunov function and switched Lyapunov function approaches.

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Samenvatting

De studie in dit proefschrift begint met het karakteriseren van vier karaktereigenschappen van reguliere lineaire schakelende systemen: waarneembaarheid, bepaalbaarheid, bereikbaarheid en controleerbaarheid. Noodzakelijke en voldoende voorwaarden voor deze eigenschappen zijn tot stand gekomen via geometrische benaderingen onder vaste schakelsignalen. Hoewel de analyse van deze kenmerken voor regulieren lineaire schakelende systemen wordt behandeld in bestaande onderzoeken aangaande willekeurige schakelsignalen, zijn karakteriseringen voor vaste schakelsignalen significant vanwege hun minder beperkende voorwaarden. Verder is onderzocht of de eigenschappen van reguliere lineaire schakelende systemen onafhankelijk zijn van de schakeltijden, dit wordt tevens gezien als het belangrijkste nieuwe aspect van dit onderzoek. Uit het onderzoek blijkt dat deze eigenschappen voor eendimensionale schakelende systemen bij langzaam schakelen onafhankelijk zijn van de schakeltijden. Daarbij geldt, voor eenmalig schakelende systemen, dat de eigenschappen waarneembaarheid, bereikbaarheid en controleerbaarheid onafhankelijk zijn van de schakeltijden. Hoewel het niet (volledig) is bewezen, wordt op basis van observaties vermoed dat de bepaalbaarheid van eenmalig schakelende systemen ook onafhankelijk is van de schakeltijden; dit vermoeden wordt ondersteunt door het feit dat er geen tegenvoorbeelden zijn waargenomen.

In tegenstelling to reguliere dynamische systemen die van nature well-posed zijn, zijn singuliere dynamische systemen over het algemeen niet well-posed door de aanwezigheid van drie mogelijke oplosbaarheidsproblemen: inconsistente begincondities, geen uniciteit van oplossingen en geen causaliteit. Het eerste probleem kan optreden doordat, in het algemeen, niet alle beginwaarden in de hele ruimte een oplossing hebben. Het tweede probleem doet zich voor wanneer de algebraïsche beperking geen of juist meerdere oplossingen heeft en niet-causaliteit treedt op wanneer toekomstige toestanden van het systeem vroegere toestanden bepalen.

Dit proefschrift beschouwt de well-posedness van singuliere systemen en houdt zich met name bezig met de oplosbaarheidsproblemen, voor zowel schakelende als niet-schakelende systemen. Deze oplosbaarheidsstudie wordt

uitgevoerd voor homogene singuliere lineaire (schakelende) systemen, inhomogene singuliere lineaire (schakelende) systemen en homogene singuliere nonlineaire (schakelende) systemen.

Het tweede deel van het proefschrift focust zich op singuliere lineaire systemen. Oplosbaarheidsbegrippen voor de well-posedness van singuliere lineaire systemen worden gedefinieerd. Deze oplosbaarheidsbegrippen garanderen het bestaan en uniciteit van oplossingen voor alle consistente begincondities en causaliteit. Aangaande systemen met ingangen, in tegenstelling tot reguliere systemen die een oplossing hebben voor elke willekeurige ingangssequentie, hebben singuliere systemen niet voor elke willekeurige ingangssequentie een oplossing. Een singulier systeem heeft namelijk alleen een oplossing wanneer de ingangssequentie voldoet aan de opgelegde algebraïsche restrictie. Daarom worden er voor niet-schakelende systemen drie soorten oplosbaarheidsbegrippen voorgesteld met betrekking tot de ingangssequentie van de systemen: ten eerste, oplosbaarheid wordt gedefinieerd met betrekking tot een vast ingangssequentie, ten tweede, zwakke oplosbaarheid is het bestaan van een ingangssequentie zodanig dat het systeem oplosbaar is ten opzichte van die ingangssequentie en ten derde, sterke oplosbaarheid vereist oplosbaarheid met betrekking tot elke mogelijke ingangssequentie. Voor schakelende systemen worden deze begrippen eerst gedefinieerd met betrekking tot een vast schakelsignaal, vervolgens worden de begrippen gedefinieerd met betrekking tot een vaste modussequentie met willekeurige schakelsignalen.

Noodzakelijke en voldoende voorwaarden voor de well-posedness van elke afzonderlijke systeem klasse zijn afgeleid. Alle oplosbaarheidskarakteriseringen worden gedaan met behulp van de zogenaamde index-1 notie en zijn varianten. Voor niet-schakelende systemen gebeurt deze karakterisering met behulp van de index-1 notie voor homogene systemen en de strikte index-1 notie voor inhomogene systemen. Deze index-1 noties worden vervolgens gegeneraliseerd naar de zogenaamde schakelende (strikte) index-1 notie voor een vast schakelsignaal, sequentiële (strikte) index-1 notie voor een vaste modussequentie met willekeurige schakeltijden, en gezamenlijke (strikte) index-1 notie voor willekeurige schakelsignalen. Oplosbaarheid wordt vervolgens gekarakteriseerd met behulp van die index-1 noties en een notie van oplosbaarheid komt vervolgens overeen met een specifieke index-1 notie.

Bovendien garanderen alle oplosbaarheidsbegrippen, naast het oplossen van de hierboven beschreven oplosbaarheidsproblemen, ook het bestaan van surrogaat systemen – reguliere systemen die zich gelijkwaardig gedragen als de overeenkomstige singuliere systemen – voor elke overeenkomstige singuliere systeem klasse. Surrogaat systemen zijn met succes opgezet voor oplosbare systemen. Door gebruik te maken van surrogaat systemen kunnen de eigen-

schappen waarneembaarheid, bepaalbaarheid, bereikbaarheid en controleerbaarheid gekarakteriseerd worden voor zowel schakelende als niet-schakelende singuliere lineaire systemen. Geometrische methoden worden vervolgens gebruikt om noodzakelijke en voldoende voorwaarden op te stellen voor deze eigenschappen. Bovendien wordt, net als in de studie naar reguliere systemen, gekeken of deze fundamentele eigenschappen voor singuliere lineaire schakelende systemen die langzaam schakelen onafhankelijk zijn van de schakeltijden van de systemen. De resultaten hiervan laten zien dat de waarneembaarheid van twee-dimensionale en eenmalig schakelende systemen en de bepaalbaarheid van twee-dimensionale systemen onafhankelijk zijn van de schakeltijden. Er zijn geen bewijzen afgeleid voor de andere eigenschappen, maar er zijn ook geen tegenvoorbeelden gevonden. Niettemin wordt aangenomen dat de bepaalbaarheid van eenmalig schakelende systemen, de bereikbaarheid van zowel twee-dimensionale als eenmalig schakelende systemen, en de bestuurbaarheid van zowel twee-dimensionale als eenmalig schakelende systemen onafhankelijk zijn van schakeltijden.

Het laatste deel van het proefschrift richt zich op schakelende en niet-schakelende singuliere nonlinaire systemen. Noodzakelijke en voldoende voorwaarden voor oplosbaarheid zijn gegeven en de bijbehorende surrogaat systemen worden tot stand gebracht. Door gebruik te maken van deze surrogaat systemen worden Lyapunov en incrementele stabiliteitseigenschappen geanalyseerd. Tenslotte worden voldoende (en noodzakelijke) voorwaarden afgeleid via enkele en schakelende Lyapunov functies.

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Ringkasan

Studi dalam tesis ini dimulai dengan pengakarakterisasian empat sifat-sifat fundamental (dasar) untuk sistem linier biasa berganti (*ordinary linear switched systems*): keteramatan, ketertentuan, keterjangkauan, dan keterkendalian. Syarat perlu dan syarat cukup untuk sifat-sifat tersebut telah ditemukan melalui pendekatan geometri untuk sinyal-sinyal pergantian yang diketahui dan pasti.

Meskipun analysis untuk fitur-fitur tersebut untuk sistem linier biasa berganti telah dilakukan dalam studi-studi yang sudah ada untuk sinyal-sinyal pergantian sebarang, karakterisasi untuk sinyal pergantian yang pasti cukup signifikan dikarenakan kondisi-kondisinya lebih longgar. Lebih lanjut, studi-studi untuk menyelidiki apakah sifat-sifat tersebut bergantung atau tidak terhadap waktu-waktu pergantian juga dipelajari; ini merupakan aspek kebaruan utama dari studi untuk sistem linier biasa berganti. Untuk sistem satu dimensi dengan waktu pergantian yang cukup lambat, keempat sifat tersebut tidak bergantung terhadap waktu pergantian. Sementara itu, tiga sifat, keteramatan, keterjangkauan, dan keterkendalian dari sistem dengan pergantian tunggal dibawah waktu-waktu pergantian yang cukup lambat juga tidak bergantung pada waktu pergantian. Meskipun belum dibuktikan secara lengkap, dari hasil studi pada pengkarakterisasian, dikonjekturkan bahwa ketertentuan dari sistem-sistem dengan pergantian tunggal juga tidak bergantung pada waktu pergantian, dan contoh penyangkal juga belum ditemukan, yang juga menunjang konjektur tersebut.

Tidak seperti sistem-sistem dinamik biasa yang secara alami bersifat *well-posed*, sistem-sistem dinamik singular, pada umumnya, tidak *well-posed* dikarenakan ia memuat tiga isu solvabilitas: nilai-nilai awal inkonsisten, solusi-solusi tidak unik, dan ketidakkausalan. Isu pertama dapat muncul karena pada umumnya tidak semua nilai-nilai awal di keseluruhan ruang memiliki solusi sedangkan isu kedua terjadi ketika kendala aljabar tidak memiliki solusi atau memiliki solusi lebih dari satu. Sementara itu, isu ketiga berhadapan dengan situasi dimana state-state di waktu mendatang menentukan state-state di waktu lampau.

Tesis ini berfokus terutama pada studi *well-posedness* untuk sistem-sistem singular yang berhadapan dengan isu-isu solvabilitas tersebut, baik untuk sistem

tanpa pergantian atau dengan pergantian. Studi tentang solvabilitas mencakup kelas-kelas sistem linier singular homogen (berganti), sistem linier singular tidak homogen (berganti), dan sistem nonlinier singular homogen (berganti).

Bagian kedua dari tesis ini berfokus pada sistem liner singular. Gagasan-gagasan solvabilitas untuk *well-posedness* dari sistem-sistem linier singular didefinisikan; gagasan-gagasan tersebut menjamin eksistensi dan keunikan solusi untuk setiap nilai awal konsisten dan kekausalan. Untuk sistem-sistem dengan input, tidak seperti sistem-sistem biasa yang selalu memiliki solusi unik untuk setiap barisan masukan, sistem-sistem singular tidak selalu memiliki solusi untuk sebarang barisan masukan, yaitu, sistem memiliki solusi hanya untuk masukan-masukan yang memenuhi kendala-kendala aljabar pada sistem. Sehingga, untuk sistem-sistem tidak berganti, tiga gagasan solvabilitas diusulkan menyesuaikan terhadap barisan-barisan masukan. Pertama, solvabilitas didefinisikan terhadap barisan masukan pasti. Yang kedua adalah gagasan solvabilitas lemah dimana hanya eksistensi suatu barisan masukan disyaratkan sedemikian sehingga sistemnya dapat diselesaikan terhadap barisan masukan tersebut. Yang ketiga adalah gagasan solvabilitas kuat dimana solvabilitas untuk semua barisan masukan disyaratkan. Untuk sistem-sistem berganti, gagasan-gagasan tersebut pertama didefinisikan terhadap sinyal pergantian pasti, dan gagasan-gagasan selanjutnya didefinisikan terhadap barisan mode pasti (dengan sebarang waktu pergantian) dan sebarang sinyal pergantian.

Syarat perlu dan cukup untuk *well-posedness* dari setiap kelas sistem singular telah didapatkan. Semua karakterisasi solvabilitas diselesaikan dengan bantuan gagasan indeks-1 dan variasinya. Untuk sistem-sistem tanpa pergantian, pengkarakterisasian didapatkan dengan bantuan gagasan indeks-1 untuk sistem-sistem homogen dan strik indeks-1 untuk sistem-sistem nonhomogen. Gagasan-gagasan indeks-1 tersebut kemudian diperumum menjadi (strik) indeks-1 berganti untuk sinyal pergantian pasti,(strik) indeks-1 sekuensial untuk barisan mode pasti dengan sebarang waktu pergantian, dan (strik) indeks-1 bersama untuk sinyal pergantian sebarang. Solvabilitas kemudian dikarakterisasi dengan bantuan gagasan-gagasan indeks-1 tersebut, dan sebuah gagasan solvabilitas bersesuaian dengan sebuah gagasan indeks-1.

Lebih lanjut, disamping menyelesaikan isu-isu solvabilitas yang dikemukakan diatas, semua gagasan solvabilitas juga menjamin eksistensi sistem-sistem pengganti–sistem-sistem biasa yang memiliki perilaku ekuivalen dengan sistem-sistem singular yang bersesuaian–untuk setiap kelas sistem singular yang bersesuaian. Sistem-sistem pengganti berhasil dibentuk untuk sistem-sistem yang *solvable*. Dengan memanfaatkan sistem-sistem pengganti tersebut, keteramatan, ketertentuan, keterjangkauan, dan keterkendalian kemudian dikarakterisasi untuk sistem-sistem singular linier baik tanpa pergantian maupun den-

gan pergantian. Syarat perlu dan cukup untuk fitur-fitur tersebut didapatkan melalui pendekatan geometrik. Lebih lanjut, sebagaimana dengan studi pada bagian sistem-sistem biasa, studi-studi untuk menyelidiki apakah sifat-sifat fundamental tersebut bergantung pada waktu pergantian juga dilakukan untuk sistem-sistem linier singular berganti dengan waktu pergantian yang cukup lambat. Hasil menunjukkan bahwa keteramatan darisistem-sistem dua dimensi dan sistem-sistem dengan pergantian tunggal tidak bergantung pada waktu pergantian. Bukti untuk sifat-sifat lainnya belum didapatkan sebagaima contoh penyangkal belum ditemukan, tetapi, dikonjekturkan bahwa ketertentuuan dari sistem-sistem dengan pergantian tunggal, keterjangkauan dari sistem-sistem dua dimensi dan sistem-sistem dengan pergantian tunggal, dan keterkendalian dari sistem-sistem dua dimensi dan sistem-sistem dengan pergantian tunggal juga tidak bergantung pada waktu pergantian.

Bagian terkahir dari tesis ini berhadapan dengan sistem-sistem singular nonlinier, baik tanpa pergantian dan dengan pergantian. Syarat perlu dan cukup untuk solvabilitas diberikan, dan lebih lanjut, sistem-sistem pengganti yang bersesuaian juga dibentuk. Dengan memanfaatkan sistem-sistem pengganti, stabilitas Lyapunov dan *incremental* dianalisis. Syarat cukup (dan perlu) juag didapatkan melalui pendekatan-pendekatan fungsi Lyapunov tunggal dan berganti.

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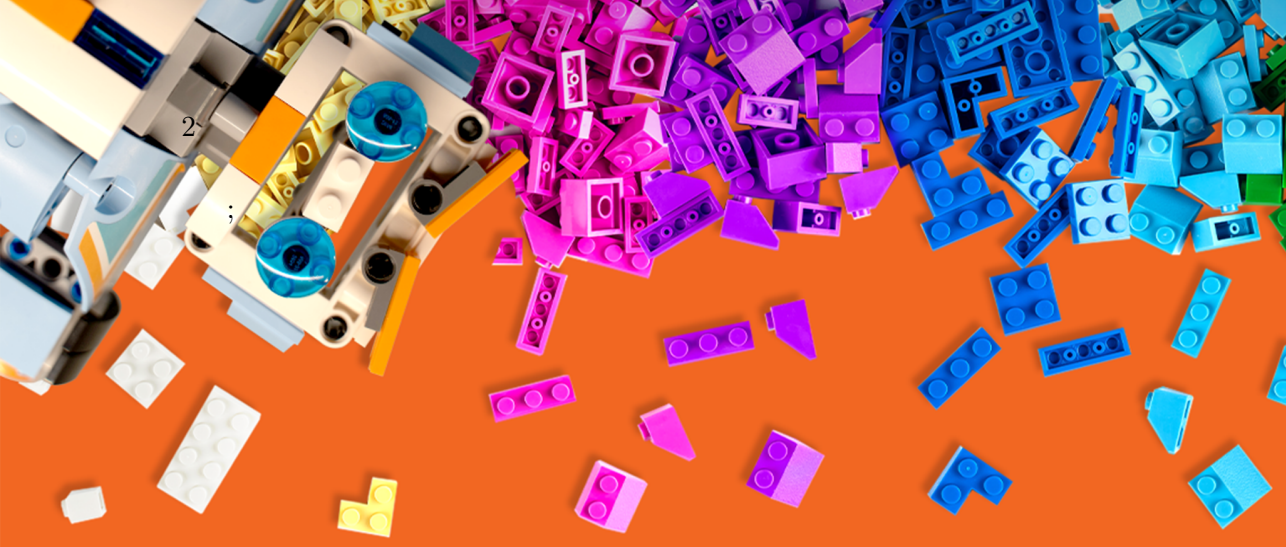

Biography

Sutrisno received his B.Sc. degree in Mathematics from Universitas Diponegoro, Semarang, Indonesia. He obtained his M.Sc. degree in Mathematics with a focus on Systems and Control from Universitas Gadjah Mada, Yogyakarta, Indonesia. Since 2014, he has been affiliated with the Dept. of Mathematics, Universitas Diponegoro.

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His research interests include mathematical modeling, systems & control both in theory and applications, singular systems, and optimization.