

Title	Intersection Dimension of Bipartite Graphs
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Citation	Lecture Notes in Computer Science, 8402: 323-340
Issue Date	2014-04-11
Type	Journal Article
Text version	author
URL	<a href="http://hdl.handle.net/10119/13763">http://hdl.handle.net/10119/13763</a>
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Description	Theory and Applications of Models of Computation, 11th Annual Conference, TAMC 2014, Chennai, India, April 11-13, 2014. Proceedings

# Intersection Dimension of Bipartite Graphs

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## ABSTRACT

We introduce a concept of intersection dimension of a graph with respect to a graph class. This generalizes Ferrers dimension, boxicity, and poset dimension, and leads to interesting new problems. We focus in particular on bipartite graph classes defined as intersection graphs of two kinds of geometric objects. We relate well-known graph classes such as interval bigraphs, two-directional orthogonal ray graphs, chain graphs, and (unit) grid intersection graphs with respect to these dimensions. As an application of these graph-theoretic results, we show that the recognition problems for certain graph classes are NP-complete.

## Keywords

Ferrers dimension, Boxicity, Unit grid intersection graph, Segment-ray graphs, Orthogonal ray graph, NP-hardness.

## 1. INTRODUCTION

Given a family  $\mathcal{F}$  of sets, the *intersection graph* of  $\mathcal{F}$  is the graph in which each set in  $\mathcal{F}$  is a vertex, and two vertices are adjacent if and only if the corresponding sets intersect. A typical example, when  $\mathcal{F}$  is a family of intervals on a line, yields the well-known class of *interval graphs*. Interval graphs have linear time recognition algorithms [3, 11], and nice forbidden structure characterizations. (For instance, the theorem of Lekkerkerker and Boland [24] characterizes interval graphs by the absence of induced cycles of length four and five, and the absence of asteroidal triples.)

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TAMC 2014 Chennai, India

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It is natural to study a bipartite version of intersection graphs: given two families  $\mathcal{F}$  and  $\mathcal{F}'$  of sets, the *intersection bigraph* of  $\mathcal{F}, \mathcal{F}'$  is the bipartite graph in which each set in  $\mathcal{F}$  is a red vertex, each set in  $\mathcal{F}'$  is a blue vertex, and a red vertex is adjacent to a blue vertex if and only if the corresponding sets intersect. When both  $\mathcal{F}$  and  $\mathcal{F}'$  are families of intervals on a line, we obtain *interval bigraphs* studied in [25, 31]. We denote the class of interval bigraphs by IBG. While the recognition of interval bigraphs is polynomial (in time  $O(n^{16})$  [25]), there is no efficient algorithm known, and no characterization in terms of forbidden substructures.<sup>1</sup> It turns out that there are better bipartite analogues of interval graphs. A *two-directional orthogonal ray graph*, or 2DOR graph, is an intersection bigraph of a family  $\mathcal{F}$  of upward rays, and a family  $\mathcal{F}'$  of rightward rays, in the plane [33]. These graphs were introduced in connection with defect tolerance schemes for nano-programmable logic arrays [29, 37]. There are several reasons these 2DOR graphs might be considered better bipartite analogues of interval graphs, including an ordering characterization [33, 21], and a Lekkerkerker-Boland type characterization [12], both analogous to the characterizations for interval graphs. Other forbidden structure characterizations of the class of 2DOR graphs can be found in [19, 20, 12].

Several other graph classes can be defined as intersection bigraphs of two families  $\mathcal{F}, \mathcal{F}'$ . When both  $\mathcal{F}$  and  $\mathcal{F}'$  are inclusion-free families of intervals on a line, we obtain the class of *proper interval bigraphs* which turns out to be the same as the better known class BPG of *bipartite permutation graphs* [20], see below. When  $\mathcal{F}$  is a family of points, and  $\mathcal{F}'$  a family of rightward rays, in a line, we obtain the class CHAIN of *chain graphs* (cf. below). When  $\mathcal{F}$  is a family of vertical segments, and  $\mathcal{F}'$  a family of horizontal segments, in the plane, we obtain the class GIG of *grid intersection graphs*. Several other examples are included in the paper.

We note that the following inclusions are well known or

<sup>1</sup>Recently, Rafiey [28] and Takaoka, Tayu, and Ueno [38] have independently reported faster algorithms for recognizing interval bigraphs.

easy to derive

$$\text{CHAIN} \subseteq \text{BPG} \subseteq \text{IBG} \subseteq \text{2DOR} \subseteq \text{GIG}.$$

We now introduce our concept of intersection dimension. Let  $G = (V, E)$  and  $G' = (V', E')$  be two graphs. The *intersection*  $G \cap G'$  of  $G$  and  $G'$  is the graph  $(V \cap V', E \cap E')$ . For two graph classes  $\mathcal{C}$  and  $\mathcal{C}'$ , we define the *pairwise intersection* of  $\mathcal{C}$  and  $\mathcal{C}'$  as  $\mathcal{C} \bowtie \mathcal{C}' = \{G \cap G' : G \in \mathcal{C}, G' \in \mathcal{C}'\}$ . We also write  $\mathcal{C}^k = \{G_1 \cap G_2 \cap \dots \cap G_k : G_i \in \mathcal{C} \text{ for } 1 \leq i \leq k\}$ . If both  $\mathcal{C}$  and  $\mathcal{C}'$  are closed under taking induced subgraphs, it is easy to check that  $\mathcal{C} \bowtie \mathcal{C}' = \{G \cap G' : G \in \mathcal{C}, G' \in \mathcal{C}', V(G) = V(G')\}$ . Since every graph class in this paper is closed under taking induced subgraphs, we shall from now on use the latter equality, and assume that the vertex sets of the two graphs are the same, when defining the pairwise intersection of graph classes.

The *dimension* of a graph  $G$  with respect to the graph class  $\mathcal{C}$  is the minimum  $k$  such that  $G \in \mathcal{C}^k$ . In the discussion below we shall point out how this definition generalizes Ferrers dimension, boxicity, cubicity, and poset dimension. We are particularly interested in expressing one graph class as a power of another graph class.

It turns out that there are several natural statements of this kind. Among other results we will show that  $\text{2DOR} = \text{CHAIN}^2$ ,  $\text{GIG} \subseteq \text{CHAIN}^4$ , and  $\text{UGIG} = \text{BPG}^2$ . We will also show that several of these inclusions are proper. See Fig. 1 for the summary of our results.

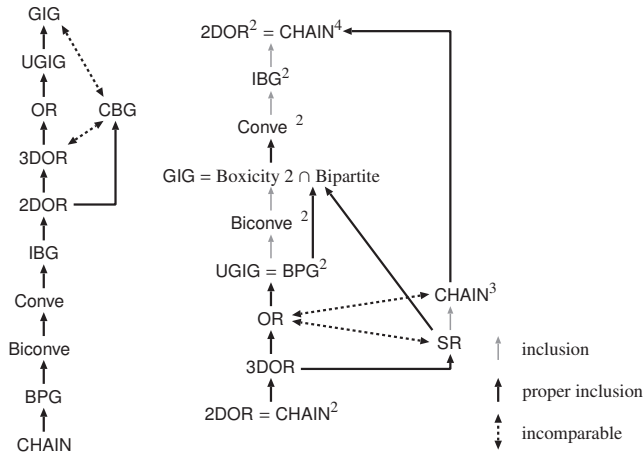


Figure 1: (Left) Known hierarchy. (Right) New hierarchy based on intersection dimensions.

## 2. PRELIMINARIES

A graph  $G = (V, E)$  is a *bipartite graph* (or a *bigraph* for short) with bipartition  $(X, Y)$  if  $V$  is partitioned into  $X$  and  $Y$  in such a way that each edge of  $G$  has one endpoint in  $X$  and the other in  $Y$ . We denote such a bigraph by  $(X, Y; E)$ . A *biadjacency matrix*  $M_B$  of a bigraph  $B = (X, Y; E)$  is a 0-1 matrix with the rows indexed by the vertices of  $X$  and the columns indexed by the vertices of  $Y$  such that  $\{x, y\} \in E$  if and only if the corresponding entry of  $M_B$  is 1. For  $m \times n$  0-1 matrices  $M'$  and  $M''$ , their intersection  $M = M' \cap M''$  is the 0-1 matrix such that  $M_{i,j} = 1$  if and only if  $M'_{i,j} = 1$  and  $M''_{i,j} = 1$ . The *neighborhood* of a vertex  $v$  in a graph  $G$ , denoted  $N_G(v)$ , is the vertices adjacent to  $v$  in  $G$ .

## 2.1 Graph classes

Here we define the graph classes we deal with in this paper. We also introduce some important properties of them. For their inclusion relations and other known results for them, the readers can refer to the standard textbooks in this field [4, 15, 36].

For a graph class  $\mathcal{C}$ , the *recognition problem* of  $\mathcal{C}$  is the problem deciding whether a given graph belongs to  $\mathcal{C}$ .

### 2.1.1 Chain graphs and Ferrers diagrams

A bipartite graph  $B = (X, Y; E)$  is a *chain graph* if there is an ordering  $(x_1, x_2, \dots, x_p)$  on  $X$  such that  $N_B(x_1) \supseteq N_B(x_2) \supseteq \dots \supseteq N_B(x_p)$ . It is easy to see that if there exists such an ordering on  $X$ , then there exists an ordering  $(y_1, y_2, \dots, y_q)$  on  $Y$  such that  $N_B(y_1) \supseteq N_B(y_2) \supseteq \dots \supseteq N_B(y_q)$ . Chain graphs are also known as *difference graphs* and *Ferrers bigraphs*. It is known that chain graphs are exactly  $2K_2$ -free bigraphs [16]. The class of chain graphs is denoted by CHAIN.

A 0-1 matrix has the *Ferrers property* if its rows and columns can be reordered so that 1's in each row and column appear consecutively with the rows left-justified and the columns top-justified. The reordered matrix is called a *Ferrers diagram*. It is easy to see that a matrix has the Ferrers property if and only if it has none of the following  $2 \times 2$  matrices as a submatrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (1)$$

Since chain graphs are exactly the  $2K_2$ -free bigraphs, it is easy to see that chain graphs are exactly the bigraphs whose biadjacency matrices have the Ferrers property.

### 2.1.2 Bipartite permutation graphs, convex graphs, biconvex graphs, interval bigraphs, and chordal bipartite graphs

A graph  $G = (V, E)$  with  $V = \{1, 2, \dots, n\}$  is a *permutation graph* if there is a permutation  $\pi$  over  $V$  such that  $\{i, j\} \in E(G)$  if and only if  $(i - j)(\pi(i) - \pi(j)) < 0$ . A graph is a *bipartite permutation graph* if it is bipartite and a permutation graph. The class of bipartite permutation graphs is denoted by BPG. Several equivalent definitions of the class BPG are collected in [20].

An ordering  $<$  of  $X$  in a bipartite graph  $B = (X, Y; E)$  has the *adjacency property* if for every vertex  $y$  in  $Y$ ,  $N(y)$  consists of vertices that are consecutive in the ordering  $<$  of  $X$ . A bipartite graph  $(X, Y; E)$  is *convex* if there is an ordering of  $X$  or  $Y$  that fulfills the adjacency property. A bipartite graph  $(X, Y; E)$  is *biconvex* if there are orderings of  $X$  and  $Y$  that fulfill the adjacency property. We denote the classes of convex bipartite graphs and biconvex bipartite graphs by Convex and Biconvex, respectively.

A *bi-interval representation* of a bigraph  $B = (U, V; E)$  is a pair  $(\mathcal{I}_U, \mathcal{I}_V)$  of sets of closed intervals such that  $\mathcal{I}_U = \{I_u = [\ell_u, r_u] : u \in U\}$  and  $\mathcal{I}_V = \{I_v = [\ell_v, r_v] : v \in V\}$ , and  $\{u, v\} \in E$  for  $u \in U$  and  $v \in V$  if and only if  $I_u \cap I_v \neq \emptyset$ . A bi-interval representation  $(\mathcal{I}_U, \mathcal{I}_V)$  is *unit* if for each interval  $[\ell, r] \in \mathcal{I}_U \cup \mathcal{I}_V$ ,  $r - \ell = 1$ .

A bigraph is a *chordal bipartite graph* if every induced cycle is of length four. The class of chordal bipartite graphs is denoted by CBG.

### 2.1.3 Orthogonal ray graphs

A bipartite graph  $B = (X, Y; E)$  is an *orthogonal ray graph* if there is a pair  $(\mathcal{R}_X, \mathcal{R}_Y)$  of families of rays (or half-lines) such that  $\mathcal{R}_X = \{R_x : x \in X\}$  is a family of pairwise non-intersecting horizontal rays,  $\mathcal{R}_Y = \{R_y : y \in Y\}$  is a family of pairwise non-intersecting vertical rays, and  $\{x, y\} \in E$  if and only if  $R_x$  and  $R_y$  intersect. We call such a pair  $(\mathcal{R}_X, \mathcal{R}_Y)$  an *orthogonal ray representation* of  $B$ . We denote the class of orthogonal ray graphs by OR.

Note that in a representation of an orthogonal ray graph horizontal rays can go rightward and leftward and vertical rays can go upward and downward. If we restrict horizontal rays to be only rightwards, then we have *3-directional orthogonal ray graphs*. Furthermore, if we restrict horizontal rays to be only rightwards and vertical rays to be only upwards, then we have *2-directional orthogonal ray graphs*. We denote the classes of 3-directional orthogonal ray graphs and 2-directional orthogonal ray graphs by 3DOR and 2DOR, respectively.

For the class 2DOR, several nice characterizations are known (see e.g. [21, 12, 30, 31, 32, 19, 33]). Among those characterizations, the followings are useful for our purpose. In this language they appear in [32, 33], in an equivalent graph theoretic form they are given in [21, 19].

**THEOREM 2.1.** *For a bigraph  $B$ , the following conditions are equivalent:*

1.  $B$  is a 2-directional orthogonal ray graph;
2.  $B$  is  $\gamma$ -freeable; that is, the rows and columns of a biadjacency matrix of  $B$  can be independently permuted so that no 0 has a 1 both below it and to its right;
3.  $B$  is of Ferrers dimension at most 2. (The Ferrers dimension of a bigraph is defined in Section 2.1.10.)

There are other equivalent characterizations of the class 2DOR, as suggested in the introduction, in terms of absence of induced cycles and bipartite versions of asteroids, in terms of invertible pairs, etc. [21, 12, 19].

It is known that the recognition of 2DOR can be done in polynomial time [12, 33], while it is open for 3DOR and OR. Recently, Felsner, Mertzius, and Mustatã [14] have shown that if the direction (right, left, up, or down) for each vertex is given, then it can be decided in polynomial time whether a given graph has an orthogonal ray representation in which each vertex has the given direction.

### 2.1.4 Grid intersection graphs

A bipartite graph  $B = (X, Y; E)$  is a *grid intersection graph* if there is a pair  $(\mathcal{S}_X, \mathcal{S}_Y)$  of families of segments such that  $\mathcal{S}_X = \{S_x : x \in X\}$  is a family of pairwise non-intersecting horizontal segments,  $\mathcal{S}_Y = \{S_y : y \in Y\}$  is a family of pairwise non-intersecting vertical segments, and  $\{x, y\} \in E$  if and only if  $S_x$  and  $S_y$  intersect. We call such a pair  $(\mathcal{S}_X, \mathcal{S}_Y)$  a *grid intersection representation* of  $B$ . A bipartite graph is a *unit grid intersection graph* if it has a grid intersection representation in which each segment is of length 1. We denote the classes of grid intersection graphs and unit grid intersection graphs by GIG and UGIG, respectively.

### 2.1.5 Segment-ray graphs

A bipartite graph  $B = (X, Y; E)$  is a *segment-ray graph* if there is a pair  $(\mathcal{S}_X, \mathcal{R}_Y)$  of families of segments and rays

such that  $\mathcal{S}_X = \{S_x : x \in X\}$  is a family of pairwise non-intersecting horizontal segments,  $\mathcal{R}_Y = \{R_y : y \in Y\}$  is a family of pairwise non-intersecting vertical upward rays, and  $\{x, y\} \in E$  if and only if  $S_x$  and  $R_y$  intersect. We call such a pair  $(\mathcal{S}_X, \mathcal{R}_Y)$  a *segment-ray representation* of  $B$ . We denote the class of segment-ray graphs by SR.

### 2.1.6 Recognition problems and inclusion relations

For the graph classes introduced above, the following relations are known [4, 27, 33]:

CHAIN  $\subsetneq$  BPG  $\subsetneq$  Biconvex  $\subsetneq$  Convex  $\subsetneq$  IBG  $\subsetneq$  2DOR  $\subsetneq$  3DOR  $\subsetneq$  OR  $\subsetneq$  UGIG  $\subsetneq$  GIG.

Also it is known that 2DOR  $\subsetneq$  CBG [33], and that CBG is incomparable to 3DOR and GIG [27].

It is known that the recognition problems of CHAIN [18], BPG [34], Biconvex [36], Convex [36], IBG [25], 2DOR [33], and CBG [35] can be solved in polynomial time. On the other hand, it is known that the recognition problems of GIG [23] and UGIG [26, 39] are NP-complete. The complexity of the recognition problems of 3DOR, OR, and SR is not known.

Note that even if three graph classes  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $\mathcal{C}$  satisfy  $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$  and the recognition problems of  $\mathcal{A}$  and  $\mathcal{C}$  are both polynomial-time solvable (NP-hard), it does not mean the recognition problem of  $\mathcal{B}$  is polynomial-time solvable (NP-hard, resp.).

### 2.1.7 Other graphs

The  $d$ -dimensional hypercube  $H_d$  is the graph with  $2^d$  vertices in which the vertices corresponds to the subsets of  $\{1, \dots, d\}$  and two vertices are adjacent if and only if the symmetric difference of the corresponding sets is of size 1.

Let  $K_{a,b}$  denote the complete bipartite graph having  $a$  vertices in one side and  $b$  vertices in the other side. We denote by  $K_{n,n} - nK_2$  the graph obtained by removing a perfect matching from the complete bipartite graph  $K_{n,n}$ .

### 2.1.8 Boxicity and cubicity

An *interval graph* is the intersection graph of closed intervals on the real line. A *unit interval graph* is the intersection graph of closed unit intervals on the real line. We denote the classes of interval graphs and unit interval graphs by INT and UINT, respectively.

The *boxicity* of a graph  $G$  is the minimum integer  $k$  such that  $G \in \text{INT}^k$ , and the *cubicity* of  $G$  is the minimum integer  $k$  such that  $G \in \text{UINT}^k$ . It is known that given a graph, deciding whether its boxicity (or cubicity) is at most 2 is NP-complete [23, 5].

### 2.1.9 Bigraph intersection dimension

For bipartite graph classes, if one of them is additionally closed under disjoint union, we may assume that the bipartitions of  $G$  and  $G'$  are the same when taking their intersection. More precisely, we have the following lemma.

**LEMMA 2.2.** *Let  $\mathcal{B}$  and  $\mathcal{B}'$  be bipartite graph classes. If at least one of them is closed under disjoint union and taking induced subgraphs, then  $\mathcal{B} \boxtimes \mathcal{B}' = \{(X, Y; E) \cap (X, Y; E') : (X, Y; E) \in \mathcal{B}, (X, Y; E') \in \mathcal{B}'\}$ .*

**PROOF.** Let  $\mathcal{C} = \{(X, Y; E) \cap (X, Y; E') : (X, Y; E) \in \mathcal{B}, (X, Y; E') \in \mathcal{B}'\}$ . Clearly,  $\mathcal{C} \subseteq \mathcal{B} \boxtimes \mathcal{B}'$ . In the following, we show that  $\mathcal{B} \boxtimes \mathcal{B}' \subseteq \mathcal{C}$ . By symmetry, we may assume that  $\mathcal{B}'$  is closed under disjoint union and taking induced subgraphs.



Let  $H = (X, Y; E) \in \mathcal{B}$  and  $H' = (X', Y'; E') \in \mathcal{B}'$ . Now let  $H'' = (X, Y; E' \cap \{\{x, y\} : x \in X, y \in Y\})$ . It is easy to see that  $H \cap H' = H \cap H''$ . Observe that  $H''$  is the disjoint union of two induced subgraphs of  $H'$ , where one is induced by  $(X \cap X', Y \cap Y')$  and the other by  $(X \cap Y', X \cap Y')$ . Since  $\mathcal{B}'$  is closed under disjoint union and taking induced subgraphs, it follows that  $H'' \in \mathcal{B}'$ . Since  $H \cap H' = H \cap H''$ , we have  $H \cap H' \in \mathcal{C}$ .  $\square$

Unfortunately, CHAIN is not closed under disjoint union. For example,  $K_2$  is a chain graph but  $2K_2$  is not. It is the only exception in this paper. Fortunately, we have the following lemma for chain graphs.

LEMMA 2.3.  $\text{CHAIN}^2 = \{(X, Y; E) \cap (X, Y; E') : (X, Y; E), (X, Y; E') \in \text{CHAIN}\}$ .

PROOF. Let  $\mathcal{C} = \{(X, Y; E) \cap (X, Y; E') : (X, Y; E), (X, Y; E') \in \text{CHAIN}\}$ . Clearly,  $\mathcal{C} \subseteq \text{CHAIN}^2$ . In the following, we show that  $\text{CHAIN}^2 \subseteq \mathcal{C}$ .

Let  $H_1 = (X_1, Y_1; E_1) \in \text{CHAIN}$  and  $H_2 = (X_2, Y_2; E_2) \in \text{CHAIN}$ . Now let  $H'_1 = (X_1, Y_1; E'_1)$  and  $H'_2 = (X_1, Y_1; E'_2)$ , where

$$\begin{aligned} E'_1 &= E_1 \cup \{\{x, y\} : x \in X_1 \cap X_2, y \in Y_1 \cap X_2\} \\ &\quad \setminus \{\{x, y\} : x \in X_1 \cap Y_2, y \in Y_1 \cap Y_2\}, \\ E'_2 &= E_2 \cup \{\{x, y\} : x \in X_1 \cap Y_2, y \in Y_1 \cap Y_2\}, \\ &\quad \setminus \{\{x, y\} : x \in X_1 \cap X_2, y \in Y_1 \cap X_2\}. \end{aligned}$$

See Fig. 2. It is not difficult to see that  $H_1 \cap H_2 = H'_1 \cap H'_2$ . Observe that both  $H'_1$  and  $H'_2$  are chain graphs. Therefore,  $H_1 \cap H_2 = H'_1 \cap H'_2 \in \mathcal{C}$ .  $\square$

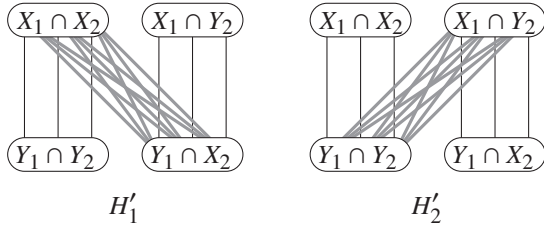


Figure 2: Intersection of two chain graphs.

By Lemmas 2.2 and 2.3, we can assume that the bipartitions of two graphs are the same when we are defining the pairwise intersection of two graph classes, since, in this paper, either one of them is closed under disjoint union or both of them are the class of chain graphs.

### 2.1.10 Ferrers dimension

The *Ferrers dimension*  $\text{fd}(B)$  of a bigraph  $B$  is the smallest number of Ferrers bigraphs whose intersection is  $B$ . That is,  $\text{fd}(B)$  is the minimum integer  $k$  such that  $B \in \text{CHAIN}^k$ . If  $B = (X, Y; E)$  and  $\text{fd}(B) = k$ , then there are Ferrers bigraphs  $B_i = (X, Y; E_i)$  for  $1 \leq i \leq k$  such that  $B = \bigcap_{1 \leq i \leq k} B_i$ . That is, we can assume all the graphs  $B$  and  $B_i$ ,  $1 \leq i \leq k$  have the same bipartition.

A *Ferrers digraph*  $D = (V, A)$  is a digraph whose adjacency matrix has the Ferrers property. The *Ferrers dimension*  $\text{fd}(D)$  of a digraph  $D$  is the smallest number of Ferrers digraphs whose intersection is  $D$ .

### 2.1.11 Poset dimension

The *poset dimension*  $\text{pd}(P)$  of a poset  $P$  is the minimum integer  $k$  such that there exist  $k$  linear extensions of  $P$  such that for any two elements  $x, y$  of  $P$ ,  $x < y$  in  $P$  if and only if  $x < y$  in all the linear extensions. The *Ferrers dimension*  $\text{fd}(P)$  of a poset  $P$  is the Ferrers dimension of the digraph defined in such way that the vertices are the elements of  $P$  and there is an arc  $(u, v)$  if and only if  $u < v$ . Cogis [10] showed that for any poset  $P$ ,  $\text{fd}(P) = \text{pd}(P)$ .

A poset is of *height 2* if every element is either a minimal element or a maximal element. The *underlying graph* of a height-2 poset is the bigraph  $B = (X, Y; E)$  such that  $X$  is the set of minimal elements,  $Y$  is the set of maximal elements, and  $\{x, y\} \in E$  if and only if  $x < y$ . It is easy to see that any bigraph is the underlying graph of some poset of height 2.

## 3. $(P, Q; D)$ -BIGRAPHS

We introduce the notion of  $(P, Q; D)$ -bigraphs, where a bigraph  $B = (U, V, E)$  is said to be an  $(P, Q; D)$ -bigraph if and only if for some domain  $D$  (e.g., the real number line  $\mathbb{R}$ ) each vertex in  $u \in U$  can be represented as a type  $P$  subset  $P_u$  of  $D$  and each vertex  $v \in V$  can be represented as a type  $Q$  subset  $Q_v$  of  $D$  such that for every  $u \in U, v \in V, uv \in E$  if and only if  $P_u \cap Q_v \neq \emptyset$ . For example, in this setting, interval bigraphs are (interval, interval,  $\mathbb{R}$ )-bigraphs. We will use  $(P, Q; D)$  to denote the class of  $(P, Q; D)$ -bigraphs.

Our discussion will focus on the cases when  $P, Q$  are the following subsets of  $\mathbb{R}$ : points, rays, unit-intervals, and intervals; and the following axis-aligned subsets of  $\mathbb{R}^2$ : points, rays, unit-segments, segments, squares, and rectangles. Note: for rays, we will use  $\rightarrow, \downarrow, \leftarrow,$  and  $\uparrow$  to denote the *rightward, downward, leftward,* and *upward* rays respectively. Moreover, when we refer to a ray  $r$  (rather than using a specific arrow),  $r$  can be any axis-aligned ray from the domain.

### 3.1 $(P, Q; \mathbb{R})$ -Bigraphs

We begin with some easy observations characterizing CHAIN, Convex, and Biconvex bigraphs as  $(P, Q; D)$ -bigraphs (see Proposition 3.1). This is followed by a couple essential lemmas that we will use to relate  $(P, Q, \mathbb{R})$ -bigraphs to  $(P', Q', \mathbb{R}^2)$ -bigraphs.

PROPOSITION 3.1. For a bigraph  $B = (X, Y, E)$ :

1.  $B$  is CHAIN if and only if  $B$  is (point,  $\rightarrow$ ;  $\mathbb{R}$ ).
2.  $B$  is Convex if and only if  $B$  is (point, interval;  $\mathbb{R}$ ).
3.  $B$  is Biconvex if and only if  $B$  is both (point, interval;  $\mathbb{R}$ ) and (interval, point;  $\mathbb{R}$ ).

PROOF. These follow easily by definition.  $\square$

It is also known that a bigraph is a bipartite permutation graph (BPG) if and only if it is a unit-interval bigraph [20]; i.e.,  $\text{BPG} = (\text{unit-interval, unit-interval; } \mathbb{R})$ . Interestingly, we observe that (unit-interval, unit-interval;  $\mathbb{R}$ )-bigraphs actually have a simpler representation. Specifically, (unit-interval, unit-interval;  $\mathbb{R}$ ) = (point, unit-interval;  $\mathbb{R}$ ) and we prove this via the following more general lemma.

LEMMA 3.2. For a bigraph  $B = (U, V; E)$  and any  $Q \in \{\rightarrow, \text{ray, unit-interval, interval}\}$ ,  $B \in (\text{unit-interval, } Q; \mathbb{R})$  if and only if  $B \in (\text{point, } Q; \mathbb{R})$ .



**Figure 3: The path on seven vertices ( $P_7$ ) and a (point, ray; $\mathbb{R}$ ) representation of it. Note:  $P_7$  is not both (point, ray; $\mathbb{R}$ ) and (ray, point; $\mathbb{R}$ ) since the neighborhoods of  $a$ ,  $b$ , and  $c$  are pairwise incomparable.**

PROOF. Notice that for any choice of  $Q$  each element of  $V$  is represented as an interval. Let  $(\mathcal{I}_U, \mathcal{I}_V)$  be a (unit-interval,  $Q; \mathbb{R}$ ) representation of  $B$ . Let  $I_u = [\ell_u, \ell_u + 1] \in \mathcal{I}_U$  and  $I_v = [\ell_v, r_v] \in \mathcal{I}_V$  be intervals corresponding to  $u \in U$  and  $v \in V$ , respectively. It is easy to see that  $I_u$  and  $I_v$  intersect if and only if either  $\ell_u \in I_v$  or  $\ell_v - \ell_u \in [0, 1]$ .

We define the following (point,  $Q; \mathbb{R}$ ) representation  $(\mathcal{I}'_U, \mathcal{I}'_V)$  as:

$$\begin{aligned} \mathcal{I}'_U &= \{[\ell_u] : [\ell_u, \ell_u + 1] \in \mathcal{I}_U\}, \\ \mathcal{I}'_V &= \{[\ell_v - 1, r_v] : [\ell_v, r_v] \in \mathcal{I}_V\}. \end{aligned}$$

Obviously  $(\mathcal{I}'_U, \mathcal{I}'_V)$  represents  $B$ , since  $\ell_u \in [\ell_v - 1, r_v]$  if and only if either  $\ell_u \in I_v$  or  $\ell_v - \ell_u \in [0, 1]$ . It is easy to see that we now have a (point,  $Q; \mathbb{R}$ ) representation of  $B$ .  $\square$

Lemma 3.2 allows us to equate several  $(P, Q; \mathbb{R})$  classes. These are given in the following two corollaries.

**COROLLARY 3.3.** *For each  $Q \in \{\rightarrow, \text{ray}, \text{unit-interval}, \text{interval}\}$ , the following classes of bigraphs are the same: (point,  $Q; \mathbb{R}$ ),  $(\rightarrow, Q; \mathbb{R})$ , (ray,  $Q; \mathbb{R}$ ), (unit-interval,  $Q; \mathbb{R}$ ).*

**COROLLARY 3.4.** *For each  $P, Q \in \{\text{point}, \rightarrow, \leftarrow, \text{unit-interval}\}$ , a bigraph  $B$  is  $(P, Q; \mathbb{R})$  if and only if  $B$  is  $(Q, P; \mathbb{R})$ .*

Notice that the statement of Corollary 3.4 does not allow either of  $P$  or  $Q$  to be ray-type sets. This is because Lemma 3.2 cannot be used to give us the desired biconvexity-like when rays are allowed for a given set. However, by Lemma 3.2, we can transform any (ray, ray; $\mathbb{R}$ ) representation into a (point, ray; $\mathbb{R}$ ) representation. Thus, (ray, ray; $\mathbb{R}$ ) is a subset of the bigraphs which are both (point, ray; $\mathbb{R}$ ) and (ray, point; $\mathbb{R}$ ). One open question would be whether these are the same.

Moreover, the graph ( $P_7$ ) given in Figure 3 is (point, ray; $\mathbb{R}$ ) but not both (point, ray; $\mathbb{R}$ ) and (ray, point; $\mathbb{R}$ ). This is easy to see since no three vertices in the same partition (say,  $X$ ) can have pairwise incomparable neighborhoods; i.e., two of the three must be represented by rays in the same direction and thus must have nested neighborhoods. Moreover, the graph in Figure 3 has  $a, b, c \in X$  such that their neighborhoods are pairwise incomparable. This is formalized in the following proposition.

**PROPOSITION 3.5.** *If a bigraph  $B = (X, Y; E)$  is (ray, point; $\mathbb{R}$ ) where each  $x \in X$  is a ray then for every  $\{x, x', x''\} \subseteq X$  and every  $y \in Y$ , there exists  $x^* \in \{x, x', x''\}$  and  $x^{**} \in \{x, x', x''\} \setminus \{x^*\}$  such that  $N(x^*) \subseteq N(x^{**})$  or  $N(x) \subseteq N(x'')$ .*

### 3.2 $(P, Q; \mathbb{R}^2)$ -Bigraphs

In this subsection we consider the domain  $\mathbb{R}^2$  and describe several classes of bigraphs as the intersection of one dimensional bigraph classes (i.e., as  $(P, Q; \mathbb{R}) \bowtie (P', Q'; \mathbb{R})$ ). Notice that, for  $P, Q \in \{\text{point}, \text{unit-interval}, \text{interval}\}$   $(P, Q; \mathbb{R})$  is hereditary and closed under disjoint union. Thus, by Lemma 2.2, for  $P, Q \in \{\text{point}, \text{unit-interval}, \text{interval}\}$  and any choices of  $P'$  and  $Q'$ ,  $B = (X, Y; E)$  is  $(P, Q; \mathbb{R}) \bowtie (P', Q'; \mathbb{R})$  if and only if  $B = (X, Y; E \cap E')$  for  $(X, Y; E) \in (P, Q; \mathbb{R})$  and  $(X, Y; E') \in (P', Q'; \mathbb{R})$ .

**THEOREM 3.6.**  $\text{UGIG} = \text{BPG}^2 = (\text{point}, \text{unit-interval}; \mathbb{R}^2)$ .

PROOF. First we show that  $\text{UGIG} \subseteq \text{BPG}^2$ . Let  $G = (U, V; E) \in \text{UGIG}$  and  $\mathcal{R} = (\mathcal{U}, \mathcal{V})$  be a unit grid representation of  $G$ , where the horizontal segments  $\mathcal{U}$  represent the vertices in  $U$  and the vertical segments  $\mathcal{V}$  represent the vertices in  $V$ . That is,  $\mathcal{U} = \{\{y_u\} \times [x_u, x_u + 1] : u \in U\}$ ,  $\mathcal{V} = \{\{y_v, y_v + 1\} \times \{x_v\} : v \in V\}$ , and  $E = \{\{u, v\} : u \in U, v \in V, y_u \in [y_v, y_v + 1], x_v \in [x_u, x_u + 1]\}$ . From  $\mathcal{U}$ , we construct two point-unit bi-interval representations  $\mathcal{R}'$  and  $\mathcal{R}''$  as follows:

$$\begin{aligned} \mathcal{R}' &= (\{y_u : u \in U\}, \{\{y_v, y_v + 1\} : v \in V\}), \\ \mathcal{R}'' &= (\{x_v : v \in V\}, \{\{x_u, x_u + 1\} : u \in U\}). \end{aligned}$$

By Lemma 3.2,  $\mathcal{R}'$  and  $\mathcal{R}''$  represent the bipartite permutation graphs  $G' = (U, V; E')$  and  $G'' = (U, V; E'')$ , respectively, where

$$\begin{aligned} E' &= \{\{u, v\} : u \in U, v \in V, y_u \in [y_v, y_v + 1]\}, \text{ and} \\ E'' &= \{\{u, v\} : u \in U, v \in V, x_v \in [x_u, x_u + 1]\}. \end{aligned}$$

Since  $\{u, v\} \in E' \cap E''$  for  $u \in U$  and  $v \in V$  if and only if  $y_u \in [y_v, y_v + 1]$  and  $x_v \in [x_u, x_u + 1]$ , we have  $E = E' \cap E''$ . Therefore,  $G = G' \cap G''$ .

Next we show that  $\text{BPG}^2 \supseteq \text{UGIG}$ . Let  $G' = (U, V; E')$  and  $G'' = (U, V; E'')$  be bipartite permutation graphs. Let  $\mathcal{R}'$  and  $\mathcal{R}''$  be point-unit bi-interval representations of  $G'$  and  $G''$ , respectively, such that  $U$  is the point set of  $\mathcal{R}'$  and the unit interval set of  $\mathcal{R}''$ . Such representations exist by Corollary 3.3. Let  $u \in U$ , and let  $p_u$  and  $[\ell_u, \ell_u + 1]$  be the point in  $\mathcal{R}'$  and the unit interval in  $\mathcal{R}''$  representing the vertex  $u$ . We assign the unit horizontal segment  $\{p_u\} \times [\ell_u, \ell_u + 1]$  to  $u$ . Similarly, for a vertex  $v \in V$  with the unit interval  $[\ell_v, \ell_v + 1]$  in  $\mathcal{R}'$  and the point  $p_v$  in  $\mathcal{R}''$ , we assign the unit vertical segment  $[\ell_v, \ell_v + 1] \times \{p_v\}$ . The obtained unit grid representation represents  $G = G' \cap G''$ , since  $\{p_u\} \times [\ell_u, \ell_u + 1]$  and  $[\ell_v, \ell_v + 1] \times \{p_v\}$  intersect if and only if  $p_u \in [\ell_v, \ell_v + 1]$  and  $p_v \in [\ell_u, \ell_u + 1]$ .  $\square$

Using Theorem 3.6 and Corollary 3.4 the following is immediate.

**COROLLARY 3.7.**  $(\text{unit-square}, \text{unit-square}; \mathbb{R}^2) = (\text{point}, \text{unit-interval}; \mathbb{R}^2) = \text{UGIG}$ .

The corollary above implies that a bipartite graph of cubicity-2 is UGIG. It is easy to see that the star  $K_{1,5}$  is UGIG, but its cubicity is more than 2. Therefore, we have the following corollary, which is a nice complement to the fact  $\text{Boxicity-2} \cap \text{Bipartite} = \text{GIG}$  [2].

**COROLLARY 3.8.**  $\text{Cubicity-2} \cap \text{Bipartite} \subsetneq \text{UGIG}$ .

The proof of the following theorem is an easy modification of the proof of Theorem 3.6. The relation  $\text{GIG} \neq \text{Convex}^2$  is shown by Fig. 5.

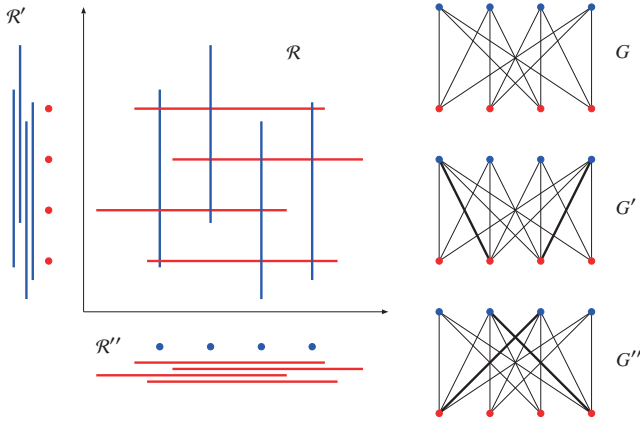


Figure 4: UGIG = BPG<sup>2</sup>.

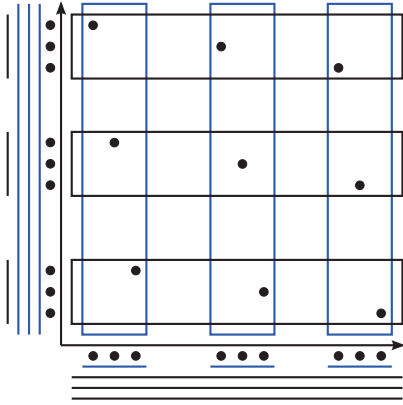


Figure 5: A (point, interval)<sup>2</sup> representation of the full subdivision  $H$  of  $K_{3,3}$ ; i.e.,  $H \in \text{Convex}^2$ . On the other hand,  $H \notin \text{GIG}$ , since it is the full subdivision of a non-planar graph, and thus not a string graph.

THEOREM 3.9.  $\text{Biconvex}^2 \subseteq (\text{Biconvex} \otimes \text{Convex}) \subseteq \text{GIG} \subseteq \text{Convex}^2$ .

Since  $\text{Convex} \subset \text{2DOR}$ , it holds that  $\text{GIG} \subseteq \text{2DOR}^2 = \text{CHAIN}^4$ . Therefore, every grid intersection graph has Ferrers dimension at most 4.

COROLLARY 3.10. *The recognition problems of BPG<sup>2</sup>, Biconvex<sup>2</sup>, and Biconvex  $\otimes$  Convex are NP-complete.*

PROOF. The problems are in NP since the recognition problems of BPG and Biconvex are polynomial-time solvable and the intersection of two graphs can be computed in polynomial time.

Mustață and Pergel [26] showed that the recognition problem is NP-hard for any graph class  $\mathcal{C}$  satisfying  $\text{UGIG} \subseteq \mathcal{C} \subseteq \text{GIG}$ . By Theorems 3.6 and 3.9 and the fact that  $\text{BPG} \subseteq \text{Biconvex}$ , it follows that  $\text{UGIG} = \text{BPG}^2 \subseteq \text{Biconvex}^2 \subseteq \text{GIG}$ . Therefore, the recognition problems are NP-hard for BPG<sup>2</sup> and Biconvex<sup>2</sup>.  $\square$

## 4. SEGMENT-RAY GRAPHS

Let  $F$  be a matrix with entries 0, 1, \*, where \* means “don’t care.” A matrix  $M$  is  $F$ -free if  $M$  does not have  $F$  as a

submatrix ignoring \*-entries. A bipartite graph is  $F$ -freeable if it has a  $F$ -free biadjacency matrix.

It is known that a bipartite graph is a chordal bipartite graph if and only if it is  $\Gamma$ -freeable (see [22]), a 2-directional orthogonal ray graph if and only if it is  $\gamma$ -freeable [33], and a grid intersection graph if and only if it is cross-freeable [17], where the forbidden matrices are defined as follows:

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}, \quad \text{cross} = \begin{pmatrix} * & 1 & * \\ 1 & 0 & 1 \\ * & 1 & * \end{pmatrix}.$$

In this section, using the following matrix  $V$ , we characterize segment-ray graphs:

$$V = \begin{pmatrix} 1 & 0 & 1 \\ * & 1 & * \end{pmatrix}.$$

Obviously, a matrix is cross-free if it is  $V$ -free, and  $V$ -free if it is  $\gamma$ -free.

The proof of the following proof is similar to the proofs of the cross-free characterization of GIG [17] and the  $\gamma$ -free characterization of 2DOR [33].

THEOREM 4.1. *A bipartite graph is a segment-ray graph if and only if it is  $V$ -freeable.*

PROOF. For the only-if part, let  $B = (U, V; E)$  be a segment-ray graph and  $\mathcal{R}$  be its segment-ray representation such that each vertex in  $U$  corresponds to a horizontal segment in  $\mathcal{R}$ , and each vertex in  $V$  corresponds to a vertical upward ray in  $\mathcal{R}$ . Let  $M$  be the bipartite adjacency matrix of  $B$  with the rows indexed by  $U$  and the columns indexed by  $V$ . Let  $S_u$  be the segment corresponding to  $u \in U$  with  $y$ -coordinate  $b$ , and  $R_v$  be the ray corresponding to  $v \in V$  with  $x$ -coordinate  $a$ . If  $S_u$  intersects with rays on both sides of  $x = a$  and  $R_v$  intersects with a segment below  $y = b$ , then  $S_u$  and  $R_v$  must intersect at  $(a, b)$ . Thus we can make  $M$   $V$ -free by permuting the columns in nondecreasing order of the  $x$ -coordinates of the corresponding rays and the rows in nonincreasing order of the  $y$ -coordinates of the corresponding segments.

For the if part, let  $B = (U, V; E)$  be a bipartite graph and  $M$  be its  $V$ -free bipartite adjacency matrix with the rows indexed by  $U$  and the columns indexed by  $V$ . For each  $u \in U$ , we put the horizontal segment with end points  $(i, j_1)$  and  $(i, j_2)$ , where  $i$  is the row index of  $u$  and  $j_1, j_2$  are the smallest and largest indices such that  $M_{i,j} = 1$ . For each  $v \in V$ , we put the vertical upward ray from the starting point  $(i, j)$ , where  $j$  is the column index of  $v$  and  $i$  is the largest index such that  $M_{i,j} = 1$ . For any two vertices  $u \in U$  and  $v \in V$ , it is clear that the corresponding segment and ray intersect if the vertices are adjacent. Conversely, if  $u$  and  $v$  are not adjacent, then the corresponding segment and ray cannot intersect since  $M$  is  $V$ -free.  $\square$

Now we show that every segment-ray graph has Ferrers dimension at most 3. To this end, we need the following simple fact.

LEMMA 4.2. *An  $m \times n$  0-1 matrix  $M$  is  $V$ -free if and only if for each entry  $(i, j)$  with  $M_{i,j} = 0$  at least one of the following holds:*

1.  $M_{i,k} = 0$  for all  $1 \leq k \leq j$ ;
2.  $M_{i,k} = 0$  for all  $j \leq k \leq n$ ;
3.  $M_{k,j} = 0$  for all  $i \leq k \leq m$ .

**THEOREM 4.3.** *Every segment-ray graph has Ferrers dimension at most 3.*

**PROOF.** Let  $B$  be a segment-ray graph and  $M$  be its  $V$ -free bipartite adjacency matrix. Let  $M^{(1)}, M^{(2)}, M^{(3)}$  be the following 0-1 matrices of the same size with  $M$ :

- $M_{i,j}^{(1)} = 0$  if and only if  $M_{i,k} = 0$  for all  $1 \leq k \leq j$ ;
- $M_{i,j}^{(2)} = 0$  if and only if  $M_{i,k} = 0$  for all  $j \leq k \leq n$ ;
- $M_{i,j}^{(3)} = 0$  if and only if  $M_{k,j} = 0$  for all  $i \leq k \leq m$ .

It is easy to see that  $M^{(1)}, M^{(2)}, M^{(3)}$  have the Ferrers property. By Lemma 4.2, it holds that  $M^{(1)} \cap M^{(2)} \cap M^{(3)} = M$ . This completes the proof.  $\square$

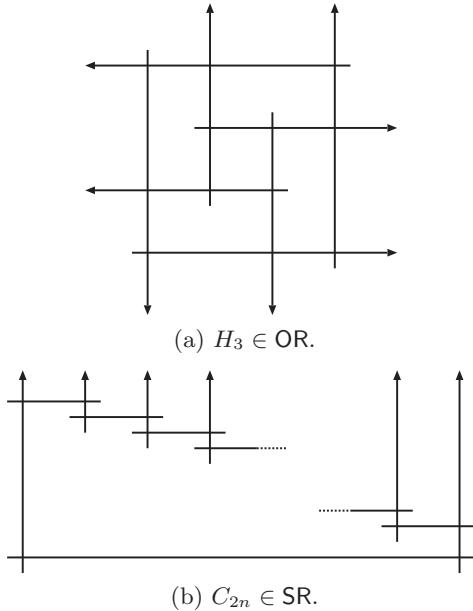
Note that the upper bounds of the Ferrers dimension for GIG ( $\leq 4$ ) and 2DOR ( $\leq 2$ ) can be shown in similar ways by using the forbidden submatrix characterizations.

**COROLLARY 4.4.** *OR is incomparable to both CHAIN<sup>3</sup> and SR.*

**PROOF.** By Theorem 4.3, it holds that  $SR \subseteq CHAIN^3$ . Hence it suffices to show that  $OR \not\subseteq CHAIN^3$  and  $SR \not\subseteq OR$ . Fig. 6(a) shows that  $H_3 \in OR$ . From the definitions, it holds that  $H_3 = K_{4,4} - 4K_2$ . It is known that  $fd(K_{n,n} - nK_2) = n$  [40, 41], and thus  $fd(H_3) = 4$ . Thus  $OR \not\subseteq CHAIN^3$ . It is known that  $C_{2n} \notin OR$  if  $n > 6$  [33]. On the other hand, it is easy to see that  $C_{2n} \in SR$  for any  $n$  (see Fig. 6(b)). Thus  $SR \not\subseteq OR$ .  $\square$

**COROLLARY 4.5.** *SR is a proper subset of GIG.*

**PROOF.** From the definition, SR is a subset of GIG. Since  $H_3 \in OR \subset GIG$  and  $H_3 \notin CHAIN^3 \supseteq SR$ , it holds that  $SR \neq GIG$ .  $\square$



**Figure 6:** Examples showing incomparabilities.

## 5. BOXICITY AND FERRERS DIMENSION

Chatterjee and Ghosh [9] presented some relations between the boxicity of undirected graphs and the Ferrers dimension of the directed graphs obtained somehow from the undirected graphs. Here we present a similar but more direct relation between the boxicity and the Ferrers dimension of bigraphs.

If  $fd(B) = 1$ , then  $box(B) \leq 2$ . This is because,  $fd(B) = 1$  implies that  $B$  is a chain graph, and thus  $B$  is a grid intersection graph [27]. This bound is tight since  $fd(K_{n,n}) = 1$  and  $box(K_{n,n}) = 2$  for every  $n \geq 2$ .

**THEOREM 5.1.** *Let  $B$  be a bigraph with  $fd(B) \geq 2$ . It holds that*

$$box(B) \leq fd(B) \leq 2box(B).$$

**PROOF.** Adiga, Bhowmick, and Chandran [1] showed that for a poset  $Q$  of height 2 and its underlying graph  $H$  it holds that  $box(H) \leq pd(Q) \leq 2box(H)$  if  $pd(Q) \geq 2$ . (Recently Felsner [13] has shown a more general result.) Since  $fd(Q) = pd(Q)$  [10], it holds that  $box(H) \leq fd(Q) \leq 2box(H)$  if  $fd(Q) \geq 2$ .

Let  $P$  be a poset that has  $B$  as the underlying graph. From the argument above, it follows that  $box(B) \leq fd(P) \leq 2box(B)$  if  $fd(P) \geq 2$ . Hence it suffices to show that  $fd(P) = fd(B)$ .

Let  $M_B$  be a bipartite adjacency matrix of  $B$ . Then, an adjacency matrix  $M_P$  of the digraph corresponding to  $P$  can be represented by the following form:

$$M_P = \begin{pmatrix} M_B & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Thus it is easy to see that  $fd(P) \geq fd(B)$  as  $M_B$  is a submatrix of  $M_P$ . On the other hand, let  $B_1, \dots, B_{fd(B)}$  be Ferrers bigraphs that satisfy  $B = \bigcap_{1 \leq i \leq fd(B)} B_i$ . Let  $M_{B_i}$  be the bipartite adjacency matrix of  $B_i$  in which the rows and columns are ordered as in  $M_B$ . Now we define  $M_{P_i}$  as follows:

$$M_{P_i} = \begin{pmatrix} M_{B_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Clearly  $M_P = \bigcap_{1 \leq i \leq fd(B)} M_{P_i}$ , and each  $M_{P_i}$  has the Ferrers property. This implies that  $fd(P) \leq fd(B)$ .  $\square$

The upper bound in Theorem 5.1 is tight. It is known that  $box(K_{n,n} - nK_2) = \lceil n/2 \rceil$  [6] and  $fd(K_{n,n} - nK_2) = n$  [40, 41].

Bellatoni, Hartman, Przytycka, and Whitesides [2] showed that the grid intersection graphs are exactly the bigraphs of boxicity at most 2. This implies that the Ferrers dimension of a grid intersection graph is at most 4. We show that the converse is not true.

**THEOREM 5.2.** *GIG  $\subsetneq$  CHAIN<sup>4</sup>.*

**PROOF.** We show that  $H_4 \in CHAIN^4 \setminus GIG$ . Chang and West [8] showed that  $H_4$  cannot be represented as the intersection graph of axis-parallel rectangles in the plane. This implies that  $H_4 \notin GIG$ . Let  $M$  and  $M'$  be the following



matrices:

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix},$$

$$M' = \begin{pmatrix} a & 1 & 1 & 1 & 1 & a & a & a \\ 1 & b & 1 & 1 & b & 1 & b & b \\ 1 & 1 & c & 1 & c & c & 1 & c \\ 1 & 1 & 1 & d & d & d & d & 1 \\ 1 & b & c & d & d & 1 & 1 & 1 \\ a & 1 & c & d & 1 & c & 1 & 1 \\ a & b & 1 & d & 1 & 1 & b & 1 \\ a & b & c & 1 & 1 & 1 & 1 & a \end{pmatrix}.$$

The matrix  $M$  is a biadjacency matrix of  $H_4$ , and  $M'$  has the same 1-entries as  $M$  but has one of  $a$ ,  $b$ ,  $c$ , and  $d$  for each 0-entry of  $M$ . For  $x \in \{a, b, c, d\}$ , let  $M_x$  be the 0-1 matrix obtained from  $M'$  by replacing all  $x$  with 0 and replacing all other non-numeric entries with 1. It is easy to see that  $M_x$ , for all  $x \in \{a, b, c, d\}$ , has none of the forbidden  $2 \times 2$  matrices in (1) as a submatrix, and thus has the Ferrers property. Since  $M = M_a \cap M_b \cap M_c \cap M_d$ , it holds that  $H_4 \in \text{CHAIN}^4$ .  $\square$

Chandran, Francis, and Mathew [7] showed that boxicity is unbounded for chordal bipartite graphs. Thus we have the following.

**COROLLARY 5.3.** *Ferrers dimension is unbounded for chordal bipartite graphs.*

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