Noname manuscript No.
(will be inserted by the editor)

Coverage with k-Transmitters in the Presence of Obstacles

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the date of receipt and acceptance should be inserted later

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Abstract For a fixed integer $k \geq 0$, a k-transmitter is an omnidirectional wireless transmitter with an infinite broadcast range that is able to penetrate up to k "walls", represented as line segments in the plane. We develop lower and upper bounds for the number of k-transmitters that are necessary and sufficient to cover a given collection of line segments, polygonal chains and polygons.

Keywords Coverage · guarding · transmitters · art gallery · visibility

1 Introduction

Illumination and guarding problems generalize the well-known art gallery problem in computational geometry [19, 20]. The task is to determine a minimum number of guards that are sufficient to guard, or "illuminate" a given region under specific constraints. The region under surveillance may be a polygon, or may be the entire plane with polygonal or line segment obstacles. The placement of guards may be restricted to vertices (*vertex* guards) or edges (*edge* guards) of the input polygon(s), or may be unrestricted (*point* guards). The guards may be *omnidirectional*, illuminating all directions equally, or may be represented as *floodlights*, illuminating a certain angle in a certain direction.

Inspired by advancements in wireless technologies and the need to offer wireless services to clients, Fabila-Monroy et al. [13] and Aichholzer et al. [2] introduce a new variant of the illumination problem, called *modem* illumination. In this problem, a guard is modeled as an omnidirectional wireless modem with an infinite broadcast range and the power to penetrate up to k "walls" to reach a client, for some fixed integer $k \geq 0$. Geometrically, walls are most often represented as line segments in the plane. In this paper, we refer to such a guard as a *k-transmitter*, and we speak of *covering* (rather than illuminating or guarding). We address the general problem introduced in [13, 2], reformulated as follows:

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† Partly supported by the ESF EUROCORES programme EUROGIGA-ComPoSe IP04-MICINN Project EUI-EURC-2011-4306, and projects MTM2009-07242 and Gen. Cat. DGR 2009SGR1040.

k-Transmitter Problem: Given a set of obstacles in the plane, a target region, and a fixed integer $k \geq 0$, how many k-transmitters are necessary and sufficient to cover that region?

We consider instances of the k-transmitter problem in which the obstacles are line segments or simple polygons, and the target region is a collection of line segments, or a polygonal region, or the entire plane. In the case of plane coverage, we assume that transmitters may be embedded in the wall, and therefore can reach both sides of the wall at no cost. In the case of polygonal region coverage, we favor the placements of transmitters *inside* the region itself; therefore, when we talk about a *vertex* transmitter, the implicit assumption is that the transmitter is placed just inside the polygonal region, and so must penetrate one wall to reach the exterior.

1.1 Previous Results

For a comprehensive survey on the art gallery problem and its variants, we refer the reader to [19,20]. Also see [12,10,7] for results on the *wireless localization* problem, which asks for a set of *0*-transmitters that need not only cover a given region, but also enable mobile communication devices to prove that they are inside or outside the given region. In this section, we focus on summarizing existing results on the k-transmitter problem and a few related issues.

For $k = 0$, the k-transmitter problem for simple polygons is settled by the Art Gallery Theorem [8], which states that $\lfloor \frac{n}{3} \rfloor$ guards are sufficient and sometimes necessary to guard a polygonal region with n vertices. Finding the minimum number of *0*-transmitters that can guard a given polygon is NP-hard [18, 19]. For $k > 0$, Aichholzer et al. [13,2] study the k-transmitter problem in which the target region is represented as a monotone polygon or a monotone orthogonal polygon with n vertices. They show that $\frac{n}{2k}$ k-transmitters are sufficient, and $\lceil \frac{n}{2k+4} \rceil$ k-transmitters are sometimes necessary[1] to cover a monotone polygon. They also show that $\lceil \frac{n}{2k+4} \rceil$ k-transmitters are sufficient and necessary to cover any monotone orthogonal polygon. The authors also study simple polygons, orthogonal polygons and arrangements of lines in the context of very powerful transmitters, i.e, k-transmitters where k may grow as a function of n. For example, they show that any simple polygon with n vertices can always be covered with one transmitter of power $\lceil \frac{2n+1}{3} \rceil$, and this bound is tight up to an additive constant. In the case of orthogonal polygons, one $\lceil \frac{n}{3} \rceil$-transmitter is sufficient to cover the entire polygon. The problem of covering the plane with a single k-transmitter has been also considered in [15], where it is proved that there exist collections of n pairwise disjoint equal-length segments in the Euclidean plane such that, from any point, there is a ray that meets at least $2n/3$ of them (roughly). While the focus in [13, 2,15] is on finding a small number of high power transmitters, our focus in this paper is primarily on lower power transmitters. This direction of research

[1] The bound $\lceil n/(2k + 2) \rceil$ stated in Theorem 7 from [2] is a typo.

is partly motivated by practical applications, generally related to low k values, and partly by the celebrated art gallery theorem on guarding polygons, which corresponds to $k = 0$. The next natural step is to improve upon existing bounds for $k = 0$, by allowing $k = 1$ (for the case of line segments), or $k = 2$ (for transmitters placed interior to a polygon), before moving on to arbitrarily large values for k.

The concept of visibility through k segments has also appeared in other works. Dean et al. [11,17,14] study *vertical bar k-visibility*, where k-visibility goes through k segments. Aichholzer et al. [1] introduce and study the notion of *k-convexity*, where a diagonal may cross the boundary at most $2(k-1)$ times.

1.2 Our Results

We consider several instances of the k-transmitter problem. If obstacles are disjoint segments in the plane, where each segment has one of two slopes, and the target region is the entire plane, we show that $\lceil \frac{1}{2}((5/6)^{\log(k+1)}n + 1)\rceil$ k-transmitters are always sufficient and $\lceil \frac{n+1}{2k+2} \rceil$ k-transmitters are sometimes necessary to cover the target region. We generalize this result to the case where each segment has one of γ slopes, for some fixed integer $\gamma > 0$, and show that $\frac{2n}{3}(1 - \frac{1}{\lceil 10\gamma/3 \rceil})^{\log(k+1)}$ k-transmitters suffice to cover the target region. If the target region is the plane and the obstacles are lines and line segments that form a guillotine subdivision (defined in §2.2), then $\frac{n+1}{2}$ *1*-transmitters suffice to cover the target region. We next consider the case where the obstacles consist of a set of nested convex polygons. If the target region is the boundaries of these polygons, then $\lfloor \frac{n}{7} \rfloor + 3$ *2*-transmitters are always sufficient to cover it. On the other hand, if the target region is the entire plane, then $\lfloor \frac{n}{6} \rfloor + 3$ *2*-transmitters suffice to cover it, and $\lfloor \frac{n}{14} \rfloor + 1$ *2*-transmitters are sometimes necessary. All these results (detailed in §2) use point transmitters, with the implicit assumption that transmitters on a boundary segment are embedded in the segment and can reach either side of the segment at no cost.

In §3 we move on to the case where the target region is the interior of a simple polygon. In this case, we restrict the placement of vertex and edge transmitters to the interior of the polygon. We show that $\frac{n}{6}$ *2*-transmitters are sometimes necessary to cover the interior of a simple polygon. In §3.2 we introduce a class of spiral polygons, which we refer to *spirangles*, and show that $\lfloor \frac{n}{8} \rfloor$ *2*-transmitters are sufficient, and sometimes necessary, to cover the interior of a spirangle polygon. In the case of arbitrary spiral polygons, we derive an upper bound of $\lfloor \frac{n}{4} \rfloor$ *2*-transmitters, matching the upper bound for monotone polygons from [2].

2 Coverage of Plane with Obstacles

We begin with the problem of covering the entire plane with transmitters, in the presence of obstacles that are disjoint segments (§2.1), a guillotine subdivision (§2.2), or a set of nested convex polygons (§2.3). Throughout this manuscript, by a segment, we mean a line segment. There is no restriction on the placement of transmitters (on or off a segment). In the case of a transmitter located on a line segment itself, the assumption is that the segment does not act as on obstacle for that transmitter, in other words, that the transmitter has the power of a k-transmitter on both sides of the segment.

2.1 Disjoint Line Segments

We begin with an overview of our approach for the case of *1*-transmitters. Let S be a set of n disjoint segments in the plane. The main idea is to remove from S a set I of segments that are *independent* in the sense that no line goes through two of them consecutively, i.e. no two segments of I are weakly visible in S. We then take a set of conventional transmitters (i.e. *0*-transmitters) for the remaining segments $S \setminus I$. By upgrading these transmitters to *1*-transmitters we cover the whole plane with respect to the original segments S, as justified in Lemma 1 below. Thus one ingredient of the proof will be a bound on the size of a set of *0*-transmitters. For each $m \leq n$, let $\emptyset(m)$ denote the number of *0*-transmitters sufficient to cover the plane in the presence of any subset of m segments of S. The bounds we use are reported in Theorem 3 below.

To carry out the above plan, we must guarantee a large independent set I. We will do this by coloring an appropriately defined graph and taking the largest color class. More precisely, we will construct a plane graph (an embedded planar graph) in which we want one color class of a *cyclic coloring*: a coloring of the vertices such that any two vertices incident to the same face have different colors. The minimum number of colors required in a cyclic coloring depends on the maximum *face degree*, Δ^* (where the *degree* of a face is the number of edges incident to it). Let $cc(\Delta^*)$ denote the maximum, over all plane graphs of face degree Δ^*, of the minimum number of colors in a cyclic coloring of the graph. Cyclic coloring of plane graphs is a well-studied problem in graph theory. The bounds on $cc(\Delta^*)$ that we use are reported in Theorem 2 below.

To complete our overview, we give a bit more detail on the connection between cyclic coloring and our problem of finding an independent set of segments in S, and we explain how our bounds depend on the number of distinct slopes of the line segments in S. We first extend (in an arbitrary order) each end of each segment of S until it extends to infinity or hits another segment or an extension of one. The resulting subdivision of the plane, $X(S)$, has $n+1$ faces. Observe that if a set of k-transmitters covers the plane with respect to the extended segments $X(S)$ then it covers the plane with respect to the original segments S. From the planar subdivision $X(S)$, we will define a plane graph with vertex set S, and with the property that two segments of $X(S)$ are weakly visible in $X(S)$ if and only if they are incident to a common face in the plane graph. Thus one color class of a cyclic coloring of the graph corresponds to an independent set of segments in $X(S)$, which was our goal. Furthermore, faces of the graph will correspond to faces of the planar subdivision, so the maximum face degree in the planar subdivision will be equal to the maximum face degree in the graph.

Finally, the maximum face degree in $X(S)$ is related to the number of distinct slopes of the segments in S. The *slope* of a segment is the angle formed by the line supporting that segment with the x-axis, when moving counterclockwise from the x-axis to the supporting line. If all segments in S have one of γ slopes, for some fixed integer $\gamma > 0$, then the planar subdivision $X(S)$ has face degree $\Delta^* \leq 2\gamma$.

Having discussed all the main ingredients of our result, we now turn to the details. Our main technical result in this section is:

Theorem 1 *For a set, S, of n disjoint segments, let Δ^* denote the maximum face degree in an extension of S. Then $\emptyset(n(1 - \frac{1}{cc(\Delta^*)}))$ 1-transmitters can cover the plane in the presence of S.*

The results of this section follow from Theorem 1, so we pause to discuss the consequences of this theorem, before proving it. The smaller the value $cc(\Delta^*)$, the better the bound on the sufficient number of *1*-transmitters in Theorem 1. A similar statement holds for $\emptyset(m)$. The following are the best known bounds for the two quantities.

Theorem 2 *[16, 3–6] For all positive Δ^*, $\lfloor \frac{3}{2}\Delta^* \rfloor \leq cc(\Delta^*) \leq \lceil \frac{5}{3}\Delta^* \rceil$; and, if $\Delta^* \in \{3, 4, 5, 8\}$, then $cc(\Delta^*) \leq \lceil \frac{5}{3}\Delta^* \rceil - 1$.*

Note that the case $\Delta^* = 3$ is equivalent to the famous four color theorem. The case $\Delta^* = 4$ is Ringel's problem that was solved by Borodin [3, 4].

The following theorem states the best known bounds for $\emptyset()$.

Theorem 3 *[9, 19] Let S be a set of n disjoint segments. The plane can be covered by $\lfloor \frac{2}{3}n \rfloor$ 0-transmitters in the presence of S. If each segment in S has one of two slopes (e.g., horizontal or vertical), then $\lceil \frac{n+1}{2} \rceil$ 0-transmitters are sufficient to cover the plane in the presence of S.*

If all segments in S have one of γ slopes, for some fixed integer $\gamma > 0$, then any extension of S has face degree $\Delta^* \leq 2\gamma$. Then the following corollary follows from Theorems 1, 2 and 3 above.

Corollary 1 *Let S be a set of n disjoint segments in the plane, where each segment has one of γ fixed slopes. Then the plane can be covered by $\frac{2}{3}(1 - \frac{1}{\lceil 10\gamma/3 \rceil})n$ 1-transmitters in the presence of S.*

If all segments in S have one of two slopes, then $\Delta^* \leq 4$ and thus $cc(\Delta^*) \leq 6$ by Theorem 2. This together with the second part of Theorem 3 yields the following corollary.

Corollary 2 *Let S be a set of n disjoint segments in the plane, where each segment has one of two slopes. Then the plane can be covered by $\lceil \frac{5n+6}{12} \rceil$ 1-transmitters in the presence of S.*

To see that the bound in Corollary 2 is stronger than what can be obtained with *0*-transmitters only, consider the construction in Figure 1, which requires at least $\frac{n-2}{2}$ *0*-transmitters to cover the plane in its presence.

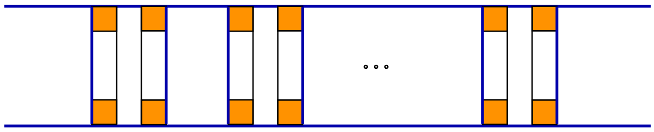


Fig. 1 A set with $n = 8k + 2$ segments that requires $4k$ *0*-transmitters. Each of the $4k$ shaded faces must have a *0*-transmitter, and no two such faces share a common point.

For the case of more than two slopes, the triangular division from Figure 2 can be used to show that the bound from Corollary 1 is stronger than what can be obtained with *0*-transmitters. Each segment has one of three slopes. Each of the $\frac{2(n-3)}{3}$ shaded triangular faces must have a *0*-transmitter interior or on its boundary, and no two shaded triangular faces share a common point. It follows that $\frac{2(n-3)}{3}$ *0*-transmitters are necessary to guard the plane in the presence of these n segments. (Compare it with the $\frac{3n}{5}$ bound obtained for $\gamma = 3$ from Corollary 1).

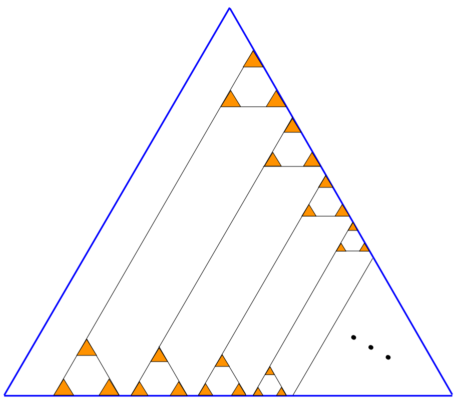


Fig. 2 At least $2(n-3)/3$ *0*-transmitters are necessary to cover the plane.

In the remainder of this section we prove Theorem 1. We start with a useful lemma, after a few definitions. Two segments $s, t \in S$ are *weakly visible* if there is a point p interior to s and a point q interior to t such that the segment pq does not properly cross any segment in S. Equivalently, for the case of extended segments, s and t are weakly visible in $X(S)$ if some face in $X(S)$ is incident to both of them. Two segments that are not weakly visible are called *independent*.

Lemma 1 *Let $I \subset S$ be a set of pairwise independent segments, and let T be a set of 0-transmitters that covers the plane in presence of $S \setminus I$. Then T is a* *set of 1-transmitters that covers the plane in presence of S. That is, the plane can be covered by $\emptyset(S \setminus I)$ 1-transmitters in the presence of S.*

Proof Suppose that a *0*-transmitter at point p, covers a point q in the presence of $S \setminus I$. Then the segment pq does not properly cross any segment in $S \setminus I$. It cannot properly cross two or more segments of I, because otherwise two such consecutive segments would be weakly visible (and therefore not independent). Thus a *1*-transmitter at p covers q in the presence of S. □

We show how to cover the plane in the presence of the extended (interior-disjoint) segments $X(S)$: if a set of k-transmitters covers the plane with respect to $X(S)$, then it covers the plane with respect to the original segment set S. Lemma 1 suggests that, in order to prove Theorem 1, all we need is to find a large set of independent segments in $X(S)$. We can obtain such a set by coloring the segments in $X(S)$ with a small number of colors, such that each pair of segments colored with the same color are independent. In the proof of Theorem 1, we will find such a coloring with the help of cyclic colorings and the related results of Theorem 2. We are now ready to prove Theorem 1.

Proof (Theorem 1) Define the following graph H: For each face f of $X(S)$, add a vertex to each edge on the boundary of f, and connect two such vertices whenever they correspond to two consecutive edges on f. Call these edges of H *type-1* edges. Also, for each extended segment in $X(S)$, connect every pair of vertices of H that are consecutive along that segment. See Figure 3. Call these edges *type-2* edges of H. Note that H is a plane graph. Let H' be

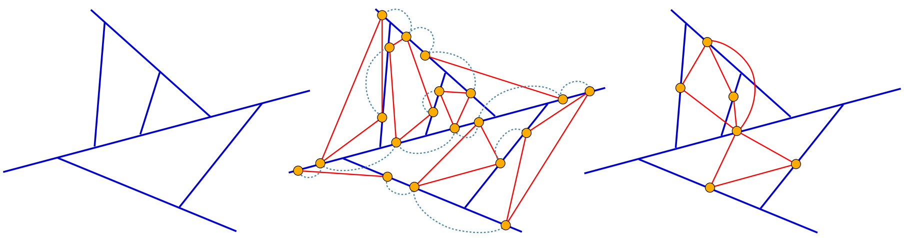


Fig. 3 (a) Extended set $X(S)$ (b) Type-1 (solid) and type-2 (dashed) edges in H (c) Graph H'.

the plane graph obtained by contracting all the type-2 edges in H. It can be verified that the largest degree of an internal face in H' is Δ^*. Thus H' has a cyclic $cc(\Delta^*)$-coloring. Two segments in $X(S)$ are independent if and only if their corresponding vertices in H' do not lie on the same face and are not adjacent. Thus $X(S)$ has a set I of at least $n/cc(\Delta^*)$ pairwise independent segments. Then Lemma 1 implies that $\emptyset(n - n/cc(\Delta^*))$ *1*-transmitters can cover the plane in presence of $X(S)$. The same bound applies immediately to S, as desired. □

The results of Corollaries 1 and 2 can be generalized to k-transmitters as follows.

Theorem 4 *Let S be a set of n disjoint segments in the plane, where each segment has one of two slopes. Then $\lceil \frac{1}{2}((5/6)^{\log(k+1)}n + 1) \rceil$ k-transmitters suffice to cover the plane in the presence of S, and $\lceil \frac{n+1}{2k+2} \rceil$ k-transmitters are sometimes necessary.*

Proof The lower bound is realized by parallel segments: one k-transmitter can only cover $2(k+1)$ of the $n+1$ regions. The proof for the upper bound builds on the proof technique for $k = 1$. We repeatedly remove independent sets from S, and extend the remaining segments after each removal. Let R_0 be S and, for $i = 1, 2, \ldots$, let S_i be a maximal independent set of segments in $X(R_{i-1})$. Let $R_i = S - (\cup_{j=1}^{i} S_j)$. Then R_i has cardinality at most $(5/6)^i n$.

Lemma 2 *If T is a set of 0-transmitters that covers the whole plane with respect to R_i, then T is a set of $(2^i - 1)$-transmitters that covers the whole plane with respect to $S = R_0$.*

Proof We prove by induction on $j = 0, \ldots, i$ that T is a set of $(2^j - 1)$-transmitters that covers the whole plane with respect to R_{i-j}. The base case $j = 0$ is true by the statement of the lemma. Assume that the inductive claim holds for $j - 1$, for some $j > 0$. Let q be a point in the plane covered by a $(2^{j-1} - 1)$-transmitter, placed at a point $p \in R_{i-j+1}$. Then the line segment pq crosses at most $2^{j-1} - 1$ segments of R_{i-j+1}, and therefore at most 2^{j-1} faces. Imagine adding back the segments of S_{i-j+1}, to obtain R_{i-j}. By definition, the segments of S_{i-j+1} are independent in R_{i-j}. This implies that the line segment pq can cross at most one segment of S_{i-j+1} in each face. The total number of segments of R_{i-j} crossed by pq is thus $2^{j-1} - 1 + 2^{j-1} = 2^j - 1$. In other words, a $(2^j - 1)$-transmitter at p in R_{i-j} covers the same area as the original $(2^{j-1} - 1)$-transmitter at p in R_{i-j+1}. □

We use this lemma to complete the proof of the theorem. Since we have the power of k-transmitters, we can continue removing independent sets up to $i = \log(k + 1)$ times ($k = 2^i - 1$). Then R_i has size $(5/6)^{\log(k+1)}n$, and the number of *0*-transmitters needed to cover the plane with respect to R_i is $\lceil \frac{1}{2}((5/6)^{\log(k+1)}n + 1) \rceil$. By Lemma 2, this is precisely the number of k-transmitters we need to cover the plane with respect to S. □

An analysis similar to the one used in the proof of Theorem 4 can be used to generalize the result of Corollary 1 to k-transmitters:

Theorem 5 *Let S be a set of n disjoint segments in the plane, where each segment has one of γ fixed slopes. Then the plane can be covered by $\frac{2n}{3}(1 - \frac{1}{\lceil 10\gamma/3 \rceil})^{\log(k+1)}$ k-transmitters in the presence of S.*

2.2 Guillotine Subdivisions

A *guillotine subdivision* S is obtained by inserting a sequence $s_1, \ldots, s_n$ of line segments (possibly rays or lines), such that each inserted segment s_i splits a face of the current subdivision S_{i-1} into two new faces yielding a new subdivision S_i. We start with one unbounded face S_0, which is the entire plane.

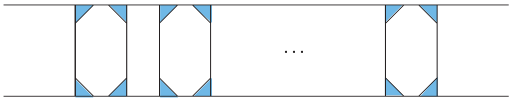


Fig. 4 A guillotine subdivision with $n = 6k + 2$ segments that requires $4k$ *0*-transmitters. Each of the $4k$ shaded faces must have a *0*-transmitter, and no two such faces share a common point.

As the example in Figure 4 shows, a guillotine subdivision with n segments can require $2(n-2)/3$ *0*-transmitters. In this section, we show that no guillotine subdivision requires more than $(n+1)/2$ *1*-transmitters. We begin with a lemma:

Lemma 3 *Let F be a face in a guillotine subdivision. If there are 1-transmitters on every face that shares an edge with F then these 1-transmitters see all of F.*

Proof Consider the segment s_i whose insertion created the face F. Before the insertion of s_i, the subdivision S_{i-1} contained a convex face that was split by s_i into two faces F and F' (Figure 5a). No further segments were inserted into F, but F' may have been further subdivided, so that there are now several faces $F'_1, \ldots, F'_k$, with $F'_j \subseteq F'$ and F'_j incident on s_i for all $j \in \{1, \ldots, k\}$ (Figure 5b).

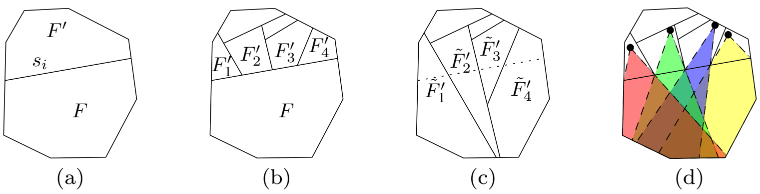


Fig. 5 The proof of Lemma 3.

We claim that the *1*-transmitters in $F'_1, \ldots, F'_k$ cover the interior of F. To see this, imagine removing s_i from the subdivision and instead, constructing a guillotine subdivision $\tilde{S}$ from the sequence $s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n$ (Figure 5c). In this case, each face F'_j in S becomes a larger face $\tilde{F}'_j$ in $\tilde{S}$ and together $\bigcup_{j=1}^{k} \tilde{F}'_j \supseteq F$. Finally, we observe that each *1*-transmitter in S in face F'_j covers at least $\tilde{F}'_j$, so together, the *1*-transmitters in $F'_1, \ldots, F'_k$ cover all of F (Figure 5d). □

Theorem 6 *Any guillotine subdivision can be covered with at most* $(n+1)/2$ *1-transmitters.*

Proof Consider the dual graph T of the subdivision. T is a triangulation with $n+1$ vertices. Let M be any maximal matching in T. Consider the unmatched vertices of T. Each such vertex is adjacent only to matched vertices (otherwise M would not be maximal). Let G be the set of *1*-transmitters obtained by placing a single *1*-transmitter on the primal edge associated with each edge $e \in M$. Then $|G| = |M| \leq (n+1)/2$. For every face F of S, F either contains a *1*-transmitter in G, or all faces that share an edge with F contain a *1*-transmitter in G. In the former case, F is obviously covered. In the latter case, Lemma 3 ensures that F is covered. Therefore, G is a set of *1*-transmitters that covers all faces of F and has size at most $(n+1)/2$. □

2.3 Nested Convex Polygons

The problems analyzed in this section are essentially two:

1. How many *2*-transmitters are always sufficient (and sometimes necessary) to cover the edges of a set of nested convex polygons?
2. How many *2*-transmitters are always sufficient (and sometimes necessary) to cover the plane in the presence of a set of nested convex polygons?

2.3.1 Some notation.

We call a set of k convex polygons $\{P_1, P_2, \ldots, P_k\}$ *nested* if $P_1 \supseteq P_2 \supseteq \cdots \supseteq P_k$. The total number of vertices of the set of polygons $\{P_1, P_2, \ldots, P_k\}$ is n.

Given such a set, we use the term *layers* for the boundaries of the polygons and *rings* for the portions of the plane between layers, i.e., the the i-th ring is $R_i = P_i - P_{i+1}$, for $i = 1, \ldots, k-1$. In addition, $R_0 = \mathbb{R} - P_1$ and $R_k = P_k$.

We assume that vertices on each layer have labels with indices increasing counterclockwise. Given a vertex $v_j \in P_i$, we call the positive angle $\angle v_{j-1}v_jv_{j+1}$ its *external visibility angle.* (Positive angles are measured counterclockwise, and negative angles are measured clockwise.) Its *internal visibility angle* is the negative angle $\angle v_{j-1}v_jv_{j+1}$.

We begin with a simple lemma that will assist us in our subsequent analysis.

Lemma 4 *Placing a 2-transmitter at every other vertex in a given layer* i *guarantees to completely cover layers* $i-3$, $i-2$, $i-1$ *and* i, *as well as rings* $i-3$, $i-2$ *and* $i-1$.

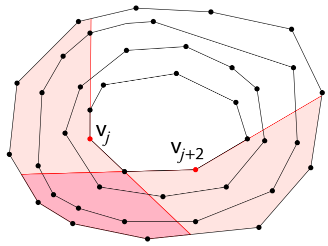


Fig. 6 External visibility angles of two vertices v_j, v_{j+2} of layer i. Only layers $i-3$, $i-2$, $i-1$ and i are shown.

Proof The fact that layer i is covered is obvious. As for the previous layers, notice that the convexity of P_i guarantees that the external visibility angles of any vertex pair v_j and v_{j+2} overlap, as illustrated in Figure 6. Since $v_j \in P_i \subseteq P_{i-1} \subseteq P_{i-2} \subseteq P_{i-3}$ and the polygons are convex, all rays from v_j within its external visibility angle traverse exactly two segments before reaching layer $i-3$. □

2.3.2 A particular case.

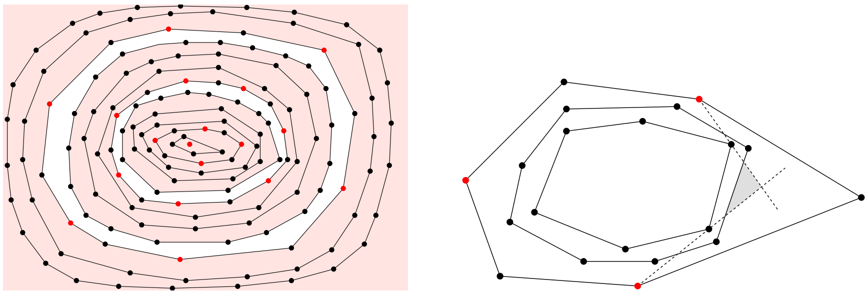
We first study the special case when all layers (convex polygons) have an even number of vertices.

Lemma 5 $\lfloor n/8 \rfloor + 1$ *2-transmitters are always sufficient to cover the edges of any nested set of convex polygons with a total of n vertices, if each of the polygons has an even number of vertices.*

Proof If the number of layers is $k \in \{1, 2, 3\}$, one transmitter trivially suffices. If $k \geq 4$, from the pigeonhole principle one of $i \in \{1, 2, 3, 4\}$ is such that the set $G_i = \{P_j \mid j \in \{1, \ldots, k\},\ j \equiv i (\text{mod } 4)\}$ has no more than $\lfloor n/4 \rfloor$ vertices (in fact, the number of vertices in G_i must be even since each layer has an even number of vertices). Place one *2*-transmitter at every other vertex of each $P_j \in G_i$, i.e., in $P_i, P_{i+4}, \ldots, P_{i+4m}$, where $m = |G_i| - 1$. From Lemma 4, the transmitters on a layer cover their own layer and the three preceding layers (if they exist). Since every fourth layer has transmitters starting with layer $i \in \{1, 2, 3, 4\}$, all layers from 1 up to $i + 4m$ are covered. On the other hand, $i+4m \in \{k, k-1, k-2, k-3\}$. If $i+4m = k$, all layers are covered; otherwise, placing one more *2*-transmitter in the interior of P_k completes the job, giving a total of at most $\lfloor n/8 \rfloor + 1$ *2*-transmitters. Figure 7a shows an example. □

As illustrated in Figure 7a, the location of the transmitters established in Lemma 5 does not guarantee that all rings are covered. Figure 7b shows a specific example that leaves some portions of the white rings uncovered.

Lemma 6 $\lfloor \frac{n}{6} \rfloor + 1$ *2-transmitters are always sufficient to cover the plane in the presence of any nested set of convex polygons with a total of n vertices, if each of the polygons has an even number of vertices.*



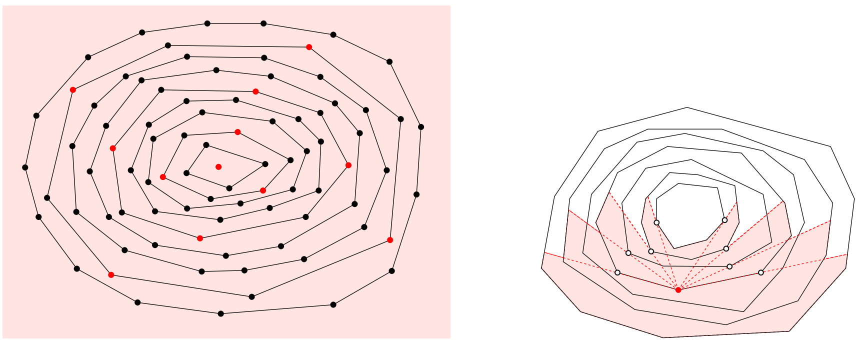
(a) (b)

Fig. 7 (a) Location of the at most $\lfloor n/8 \rfloor + 1$ *2*-transmitters to cover all the edges. The shaded rings are guaranteed to be covered. The white rings are not necessarily covered. (b) The shaded region is not covered by the *2*-transmitters located at the red vertices. Only the three involved layers are shown.

Proof An argument analogous to that of Lemma 5 establishes that the plane is entirely covered if a *2*-transmitter is located at every other vertex on each polygon in the class $G = \{P_j \mid j \in \{1, \ldots, k\},\ j \equiv i (\text{mod } 3)\}$, $i \in \{1, 2, 3\}$ having less than or equal to $\lfloor \frac{n}{3} \rfloor$ vertices, with the possible help of an additional *2*-transmitter in the interior of P_k. An example is depicted in Figure 8a. □

2.3.3 General case.

In this section we study the general case, independent of the parity (odd, even) of the vertex count in each layer.



(a) (b)

Fig. 8 (a) Location of the at most $\lfloor \frac{n}{6} \rfloor + 1$ *2*-transmitters to cover the entire plane. (b) External and internal visibility from a *2*-transmitter located in a vertex of layer i. Only layers $i-3$, $i-2$, $i-1$, i, $i+1$, $i+2$ and $i+3$ are shown.

Lemma 7 *Placing a 2-transmitter at each vertex of a given layer i guarantees to completely cover layers $i-3$, $i-2$, $i-1$, i, $i+1$, $i+2$ and $i+3$, as well as rings $i-3$, $i-2$, $i-1$, i, $i+1$ and $i+2$.*

Proof The fact that layers $i-3$, $i-2$, $i-1$, i and rings $i-3$, $i-2$ and $i-1$ are covered is a consequence of Lemma 4. As for the remaining layers and rings, notice that, in the internal visibility angle of a *2*-transmitter $v_j \in P_i$, visibility is determined by the supporting lines from v_j to layers $i+1$, $i+2$ and $i+3$, as illustrated in Figure 8b. Having a *2*-transmitter on each of the vertices of layer i, combined with the fact that all polygons are convex, guarantees total covering of layers $i+1$, $i+2$ and $i+3$ and rings i, $i+1$ and $i+2$. □

Theorem 7 $\lfloor \frac{n}{7} \rfloor + 3$ *2-transmitters are always sufficient to cover the edges of any nested set of convex polygons with a total of n vertices.*

Proof We first show that, if the number of layers is $k \in \{1,2,3,4,5,6\}$, then three *2*-transmitters suffice to cover all polygon edges. The argument is as follows. One *2*-transmitter in the interior of P_k takes care of covering the interior layers k, $k-1$ and $k-2$. The exterior layers 1, 2 and 3 can be covered using only two *2*-transmitters, as follows. Consider the leftmost and the rightmost vertices of layers 1, 2 and 3. If any of the polygon edges incident to these vertices is vertical, then rotate the entire configuration so as to avoid this situation (note that such a rotation always exists). Therefore, we may assume that each of P_1, P_2 and P_3 has exactly 2 extreme (leftmost, rightmost) vertices, for a total of 6 extreme vertices. Consider the 6 edges incident to these 6 vertices, which belong to the lower chains of the corresponding polygons. The intersection of the halfplanes they define is below P_1 and trivially not empty, so we can place one *2*-transmitter in it. Similarly, we can place another *2*-transmitter in the analogous intersection region for the edges in the upper chains. These two *2*-transmitters entirely cover layers 1, 2 and 3.

If $k \geq 7$, from the pigeonhole principle one of $i \in \{1,2,3,4,5,6,7\}$ is such that the set $G = \{P_j \mid j \in \{1,\ldots,k\},\ j \equiv i (\mathrm{mod}\ 7)\}$ has no more than $\lfloor \frac{n}{7} \rfloor$ vertices. Place one *2*-transmitter at each vertex of each $P_j \in G$. From Lemma 7, for a certain value of $m \in \mathbb{Z}$ all edges in the following layers are covered: $i-3, i-2, i-1$ (if they exist), $i, \ldots, i+7m$, $i+7m+1$, $i+7m+2$ and $i+7m+3$ (if they exist). In the worst case, the only layers that may remain uncovered are 1, 2 and 3, as well as $k-2$, $k-1$ and k. As in the case $k < 7$, three *2*-transmitters can take care of covering these layers. The total number of *2*-transmitters used is therefore at most $\lfloor \frac{n}{7} \rfloor + 3$. □

Again, as in Lemma 5, the transmitter placement from Theorem 7 guarantees that all edges are covered, while some rings remain uncovered.

Theorem 8 $\lfloor \frac{n}{6} \rfloor + 3$ *2-transmitters are always sufficient to cover the plane in the presence of any nested set of convex polygons with a total of n vertices.*

Proof The proof is similar to that of Theorem 7, but locating the *2*-transmitters at all vertices of every 6^{th} layer (as opposed to every 7^{th} layer in Theorem 7). □

2.3.4 Tighter bounds for small values of n

For small values of n, 3 extra *2*-transmitters (used in the bounds of Thms. 7 and 8) may contribute to an increase in the number of transmitters used. In this section we seek better bounds for small values of n.

Lemma 8 *The vertices of any triangulation of a given ring R_i can be colored with 3 colors, such that each triangle has a vertex of each color, by duplicating at most two vertices.*

Proof The dual graph of the triangulation is necessarily a cycle. To break the cycle, slice the triangulation along an arbitrary edge shared by two adjacent triangles, and duplicate the endpoints of that edge. Then the dual cycle turns into a path, and we can 3-color the vertices of the triangulation in a straightforward manner, starting from an arbitrary vertex, until the path of triangles gets completed. See Figure 9 for an illustration. □

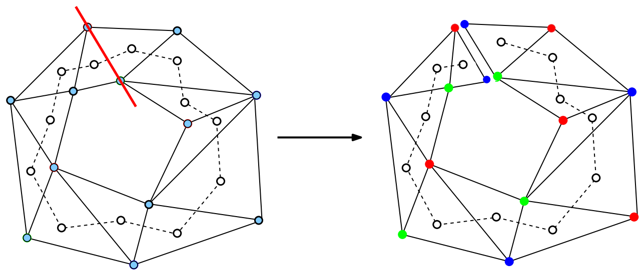


Fig. 9 Left: Slicing the cycle of triangles. The dual graph of the triangulation is shown with dashed edges and unfilled vertices. Right: 3-coloring the triangulation path.

For simplicity, we stretch the standard definition of a k-coloring of a graph, and call the coloring referred to by Lemma 8 a 3-*coloring* of a ring triangulation.

Lemma 9 *Placing one 2-transmitter at each vertex in the smallest color class of a 3-colored triangulation of a ring $R = P_i - P_{i+2}$ guarantees to completely cover layers $i-1$, i, $i+1$ and $i+2$, as well as rings $i-1$, i, and $i+1$.*

Proof The situation is illustrated in Figure 10. Rings i and $i+1$, as well as layer $i+2$ are contained in the triangulation. Placing a *2*-transmitter at each vertex of the smallest color class, ensures that each triangle will have a *2*-transmitter at one of its vertices (since each triangle has a vertex of each color). Hence we need only argue that a triangle of the triangulation can be fully covered by a *2*-transmitter placed at any one of its vertices. Let v be a vertex of a triangle T in the triangulation, and let p be any point in T. The only obstruction to v seeing p is layer $i+1$. Now because layer $i+1$ is convex, segment vp crosses layer $i+1$ at most once if v is on layer $i+2$ and at most twice if v is on layer i. Hence v can see p under *2*-transmission.

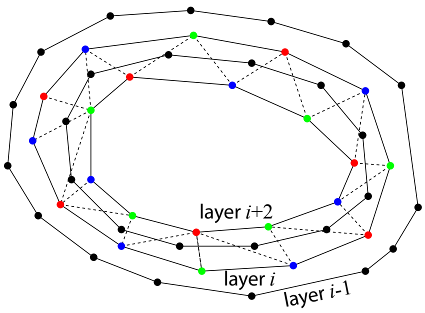


Fig. 10 Covering layers $i-1$, i, $i+1$ and $i+2$, as well as rings $i-1$, i and $i+1$, from the vertices of the least popular color.

Finally we must argue that we will also have covered ring $i-1$. Each edge $v_{j-1}v_j$ of layer i supports a triangle T of the triangulation whose third vertex, u belongs to layer $i+2$. Extend the edges uv_{j-1} and uv_j until they each hit layer $i-1$. The edge extensions define a visibility cone in ring $i-1$, as shown in Figure 11 (left). A transmitter placed at any vertex of T can see the entire

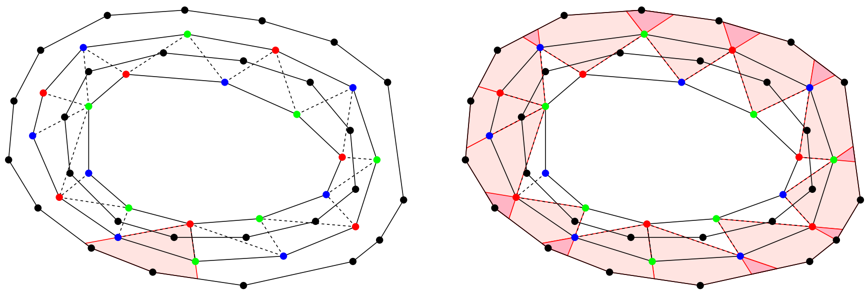


Fig. 11 Left: the visibility cone of a triangle. Right: the union of such visibility cones fully covers ring $i-1$.

cone. We showed above that it can see all of T. To see that it covers the rest of the cone, consider any point p that is in the cone but not in T. If the transmitter is at a vertex v on layer $i+2$, then it follows that v can see p from an analogous argument to the one above, except that now segment vp crosses layers $i+1$ and i exactly once each. If the transmitter is at a vertex v on layer i, then segment vp crosses no layers since it doesn't pass through T. Furthermore the union of the cones (i.e. taking the cone for each edge in layer i) fully covers ring $i-1$, as illustrated in Figure 11 (right). This shows that ring $i-1$ is also covered. $\square$

Theorem 9 $\lfloor \frac{2n}{9} \rfloor + 1$ *2-transmitters are always sufficient to cover the edges of any nested set of convex polygons with a total of n vertices.*

Proof If the number of layers is $k \in \{1, 2, 3\}$, one transmitter trivially suffices. If $k \geq 4$, from the pigeonhole principle one of $i \in \{1, 2\}$ is such that the set $G = \{P_j \mid j \in \{1, \ldots, k\},\ j \equiv i (\text{mod } 2)\}$ has no more than $\lfloor \frac{n}{2} \rfloor$ vertices. Consider only the layers in G. Triangulate every other ring in the resulting set of nested layers, starting from the first ring, using chords connecting vertices of different layers (see Figure 12).

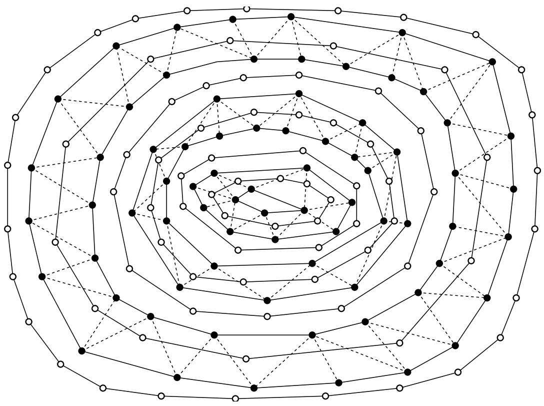


Fig. 12 Triangulating every other ring in G. Layers with filled vertices are in G, layers with unfilled vertices are not in G.

From Lemma 8, all the selected rings can be 3-colored by duplicating at most two vertices per ring. Since there are $\lfloor \frac{n}{2} \rfloor$ vertices in total, and each ring must at least have 6 vertices, at most $\lfloor \frac{n}{6} \rfloor$ vertices get duplicated, giving rise to a total of at most $\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{6} \rfloor \leq \lfloor \frac{2n}{3} \rfloor$ colored vertices. From the pigeonhole principle, the least popular of the 3 colors must have at most $\lfloor \frac{1}{3} \lfloor \frac{2n}{3} \rfloor \rfloor = \lfloor \frac{2n}{9} \rfloor$ vertices. Place one *2*-transmitter at each of these vertices, plus possibly one *2*-transmitter in the interior of P_k.

Let us now prove that these *2*-transmitters cover the entire set of layers. Notice that each of the triangulated rings is formed by some layers i and $i+2$. From Lemma 9, layers $i-1$, i, $i+1$ and $i+2$ are covered. Layers lying in the exterior or in the interior of the configuration of rings must also be taken care of. Notice that at most one layer (not belonging to G) can lie in the exterior of a triangulated ring, and Lemma 9 guarantees that this layer is covered. As for the interior, in the worst case G may end up with three uncovered layers (see Figure 13a). In this case, one more *2*-transmitter located in the interior of P_k will complete the job. □

Theorem 9 guarantees that the entire set of layers is covered, however some of the rings may not be fully covered. We achieve different bounds for the case of covering the entire plane in Theorem 10.

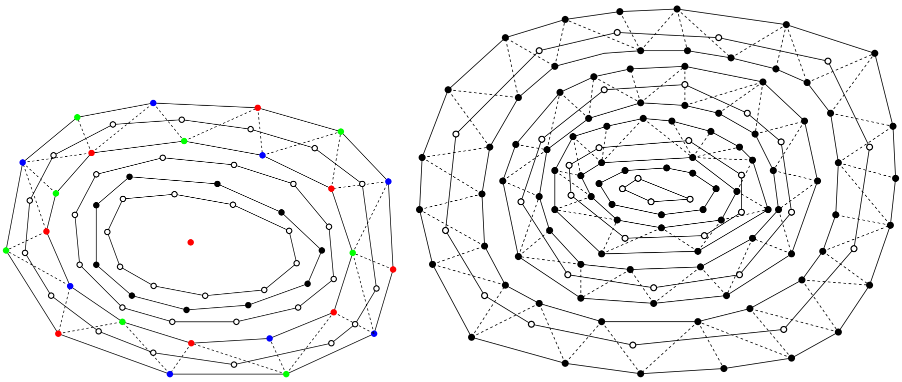


Fig. 13 (a) Covering the interior layers. (b) Triangulating the rings of G (filled vertices) containing the layers of H (unfilled vertices).

Theorem 10 $\lfloor \frac{8n}{27} \rfloor + 1$ *2-transmitters are always sufficient to cover the plane in the presence of any nested set of convex polygons with a total of n vertices.*

Proof The proof is a slight modification of that of Theorem 9. In this case, we consider the class $H = \{P_j \mid j \in \{1, \ldots, k\},\ j \equiv i (\text{mod } 3)\}$ for $i \in \{1, 2, 3\}$ having at least $\lceil \frac{n}{3} \rceil$ vertices, and let G be the set of the remaining layers. The rings to be triangulated are those of G embedding the layers of H in their interior (refer to Figure 13b). In this case, G contains at most $\lfloor \frac{2n}{3} \rfloor$ vertices and the coloring of the triangulations of the rings may require the duplication of at most two points per layer. Hence, the number of vertices of the smallest color class is less or equal than $\lfloor \left(\lfloor \frac{2n}{3} \rfloor + 2\lfloor \frac{1}{6} \frac{2n}{3} \rfloor\right) \frac{1}{3} \rfloor \leq \lfloor \frac{8n}{27} \rfloor$.

Again, the layers lying in the exterior of the configuration of rings cannot produce an occlusion to transmission, since there cannot be more than one. Hence, R_0 is covered. As for the most interior rings, one more *2*-transmitter, located in the interior of P_k, guarantees that they are covered. □

2.3.5 Lower bounds

Lemma 10 *For any nested set H of convex polygons with a total of n vertices,* $\lfloor \frac{n}{14} \rfloor$ *2-transmitters are sometimes necessary to cover the edges of the polygons in H, and* $\lfloor \frac{n}{14} \rfloor + 1$ *2-transmitters are sometimes necessary to cover the plane in the presence of H.*

Proof These lower bounds are established by the example from Figure 14, which shows seven nested regular t-gons, with t even (so $n = 7t$). Consider the set S of midpoints of alternating edges of the middle convex layer (marked u_i in Figure 14). The gap between adjacent layers controls the size of the visibility regions of the points in S (by symmetry, all visibility regions have identical size). A small enough gap guarantees that the visibility regions of the points in S are all disjoint, as illustrated in Figure 14 for $t = 10$. (Note, however, that

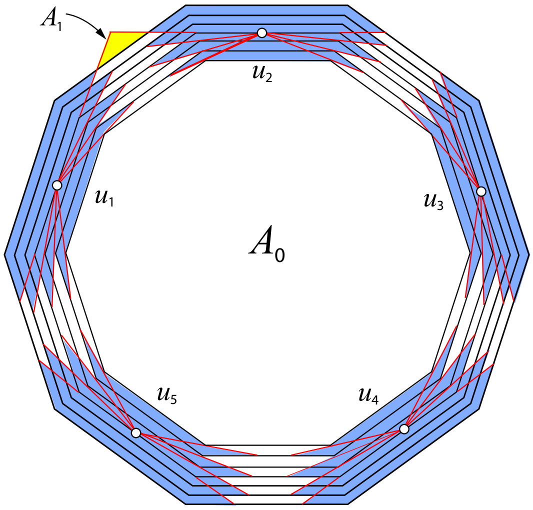


Fig. 14 $\lfloor \frac{n}{14} \rfloor + 1$ *2*-transmitters are necessary to cover the plane in the presence of these seven nested convex layers.

this claim holds for any $t \geq 4$.) This means that at least $t/2$ *2*-transmitters are necessary to cover all points in S (one transmitter in the visibility region of each point). So the number of *2*-transmitters necessary to cover all edges is at least $t/2 = n/14$.

Consider now the small area marked A_1 in Figure 14, bounded by two visibility rays from u_1 and u_2, and exterior to the outmost convex layer. Because the visibility regions of u_1 and u_2 are disjoint, A_1 is non-empty. Note that, from among all *2*-transmitters covering S, only the transmitters covering u_1 and u_2 can potentially cover A_1 as well. If neither of these two transmitters covers A_1, then one extra transmitter is necessary to cover A_1, thus establishing the lower bound of the lemma. If at least one of these two transmitters covers A_1, then such a transmitter cannot cover the region A_0 interior to the innermost convex layer, because A_0 and A_1 are separated by 7 closed polygonal walls. Also note that there are exactly $t/2$ triangular regions A_i (one A_i clockwise to each u_i), and that a transmitter covering u_i cannot simultaneously cover both A_i and A_{i-1} (here we use A_0 as an alias for $A_{t/2}$). These imply that $t/2 + 1$ transmitters are necessary to cover all of u_i, A_i, plus the interior region A_0. This establishes the second lower bound of the lemma. □

Table 1 summarizes our results on nested convex polygons.

3 Coverage of Simple Polygons

This section addresses the problem of covering a polygonal region P with *2*-transmitters placed *interior* to P. Therefore, when we talk about a *vertex* or

Target Region	Number of *2*-transmitters			
	Upper Bound		Lower Bound	
Edges	$\min\{\lfloor n/7\rfloor + 3, \lfloor 2n/9\rfloor + 1\}$	(Thms. 7,9)	$\lfloor n/14\rfloor$	(Lem. 10)
Plane	$\min\{\lfloor n/6\rfloor + 3, \lfloor 8n/27\rfloor + 1\}$	(Thms. 8,10)	$\lfloor n/14\rfloor + 1$	(Lem. 10)

Table 1 Covering results for nested convex polygons.

an *edge* transmitter, the implicit assumption is that the transmitter is placed just inside the polygonal region, and so must penetrate one wall to reach the exterior.

3.1 Lower Bounds For Covering Polygons

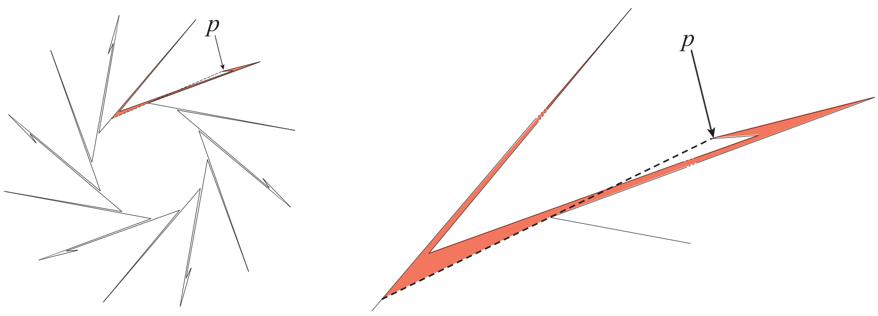


Fig. 15 A family of polygons requiring at least $n/6$ interior *2*-transmitters to cover. For labeled point p located in the tip of a barb (shown magnified on the right with the arms shortened), the locus of all interior points from which a *2*-transmitter can cover p is shown shaded.

Theorem 11 *There are simple polygons that require at least* $\frac{n}{6}$ *2-transmitters to cover when transmitters are restricted to the interior of the polygon.*

Proof Figure 15 shows the construction for a $n = 36$ vertex polygon, which generalizes to $n = 6m$, for any $m \geq 2$. It is a pinwheel whose $n/3$ arms alternate between spikes and barbs. Consider an interior point p at the tip of a barb. The locus of all interior points from which a *2*-transmitter can cover p includes the spike counter-clockwise from the barb, the barb containing p, and a small section of the pinwheel center. This region is shown shaded for the point p labeled in Figure 15. Observe that this shaded region is disjoint from the analogous regions associated with the other barb tips. Hence no two barb tips can be covered by the same *2*-transmitter. Since there are $n/6$ barbs, the lower bound is obtained. □

3.2 Spirangles

Two edges are *homothetic* if one edge is a scaled and translated image of the other. For any integer $t > 2$, a t-*spirangle* is a polygonal chain $A = a_1, a_2, ..., a_m$ that spirals inward about a center point such that every t edges it completes a 2π turn, and each edge pair $a_i a_{i+1}, a_{i+t} a_{i+1+t}$ is homethetic, for $1 \le i \le m-t$. (The condition $t > 2$ reflects the fact that two edges cannot complete a 2π turn.)

We assume that the spiral direction is clockwise. A t-sided convex polygon may be thought of as generating a family of t-spirangles where the i^{th} edge of each spirangle is parallel to the $(i \mod t)^{th}$ edge of the polygon, for $i = 0, 1, 2, \ldots$. See Figure 16a for a 4-spirangle example and a polygon generating it.

A *homothetic t-spirangle* polygon P is a simple polygon whose boundary consists of two nested t-spirangles $A = a_1, a_2, ..., a_m$ and $B = b_1, b_2, ..., b_m$ generated by the same t-sided convex polygon, plus two additional edges $a_1 b_1$ and $a_m b_m$ joining their endpoints. We assume that chain B is nested inside of chain A, as shown in Figure 16b. We refer to A as the convex chain and B as the reflex chain in reference to the type of vertices found on each.

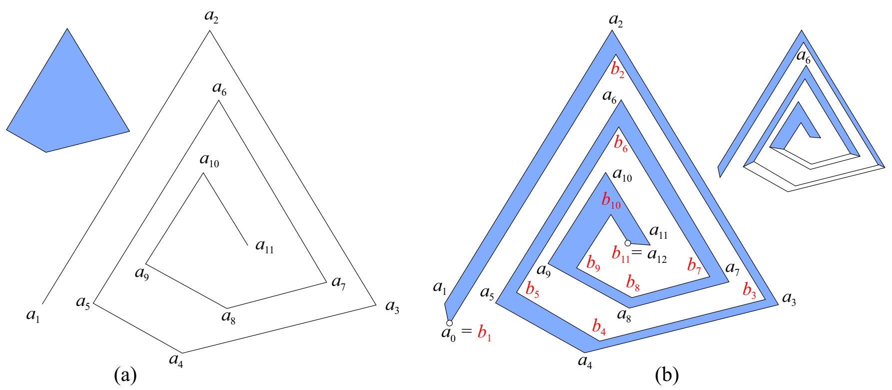


Fig. 16 Definitions (a) A 4-spirangle and corresponding convex polygon (b) Edge-homothetic spiral polygon (left) and quadrilaterals entirely visible to a 2-transmitter placed at a_6 (right).

Property 1 Let P be a homothetic spirangle polygon, composed of a convex spirangle $A = a_1, a_2, \ldots,$ and a reflex spirangle $B = b_1, b_2, \ldots$. Then a_i and b_i see each other, and the set of diagonals $\{a_i b_i \mid i = 1, 2, \ldots\}$, induces a partition of P into quadrilaterals. Furthermore, the visibility region of the 2-transmitter placed at a_i includes six quadrilaterals: two quadrilaterals adjacent to $a_{i-t} b_{i-t}$, two adjacent to $a_i b_i$, and two adjacent to $a_{i+t} b_{i+t}$. See right of Figure 16b.

Theorem 12 $\lfloor \frac{n}{8} \rfloor$ *2-transmitters are sufficient, and sometimes necessary, to cover a homothetic t-spirangle polygon P with n vertices.*

Proof Recall that $t \geq 3$ (by definition) and $n \geq 2(t+1) \geq 8$, therefore $\lfloor n/8 \rfloor \geq 1$. The algorithm that places transmitters at vertices of P to cover the interior of P is fairly simple, and is outlined in Table 2.

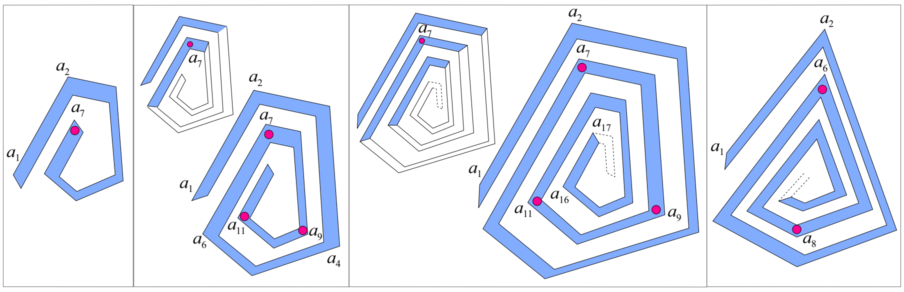
Homothetic t-Spirangle Polygon Cover(P)

Let $A = a_1, a_2, \ldots a_m$ be the convex spirangle of P, with a_1 outermost.
Let $B = b_1, b_2, \ldots b_m$ be the reflex spirangle of P.

1. If $m \leq t+2$ (or equivalently, the total turn angle of A is $\leq 2\pi$):
   Place one transmitter at a_m, and return (see Figure 17a).
2. Place the first transmitter at vertex a_{t+2} (see a_7 in Figure 17b).
3. Starting at a_{t+2}, place transmitters at every other vertex of A, up to a_{2t+1} (i.e., for a 2π turn angle of A, but excluding a_{2t+2}).
4. Let a_j be the vertex hosting the last transmitter placed in step 3.
   ($j = 2t+1$ for t odd, $j = 2t$ for t even.)
   Let P_1 be the subpolygon of P induced by vertices $a_1, \ldots, a_{j+t+1}$ and $b_1, \ldots, b_{j+t+1}$ (shaded left of Figure 17b).

   Recurse on $P \setminus P_1$: Homothetic t-Spirangle Polygon Cover($P \setminus P_1$).

Table 2 Covering the interior of a homothetic spirangle polygon with *2*-transmitters.



(a) (b) (c) (d)

Fig. 17 Covering spirangles with *2*-transmitters: (a) A t-spirangle ($t = 5$) with $2t+4$ edges covered with one transmitter. (b) A t-spirangle ($t = 5$) with $8t$ edges. (c) A t-spirangle ($t = 5$) with $6t+4$ edges covered with $t/2+1$ transmitters. (d) A t-spirangle ($t = 4$) with $6t$ edges covered with $t/2$ transmitters.

We now turn to proving that the algorithm described in Table 2 covers the interior of P. If the total turn angle of A is no greater than 2π, then one *2*-transmitter placed at an innermost vertex suffices, as illustrated in Figure 17a. Such a transmitter can reach any point interior to P by passing through at most two edges of P. If the total turn angle of A is greater than 2π, the algorithm skips the first 2π turn, places transmitters at every other vertex of the second 2π turn, then skips the third 2π turn before recursing. This procedure is depicted in Figures 17b and 17c. By Property 1, a transmitter placed at a vertex a_i covers all six quadrilaterals incident to a_{i-t}, a_i and a_{i+t} (see left of Figure 17c). It follows that the entire P gets covered.

To obtain an upper bound on the number of transmitters, we charge four quadrilaterals (eight spirangle edges) to each transmitter a_i – those adjacent to a_ib_i and $a_{i-t}b_{i-t}$ (see top of Figure 17b). It may appear that we could charge to a_i the two quadrilaterals adjacent to $a_{i+t}b_{i+t}$ as well, however it may be that the spirangle does not extend this far (i.e., the total turn angle of the spirangle is less than 6π).

In any iteration of the recursion, the last transmitter may be charged with one quadrilateral that has already been charged to the first transmitter (see a_7, a_{11} in Figure 17b, both of which are charged with quadrilateral $a_6a_7b_7b_6$). Moreover, transmitters placed in the final iteration may likewise be charged with quadrilaterals already charged to transmitters in a previous iteration. However, since transmitters are placed at every other vertex, and since each iteration (except possibly the last) skips the first 2π turn, no other such collisions may occur. Then each transmitter is in charge of precisely eight edges, yielding a bound of $\lfloor \frac{n}{8} \rfloor$ transmitters.

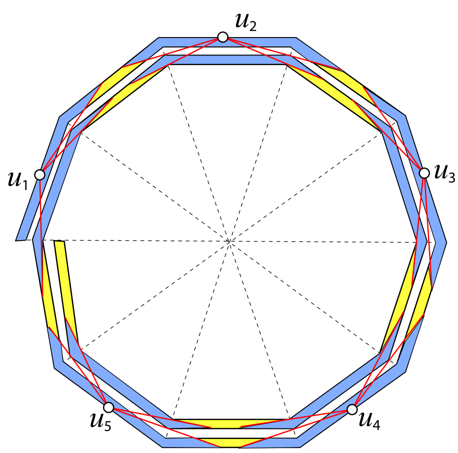


Fig. 18 $\lfloor \frac{n}{8} \rfloor + 1$ 2-transmitters are necessary to cover the edges of this spirangle polygon.

The fact that this bound is tight is established by the spirangle polygon example from Figure 18, which shows a 4π turn spirangle polygon P corresponding to a t-sided regular polygon. The total number of vertices of P is $n = 4t + 2$. This is a worst-case scenario in which transmitters do not get the chance to use their full coverage potential, since the total turn angle of the spirangle is between 2π and 6π.

The argument here is similar to the one used in the proof of Lemma 10. Consider the set S of midpoints of alternating outermost edges (marked u_i in Figure 18b). The gap between the turns controls the size of the visibility regions of the points in S. A small enough gap guarantees that the visibility regions of the points in S are all disjoint, meaning that at least $t/2$ *2*-transmitters are necessary to cover all points in S (one transmitter in the visibility region of each point). So the number of *2*-transmitters necessary to cover all edges is $t/2 = n/8$. □

The following lemma establishes a lower bound for the case when the total turn angle of the spirangle is arbitrarily large.

Lemma 11 *There are homothetic* 3*-spirangle polygons that require* $\lceil \frac{n}{10} \rceil$ *2-transmitters.*

Proof This lower bound is established by the triangular spirangle polygon P from Figure 19b. We show inductively that at least $\lceil \frac{n}{10} \rceil$ *2*-transmitters are necessary to cover the interior of P. Let $A = a_1, \ldots, a_m$ be the convex 3-spirangle of P, with a_1 an outermost vertex. Similarly, let $B = b_1, b_2, \ldots, b_m$ be the reflex 3-spirangle of P. For $i = 0, 1, \ldots$, define layer L_i to be the spirangle subpolygon of P induced by the subchains $(a_{3i+1}, a_{3i+2}, a_{3i+3}, a_{3i+4})$ and $(b_{3i+1}, b_{3i+2}, b_{3i+3}, b_{3i+4})$. Thus, adjacent layers share two vertices, one a-vertex and one b-vertex.

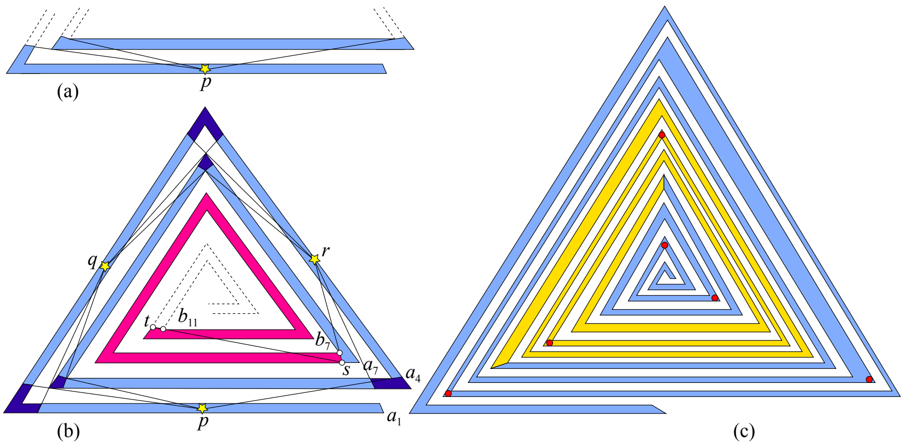


Fig. 19 Homothetic 3-spirangles require $\lceil \frac{n}{10} \rceil$ transmitters (a) Visibility area V(p) (b) Maximum area covered by transmitters visible to p, q, and r (c) Coverage by the algorithm from Table 2.

Consider now three points p, q, r placed halfway along the three outer edges of layer L_0. The locus of all points visible from p, denoted $V(p)$, can be obtained by extending from p tangents to the convex and reflex chains of L_1. These tangents delimit the area $V(p)$, shaded in Figure 19a. Note that $V(p)$, $V(q)$ and $V(r)$ have pairwise non-empty intersections (shaded in a darker color in Figure 19b), however the three of them share no common point. This implies that at least two transmitters are necessary to cover all three of p, q and r, and these transmitters must be placed in the area $V(p, q, r) = V(p) \cup V(q) \cup$

$V(r)$. We take one step further and delineate the visibility region $V^2(p,q,r)$ of all points in $V(p,q,r)$. Note that $V^2(p,q,r)$ can be obtained by restricting our attention to vertices of $V(p,q,r)$. Using the same approach of extending tangents from vertices of $V(p,q,r) \setminus L_0$ to the reflex and convex chains of L_2, we determine that $V(p,q,r)$ can see the entire layer L_2, plus a small piece of layer L_3 extending past the diagonal $a_{11}b_{11}$ (see entire shaded area in Figure 19b). The actual size of this L_3 piece is irrelevant to our analysis. The important observation is that the removal of $V^2(p,q,r)$ leaves an edge-homothetic spiral polygon with $n-20$ edges.

We have established p, q and r require at least two transmitters placed in the area $V(p,q,r)$, and that those transmitters can cover no points outside of $V^2(p,q,r)$. Inductively, we can argue that $P \setminus V^2(p,q,r)$ requires $\lceil \frac{n-20}{10} \rceil = \lceil \frac{n}{10} \rceil - 2$ transmitters. Summing up these transmitters with the two transmitters placed in the area $V(p,q,r)$, yields the lower bound claimed by the theorem. Figure 19c shows the coverage of a 3-spirangle polygon with $\lceil \frac{n}{10} \rceil$ transmitters, produced by the upper bound algorithm from Table 2. □

3.3 Arbitrary Spirals

A spiral polygon P consists of a clockwise convex chain and a clockwise reflex chain that meet at their endpoints. A trivial $\lfloor \frac{n}{4} \rfloor$ upper bound for the number of *2*-transmitters that are sufficient to cover P can be obtained as follows. Pick the chain Γ of P with fewer vertices (i.e., Γ is the reflex chain of P, if the number of reflex vertices is less than the number of convex vertices, and the convex chain of P otherwise). Then simply place one *2*-transmitter at every other vertex of Γ. By definition, the visibility ray from one *2*-transmitter can cross the boundary of P at most twice. Note however that, even under the restriction that transmitters be placed interior of P, the visibility ray of one transmitter can leave and re-enter P, as depicted in Figure 20a for transmitter labeled a. Then arguments similar to the ones used in Lemma 4 show that the union of the external visibility angles of all these *2*-transmitters cover the entire spiral. So we have the following result:

Lemma 12 $\lfloor \frac{n}{4} \rfloor$ *2-transmitters placed interior to an arbitrary polygonal spiral P are sufficient to cover P.*

We remark on two special situations. In the case of transmitters placed at every other reflex vertex of P, *0*-transmitters are sufficient to cover the interior of P; and in the case of transmitters placed at every other convex vertex of P, *1*-transmitters are sufficient to cover P, *if* they are placed *outside* of P.

An improved upper bound can be established for *non-degenerate* spirals, which we define as spirals in which each 2π-turn of each of the convex and reflex chain of P is homothetic to a convex polygon (i.e., it contains at least 3 vertices).

Lemma 13 *Let P be a polygonal spiral whose every 2π turn subchain has at least 3 vertices. Then $\lceil \frac{2n}{9} \rceil$ 2-transmitters placed interior to P are sufficient to cover the interior of P (in fact, the entire plane).*

Proof We distinguish two situations, depending on the relative number of reflex and convex vertices. If the number of convex vertices does not exceed $\lfloor \frac{4n}{9} \rfloor$, then we place a *2*-transmitter at every other convex vertex of P, for a total number of $\lfloor \frac{2n}{9} \rfloor$ *2*-transmitters. Arguments similar to the ones above show that the entire plane is covered in this case.

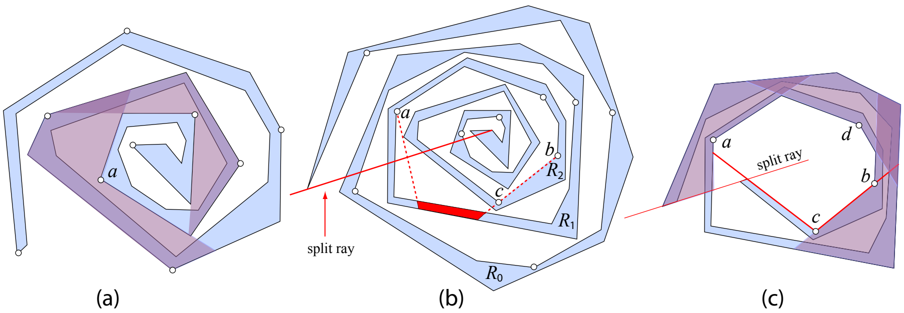


Fig. 20 Transmitters marked with a small circle (a) External visibility angle of a (b) The dark area is not covered by a and b (c) P is covered.

If the number of convex vertices is greater than $\lfloor \frac{4n}{9} \rfloor$, then the number of reflex vertices is at most $\lfloor \frac{5n}{9} \rfloor$. In this case, we partition P into "layers" P_1, P_2, P_3, ..., using a *split ray* that starts at the last (innermost) vertex and passes through the first (outermost) vertex of the reflex chain of P. Let R_i be the reflex chain of P_i. See Figure 20b. We divide these reflex chains into two sets $S_i = \{R_j \mid j \equiv i (\mathrm{mod}\ 2)\}$, for $i = 0, 1$. By the pigeonhole principle, one of these sets (call it S) has no more than $\lfloor \frac{5n}{18} \rfloor$ vertices. We place *2*-transmitters at every other reflex vertex of each chain $R_j \in S$, starting with the first vertex of R_j; if R_j has an even number of vertices, we add one extra *2*-transmitter at the last vertex of R_j. We claim that the transmitters placed on R_j cover the layers P_j and P_{j-1} (if $j > 1$).

To see this, note that the visibility angles of the *2*-transmitters placed at every other vertex of R_j overlap so that collectively they cover a contiguous region of each of P_j and P_{j-1}, starting at the split ray and extending clockwise (see Figure 20c). If R_j has an odd number of vertices, then the visibility angles of the first and last transmitters on R_j also overlap so that P_j and P_{j-1} are entirely covered. Otherwise, there may be end pieces of P_j and P_{j-1} that remain uncovered, unless an extra transmitter is placed at the last vertex of R_j (see, for example, the chain R_2 with 6 reflex vertices from Figure 20b, in which the transmitters a and b do not cover the dark region of P_1). Let c be the last vertex of R_j. The edge of P extending clockwise from c must cross the split ray, since R_j starts and ends on the split ray (by definition). This implies that the visibility angle of the 2-transmitter at c overlaps the visibility angle of the first transmitter on R_j, and the apex of the shared angle is on the other side of the split ray. This shows that c and the first transmitter on R_j cover a contiguous region of P_j and P_{j-1}, and similarly c and the previous transmitter on R_j cover a contiguous region of P_j and P_{j-1}. Therefore, P_j and P_{j-1} are entirely covered.

The total number of 2-transmitters used is $\lfloor \frac{5n}{36} \rfloor + \lceil \frac{\ell}{2} \rceil$, where ℓ is the number of layers. By our non-degeneracy assumption, each layer has at least 6 vertices (at least 3 reflex vertices and at least 3 convex vertices), which implies $\ell \le \frac{n}{6}$ (the last innermost layer could be covered with a single 2-transmitter, so we do not count it here). This gives us a total of at most $\lfloor \frac{5n}{36} \rfloor + \lceil \frac{n}{12} \rceil$, which is upper bounded by $\lceil \frac{2n}{9} \rceil$. □

4 Conclusion

In this paper we study the problem of covering ("guarding") a target region in the plane with k-transmitters, in the presence of obstacles. We develop lower and upper bounds for the problem instance in which the target region is the plane, and the obstacles are lines and line segments, a guillotine subdivision, or nested convex polygons. We also develop lower and upper bounds for the problem instance in which the target region is the set of rings created by nested convex polygons, or the interior of a spiral polygon. Our work leaves many interesting problems open. The two main open problems are (1) developing an upper bound on the number of k-transmitters required to cover the interior of a simple polygon, and (2) generalizing the lower bound result of Theorem 11 for arbitrary k. Along the way, progress can be made on closing the three gaps left open by our work: (i) the gap between the lower and upper bound for the case of disjoint line segments in the plane (Theorem 5), (ii) the gap between the $\lfloor \frac{n}{14} \rfloor$ lower bound and the $\lfloor \frac{n}{6} \rfloor$ upper bound for the case of nested convex layers, and (iii) the gap between the $\lfloor \frac{n}{8} \rfloor$ lower bound and the $\lfloor \frac{n}{4} \rfloor$ upper bound for spiral polygons. Investigating the k-transmitter problem for other classes of polygons (such as orthogonal polygons) also remains open.

Acknowledgements We thank Joseph O'Rourke for the pinwheel example from Figure 15 and for initiating this line of work.

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