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THE JOINT MOMENT GENERATING FUNCTION OF QUADRATIC FORMS IN MULTIVARIATE AUTOREGRESSIVE SERIES

The Case with Deterministic Components

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Let $\{X_t\}$ follow a discrete Gaussian vector autoregression with deterministic components. We derive the exact finite-sample joint moment generating function (MGF) of the quadratic forms that form the basis for the sufficient statistic. The formula is then specialized to the limiting MGF of functionals involving multivariate and univariate Ornstein–Uhlenbeck processes, drifts, and time trends. Such processes arise asymptotically from more general non-Gaussian processes and also from the Gaussian $\{X_t\}$ and have also been used in areas other than time series, such as the "goodness of fit" literature.

1. INTRODUCTION

Let the $k \times 1$ vector of discrete time series $\{X_t\}_1^T$ be generated by the vector autoregression (VAR)

$$X_t = \sum_{j=0}^{p} \mu_j t^j + AX_{t-1} + \varepsilon_t \equiv \mu\tau_t + AX_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim NID(0, \Omega), \tag{1}$$

where $\mu \equiv [\mu_0, \mu_1, \dots, \mu_p]$ is $k \times (p+1)$, $\tau_t \equiv [1, t, \dots, t^p]'$ is $(p+1) \times 1$, A is $k \times k$, X_0 is a known constant vector, and Ω is $k \times k$ positive definite. The moment generating function (MGF) derivations given subsequently are not affected by the value of Ω, which we take for simplicity to be I_k, the identity matrix of order k. Also, because the process includes a drift term, we can take

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$X_0 = \mathbf{0}$ without loss of generality. For example, defining $\{Y_t\} \equiv \{X_t - X_0\}$ and using (1),

$$(Y_t + X_0) = \mu\tau_t + A(Y_{t-1} + X_0) + \varepsilon_t,$$

leading to the VAR

$$Y_t = (\mu\tau_t + (A - I_k)X_0) + AY_{t-1} + \varepsilon_t \equiv \tilde{\mu}\tau_t + AY_{t-1} + \varepsilon_t,$$

where $Y_0 = 0$, $\tilde{\mu} \equiv [\tilde{\mu}_0, \mu_1, \ldots, \mu_p]$, and $\tilde{\mu}_0 \equiv \mu_0 + (A - I_k)X_0$.

The likelihood of this model can be written as

$$\begin{aligned} &L(\mu, A; X_0, \ldots, X_T) \\ &\quad = (2\pi)^{-(Tk/2)}|\Omega|^{-(T/2)} \\ &\quad\quad \times \operatorname{etr}\left[-\frac{1}{2}\Omega^{-1}\sum_{t=1}^{T}(X_t - \mu\tau_t - AX_{t-1})(X_t - \mu\tau_t - AX_{t-1})'\right], \end{aligned} \tag{2}$$

where $|\cdot| \equiv \det(\cdot)$ and $\operatorname{etr}(\cdot) \equiv \exp[\operatorname{tr}(\cdot)]$ and the corresponding sufficient statistic is extracted from

$$\sum X_t[X_t' \quad \tau_t' \quad X_{t-1}'] \quad \text{and} \quad \sum \begin{bmatrix} \tau_t \\ X_{t-1} \end{bmatrix} [\tau_t' \quad X_{t-1}'],$$

where henceforth all the summations are from $t = 1, \ldots, T$ except when stated explicitly. There are two obvious reductions in our special setting to

$$\left(\sum X_t\tau_t', \sum X_{t-1}\tau_t', \sum X_t X_{t-1}', \sum X_{t-1}X_{t-1}'\right),$$

unlike in the full linear model. First, $\sum \tau_t\tau_t'$ is deterministic and need not appear. Second, $\sum X_t X_t'$ and $\sum X_{t-1}X_{t-1}'$ differ by $X_T X_T'$, which is already obtainable from the first element X_T of $\sum X_t\tau_t' - \sum X_{t-1}\tau_t'$. There remains one final and more subtle simplification. To this end, note that

$$\sum X_t\tau_t' - \sum X_{t-1}\tau_t' = X_T\tau_{T+1}' + \sum X_t(\tau_t' - \tau_{t+1}')$$

and

$$\sum X_t(\tau_{t+1}' - \tau_t') = \sum X_t[0, \quad 1, \quad 2t+1, \quad 3t^2 + 3t + 1, \quad \ldots]$$

is a function of $\sum X_t\tau_t'$ only. (When $p = 0$, the term is the null function and may be omitted altogether.) The reduction is therefore to replace $\sum X_t\tau_t'$ by X_T, and the sufficient statistic is

$$\left(X_T, \sum X_{t-1}\tau_t', \sum X_t X_{t-1}', \sum X_{t-1}X_{t-1}'\right).$$

The sufficient statistic is minimal if one furthermore excludes terms that are repeated in the symmetric matrix $\sum X_{t-1}X_{t-1}'$. The elimination matrix could be used to remove the redundant terms, but this is not necessary for the MGF that follows where the off-diagonal elements of parameter matrices corresponding to symmetric matrices are customarily scaled by $\frac{1}{2}$ anyway.

For the purpose of the asymptotic analysis in Section 3, it is more convenient to work with the basis for the sufficient statistic

$$S \equiv (S_1, S_2, S_3, S_4) \equiv \left(X_T, \sum X_{t-1}\tau_t', \sum \varepsilon_t X_{t-1}', \sum X_{t-1}X_{t-1}'\right), \tag{3}$$

which is a $1-1$ transform of the sufficient statistic, by (1).

Distributional results for such models are patchy and are summarized in Tanaka (1996). See also Nielsen (1997), Rothenberg (1999), Gönen, Puri, Ruymgaart, and van Zuijlen (1999). In this paper, we present the exact MGF of the general S in Section 2. This extends earlier work by Abadir and Larsson (1996) and opens the way for a systematic study of the effects of including drifts and trends in VAR models. For example, results along the lines of Abadir, Hadri, and Tzavalis (1999) may now be investigated. In Abadir and Larsson (1996), the marginal MGF for the different basis

$$\left(X_T X_T', \sum \varepsilon_t X_{t-1}', \sum X_{t-1}X_{t-1}'\right)$$

was derived because there were no deterministic components there, hence the irrelevance of the sign of X_T, from a distributional viewpoint, and its inclusion through $X_T X_T'$.

In Section 3, we specialize the MGF to the asymptotic case, which happens to allow the process (1) to cover more general error structures $\{\varepsilon_t\}$. We do so while focusing on the case $\mu = \mathbf{0}$. The result is the joint MGF of functionals involving Ornstein–Uhlenbeck processes, drifts, and time trends, all of which had no known joint MGF's except for some (not all) of the univariate special cases. Other potential uses for our results can be found in the literature on goodness of fit, where these functionals arise (see, e.g., d'Agostino and Stephens, 1986).

All the proofs are collected in the Appendix. As for the general notation, we follow the one summarized in the appendix of Abadir and Larsson (1996). Additionally, the change of a variable of integration that maps $u \mapsto v \equiv \lambda(u)$, for some function $\lambda(\cdot)$, will be written in the inverse-mapping form $u \leftrightarrow \lambda^{-1}(v)$, whereby u is replaced by $\lambda^{-1}(v)$ in the integrand.

2. THE MOMENT GENERATING FUNCTION

Consider the block-tridiagonal nonsingular $Tk \times Tk$ symmetric matrix

$$D_s = \begin{bmatrix} P & Q' & \mathbf{0} & \dots & \mathbf{0} \\ Q & P & Q' & \ddots & \vdots \\ \mathbf{0} & \ddots & \ddots & \ddots & \mathbf{0} \\ \vdots & \ddots & Q & P & Q' \\ \mathbf{0} & \dots & \mathbf{0} & Q & M \end{bmatrix}, \tag{4}$$

where M, P, Q are k-square full-rank matrices with M and P symmetric and $\mathbf{0}$ denotes null matrices of appropriate dimensions. The following lemma is extracted from the proofs in Abadir and Larsson (1996) and will be used in our derivations here.

LEMMA 1. *The determinant of D_s is given by*

$$|D_s| = |-Q|^{T-1} \left| [I_k \quad \mathbf{0}] \begin{bmatrix} -PQ^{-1} & -I_k \\ Q'Q^{-1} & \mathbf{0} \end{bmatrix}^{T-1} \begin{bmatrix} M \\ -Q' \end{bmatrix} \right|.$$

We need to derive one further lemma about D_s before proceeding to obtain the main result of this section. Following Abadir and Larsson (1996, Theorem 2.1), define

$$C_0 \equiv I_k$$

$$C_1 \equiv PQ^{-1},$$

$$C_j \equiv C_{j-1} PQ^{-1} - C_{j-2} Q'Q^{-1}, \quad j \in \mathcal{Z},$$

whose solution is

$$[C_j \quad C_{j-1}] = [I_k \quad \mathbf{0}] \begin{bmatrix} PQ^{-1} & I_k \\ -Q'Q^{-1} & \mathbf{0} \end{bmatrix}^{j} = -Q[\mathbf{0} \quad (Q')^{-1}] \begin{bmatrix} PQ^{-1} & I_k \\ -Q'Q^{-1} & \mathbf{0} \end{bmatrix}^{j+1} \tag{5}$$

for any integer (including negative) j. Then we have the following lemma.

LEMMA 2. *The typical block of the inverse of D_s is given by*

$$\begin{aligned} D_s^{T-t+1+j, T-t+1} &= (-1)^j Q^{-1} (C_{t-j-2} M - C_{t-j-3} Q') \\ &\quad \times \sum_{n=0}^{T-t} (C_{t+n-1} M - C_{t+n-2} Q')^{-1} Q' (M C'_{t+n-2} - Q C'_{t+n-3})^{-1} \\ &\quad \times (M C'_{t-2} - Q C'_{t-3}) (Q')^{-1}, \end{aligned}$$

where $t = 1, \ldots, T$ and $j = 0, \ldots, t-1$ and the superscripts refer to (block-) row and column numbers, respectively, starting from the top left corner. The terms above the diagonal blocks are obtained by $D_s^{T-t+1, T-t+1+j} = (D_s^{T-t+1+j, T-t+1})'$.

This lemma is general and, as pointed out by the referee, can be used in problems that are not necessarily related to our work (or to statistics).

We can now derive the main result of this section.

THEOREM 3. *The* MGF *of S is*

$$\varphi_{T,\mu}(u_1, U_2, U_3, U_4) = \varphi_{T,\mathbf{0}}(\mathbf{0}, \mathbf{0}, U_3, U_4) \operatorname{etr}\left(-\tfrac{1}{2} \mu' \mu \sum \tau_t \tau_t'\right) \exp\left(\tfrac{1}{2} \zeta' D_s^{-1} \zeta\right),$$

where (u_1, U_2, U_3, U_4) *correspond to* (S_1, S_2, S_3, S_4), *respectively*, U_4 *is symmetric*, *and*

$$M \equiv I_k$$

$$P \equiv I_k - 2U_4 + U_3 A + A'U_3' + A'A = P'$$

$$Q \equiv -A - U_3'$$

$$\varphi_{T,\mathbf{0}}(\mathbf{0},\mathbf{0},U_3,U_4) = \det(-Q)^{(1-T)/2} \times \det\left([I_k \quad \mathbf{0}] \begin{bmatrix} -PQ^{-1} & -I_k \\ Q'Q^{-1} & \mathbf{0} \end{bmatrix}^{T-1} \begin{bmatrix} I_k \\ -Q' \end{bmatrix} \right)^{-(1/2)}.$$

Finally, D_s^{-1} *is defined explicitly in Lemma* 2, *and* $\zeta \equiv (\zeta_1', \ldots, \zeta_T')'$ *with*

$$\zeta_t' \equiv \tau_{t+1}'(U_2 - \mu' U_3' - \mu' A) + \tau_t' \mu', \qquad t = 1, \ldots, T-1,$$

$$\zeta_T' \equiv u_1' + \tau_T' \mu'.$$

The theorem can be made more explicit in a variety of directions, depending on the required application. For example, the $(p+1) \times (p+1)$ matrix $\sum \tau_t \tau_t'$ can be written as

$$\sum \tau_t \tau_t' = \sum \begin{bmatrix} 1 & t & \cdots \\ t & t^2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} T & \dfrac{T(T+1)}{2} & \cdots \\ \dfrac{T(T+1)}{2} & \dfrac{T(T+1)(2T+1)}{6} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \sim \left[\quad \frac{T^{i+j-1}}{i+j-1} \quad \right],$$

where i, j are the row and column numbers, respectively. This theorem can also be simplified to a variety of published univariate asymptotic special cases. However, more important, we can specialize this theorem to a general asymptotic nearly nonstationary case that arises frequently in connection with limiting distributions in time series. This is the purpose of the next section.

3. THE NEARLY NONSTATIONARY LIMITING CASE

Let $\mu = \mathbf{0}$ and $A = I_k + (1/T)H$, where H is an arbitrary $k \times k$ matrix. Then, defining

$$\Gamma \equiv \text{diag}(1, T^{-1}, \ldots, T^{-p}),$$

it can be shown that

$$\left(\frac{1}{\sqrt{T}}X_T, \frac{1}{\sqrt{T^3}}\sum X_{t-1}\tau_t'\Gamma, \frac{1}{T}\sum \varepsilon_t X_{t-1}', \frac{1}{T^2}\sum X_{t-1}X_{t-1}'\right)$$
$$\xrightarrow{d}\left(J_H(1), \int_0^1 J_H(r)[1,r,\ldots,r^p]\mathrm{d}r, \left(\int_0^1 J_H(r)\,\mathrm{d}W(r)'\right)',\right.$$
$$\left.\int_0^1 J_H(r)J_H(r)'\,\mathrm{d}r\right)$$
$$\equiv(\tilde{S}_1,\tilde{S}_2,\tilde{S}_3,\tilde{S}_4)\equiv\tilde{S}, \tag{6}$$

where $\xrightarrow{d}$ denotes convergence in distribution, $W(r)$ is the standard k-dimensional Wiener process on $r\in[0,1]$, and $J_H(r)$ is the corresponding Ornstein–Uhlenbeck process defined by

$$J_H(r)\equiv\int_0^r \exp[(r-s)H]\,\mathrm{d}W(s)$$

when $X_0=\mathbf{0}$. These limiting distributions hold under less restrictive distributional assumptions on $\{\varepsilon_t\}$.

In view of (6), the limiting MGF of interest becomes

$$\begin{aligned}\phi_H(u_1,U_2,U_3,U_4)&\equiv\mathrm{E}[\mathrm{etr}(u_1'\tilde{S}_1+U_2\tilde{S}_2+U_3\tilde{S}_3+U_4\tilde{S}_4)]\\&=\lim_{T\to\infty}\varphi_{T,\mathbf{0}}\left(\frac{1}{\sqrt{T}}u_1, \frac{1}{\sqrt{T^3}}\Gamma U_2, \frac{1}{T}U_3, \frac{1}{T^2}U_4\right),\end{aligned}$$

which is the joint MGF of $\tilde{S}\equiv(\tilde{S}_1,\tilde{S}_2,\tilde{S}_3,\tilde{S}_4)$. When $H=\mathbf{0}$, $J_{\mathbf{0}}(r)=W(r)$ and the resulting MGF is denoted by $\phi(u_1,U_2,U_3,U_4)\equiv\phi_{\mathbf{0}}(u_1,U_2,U_3,U_4)$, and the functionals contain no stochastic components other than Wiener processes.

COROLLARY 4. *The joint* MGF *of* $\tilde{S}$ *is*

$$\phi_H(u_1,U_2,U_3,U_4)=\exp[\tfrac{1}{2}(\ell_1+\ell_2+\ell_3)]\,\mathrm{etr}[-\tfrac{1}{2}(H'+U_3)]/\sqrt{\det(g(1))},$$

where

$$F\equiv H'H+H'U_3'+U_3H-2U_4,$$

$$G\equiv H'-H+U_3-U_3',$$

$$\ell_1\equiv u_1'\int_0^1 (g(1-z)'g(1-z))^{-1}\,\mathrm{d}z\,u_1,$$

$$\ell_2\equiv 2u_1'\int_0^1 (g(1-z)'g(1-z))^{-1}f(z)\,\mathrm{d}z,$$

$$\ell_3 \equiv \int_0^1 f(z)'(g(1-z)'g(1-z))^{-1}f(z)\,\mathrm{d}z,$$

$$g(z) \equiv [I_k \quad \mathbf{0}]\exp\left(z\begin{bmatrix} G & I_k \\ F & \mathbf{0}\end{bmatrix}\right)\begin{bmatrix} I_k \\ -H' - U_3\end{bmatrix},$$

$$f(z)' \equiv \sum_{i=0}^{p}[u_{i1} \quad \dots \quad u_{ik}, \quad 0']\sum_{j=0}^{i}(-i)_j(-1)^j$$
$$\times\left(z^{i-j}\exp\left((1-z)\begin{bmatrix} G & I_k \\ F & \mathbf{0}\end{bmatrix}\right) - I_{2k}\right)$$
$$\times\begin{bmatrix} G & I_k \\ F & \mathbf{0}\end{bmatrix}^{-j-1}\begin{bmatrix} I_k \\ -H' - U_3\end{bmatrix},$$

with z being a scalar, *u_{in} being the typical element* (*row i*, *column n*) of U_2, *and* $(-i)_j \equiv \prod_{n=0}^{j-1}(n-i)$ *denoting the Pochhammer symbol.*

We have not worked out the integrals in z for the general case, as they are more easily manipulated numerically (the integration is over the interval (0,1) in one dimension) in applications. Note that *any* function of an $n \times n$ matrix can be written as a polynomial of degree $n-1$ in the matrix, by the Cayley–Hamilton theorem, so that infinite series of matrices are not required numerically. This comment applies to $\exp(\cdot)$ and also to matrix functions such as $(g(\cdot)g(\cdot)')^{-1}$.

In $f(\cdot)$, the finite series in j is Tricomi's confluent hypergeometric function. Little additional insight is gained from using the hypergeometric formulation here, so we have refrained from doing so.

We now illustrate our corollary by specializing it to the univariate case with general trend components. In the univariate case, $G = 0$ and, furthermore, put $F = f$, $H = h$, and $U_3 = u_3$ to stress that these quantities have become scalars. Because

$$\begin{bmatrix} G & I_1 \\ F & 0\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ f & 0\end{bmatrix},$$

it is now possible to evaluate the exponential of this matrix explicitly. To this end, a series expansion yields

$$\exp\left(z\begin{bmatrix} 0 & 1 \\ f & 0\end{bmatrix}\right) = \begin{bmatrix} \cosh(z\sqrt{f}) & \frac{1}{\sqrt{f}}\sinh(z\sqrt{f}) \\ \sqrt{f}\sinh(z\sqrt{f}) & \cosh(z\sqrt{f})\end{bmatrix},$$

implying

$$g(z) = \cosh(z\sqrt{f}) - v\sinh(z\sqrt{f}), \qquad v \equiv \frac{h+u_3}{\sqrt{f}}. \tag{7}$$

In connection with ℓ_1, ℓ_2, ℓ_3 of Corollary 4, we have the following further reductions.

COROLLARY 5. *In the univariate case* $(k=1)$, *we have*

$$\ell_1 = u_1^2 \frac{\sinh\sqrt{f}}{g(1)\sqrt{f}},$$

$$\ell_2 = -\frac{u_1}{g(1)f}\sum_{i=0}^{p} u_{i1}\left(\frac{i!(1+(-1)^i)}{\left(\sqrt{f}\right)^i} - \sum_{j=0}^{i}\frac{(-i)_j}{\left(\sqrt{f}\right)^j}\left(e^{\sqrt{f}} + (-1)^j e^{-\sqrt{f}}\right)\right),$$

$$\ell_3 = \frac{1}{2g(1)f}\sum_{i=0}^{p}\sum_{j=0}^{p} u_{i1}u_{j1}\gamma_{ij},$$

where u_{i1} *is the typical element of the vector* u_2 *and*

$$\begin{aligned}\gamma_{ij} = -\frac{i!(1+(-1)^i)}{\left(\sqrt{f}\right)^{i+1}}&\left[\frac{j!\left((1-v)e^{\sqrt{f}} - (-1)^j(1+v)e^{-\sqrt{f}}\right)}{\left(\sqrt{f}\right)^j}\right.\\ &\left.+\sum_{n=0}^{j}\frac{(-j)_n((1+v)-(-1)^n(1-v))}{\left(\sqrt{f}\right)^n}\right]\\ +\sum_{m=0}^{i}\frac{(-i)_m}{\left(\sqrt{f}\right)^m}&\left[\frac{(1-v)e^{\sqrt{f}} + (-1)^m(1+v)e^{-\sqrt{f}}}{j+i-m+1}\right.\\ &+\frac{(-1)^m(j+i-m)!\left((1-v)e^{\sqrt{f}} - (-1)^{i+j}(1+v)e^{-\sqrt{f}}\right)}{(2\sqrt{f})^{j+i-m+1}}\\ &\left.+\sum_{n=0}^{j+i-m}\frac{(m-i-j)_n}{(2\sqrt{f})^{n+1}}\left((1+v)e^{\sqrt{f}} - (-1)^{m+n}(1-v)e^{-\sqrt{f}}\right)\right].\end{aligned}$$

It is worth observing that, in the asymptotics for the univariate case, matters simplify further. This is because, from (6),

$$\begin{aligned}\tilde{S}_3 &= \int_0^1 J_h(r)\,\mathrm{d}W(r) = \frac{J_h(1)^2-1}{2} - h\int_0^1 J_h(r)^2\,\mathrm{d}r\\ &= \frac{\tilde{S}_1^2-1}{2} - h\tilde{S}_4,\end{aligned}$$

so that we may set $u_3 = 0$. This, however, induces no simplifications for the preceding derivations, other than setting $v \equiv h/\sqrt{f}$ (i.e., $u_3 = 0$) in (7). For the vector case, inequalities such as

$$\int_0^1 J_h(r)\,\mathrm{d}W(r)' \neq \left(\int_0^1 J_h(r)\,\mathrm{d}W(r)'\right)'$$

rule out such manipulations.

Some special cases follow directly from these corollaries by setting some components of $\boldsymbol{u}.$ to zero, hence obtaining marginal MGF's. For example, when only a constant is fitted to the model, we get the MGF of sufficient statistics associated with de-meaned Brownian motions which are particularly useful in connection with the goodness-of-fit literature (see, e.g., d'Agostino and Stephens, 1986). In this case, the joint MGF of the sufficient statistic

$$\left(W(1), \int_0^1 W(r)\,\mathrm{d}r, \int_0^1 W(r)^2\,\mathrm{d}r\right)$$

is

$$\phi_{\mathbf{0}}(u_1,u_2,0,u_4) = \frac{\exp[\frac{1}{2}(\ell_1+\ell_2+\ell_3)]}{\sqrt{\cosh(\sqrt{-2u_4})}}$$

with the ordering of u_1, u_2, u_4 corresponding to the respective variates given earlier, and

$$\ell_1 = \frac{u_1^2}{\sqrt{-2u_4}}\tanh(\sqrt{-2u_4}),$$

$$\ell_2 = \frac{u_1 u_2}{u_4}\left(\frac{1}{\cosh(\sqrt{-2u_4})} - 1\right),$$

$$\ell_3 = \frac{u_2^2}{2u_4}\left(\frac{\tanh(\sqrt{-2u_4})}{\sqrt{-2u_4}} - 1\right).$$

The marginal MGF of a famous related stochastic integral has been obtained by Anderson and Darling (1952) and used further by Abadir and Paruolo (1997); see also Rothenberg (1999).

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APPENDIX: PROOFS

Proof of Lemma 1. See Theorem 2.3 of Abadir and Larsson (1996).

Proof of Lemma 2. Let

$$D_s^{-1} = \left[\begin{array}{c|c|c|cc} \psi_T & \Psi_T' & & & \\ \hline \Psi_T & \psi_{T-1} & \Psi_{T-1}' & & \\ \cline{2-5} & \Psi_{T-1} & \ddots & \ddots & \\ \cline{3-5} & & \ddots & \psi_2 & \Psi_2' \\ \cline{4-5} & & & \Psi_2 & \psi_1 \end{array}\right],$$

where ψ_t is $k \times k$ and Ψ_t is the $(t-1)k \times k$ matrix of *all* the blocks below ψ_t, therefore expanding with t. Abadir and Larsson (1996, B_T of Theorems 2.2 and 2.3 there) derive ψ_T by recursive partitioned inverse, but here we need to derive the whole matrix.

Define $D_s(t)$ as the matrix $D_s(T) \equiv D_s$ but of dimensions $tk \times tk$ instead of $Tk \times Tk$ and denote the leftmost upper block of $D_s^{-1}(t)$ by B_t. Notice that $B_t \neq \psi_t$ except for $t = T$. First, partition D_s into

$$D_s = \left[\begin{array}{c|cc} P & Q' & \mathbf{0} \\ \hline Q & & \\ \mathbf{0} & \multicolumn{2}{c}{D_s(T-1)} \end{array}\right].$$

Then, using the partitioned inverse formula, the first component of the second diagonal block is

$$\begin{aligned} \psi_{T-1} &= [I_k \quad \mathbf{0}]\left(D_s^{-1}(T-1) + D_s^{-1}(T-1)\begin{bmatrix} I_k \\ \mathbf{0} \end{bmatrix} Q\psi_T Q'[I_k \quad \mathbf{0}]D_s^{-1}(T-1)\right)\begin{bmatrix} I_k \\ \mathbf{0} \end{bmatrix} \\ &= B_{T-1} + B_{T-1}Q\psi_T Q'B_{T-1}. \end{aligned}$$

Second, partition D_s into

$$D_s = \left[\begin{array}{cc|ccc} P & Q' & \mathbf{0} & \mathbf{0} & \dots \\ Q & P & Q' & \mathbf{0} & \dots \\ \hline \mathbf{0} & Q & & & \\ \mathbf{0} & \mathbf{0} & \multicolumn{3}{c}{D_s(T-2)} \\ \vdots & \vdots & & & \end{array}\right]$$

and repeat a similar operation to get

$$\psi_{T-2} = [I_k \quad \mathbf{0}]\left(D_s^{-1}(T-2) + D_s^{-1}(T-2)\begin{bmatrix} \mathbf{0} & Q \\ \mathbf{0} & \mathbf{0} \\ \vdots & \vdots \end{bmatrix} \right.$$

$$\left. \times \begin{bmatrix} \cdot & \cdot \\ \cdot & \psi_{T-1} \end{bmatrix}\begin{bmatrix} \mathbf{0} & \mathbf{0} & \dots \\ Q' & \mathbf{0} & \dots \end{bmatrix} D_s^{-1}(T-2)\right)\begin{bmatrix} I_k \\ \mathbf{0} \end{bmatrix}$$

$$= B_{T-2} + B_{T-2}Q\psi_{T-1}Q'B_{T-2}$$

as before. By induction, this relation

$$\psi_{t-1} = B_{t-1} + B_{t-1}Q\psi_t Q'B_{t-1}$$

holds for all partitions of D_s because $B_1 = M^{-1}$. As a result,

$$\psi_t = B_t + B_tQB_{t+1}Q'B_t + \cdots + B_tQB_{t+1}\dots QB_TQ'\dots B_{t+1}Q'B_t. \tag{A.1}$$

For the explicit solution of this formula, we need to work out $B_tQ\dots B_{t+j}Q$ using

$$B_tQ = [\mathbf{0} \quad (Q')^{-1}]\begin{bmatrix} PQ^{-1} & I_k \\ -Q'Q^{-1} & \mathbf{0} \end{bmatrix}^{t-1}\begin{bmatrix} M \\ -Q' \end{bmatrix}$$

$$\times \left([\mathbf{0} \quad (Q')^{-1}]\begin{bmatrix} PQ^{-1} & I_k \\ -Q'Q^{-1} & \mathbf{0} \end{bmatrix}^{t}\begin{bmatrix} M \\ -Q' \end{bmatrix}\right)^{-1}, \tag{A.2}$$

which we deduce from Abadir and Larsson (1996, Theorem 2.3). We have reformulated the power of the first $(2k) \times (2k)$ matrix in their formula for B_t, so that the required typical product simplifies sequentially to

$$B_tQ\dots B_{t+n}Q = [\mathbf{0} \quad (Q')^{-1}]\begin{bmatrix} PQ^{-1} & I_k \\ -Q'Q^{-1} & \mathbf{0} \end{bmatrix}^{t-1}\begin{bmatrix} M \\ -Q' \end{bmatrix}$$

$$\times \left([\mathbf{0} \quad (Q')^{-1}]\begin{bmatrix} PQ^{-1} & I_k \\ -Q'Q^{-1} & \mathbf{0} \end{bmatrix}^{t+n}\begin{bmatrix} M \\ -Q' \end{bmatrix}\right)^{-1}$$

$$= Q^{-1}(C_{t-2}M - C_{t-3}Q')(C_{t+n-1}M - C_{t+n-2}Q')^{-1}Q \tag{A.3}$$

by (5). We also need to work out $Q'B_{t+n-1}\dots Q'B_t$ to solve (A.1) explicitly. Because the $B_{\cdot}$ matrices are symmetric, $Q'B_{t+n-1}\dots Q'B_t = (B_tQ\dots B_{t+n-1}Q)'$, and using the transpose of (A.3) with $n-1$ in place of n,

$$Q'B_{t+n-1}\dots Q'B_t = Q'(MC'_{t+n-2} - QC'_{t+n-3})^{-1}(MC'_{t-2} - QC'_{t-3})(Q')^{-1}.$$

Combining it with (A.3),

$$B_t QB_{t+1}\ldots QB_{t+n}Q'\ldots B_{t+1}Q'B_t$$
$$= Q^{-1}(C_{t-2}M - C_{t-3}Q')(C_{t+n-1}M - C_{t+n-2}Q')^{-1}$$
$$\times Q'(MC'_{t+n-2} - QC'_{t+n-3})^{-1}(MC'_{t-2} - QC'_{t-3})(Q')^{-1},$$

which upon substitution in (A.1) gives

$$\psi_t = \sum_{n=0}^{T-t} Q^{-1}(C_{t-2}M - C_{t-3}Q')(C_{t+n-1}M - C_{t+n-2}Q')^{-1}$$
$$\times Q'(MC'_{t+n-2} - QC'_{t+n-3})^{-1}(MC'_{t-2} - QC'_{t-3})(Q')^{-1}, \tag{A.4}$$

as the required diagonal blocks.

Now we turn to the off-diagonal blocks of $D_s^{-1}(T)$. By symmetry of $D_s^{-1}(T)$, we only need the blocks Ψ_t that are below any typical diagonal block ψ_t. For this, we need to calculate the complete first block-column of $D_s^{-1}(t)$, which we denote by

$$D_s^{-1}(t)\begin{bmatrix} I_k \\ \mathbf{0} \end{bmatrix} \equiv \begin{bmatrix} B_t \\ \hline \mathbf{B}_t \end{bmatrix}.$$

By the recursive partitioned inversion of $D_s^{-1}(t)$ for successive t,

$$\mathbf{B}_t = -D_s^{-1}(t-1)\begin{bmatrix} Q \\ \mathbf{0} \end{bmatrix} B_t = -\begin{bmatrix} B_{t-1} \\ \hline \mathbf{B}_{t-1} \end{bmatrix} QB_t = \begin{bmatrix} -B_{t-1}QB_t \\ +B_{t-2}QB_{t-1}QB_t \\ -B_{t-3}QB_{t-2}QB_{t-1}QB_t \\ \vdots \end{bmatrix}.$$

Similarly,

$$\Psi_T = -D_s^{-1}(T-1)\begin{bmatrix} Q \\ \mathbf{0} \end{bmatrix}\psi_T = -\begin{bmatrix} B_{T-1} \\ \hline \mathbf{B}_{T-1} \end{bmatrix} Q\psi_T,$$

and by repeated use of the partitioned inverse formula as we did earlier for ψ_t,

$$\Psi_{T-1} = -D_s^{-1}(T-2)\begin{bmatrix} \mathbf{0} & Q \\ \mathbf{0} & \mathbf{0} \\ \vdots & \vdots \end{bmatrix}\begin{bmatrix} . & . \\ . & \psi_{T-1} \end{bmatrix}\begin{bmatrix} \mathbf{0} \\ I_k \end{bmatrix} = -D_s^{-1}(T-2)\begin{bmatrix} Q\psi_{T-1} \\ \hline \mathbf{0} \end{bmatrix}$$
$$= -\begin{bmatrix} B_{T-2} \\ \hline \mathbf{B}_{T-2} \end{bmatrix} Q\psi_{T-1}.$$

By induction,

$$\Psi_t = -\begin{bmatrix} B_{t-1} \\ \hline \mathbf{B}_{t-1} \end{bmatrix} Q\psi_t = \begin{bmatrix} -B_{t-1}Q\psi_t \\ +B_{t-2}QB_{t-1}Q\psi_t \\ -B_{t-3}QB_{t-2}QB_{t-1}Q\psi_t \\ \vdots \end{bmatrix}. \tag{A.5}$$

Changing $t \leftrightarrow t-j$ and $n \leftrightarrow j-1$ in (A.3), we have

$$B_{t-j}Q\ldots B_{t-1}Q = Q^{-1}(C_{t-j-2}M - C_{t-j-3}Q')(C_{t-2}M - C_{t-3}Q')^{-1}Q, \tag{A.6}$$

which together with (A.4) and (A.5) gives the stated result. ■

Proof of Theorem 3. The MGF of $S \equiv (S_1, S_2, S_3, S_4) \equiv (X_T, \sum X_{t-1}\tau_t', \sum \varepsilon_t X_{t-1}', \sum X_{t-1}X_{t-1}')$ is

$$\begin{aligned}
&\varphi_{T,\mu}(u_1, U_2, U_3, U_4)\\
&\quad \equiv \mathrm{E}\Big[\exp\Big(u_1' X_T + \sum \tau_t' U_2 X_{t-1} + \sum X_{t-1}' U_3 \varepsilon_t + \sum X_{t-1}' U_4 X_{t-1}\Big)\Big]\\
&\quad = (2\pi)^{-(Tk/2)} \int \exp\Bigg(u_1' x_T + \sum x_{t-1}' U_2' \tau_t + \sum x_{t-1}' U_3 \varepsilon_t\\
&\qquad\qquad + \sum x_{t-1}' U_4 x_{t-1} - \frac{1}{2}\sum \varepsilon_t' \varepsilon_t\Bigg)(\mathrm{d}x)
\end{aligned} \tag{A.7}$$

by using (2) with $\Omega = I_k$ and where the integral is over $\mathcal{R}^{Tk}$ with $(\mathrm{d}x)$ being the exterior product $\mathrm{d}x_{11}\mathrm{d}x_{12}\ldots\mathrm{d}x_{kT}$. Using the VAR formulation (1) to substitute for ε_t, we can decompose these sums into deterministic and stochastic components

$$\sum x_{t-1}' U_3 \varepsilon_t = \sum x_{t-1}' U_3 (x_t - \mu\tau_t - A x_{t-1}),$$

$$\begin{aligned}
-\frac{1}{2}\sum \varepsilon_t' \varepsilon_t &= -\frac{1}{2}\sum (x_t - \mu\tau_t - A x_{t-1})'(x_t - \mu\tau_t - A x_{t-1})\\
&= -\frac{1}{2}\sum \tau_t' \mu' \mu \tau_t\\
&\quad + \left(\sum x_t' \mu\tau_t - \frac{1}{2}\sum x_t' x_t + \sum x_{t-1}' A'\left(x_t - \mu\tau_t - \frac{1}{2}A x_{t-1}\right)\right)
\end{aligned}$$

Substituting back into (A.7), rearranging, then using $x_0 = \mathbf{0}$, these give

$$\begin{aligned}
&\varphi_{T,\mu}(u_1, U_2, U_3, U_4)\\
&\quad = (2\pi)^{-(Tk/2)} \exp\left(-\frac{1}{2}\sum \tau_t' \mu' \mu \tau_t\right)\\
&\qquad \times \int \exp\Bigg[u_1' x_T + \sum x_t' \mu\tau_t - \frac{1}{2}\sum x_t' x_t\\
&\qquad\qquad + \sum x_{t-1}' \Bigg((U_2' - U_3\mu - A'\mu)\tau_t + (U_3 + A')x_t\\
&\qquad\qquad\qquad + \left(U_4 - U_3 A - \frac{1}{2}A'A\right)x_{t-1}\Bigg)\Bigg](\mathrm{d}x).
\end{aligned} \tag{A.8}$$

Define $x \equiv \text{vec}(x_1, \ldots, x_T)$, and D_s as in (4) with M, P, Q stated in the theorem. Then, we can rewrite (A.8) as

$$\begin{aligned}
&\varphi_{T,\mu}(u_1, U_2, U_3, U_4) \\
&\quad = (2\pi)^{-(Tk/2)} \text{etr}\left(-\frac{1}{2}\mu'\mu \sum \tau_t \tau_t'\right) \int \exp\left(\zeta' x - \frac{1}{2} x' D_s x\right)(\mathrm{d}x) \\
&\quad = |D_s|^{-(1/2)} \text{etr}\left(-\frac{1}{2}\mu'\mu \sum \tau_t \tau_t'\right) \exp\left(\frac{1}{2}\zeta' D_s^{-1} \zeta\right) \\
&\quad = \varphi_{T,\mathbf{0}}(\mathbf{0}, \mathbf{0}, U_3, U_4) \text{etr}\left(-\frac{1}{2}\mu'\mu \sum \tau_t \tau_t'\right) \exp\left(\frac{1}{2}\zeta' D_s^{-1} \zeta\right),
\end{aligned} \tag{A.9}$$

where we have completed the square and integrated x out and where $\varphi_{T,\mathbf{0}}(\mathbf{0}, \mathbf{0}, U_3, U_4)$ is obtained by Lemma 1. ■

Proof of Corollary 4. From Theorem 3,

$$\begin{aligned}
\phi_H(u_1, U_2, U_3, U_4) &= \lim_{T\to\infty} \varphi_{T,\mathbf{0}}\left(\frac{1}{\sqrt{T}} u_1, \frac{1}{\sqrt{T^3}} \Gamma U_2, \frac{1}{T} U_3, \frac{1}{T^2} U_4\right) \\
&= \lim_{T\to\infty} \varphi_{T,\mathbf{0}}\left(\mathbf{0}, \mathbf{0}, \frac{1}{T} U_3, \frac{1}{T^2} U_4\right) \exp\left(\frac{1}{2}\zeta' D_s^{-1} \zeta\right).
\end{aligned}$$

The limit of the first component is available from Corollary 3.2 of Abadir and Larsson (1996) as

$$\begin{aligned}
\lim_{T\to\infty} \varphi_{T,\mathbf{0}}\left(\mathbf{0}, \mathbf{0}, \frac{1}{T} U_3, \frac{1}{T^2} U_4\right) &= \text{etr}\left[-\frac{1}{2}(H' + U_3)\right] \\
&\quad \times \left| [I_k \quad \mathbf{0}] \exp\left(\begin{bmatrix} G & I_k \\ F & \mathbf{0} \end{bmatrix}\right) \begin{bmatrix} I_k \\ -H' - U_3 \end{bmatrix} \right|^{-1/2}.
\end{aligned}$$

Define $E \equiv G - (1/T)G(H + U_3')$. Then for the second component, $\lim_{T\to\infty} \exp(\frac{1}{2}\zeta' D_s^{-1} \zeta)$, we note the reductions

$$\begin{aligned}
M &= I_k \\
P &= 2I_k + \frac{1}{T}(H + H' + U_3 + U_3') + \frac{1}{T^2} F \\
-Q &= I_k + \frac{1}{T}(H + U_3') \\
-Q^{-1} &= I_k - \frac{1}{T}(H + U_3') + \frac{1}{T^2}(H + U_3')^2 + O\left(\frac{1}{T^3}\right) \qquad \text{(A.10)} \\
-PQ^{-1} &= 2I_k + \frac{1}{T} E + \frac{1}{T^2} F + O\left(\frac{1}{T^3}\right) \\
Q'Q^{-1} &= I_k + \frac{1}{T} E + O\left(\frac{1}{T^3}\right),
\end{aligned}$$

and dropping lower-order terms henceforth for clarity of exposition, (5) gives

$$
\begin{aligned}
(-1)^q [C_q \quad C_{q-1}] \begin{bmatrix} M \\ -Q' \end{bmatrix} &= [I_k \quad \mathbf{0}] \begin{bmatrix} -PQ^{-1} & -I_k \\ Q'Q^{-1} & \mathbf{0} \end{bmatrix}^q \begin{bmatrix} M \\ -Q' \end{bmatrix} \\
&\sim [I_k \quad \mathbf{0}] \begin{bmatrix} 2I_k + \frac{1}{T} E + \frac{1}{T^2} F & -I_k \\ I_k + \frac{1}{T} E & \mathbf{0} \end{bmatrix}^q \\
&\quad \times \begin{bmatrix} I_k \\ I_k + \frac{1}{T}(H' + U_3) \end{bmatrix} \\
&\sim [I_k \quad \mathbf{0}] \exp\left(\frac{q}{T} \begin{bmatrix} G & I_k \\ F & \mathbf{0} \end{bmatrix} \right) \begin{bmatrix} I_k \\ -H' - U_3 \end{bmatrix} \equiv g\left(\frac{q}{T}\right).
\end{aligned} \tag{A.11}
$$

The latter step follows because, for any fixed $(2k) \times (2k)$ matrix Ξ, and $q \in \mathcal{N}$ an increasing function of T of maximal order $O(T)$,

$$
\begin{aligned}
\lim_{T\to\infty} \left(I_{2k} + \frac{1}{T} \Xi \right)^q &= \lim_{T\to\infty} \exp\left[q \log\left(I_{2k} + \frac{1}{T} \Xi \right) \right] \\
&= \lim_{T\to\infty} \exp\left[q\left(\frac{1}{T} \Xi + O\left(\frac{1}{T^2} \right) \right) \right] = \lim_{T\to\infty} \exp\left[\frac{q}{T} \Xi \right],
\end{aligned}
$$

so that we could use the same steps of Corollary 3.2 of Abadir and Larsson (1996). Then, (A.11) allows us to write the required blocks of D_s^{-1} of Lemma 2 for large T, with $t = 1, \dots, T$ and $j = 0, \dots, t-1$, as

$$
D_s^{T-t+1+j, T-t+1} \sim g\left(\frac{t-j}{T} \right) \sum_{n=0}^{T-t} \left(g\left(\frac{t+n}{T} \right)' g\left(\frac{t+n}{T} \right) \right)^{-1} g\left(\frac{t}{T} \right)'.
$$

We need the limit

$$
\begin{aligned}
\lim_{T\to\infty} \zeta' D_s^{-1} \zeta &\equiv \lim_{T\to\infty} \sum_{t=1}^{T} \sum_{j=0}^{t-1} (1 + \operatorname{sgn}(j)) \zeta'_{T-t+1+j} D_s^{T-t+1+j, T-t+1} \zeta_{T-t+1} \\
&= \lim_{T\to\infty} \sum_{t=1}^{T} \sum_{j=0}^{t-1} (1 + \operatorname{sgn}(j)) \zeta'_{T-t+1+j} g\left(\frac{t-j}{T} \right) \\
&\quad \times \sum_{n=0}^{T-t} \left(g\left(\frac{t+n}{T} \right)' g\left(\frac{t+n}{T} \right) \right)^{-1} g\left(\frac{t}{T} \right)' \zeta_{T-t+1} \\
&= \lim_{T\to\infty} \sum_{t=1}^{T} \sum_{j=0}^{t-1} (1 + \operatorname{sgn}(j)) \zeta'_{T-t+1+j} g\left(\frac{t-j}{T} \right) \\
&\quad \times \sum_{n=0}^{T-t} \left(g\left(1 - \frac{n}{T} \right)' g\left(1 - \frac{n}{T} \right) \right)^{-1} g\left(\frac{t}{T} \right)' \zeta_{T-t+1},
\end{aligned} \tag{A.12}
$$

where sgn$(\cdot)$ is the signum (sign) function and we have reversed the sum in n by replacing $n \leftrightarrow T - t - n$. In the context of this corollary,

$$\zeta_t' = \frac{1}{\sqrt{T^3}}\,\tau_{t+1}'\Gamma U_2, \qquad t = 1,\ldots,T-1,$$

$$\zeta_T' = \frac{1}{\sqrt{T}}\,u_1',$$

where $\tau_{t+1}'\Gamma$ is a normalized bounded matrix for any t, T. For large T, the sums become integrals, and we convert successively

$$\frac{t}{T} \leftrightarrow x \in (0,1),$$

$$\frac{j}{T} \leftrightarrow y \in (0,x),$$

$$\frac{n}{T} \leftrightarrow z \in (0,1-x),$$

and

$$\tau_{t+1}'\Gamma \sim \left[1, \frac{t}{T}, \ldots, \left(\frac{t}{T}\right)^p\right] \leftrightarrow [1, x, \ldots, x^p] \equiv h(x)'. \tag{A.13}$$

With these normalizations, the quadratic terms in ζ_t for $t = 1,\ldots,T-1$ (which are obtained when $j = 0$) vanish at a rate of T^{-1}. The remaining quadratic term is the one in ζ_T, which is obtained when $j = 0$ and $t = 1$ in (A.12) and which we denote by ℓ_1 in the limit. The cross-product terms are split in two, namely, when $j = t - 1 > 0$ in (A.12) and we have products involving ζ_T' and ζ_{T-t+1}, whose limit we denote by ℓ_2; and the remaining terms whose limit we denote by ℓ_3. This gives

$$\lim_{T\to\infty} \zeta' D_s^{-1} \zeta = \ell_1 + \ell_2 + \ell_3,$$

where

$$\ell_1 \equiv u_1' \int_0^1 (g(1-z)'g(1-z))^{-1}\,\mathrm{d}z\, u_1,$$

$$\ell_2 \equiv 2u_1' \int_0^1 \int_0^{1-x} (g(1-z)'g(1-z))^{-1}\,\mathrm{d}z\, g(x)'U_2' h(1-x)\,\mathrm{d}x,$$

$$\begin{aligned}\ell_3 \equiv 2 &\int_0^1 \int_0^x h(1-x+y)'U_2 g(x-y)\,\mathrm{d}y \\ &\times \int_0^{1-x} (g(1-z)'g(1-z))^{-1}\,\mathrm{d}z\, g(x)'U_2' h(1-x)\,\mathrm{d}x.\end{aligned} \tag{A.14}$$

The integral in z is the least tractable of the three, because of the inversion of $g(\cdot)$. It is therefore beneficial to reverse the order of integration in ℓ_2 to

$$\ell_2 = 2u_1' \int_0^1 (g(1-z)'g(1-z))^{-1} \int_0^{1-z} g(x)'U_2'h(1-x)\,\mathrm{d}x\mathrm{d}z. \tag{A.15}$$

For ℓ_3, a change of variable of $y \leftrightarrow x - y$, followed by a change of order of integration (x and z), gives

$$\begin{aligned}\ell_3 &= 2\int_0^1 \int_0^x h(1-y)'U_2 g(y)\,\mathrm{d}y \int_0^{1-x} (g(1-z)'g(1-z))^{-1}\,\mathrm{d}z\, g(x)'U_2'h(1-x)\,\mathrm{d}x \\ &= 2\int_0^1 \int_0^{1-z} \int_0^x h(1-y)'U_2 g(y)\,\mathrm{d}y (g(1-z)'g(1-z))^{-1} g(x)'U_2'h(1-x)\,\mathrm{d}x\mathrm{d}z.\end{aligned} \tag{A.16}$$

By symmetry of this particular integrand in x and y, we have here

$$\int_0^{1-z} \int_0^x \ldots \mathrm{d}y\mathrm{d}x = \int_0^{1-z} \int_0^y \ldots \mathrm{d}x\mathrm{d}y,$$

whereas we know that, from standard changing of the order of double integrals,

$$\int_0^{1-z} \int_0^x \ldots \mathrm{d}y\mathrm{d}x \equiv \int_0^{1-z} \int_y^{1-z} \ldots \mathrm{d}x\mathrm{d}y$$

for any integrand; so that

$$2\int_0^{1-z} \int_0^x \ldots \mathrm{d}y\mathrm{d}x = \int_0^{1-z} \int_0^{1-z} \ldots \mathrm{d}x\mathrm{d}y$$

and

$$\ell_3 = \int_0^1 \left(\int_0^{1-z} g(x)'U_2'h(1-x)\,\mathrm{d}x \right)' (g(1-z)'g(1-z))^{-1} \int_0^{1-z} g(x)'U_2'h(1-x)\,\mathrm{d}x\mathrm{d}z. \tag{A.17}$$

We now work out the row vector

$$f(z)' \equiv \int_0^{1-z} h(1-x)'U_2 g(x)\,\mathrm{d}x$$

for use in (A.15) and (A.17).

From the definitions of $g(\cdot)$ and $h(\cdot)$ in (A.11) and (A.13),

$$f(z)' = \int_0^{1-z} \big[[1, (1-x), \ldots, (1-x)^p]U_2 \quad 0'\big] \exp\left(x \begin{bmatrix} G & I_k \\ F & \mathbf{0} \end{bmatrix}\right) \mathrm{d}x \begin{bmatrix} I_k \\ -H' - U_3 \end{bmatrix}$$

$$= \int_{-1}^{-z} \big[[1, (-x), \ldots, (-x)^p]U_2 \quad 0'\big] \exp\left(x \begin{bmatrix} G & I_k \\ F & \mathbf{0} \end{bmatrix}\right) \mathrm{d}x$$

$$\times \exp\left(\begin{bmatrix} G & I_k \\ F & \mathbf{0} \end{bmatrix}\right) \begin{bmatrix} I_k \\ -H' - U_3 \end{bmatrix},$$

where the latter equality follows from $x \leftrightarrow x+1$ and from the $(2k) \times (2k)$ matrix commuting with itself. Notice that the first matrix is now a row vector and we can write

$$f(z)' = \sum_{i=0}^{p} [u_{i1} \quad \ldots \quad u_{ik}, \quad 0'] \int_{-1}^{-z} (-x)^i \exp\left(x \begin{bmatrix} G & I_k \\ F & \mathbf{0} \end{bmatrix}\right) \mathrm{d}x$$

$$\times \exp\left(\begin{bmatrix} G & I_k \\ F & \mathbf{0} \end{bmatrix}\right) \begin{bmatrix} I_k \\ -H' - U_3 \end{bmatrix}$$

$$= \sum_{i=0}^{p} (-1)^i [u_{i1} \quad \ldots \quad u_{ik}, \quad 0'] \exp\left((x+1) \begin{bmatrix} G & I_k \\ F & \mathbf{0} \end{bmatrix}\right) \sum_{j=0}^{i} (-i)_j x^{i-j} \Big|_{-1}^{-z}$$

$$\times \begin{bmatrix} G & I_k \\ F & \mathbf{0} \end{bmatrix}^{-j-1} \begin{bmatrix} I_k \\ -H' - U_3 \end{bmatrix}$$

$$= \sum_{i=0}^{p} [u_{i1} \quad \ldots \quad u_{ik}, \quad 0'] \sum_{j=0}^{i} (-i)_j (-1)^j \left(z^{i-j} \exp\left((1-z) \begin{bmatrix} G & I_k \\ F & \mathbf{0} \end{bmatrix}\right) - I_{2k}\right)$$

$$\times \begin{bmatrix} G & I_k \\ F & \mathbf{0} \end{bmatrix}^{-j-1} \begin{bmatrix} I_k \\ -H' - U_3 \end{bmatrix},$$

which is the required result. ■

Proof 1 of Corollary 5. To evaluate ℓ_1, via (7) and because

$$\frac{\mathrm{d}}{\mathrm{d}z} \frac{\sinh((1-z)\sqrt{f})}{g(1-z)} = -\frac{\sqrt{f}}{g(1-z)^2}, \tag{A.18}$$

we simply have

$$\ell_1 = u_1^2 \int_0^1 \frac{\mathrm{d}z}{g(1-z)^2} = u_1^2 \frac{\sinh\sqrt{f}}{g(1)\sqrt{f}}. \tag{A.19}$$

As for ℓ_2, we retrace the steps of the proof of the previous corollary and observe that

$$U_2' h(1-x) = \sum_{i=0}^{p} u_{i1} (1-x)^i \tag{A.20}$$

to find

$$\ell_2 = 2u_1 \sum_{i=0}^{p} u_{i1} \int_0^1 \int_0^{1-x} \frac{\mathrm{d}z}{g(1-z)^2}\, g(x)(1-x)^i \,\mathrm{d}x. \tag{A.21}$$

Here, via (A.18) and using the identity

$$\sinh a \cosh b - \cosh a \sinh b = \sinh(a-b), \tag{A.22}$$

we find

$$\begin{aligned}
\ell_2 &= \frac{2u_1}{\sqrt{f}} \sum_{i=0}^{p} u_{i1} \int_0^1 \left(\frac{\sinh\sqrt{f}}{g(1)} - \frac{\sinh(x\sqrt{f})}{g(x)} \right) g(x)(1-x)^i \,\mathrm{d}x \\
&= \frac{2u_1}{g(1)\sqrt{f}} \sum_{i=0}^{p} u_{i1} \int_0^1 \sinh((1-x)\sqrt{f})(1-x)^i \,\mathrm{d}x \\
&= \frac{2u_1}{g(1)\sqrt{f}} \sum_{i=0}^{p} u_{i1} \int_0^1 x^i \sinh(x\sqrt{f}) \,\mathrm{d}x.
\end{aligned} \tag{A.23}$$

Further, successive integration by parts yields

$$\int_0^y x^i \sinh(x\sqrt{f})\,\mathrm{d}x = -\frac{i!(1+(-1)^i)}{2(\sqrt{f})^{i+1}} + \frac{y^i}{2\sqrt{f}} \sum_{j=0}^{i} \frac{(-i)_j}{(y\sqrt{f})^j} (e^{y\sqrt{f}} + (-1)^j e^{-y\sqrt{f}}), \tag{A.24}$$

giving the desired expression for ℓ_2 by letting $y = 1$.

Looking at ℓ_3, we use the proof of the preceding corollary to obtain

$$\ell_3 = \frac{1}{2g(1)f} \sum_{i=0}^{p} \sum_{j=0}^{p} u_{i1} u_{j1} \gamma_{ij}, \tag{A.25}$$

where, by (A.20) and (A.16),

$$\gamma_{ij} \equiv 4g(1)f \int_0^1 \int_0^x (1-y)^j g(y)\,\mathrm{d}y \int_0^{1-x} \frac{\mathrm{d}z}{g(1-z)^2}\, g(x)(1-x)^i \,\mathrm{d}x.$$

Now, as in (A.21)–(A.23) again,

$$\int_0^{1-x} \frac{\mathrm{d}z}{g(1-z)^2}\, g(x) = \frac{1}{g(1)\sqrt{f}} \sinh((1-x)\sqrt{f}),$$

and

$$\begin{aligned}
\gamma_{ij} &= 4\sqrt{f} \int_0^1 \int_0^x (1-y)^j g(y)\,\mathrm{d}y \sinh((1-x)\sqrt{f})(1-x)^i \,\mathrm{d}x \\
&= 4\sqrt{f} \int_0^1 \int_0^{1-x} (1-y)^j g(y)\,\mathrm{d}y \sinh(x\sqrt{f}) x^i \,\mathrm{d}x \\
&= 4\sqrt{f} \int_0^1 (1-y)^j g(y) \int_0^{1-y} \sinh(x\sqrt{f}) x^i \,\mathrm{d}x\mathrm{d}y \\
&= 4\sqrt{f} \int_0^1 y^j g(1-y) \int_0^y x^i \sinh(x\sqrt{f})\,\mathrm{d}x\mathrm{d}y,
\end{aligned} \tag{A.26}$$

by $x \leftrightarrow 1 - x$, changing the integration order and $y \leftrightarrow 1 - y$, respectively. The inner integral is given by (A.24), so that

$$\gamma_{ij} = -2\int_0^1 y^j g(1-y)\left(\frac{i!(1+(-1)^i)}{(\sqrt{f})^i} - \sum_{m=0}^{i}\frac{(-i)_m y^{i-m}}{(\sqrt{f})^m}(e^{y\sqrt{f}} + (-1)^m e^{-y\sqrt{f}})\right)\mathrm{d}y$$

$$= -\int_0^1 y^j((1-v)e^{(1-y)\sqrt{f}} + (1+v)e^{-(1-y)\sqrt{f}})$$

$$\times\left[\frac{i!(1+(-1)^i)}{(\sqrt{f})^i} - \sum_{m=0}^{i}\frac{(-i)_m}{(\sqrt{f})^m} y^{i-m}(e^{y\sqrt{f}} + (-1)^m e^{-y\sqrt{f}})\right]\mathrm{d}y$$

$$= -\frac{i!(1+(-1)^i)}{(\sqrt{f})^i}\left[(1-v)e^{\sqrt{f}}\int_0^1 y^j e^{-y\sqrt{f}}\,\mathrm{d}y + (1+v)e^{-\sqrt{f}}\int_0^1 y^j e^{y\sqrt{f}}\,\mathrm{d}y\right]$$

$$+ \sum_{m=0}^{i}\frac{(-i)_m}{(\sqrt{f})^m}\left[((1-v)e^{\sqrt{f}} + (-1)^m(1+v)e^{-\sqrt{f}})\int_0^1 y^{j+i-m}\,\mathrm{d}y\right.$$

$$+ (-1)^m(1-v)e^{\sqrt{f}}\int_0^1 y^{j+i-m}e^{-2y\sqrt{f}}\,\mathrm{d}y$$

$$\left. + (1+v)e^{-\sqrt{f}}\int_0^1 y^{j+i-m}e^{2y\sqrt{f}}\,\mathrm{d}y\right]$$

by the definition of $g(\cdot)$ in (7). Using

$$\int_0^1 y^q e^{ay}\,\mathrm{d}y = \frac{q!}{(-a)^{q+1}} + e^a\sum_{n=0}^{q}\frac{(-q)_n}{a^{n+1}}$$

where $q \in \mathcal{N} \cup \{0\}$ gives the stated result. ■

Proof 2 of Corollary 5. A more conventional proof may be obtained from exploiting Girsanov's theorem as in Perron (1991, Theorem 2) or Tanaka (1996, Ch. 4).

We want to calculate the MGF of

$$(S_1, S_2', S_3, S_4) \equiv \left(J_h(1), \int_0^1 (1, r, \ldots, r^p)J_h(r)\,\mathrm{d}r, \int_0^1 J_h(r)\,\mathrm{d}W(r), \int_0^1 J_h(r)^2\,\mathrm{d}r\right),$$

which we define as

$$\phi_h(u_1, u_2, u_3, u_4) \equiv \mathrm{E}\{\exp(u_1 S_1 + u_2' S_2 + u_3 S_3 + u_4 S_4)\}, \tag{A.27}$$

where u_1, u_3, u_4 are scalars and $u_2' \equiv (u_{20}, \ldots, u_{2p})$ is a $(p+1)$-dimensional vector. At first, apply Itô's formula to write

$$\mathrm{d}(J_h(t)^2) = 2J_h(t)\,\mathrm{d}J_h(t) + \mathrm{d}t = 2hJ_h(t)^2\,\mathrm{d}t + 2J_h(t)\,\mathrm{d}W(t) + \mathrm{d}t,$$

so that, by integrating and solving,

$$\int_0^1 J_h(t)\,\mathrm{d}W(t) = \frac{1}{2}\{J_h(1)^2 - 1\} - h\int_0^1 J_h(t)^2\,\mathrm{d}t$$

and (A.27) may be reformulated as

$$\phi_h(u_1,u_2,u_3,u_4) = \exp\left(-\frac{u_3}{2}\right)\mathrm{E}\left\{\exp\left(u_1 J_h(1) + u_2' \int_0^1 (1,t,\ldots,t^p)' J_h(t)\,\mathrm{d}t \right.\right.$$
$$\left.\left. + \frac{u_3}{2} J_h(1)^2 + (u_4 - u_3 h)\int_0^1 J_h(t)^2\,\mathrm{d}t\right)\right\}.$$

Now, following Tanaka (1996, p. 110), we define the auxiliary process $Y(t)$ through

$$\mathrm{d}Y(t) = -\beta Y(t)\,\mathrm{d}t + \mathrm{d}W(t), \quad Y(0) = 0.$$

Putting $X(t) = J_h(t)$ (with $\alpha = -h$ in Tanaka's notation) and letting μ_J and μ_Y be the probability measures governing J_h and Y, respectively, we get

$$\frac{\mathrm{d}\mu_J}{\mathrm{d}\mu_Y}(Y) = \exp\left[(h+\beta)\int_0^1 Y(t)\,\mathrm{d}Y(t) - \frac{h^2-\beta^2}{2}\int_0^1 Y(t)^2\,\mathrm{d}t\right]$$
$$= \exp\left[-\frac{h+\beta}{2} + \frac{h+\beta}{2} Y(1)^2 - \frac{h^2-\beta^2}{2}\int_0^1 Y(t)^2\,\mathrm{d}t\right],$$

the second line following from Itô's formula. Now, as in Tanaka (1996, p. 111),

$$\phi_h(u_1,u_2,u_3,u_4)$$
$$= \exp\left(-\frac{u_3}{2}\right)\mathrm{E}\left\{\exp\left(u_1 Y(1) + u_2'\int_0^1 (1,t,\ldots,t^p)' Y(t)\,\mathrm{d}t \right.\right.$$
$$\left.\left. + \frac{u_3}{2} Y(1)^2 + (u_4 - u_3 h)\int_0^1 Y(t)^2\,\mathrm{d}t\right)\frac{\mathrm{d}\mu_J}{\mathrm{d}\mu_Y}(Y)\right\}$$
$$= \exp(-\gamma)\mathrm{E}\left\{\exp\left(u_1 Y(1) + u_2'\int_0^1 (1,t,\ldots,t^p)' Y(t)\,\mathrm{d}t + \gamma Y(1)^2\right)\right\} \quad \textbf{(A.28)}$$

by choosing

$$\beta \equiv \sqrt{h^2 + 2hu_3 - 2u_4}$$

and defining

$$\gamma \equiv \frac{u_3 + h + \beta}{2}.$$

To go further, the idea is to define a process $Z(t) = Y(t) - \theta_t Y(1)$, where θ_t is a constant, chosen in such a way that $Z(t)$ becomes independent of $Y(1)$. But as is easily shown, we have that for $0 < s \le t \le 1$,

$$E\{Y(s)Y(t)\} = e^{-\beta t}\frac{\sinh(\beta s)}{\beta}, \quad \textbf{(A.29)}$$

and so

$$E\{Y(1)Z(t)\} = \frac{e^{-\beta}}{\beta}\{\sinh(\beta t) - \theta_t \sinh\beta\},$$

implying the choice (uncorrelatedness is equivalent to independence for normal processes)

$$\theta_t = \frac{\sinh(\beta t)}{\sinh\beta}.$$

Utilizing this, we may rewrite (A.28) as

$$\begin{aligned}
&\phi_h(u_1,u_2,u_3,u_4)\\
&\quad = \exp(-\gamma)\mathrm{E}\left\{\exp\left(u_1 Y(1) + u_2'\int_0^1 (1,t,\ldots,t^p)'(Z(t)+\theta_t Y(1))\,\mathrm{d}t + \gamma Y(1)^2\right)\right\}\\
&\quad = \exp(-\gamma)\mathrm{E}\left\{\exp\left[\left(u_1 + \theta_t u_2'\int_0^1 (1,t,\ldots,t^p)'\,\mathrm{d}t\right)Y(1) + \gamma Y(1)^2\right]\right\}\\
&\qquad \times \mathrm{E}\left\{\exp\left(u_2'\int_0^1 (1,t,\ldots,t^p)'Z(t)\,\mathrm{d}t\right)\right\}\\
&\quad = \exp(-\gamma)\mathrm{E}\{\exp(\eta Y(1) + \gamma Y(1)^2)\}\mathrm{E}\left\{\exp\left(u_2'\int_0^1 (1,t,\ldots,t^p)'Z(t)\,\mathrm{d}t\right)\right\},
\end{aligned} \qquad \textbf{(A.30)}$$

where

$$\eta \equiv u_1 + \frac{1}{\sinh\beta}\sum_{i=0}^{p} u_{2i}\int_0^1 t^i \sinh(\beta t)\,\mathrm{d}t. \qquad \textbf{(A.31)}$$

But $Y(1)$ is normal with mean zero and, from (A.29), variance $\sigma^2 \equiv (1 - e^{-2\beta})/(2\beta)$. Hence we have

$$\begin{aligned}
\mathrm{E}\{\exp(\eta Y(1) + \gamma Y(1)^2)\} &= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\exp\left(\eta y - \frac{(1-2\sigma^2\gamma)}{2\sigma^2}\,y^2\right)\mathrm{d}y\\
&= \frac{1}{\sqrt{1-2\sigma^2\gamma}}\exp\left(\frac{\eta^2\sigma^2}{2(1-2\sigma^2\gamma)}\right).
\end{aligned} \qquad \textbf{(A.32)}$$

Moreover, $\int_0^1 (1,t,\ldots,t^p)'Z(t)\,\mathrm{d}t$ is $(p+1)$-variate normal with mean zero and covariance matrix Σ with typical element σ_{ij}, $i,j = 0,1,\ldots,p$ where

$$\sigma_{ij} = \int_0^1\int_0^1 t^i s^j \mathrm{E}\{Z(t)Z(s)\}\mathrm{d}t\mathrm{d}s.$$

Here, via (A.29), we have for $0 < s \le t \le 1$,

$$\begin{aligned}
\mathrm{E}\{Z(t)Z(s)\} &= \mathrm{E}\{[Y(t)-\theta_t Y(1)][Y(s)-\theta_s Y(1)]\}\\
&= e^{-\beta t}\frac{\sinh(\beta s)}{\beta} - e^{-\beta}\frac{\sinh(\beta s)}{\sinh\beta}\frac{\sinh(\beta t)}{\beta} - e^{-\beta}\frac{\sinh(\beta t)}{\sinh\beta}\frac{\sinh(\beta s)}{\beta}\\
&\quad + e^{-\beta}\frac{\sinh(\beta t)}{\sinh\beta}\frac{\sinh(\beta s)}{\sinh\beta}\frac{\sinh\beta}{\beta}\\
&= e^{-\beta t}\frac{\sinh(\beta s)}{\beta} - e^{-\beta}\frac{\sinh(\beta s)}{\sinh\beta}\frac{\sinh(\beta t)}{\beta}\\
&= \frac{\sinh(\beta s)\sinh(\beta(1-t))}{\beta\sinh\beta},
\end{aligned}$$

and so

$$\begin{aligned}
\sigma_{ij} = \frac{1}{\beta\sinh\beta}\Bigg[&\int_0^1 t^i\sinh(\beta(1-t))\int_0^t s^j\sinh(\beta s)\,\mathrm{d}s\mathrm{d}t\\
&+\int_0^1 s^j\sinh(\beta(1-s))\int_0^s t^i\sinh(\beta t)\,\mathrm{d}t\mathrm{d}s\Bigg]. \qquad \textbf{(A.33)}
\end{aligned}$$

Now, because

$$\mathrm{E}\left\{\exp\left(u_2\int_0^1(1,t,\ldots,t^p)'Z(t)\,\mathrm{d}t\right)\right\} = \exp\left(\frac{1}{2}u_2'\Sigma u_2\right),$$

(A.30) and (A.32) yield

$$\phi_h(u_1,u_2,u_3,u_4) = \frac{1}{\sqrt{1-2\sigma^2\gamma}}\exp\left[-\gamma+\frac{\eta^2\sigma^2}{2(1-2\sigma^2\gamma)}-\frac{1}{2}u_2'\Sigma u_2\right]. \qquad \textbf{(A.34)}$$

To see how (A.34) corresponds to the results of Corollary 5, observe that $\beta = \sqrt{f}$, and so

$$g(1) = \cosh\beta - \frac{h+u_3}{\beta}\sinh\beta,$$

and we have

$$\sigma^2 = \frac{\sinh\beta}{\beta e^\beta}, \qquad \textbf{(A.35)}$$

so that

$$\begin{aligned}
1-2\sigma^2\gamma &= 1-2\frac{\sinh\beta}{\beta e^\beta}\frac{h+u_3+\beta}{2} = 1-\frac{\sinh\beta}{e^\beta}-\frac{h+u_3}{\beta}\frac{\sinh\beta}{e^\beta}\\
&= e^{-\beta}g(1). \qquad \textbf{(A.36)}
\end{aligned}$$

But

$$e^{-\gamma} = e^{-[(h+u_3)/\beta]}(e^{-\beta})^{1/2},$$

so that

$$\frac{1}{\sqrt{1-2\sigma^2\gamma}}\,e^{-\gamma} = \frac{1}{\sqrt{g(1)}}\,e^{-[(h+u_3)/\beta]},$$

which is the last factor of the result of Corollary 4. Furthermore, by (A.31),

$$\eta^2 = u_1^2 + \frac{2u_1}{\sinh\beta}\sum_{i=0}^{p} u_{2i}\int_0^1 t^i \sinh(\beta t)\,\mathrm{d}t$$
$$+\frac{1}{\sinh^2\beta}\sum_{i=0}^{p}\sum_{j=0}^{p} u_{2i}u_{2j}\int_0^1\int_0^1 t^i s^j \sinh(\beta t)\sinh(\beta s)\,\mathrm{d}s\mathrm{d}t, \tag{A.37}$$

so the contribution of (A.34) to the ℓ_1 coefficient is, by (A.35) and (A.36),

$$\frac{\sigma^2}{1-2\sigma^2\gamma} = \frac{\sinh\beta}{\beta g(1)}, \tag{A.38}$$

which is in accord with Corollary 5. Moreover, the u_1u_{2i} coefficient is, by (A.37) and (A.38),

$$\frac{2}{\beta g(1)}\int_0^1 t^i \sinh(\beta t)\,\mathrm{d}t,$$

which is in accord with (A.23) of the first proof of Corollary 5. Similarly, the contribution to the $u_{2i}u_{2j}$ coefficient from $\eta^2\sigma^2/(1-2\sigma^2\gamma)$ is

$$a_{ij} \equiv \frac{1}{\beta\sinh\beta g(1)}\int_0^1\int_0^1 t^i s^j \sinh(\beta t)\sinh(\beta s)\,\mathrm{d}s\mathrm{d}t.$$

In view of (A.26), we now need to prove that, for each (i,j),

$$a_{ij}+a_{ji}+\sigma_{ij}+\sigma_{ji} = \frac{\gamma_{ij}+\gamma_{ji}}{2\beta^2 g(1)} = \frac{2}{\beta g(1)}\int_0^1\int_0^t (t^i s^j + t^j s^i) g(1-t)\sinh(\beta s)\,\mathrm{d}s\mathrm{d}t. \tag{A.39}$$

To this end, we see from (A.33) that

$$\begin{aligned}\beta\sinh\beta(\sigma_{ij}+\sigma_{ji}) &= \int_0^1\int_0^t (t^i s^j + t^j s^i)\sinh(\beta(1-t))\sinh(\beta s)\,\mathrm{d}s\mathrm{d}t\\
&\quad+\int_0^1\int_0^s (t^i s^j + t^j s^i)\sinh(\beta(1-s))\sinh(\beta t)\,\mathrm{d}t\mathrm{d}s\\
&= 2\int_0^1\int_0^t (t^i s^j + t^j s^i)\sinh(\beta(1-t))\sinh(\beta s)\,\mathrm{d}s\mathrm{d}t,\end{aligned}$$

where the second step follows by reversing the roles of t and s in the second integral of the first step. Similarly,

$$\beta \sinh \beta (a_{ij} + a_{ji}) = \frac{1}{g(1)} \int_0^1 \int_0^1 (t^i s^j + t^j s^i) \sinh(\beta t) \sinh(\beta s) \, \mathrm{d}s \mathrm{d}t$$
$$= \frac{2}{g(1)} \int_0^1 \int_0^t (t^i s^j + t^j s^i) \sinh(\beta t) \sinh(\beta s) \, \mathrm{d}s \mathrm{d}t,$$

the second step following by symmetry. Hence, (A.39) follows if we can prove

$$\sinh(\beta t) \sinh(\beta s) + g(1) \sinh(\beta(1-t)) \sinh(\beta s) = g(1-t) \sinh \beta \sinh(\beta s);$$

i.e., crossing out $\sinh(\beta s)$ and using the definition of g,

$$\sinh(\beta t) + \left(\cosh \beta - \frac{h + u_3}{\beta} \sinh \beta \right) \sinh(\beta(1-t))$$
$$= \left[\cosh(\beta(1-t)) - \frac{h + u_3}{\beta} \sinh(\beta(1-t)) \right] \sinh \beta,$$

which is equivalent to

$$\sinh(\beta t) + \cosh \beta \sinh(\beta(1-t)) = \cosh(\beta(1-t)) \sinh \beta,$$

which is identically true because, for any a and b,

$$\sinh(a - b) = \sinh a \cosh b - \cosh a \sinh b.$$

Hence, (A.39) is proved. ■