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Total positivity and accurate computations with Gram matrices of Said-Ball bases

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Abstract

In this article, it is proved that Gram matrices of totally positive bases of the space of polynomials of a given degree on a compact interval are totally positive. Conditions to guarantee computations to high relative accuracy with those matrices are also obtained. Furthermore, a fast and accurate algorithm to compute the bidiagonal factorization of Gram matrices of the Said-Ball bases is obtained and used to compute to high relative accuracy their singular values and inverses, as well as the solution of some linear systems associated with these matrices. Numerical examples are included.

KEYWORDS

Bernstein bases, Bidiagonal decompositions, gram matrices, high relative accuracy, said-ball bases, totally positive matrices

1 | INTRODUCTION

Finding classes of matrices with relevant applications, for which algebraic computations can be efficiently performed to high relative accuracy (HRA), is an important research topic. This article provides new contributions in this field when considering Gram (mass) matrices of Said-Ball bases.

Hilbert matrices are typical examples of ill-conditioned matrices, meaning that they have large condition numbers, which increase significantly with the dimension of the matrices, indicating that they are nearly singular. Consequently, when considering Hilbert matrices, standard routines implementing best traditional numerical methods fail to solve accurately usual algebraic problems such as the computation of the singular values or the inverse matrix. Hilbert matrices can be considered as Gram matrices of the monomial bases of polynomials on $[0,1]$ and belong to the class of totally positive matrices (see Section 2), which satisfies a decomposition in terms of bidiagonal matrices (see References 1 and 2). In fact, in References 3 and 4, it is shown that the above mentioned matrix computations with Hilbert matrices can be performed to HRA when exploiting adequately this bidiagonal factorization. Excellent results have also been obtained when dealing with Gram matrices of other bases, such as Poisson and Bernstein bases on the interval $[0,1]$ and bases $\{t^i e^{\lambda t}\}$ (see References 4-6).

Bernstein bases are the polynomial bases most used in computer-aided geometric design (CAGD) and have optimal shape preserving and stability properties (see References 7 and 8). The mentioned optimalities are related to the fact that Bernstein bases are normalized B-bases (cf. References 9 and 10). Moreover, Bernstein bases also have important applications aside from CAGD. For example, in the resolution of elliptic and hyperbolic partial differential equations with Galerkin methods and collocation methods (cf. References 11 and 12), in optimal control theory (cf. Reference 13), or in stochastic dynamics (cf. Reference 14).

In the least-squares approximation of a given function by a linear combination of Bernstein polynomials on a interval I, the resolution of a linear system of normal equations is required. The coefficient matrix of this system is the Gram matrix of the considered Bernstein basis with respect to the inner product of the Hilbert space $L^2(I)$. As an alternative, the dual polynomials associated with the Bernstein basis can be considered, so that the linear system is trivially solved and the solution can be explicitly computed. Bernstein polynomials and their associated dual basis functions are related by a linear transformation. In particular, the inverse of the Bernstein mass matrix provides the coefficients of the dual Bernstein basis in terms of the corresponding Bernstein polynomials. The Gram matrices of Bernstein bases on $[0, 1]$ are also called Bernstein mass matrices (see References 15 and 16). Fast calculations of Bernstein mass matrices and their inverses are achieved in Reference 17. Furthermore, improvements for applications such as polynomial approximation of functions and degree reduction of Bézier curves on $[0, 1]$ are illustrated.

Another important class of polynomial bases with applications in CAGD is formed by the Said-Ball bases, which were defined by Said in Reference 18 for spaces of odd degree polynomials on $[0, 1]$ and later, extended to spaces of polynomials of even degree in Reference 19. Said-Ball polynomial bases have the same type of shape preserving properties as the Bernstein bases (see References 9,20,21) and provide an algorithm with less computational cost than the evaluation algorithm for the Bernstein basis (see Reference 19). Computations to HRA with the collocation matrices of the Said-Ball bases were achieved in References 21 and 22. In Reference 23 an explicit formula for the dual basis functions for generalized Ball bases is provided.

In our article, Gram matrices of Said-Ball bases are considered and, by taking the advantages of the transformation between the Bernstein basis and the Said-Ball basis, a bidiagonal factorization to HRA is provided. Then, using the algorithms presented in References 3,24,25, we can assure that the computation of their singular values and inverses can be performed to HRA.

The article is organized as follows. Section 2 recalls basic notations and results related to total positivity, bidiagonal factorizations and HRA. In Section 3, the total positivity of the Gram matrix of the polynomial Bernstein bases on compact intervals $[a, b]$ is proved. As a result, the total positivity of the Gram matrix of any totally positive basis of the space of polynomials of degree at most n on $[a, b]$ is deduced. Furthermore, conditions to guarantee computations to HRA with those matrices are obtained. Section 4 derives to HRA a bidiagonal factorization of the Gram matrix of Said-Ball bases of odd and even degrees. Finally, Section 5 includes numerical examples illustrating the application of the procedure proposed in Section 4 to get the accurate computation of singular values and inverses of Gram matrices of the Said-Ball basis, as well as the solution of some linear systems associated with these matrices.

2 | NOTATIONS AND AUXILIARY RESULTS

Let us suppose that U is a Hilbert space of functions on the interval $[0, T]$, $T \leq +\infty$, under a given inner product $\langle \cdot, \cdot \rangle$. Then, given linearly independent functions $v_0, \dots, v_n$ in U, the corresponding Gram matrix is the symmetric matrix

$$G(v_0, \dots, v_n) := \left(\langle v_{i-1}, v_{j-1} \rangle\right)_{1 \leq i,j \leq n+1}.$$

Let us recall that a matrix is totally positive (TP) if all its minors are nonnegative and strictly totally positive (STP) if all its minors are positive (see Reference 26). Interesting applications of TP and STP matrices are provided in References 26-28. A nonnegative matrix $A = (a_{i,j})_{1 \leq i,j \leq n}$ is stochastic if, $\sum_{j=1}^{n} a_{i,j} = 1$, for any $i = 1, \dots, n$.

In the sequel, given $A = (a_{ij})_{1 \leq i,j \leq n}$, $A[i_1, \dots, i_r | j_1, \dots, j_s]$ denotes the submatrix formed with rows $i_1, \dots, i_r$ and columns $j_1, \dots, j_s$. Moreover, $A[i_1, \dots, i_r] := A[i_1, \dots, i_r | i_1, \dots, i_r]$.

A system of functions $(u_0, \dots, u_n)$ on $I \subseteq \mathbb{R}$ is TP if all its collocation matrices $\left(u_{j-1}(t_i)\right)_{i,j=1,\dots,n+1}$, with $t_1 < \cdots < t_{n+1}$ in I, are TP. Moreover, a TP system is normalized (NTP) if $\sum_{i=0}^{n} u_i(t) = 1$, for all $t \in I$. It is well-known that NTP bases are commonly used in computer-aided geometric design since they provide shape preserving representations (see References 9 and 29).

Among all NTP bases of a space of functions, there exists a unique normalized B-basis, which is the basis with optimal shape preserving properties (cf. Reference 10).

The following characterization of a B-basis is a consequence of Corollary 3.10 and Proposition 3.11 of Reference 10, taking into account that a matrix is TP if and only if its transpose is TP.

Theorem 1. *Let $(u_0, \dots, u_n)$ be a TP basis of a vector space of functions U. Then $(u_0, \dots, u_n)$ is a B-basis if and only if for any other TP basis $(v_0, \dots, v_n)$ of U the change of basis matrix A with $(v_0, \dots, v_n)^T = A(u_0, \dots, u_n)^T$ is TP.*

Let us also recall, that a real value $x \in \mathbb{R}$ is calculated to high relative accuracy (HRA) if the relative error in its computation can be bounded as follows,

$$\frac{|x-\tilde{x}|}{|x|} < ku,$$

where u is the unit round-off and k is a positive constant, which is independent of the arithmetic precision. HRA guarantees that relative errors in the computations have the same order as the machine's precision and, consequently, implies excellent accuracy. [Correction added on 12 October 2023, after first online publication: The above formula has been corrected in this version.]

It is well known that a sufficient condition to guarantee that an algorithm can be computed to HRA is the non inaccurate cancellation (NIC) condition. This property is satisfied when the algorithm only evaluates products, quotients, sums of numbers of the same sign, subtractions of numbers of opposite sign or subtraction of initial data (cf. References 24,30,31).

A fundamental tool to achieve the factorizations provided in this article is the Neville elimination (NE), which is an alternative procedure to Gaussian elimination (see References 1,2,32). Given an $(n+1)\times(n+1)$, nonsingular matrix $A=(a_{i,j})_{1\le i,j\le n+1}$, the NE computes a sequence of matrices $A^{(1)} := A \to A^{(2)} \to \cdots \to A^{(n+1)}$, satisfying that the entries below the main diagonal in the k first columns of $A^{(k+1)}$, $1\le k\le n$, are zeros and finally, $A^{(n+1)}$ is an upper triangular matrix. From $A^{(k)} = (a_{i,j}^{(k)})_{1\le i,j\le n+1}$, the matrix $A^{(k+1)} = (a_{i,j}^{(k+1)})_{1\le i,j\le n+1}$ is computed as follows,

$$a_{i,j}^{(k+1)} := \begin{cases} a_{i,j}^{(k)}, & \text{if } 1\le i\le k, \\ a_{i,j}^{(k)} - \frac{a_{i,k}^{(k)}}{a_{i-1,k}^{(k)}} a_{i-1,j}^{(k)}, & \text{if } k+1\le i,j\le n+1 \text{ and } a_{i-1,k}^{(k)} \neq 0, \\ a_{i,j}^{(k)}, & \text{if } \ k+1\le i\le n+1 \text{ and } a_{i-1,k}^{(k)} = 0. \end{cases} \tag{1}$$

The (i,j) pivot of the NE of A is

$$p_{i,j} := a_{i,j}^{(j)}, \quad 1\le j\le i\le n+1, \tag{2}$$

and $p_{i,i}$, $i=1,\dots,n+1$, is the i-th diagonal pivot. If all pivots are nonzero, the NE of A can be performed with no row exchanges. For $1\le j<i\le n+1$, the (i,j) multiplier of the NE of A is

$$m_{i,j} := \begin{cases} a_{i,j}^{(j)}/a_{i-1,j}^{(j)} = p_{i,j}/p_{i-1,j}, & \text{if } a_{i-1,j}^{(j)} \neq 0, \\ 0, & \text{if } a_{i-1,j}^{(j)} = 0. \end{cases} \tag{3}$$

Let us recall the following characterization of STP matrices, which is derived from Theorem 4.1 of Reference 32 and the arguments of p. 116 of Reference 2.

Theorem 2. *A given matrix A is STP if and only if the Neville elimination of A and A^T can be performed without row exchanges, all the multipliers of the Neville elimination of A and A^T are positive, and the diagonal pivots of the Neville elimination of A are all positive.*

Furthermore, a nonsingular TP matrix $A=(a_{i,j})_{1\le i,j\le n+1}$ can be factorized as follows,

$$A = F_n F_{n-1}\cdots F_1 D G_1 G_2 \cdots G_n, \tag{4}$$

where D is a diagonal matrix with positive entries and F_i and G_i, $i=1,\dots,n$, are the TP, lower and upper, respectively, triangular bidiagonal matrices of the following form

$$F_i = \begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 0 & 1 & & & & \\ & & & m_{i+1,1} & 1 & & & \\ & & & & m_{i+2,2} & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & m_{n+1,n+1-i} & 1 \end{pmatrix}, \quad G_i^T = \begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 0 & 1 & & & & \\ & & & \widetilde{m}_{i+1,1} & 1 & & & \\ & & & & \widetilde{m}_{i+2,2} & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \widetilde{m}_{n+1,n+1-i} & 1 \end{pmatrix}, \tag{5}$$

(see Theorem 4.2 and the arguments of p. 116 of Reference 2). The diagonal entries of D are the positive diagonal pivots $p_{i,i}$, $i = 1, \dots, n+1$, of the NE of A and the entries $m_{i,j}$ and $\widetilde{m}_{i,j}$ are the multipliers of the NE of A and A^T, respectively. In addition, if $m_{i,j}$, $\widetilde{m}_{i,j}$ satisfy

$$m_{i,j} = 0 \quad \Rightarrow \quad m_{h,j} = 0,\ \forall\, h > i \quad \text{and} \quad \widetilde{m}_{i,j} = 0 \quad \Rightarrow \quad \widetilde{m}_{h,j} = 0,\ \forall\, h > i, \tag{6}$$

then the decomposition (4) is unique.

Let us notice that if A is nonsingular and TP, using the results in References 1,2,32, we can obtain a bidiagonal decomposition of A^{-1} as follows

$$A^{-1} = \tilde{G}_1\tilde{G}_2 \cdots \tilde{G}_n D^{-1}\tilde{F}_n\tilde{F}_{n-1} \cdots \tilde{F}_1, \tag{7}$$

where $\tilde{F}_i$ and $\tilde{G}_i$, $i = 1, \dots, n$, are the lower and upper triangular bidiagonal matrices of the form of F_i and G_i, respectively, but replacing the off-diagonal entries of matrices F_i: $\{m_{i+1,1}, m_{i+2,2}, \dots, m_{n+1,n+1-i}\}$ and matrices G_i: $\{\tilde{m}_{i+1,1}, \tilde{m}_{i+2,2}, \dots, \tilde{m}_{n+1,n+1-i}\}$ by the entries $\{-m_{i+1,i}, -m_{i+2,i}, \dots, -m_{n+1,i}\}$ and $\{-\tilde{m}_{i+1,i}, -\tilde{m}_{i+2,i}, \dots, -\tilde{m}_{n+1,i}\}$, respectively.

On the other hand, if a matrix A is nonsingular and TP, its transpose A^T is also nonsingular and TP. Clearly, A^T satisfies

$$A^T = G_n^T G_{n-1}^T \cdots G_1^T D F_1^T F_2^T \cdots F_n^T,$$

where F_i and G_i, $i = 1, \dots, n$, are the lower and upper triangular bidiagonal matrices in (4). If, in addition, A is symmetric then (6) holds and we can immediately deduce that $G_i = F_i^T$, $i = 1, \dots, n$, and then

$$A = F_n F_{n-1} \cdots F_1 D F_1^T F_2^T \cdots F_n^T, \tag{8}$$

where F_i, $i = 1, \dots, n$, are the lower triangular bidiagonal matrices described by (5), whose off-diagonal entries are the multipliers of the NE of A and D is the diagonal matrix with the diagonal pivots.

Using the matrix notation introduced in Reference 24, the bidiagonal decomposition (4) of a nonsingular TP matrix $A = (a_{i,j})_{1\le i,j\le n+1}$ can be represented by means of a matrix $BD(A) = (BD(A)_{i,j})_{1\le i,j\le n+1}$, such that

$$BD(A)_{i,j} := \begin{cases} m_{i,j}, & \text{if } i > j, \\ p_{i,i}, & \text{if } i = j, \\ \widetilde{m}_{j,i}, & \text{if } i < j. \end{cases} \tag{9}$$

The following auxiliary result can be easily proved and will be useful in the next sections.

Lemma 1. *Let $\alpha > 0$ and A be an $(n+1) \times (n+1)$ nonsingular TP matrix, whose bidiagonal factorization (4) is*

$$A = F_n F_{n-1} \cdots F_1 D G_1 G_2 \cdots G_n.$$

Then, the bidiagonal factorization (4) of $\tilde{A} := \alpha A$ is

$$\widetilde{A} = F_n F_{n-1} \cdots F_1 \widetilde{D} G_1 G_2 \cdots G_n,$$

where $\widetilde{D} = \alpha D$. Then

$$BD(\widetilde{A})_{i,j} := \begin{cases} BD(A)_{i,j}, & \text{if} \quad i \neq j, \\ \alpha BD(A)_{i,i}, & \text{if} \quad i = j. \end{cases} \tag{10}$$

Proof. The result follows taking into account that $\alpha F_i = F_i \alpha$, $i = 1, \dots, n$. ∎

Finally, let us recall that if the diagonal pivots and multipliers of the NE of a nonsingular TP matrix A and so, its bidiagonal factorization (4), are provided to HRA, then the algorithms in References 3,24,25 such as `TNSingularValues` and `TNInverseExpand` take as input argument $BD(A)$ and compute to HRA the singular values of A and its inverse matrix A^{-1} (using the algorithm presented in Reference 33). Besides, the algorithm `TNSolve` computes the solution of systems of linear equations $Ax = b$, for vectors b whose entries have alternating signs.

3 | GRAM MATRICES OF TP POLYNOMIAL BASES ON COMPACT INTERVALS

The Bernstein basis of the space $\mathbf{P}^n[0,1]$ of polynomials of degree less than or equal to n on the interval $[0,1]$ is

$$(B_0^n, \dots, B_n^n), \quad B_i^n(t) := \binom{n}{i} t^i(1-t)^{n-i}, \quad i = 0, \dots, n. \tag{11}$$

Bernstein polynomials belong to the vector space of square integrable functions, which is a Hilbert space under the following inner product

$$\langle f, g\rangle := \int_0^1 t^\alpha (1-t)^\beta f(t)g(t)\, dt, \quad \alpha, \beta > -1. \tag{12}$$

It can be checked that the Gram matrix of the Bernstein basis (11), with respect to the inner product (12), is the symmetric matrix $M = (M_{i,j})_{1\le i,j\le n+1}$, such that

$$M_{i,j} := \langle B_{i-1}^n, B_{j-1}^n\rangle = \binom{n}{i-1}\binom{n}{j-1}\frac{\Gamma(i+j+\alpha-1)\Gamma(2n-i-j+\beta+3)}{\Gamma(2n+\alpha+\beta+2)}, \quad 1 \le i,j \le n+1, \tag{13}$$

where $\Gamma(x)$ denotes the well-known Gamma function (see Reference 6).

Theorem 2 of Reference 6 proves that the Bernstein Gram matrix M described by (13) is STP. It also provides the multipliers and the diagonal pivots of its NE process.

Let us notice that if $\alpha = \beta = 0$, the Gram matrix (13) is usually called Bernstein mass matrix. Nice properties and applications of Bernstein mass matrices can be found in References 15-17.

For any compact interval $[a, b]$, the corresponding Bernstein basis of the space $\mathbf{P}^n[a, b]$ of polynomials of degree less than or equal to n on $[a, b]$ is

$$(\tilde B_0^n, \dots, \tilde B_n^n), \quad \tilde B_i^n(t) := \binom{n}{i}\left(\frac{t-a}{b-a}\right)^i\left(\frac{b-t}{b-a}\right)^{n-i}, \quad i = 0, \dots, n. \tag{14}$$

The following result generalizes Theorem 2 and Corollary 1 of Reference 6 to Bernstein bases on compact intervals $[a, b]$, with respect to the inner product

$$\langle f, g\rangle := \int_a^b (t-a)^\alpha (b-t)^\beta f(t)g(t)\, dt, \quad \alpha, \beta > -1. \tag{15}$$

Theorem 3. *Given $\alpha, \beta > -1$, the Gram matrix $\tilde M$ of the Bernstein basis $(\tilde B_0^n, \dots, \tilde B_n^n)$ in (14), with respect to the inner product (15) is STP. The bidiagonal factorization of $\tilde M$ can be represented by $BD(\tilde M) = (BD(\tilde M)_{i,j})_{1\le i,j\le n+1}$, such that*

$$BD(\tilde M)_{i,j} := \begin{cases} \dfrac{(n-i+2)(i+\alpha-1)(2n-i+\beta+3)}{(i-1)(2n-i-j+\beta+3)(2n-i-j+\beta+4)}, & \text{if } i > j, \\[2ex] (b-a)^{\alpha+\beta+1}\binom{n}{i-1}^2 \dfrac{\Gamma(i+\alpha)\Gamma(2n-2i+\beta+3)}{\Gamma(2n-i+\alpha+\beta+3)\binom{2n-i+\beta+2}{i-1}}, & \text{if } i = j, \\[2ex] \dfrac{(n-j+2)(j+\alpha-1)(2n-j+\beta+3)}{(j-1)(2n-i-j+\beta+3)(2n-i-j+\beta+4)}, & \text{if } i < j. \end{cases} \tag{16}$$

Moreover, if $\Gamma(\alpha+1)$, $\Gamma(\beta+1)$ and $\Gamma(\alpha+\beta+2)$ can be evaluated to HRA, $\tilde M$ and its inverse can be computed to HRA.

Proof. Using the change of variable $t = (\tilde t - a)/(b-a)$, it can be easily checked that

$$\tilde M_{i,j} = \int_a^b (\tilde t - a)^\alpha (b-\tilde t)^\beta \tilde B_{i-1}^n(\tilde t)\tilde B_{j-1}^n(\tilde t)\, d\tilde t = (b-a)^{\alpha+\beta+1}\int_0^1 t^\alpha(1-t)^\beta B_{i-1}^n(t)B_{j-1}^n(t)\, dt = (b-a)^{\alpha+\beta+1}\tilde M_{i,j},$$

for $1 \le i, j \le n+1$, where $M = (M_{i,j})_{1\le i,j\le n+1}$ is the Gram matrix (13) of the Bernstein basis on $[0, 1]$ with respect to the inner product (12). Then

$$\tilde{M} = (b-a)^{\alpha+\beta+1} M. \tag{17}$$

By Theorem 2 of Reference 6, M is STP and its bidiagonal factorization (4) can be described by $BD(M) = (BD(M)_{i,j})_{1\le i,j\le n+1}$ with

$$BD(M)_{i,j} := \begin{cases} \dfrac{(n-i+2)(i+\alpha-1)(2n-i+\beta+3)}{(i-1)(2n-i-j+\beta+3)(2n-i-j+\beta+4)}, & \text{if } i > j, \\ \dbinom{n}{i-1}^2 \dfrac{\Gamma(i+\alpha)\Gamma(2n-2i+\beta+3)}{\Gamma(2n-i+\alpha+\beta+3)\binom{2n-i+\beta+2}{i-1}}, & \text{if } i = j, \\ \dfrac{(n-j+2)(j+\alpha-1)(2n-j+\beta+3)}{(j-1)(2n-i-j+\beta+3)(2n-i-j+\beta+4)}, & \text{if } i < j. \end{cases} \tag{18}$$

Since $(b-a)^{\alpha+\beta+1} > 0$ and M is STP, we conclude that $\tilde{M}$ is also a STP matrix. Taking into account (18) and Lemma 1, we deduce that the entries of $BD(\tilde{M})$ are described by (16). Finally, let us observe that the off-diagonal entries of $BD(\tilde{M})$ are positive and can be computed to HRA. Moreover, if $\Gamma(\alpha+1)$, $\Gamma(\beta+1)$ and $\Gamma(\alpha+\beta+2)$ can be computed to HRA, using the following identity $\Gamma(x+n) = \Gamma(x)\prod_{k=0}^{n-1}(x+k)$, we deduce that $BD(\tilde{M})_{i,j}$ can also be computed to HRA. ∎

Note that, taking into account that $\Gamma(n+1) = n!$, for $n \in \mathbb{N} \cup \{0\}$, we can guarantee the computation to HRA of the Gram matrices of Bernstein bases corresponding to inner products (15) with $\alpha, \beta \in \mathbb{N} \cup \{0\}$. Furthermore, since $\Gamma(n+1/2) = (2n)!\sqrt{\pi}/(4^n n!)$, $n \in \mathbb{N}$, the conditions on α and β, provided in Theorem 3, also hold for other interesting cases such as $\alpha, \beta \in \{-1/2, 1/2\}$, corresponding to four Chebyshev-type weights.

Let us also observe that the diagonal entries $BD(\tilde{M})_{i,i}$, $1 \le i \le n+1$, in (16) can be easily computed since they satisfy

$$BD(\tilde{M})_{i+1,i+1} = \frac{(n-i+1)^2(i+\alpha)(2n-i+\alpha+\beta+2)(2n-i+\beta+2)}{i(2n-2i+\beta+1)(2n-2i+\beta+2)^2(2n-2i+\beta+3)} BD(\tilde{M})_{i,i},$$

for $1 \le i \le n$. In order to encourage the understanding of the numerical experimentation carried out in Section 5, we present the pseudocode of an algorithm for computing the bidiagonal decomposition (4) of the Gram Matrix of the Bernstein basis (14) to HRA, whenever $\Gamma(\alpha+1)$, $\Gamma(\beta+1)$ and $\Gamma(\alpha+\beta+2)$ can be evaluated to HRA. We can observe that Algorithm 1 has a computational cost of $O(n^2)$ arithmetic operations.

Finally, as a consequence of Theorem 3, we can deduce that the Gram matrix of any TP polynomial basis on a compact interval is also TP, as stated in the following corollary.

Corollary 1. *Let $(p_0^n, \dots, p_n^n)$ be a TP basis of the space $\mathbf{P}^n[a, b]$ and $(\tilde{B}_0^n, \dots, \tilde{B}_n^n)$ the Bernstein basis in (14). Then, the Gram matrix M_p of $(p_0^n, \dots, p_n^n)$ with respect to the inner product (15) is TP. Furthermore, if the change of basis matrix A, such that $(p_0^n, \dots, p_n^n)^T = A(\tilde{B}_0^n, \dots, \tilde{B}_n^n)^T$, can be computed to HRA, then for any $\alpha, \beta > -1$ such that $\Gamma(\alpha+1)$, $\Gamma(\beta+1)$ and $\Gamma(\alpha+\beta+2)$ can be evaluated to HRA, M_p and its inverse can be computed to HRA.*

Proof. Since the Bernstein basis $(\tilde{B}_0^n, \dots, \tilde{B}_n^n)$ is a B-basis of $\mathbf{P}^n[a, b]$, we can guarantee that the change of basis matrix A is TP (see Theorem 1). On the other hand, $M_p = A\tilde{M}A^T$, where $\tilde{M}$ is the Gram matrix of the Bernstein basis $(\tilde{B}_0^n, \dots, \tilde{B}_n^n)$. So, M_p is a TP matrix since it is the product of TP matrices (see Theorem 3.1 of Reference 26). Furthermore, if $\alpha, \beta > -1$ satisfy that $\Gamma(\alpha+1)$, $\Gamma(\beta+1)$ and $\Gamma(\alpha+\beta+2)$ can be computed to HRA, by Theorem 3, $BD(\tilde{M})$ can be computed to HRA. Finally, if the bidiagonal factorization of two nonsingular and TP matrices can be computed to HRA, using Algorithm 5.1 of Reference 3, we can also obtain to HRA the bidiagonal decomposition of the nonsingular and TP product matrix. ∎

Algorithm 1. Computation of the bidiagonal decomposition of the Gram matrix $\tilde{M}$ of the Bernstein basis (14)

Require: α, β, a, b, n
Ensure: $BDMB$ bidiagonal decomposition of $\tilde{M}$
$BDMB = \mathtt{zeros}(n+1)$
$c1 = \mathtt{zeros}(1, n+1)$
$c2 = \mathtt{zeros}(1, n+1)$
$c3 = \mathtt{zeros}(1, n+1)$
$c4 = \mathtt{zeros}(1, n+1)$
$BDMB(1,1) = \frac{(b-a)^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(2n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)}$
$c2(1) = 2n - 1 + \beta + 3$
  for $i = 2 : n+1$
    $c1(i) = i - 1$
    $c2(i) = c2(i-1) - 1$
    $c3(i) = c2(i) + 1$
    $c4(i) = (n - i + 2)(i + \alpha - 1)c2(i)$
    $BDMB(i,i) = \frac{c4(i)(n-i+2)(c2(i)+\alpha)}{c1(i)(c2(i)-i)(c3(i)-i)^2(c3(i)-i+1)} BDMB(i-1, i-1)$
    for j = 1:i-1
      $res = \frac{c4(i)}{c1(i)(c2(i)-j)(c3(i)-j)}$
      BDMB(i,j) = res
      BDMB(j,i) = res
    end
  end

4 | TOTAL POSITIVITY AND ACCURATE COMPUTATIONS WITH GRAM MATRICES OF SAID-BALL BASES

The Said-Ball basis $(s_0^n, \dots, s_n^n)$ of $\mathbf{P}^n[0,1]$ is defined by:

$$s_i^n(t) := \binom{\lfloor n/2 \rfloor + i}{i} t^i (1-t)^{\lfloor n/2 \rfloor + 1}, \quad i = 0, \dots, \lfloor (n-1)/2 \rfloor,$$
$$s_i^n(t) := \binom{\lfloor n/2 \rfloor + n - i}{n - i} t^{\lfloor n/2 \rfloor + 1} (1-t)^{n-i}, \quad i = \lfloor n/2 \rfloor + 1, \dots, n, \tag{19}$$

and, for an even n,

$$s_{n/2}^n(t) := \binom{n}{n/2} t^{n/2} (1-t)^{n/2}, \tag{20}$$

where $\lfloor a \rfloor$ denotes the greatest positive integer less than or equal to a, for a given $a > 0$.

In Theorem 1 of Reference 20 it was proved that the Said-Ball basis $(s_0^n, \dots, s_n^n)$ is a NTP basis of $\mathbf{P}^n[0,1]$ for odd degrees n. Later, in Proposition 3 of Reference 34, this property was also proved for even degrees. Taking into account that the Bernstein basis on $[0,1]$ (see (11)) is the normalized B-basis of $\mathbf{P}^n[0,1]$, we can immediately deduce from Theorem 1 that the change of basis matrix A such that

$$(s_0^n, \dots, s_n^n)^T = A(B_0^n, \dots, B_n^n)^T \tag{21}$$

is TP. Observe that, since both bases are normalized, we also have that A^T is also stochastic. Using formula (7) of Reference 19, A can be described as $A = (a_{i,j})_{1 \le i,j \le n+1}$, with

$$a_{i,j} = \begin{cases} \binom{k+i-1}{i-1} \binom{k-i+1}{j-i} / \binom{2k+1}{j-1}, & \text{if } 1 \le i \le j \le k+1, \\ \binom{3k-i+2}{2k-i+2} \binom{i-k-2}{i-j} / \binom{2k+1}{j-1}, & \text{if } k+2 \le j \le i \le 2k+2, \\ 0, & \text{otherwise,} \end{cases} \tag{22}$$

for odd degree $n = 2k+1$, $k \in \mathbb{N}$ and

$$a_{i,j} = \begin{cases} \binom{k+i-1}{i-1}\binom{k-i}{j-i} / \binom{2k}{j-1}, & \text{if } 1 \le i \le j \le k, \\ \binom{3k-i+1}{2k-i+1}\binom{i-k-2}{i-j} / \binom{2k}{j-1}, & \text{if } k+2 \le j \le i \le 2k+1, \\ 1, & \text{if } i = j = k+1, \\ 0, & \text{otherwise,} \end{cases} \tag{23}$$

for even degree $n = 2k$, $k \in \mathbb{N}$.

A corner cutting algorithm for obtaining the Bézier polygon of a polynomial curve from its control polygon with respect to the generalised Ball basis is constructed in section 3 of Reference 20. This corner cutting provides a factorization of the stochastic and TP matrix A^T in terms of TP, bidiagonal and stochastic matrices. In contrast, the following result provides the pivots and the multipliers of the NE of A and then its bidiagonal factorization (9). As we shall see, the computation of this new factorization satisfies the NIC condition and then will lead to computations to HRA when considering Gram matrices of Said-Ball bases.

Theorem 4. *For $n = 2k+1$, $k \in \mathbb{N}$, the $(2k+2) \times (2k+2)$ change of basis matrix A satisfying (21), whose entries are described by (22), admits a factorization of the form (4) such that*

$$A = F_{2k+1}F_{2k} \cdots F_1 D G_1 \cdots G_{2k} G_{2k+1}, \tag{24}$$

where F_i and G_i, $i = 1, \dots, 2k+1$, are the lower and upper triangular bidiagonal matrices given by (5) and $D = diag\left(p_{1,1}, \dots, p_{2k+2,2k+2}\right)$. This factorization can be computed to HRA. The off-diagonal entries $m_{i,j}$ and $\widetilde{m}_{i,j}$, $1 \le j < i \le 2k+2$, are given by

$$\begin{aligned} m_{i,j} &= 0, \quad 1 \le j \le k+1, \qquad m_{i,j} = \frac{2k-i+3}{3k-i+3}, \quad k+2 \le j \le 2k+1, \\ \widetilde{m}_{i,j} &= \frac{k-i+2}{2k-i+3}, \quad 2 \le i \le k+1, \qquad \widetilde{m}_{i,j} = 0, \quad k+2 \le i \le 2k+2. \end{aligned} \tag{25}$$

Moreover, the diagonal entries $p_{i,i}$, $1 \le i \le 2k+2$, are given by

$$p_{i,i} = \binom{k+i-1}{i-1} \Big/ \binom{2k+1}{i-1}, \quad 1 \le i \le k+1, \quad p_{i,i} = \binom{3k-i+2}{2k-i+2} \Big/ \binom{2k+1}{i-1}, \quad k+2 \le i \le 2k+2, \tag{26}$$

and then satisfy

$$\begin{aligned} p_{1,1} &= 1, \quad p_{i+1,i+1} = \frac{k+i}{2k-i+2}\, p_{i,i}, \quad i = 1, 2, \dots, k, \\ p_{2k+2,2k+2} &= 1, \quad p_{i,i} = \frac{3k-i+2}{i}\, p_{i+1,i+1}, \quad i = 2k+1, 2k, \dots, k+2. \end{aligned} \tag{27}$$

Proof. Let us observe that A can be seen as a diagonal block matrix whose main-diagonal blocks are $(k+1) \times (k+1)$ triangular matrices and the off-diagonal blocks are zero matrices. In particular, the first diagonal block $A[1, \dots, k+1] = (a_{i,j})_{1 \le i,j \le k+1}$ is an upper triangular matrix and the second diagonal block $A[k+2, \dots, 2k+2] = (a_{i,j})_{k+2 \le i,j \le 2k+2}$ is a lower triangular matrix.

Let $A^{(1)} := A$ and $A^{(m)} = (a_{ij}^{(m)})_{1 \le i,j \le n+1}$, $m = 2, \dots, 2k+2$, be the matrices obtained after $m-1$ steps of the NE process of A. First, taking into account that $a_{i,j} = 0$ if $j < i$ and $1 \le j \le k+1$, we deduce that $m_{i,j} = 0$, for $1 \le j \le k+1$, and

$$A = A^{(1)} = \cdots = A^{(k+2)}. \tag{28}$$

Now, by induction on $r \in \{2, \dots, k+2\}$, we are going to see that, for $k+2 \le j \le 2k+2$ and $j \le i$,

$$a_{i,j}^{(k+r)} = \binom{3k-i+2}{2k-i+2}\binom{i-k-r}{i-j} \Big/ \binom{2k+1}{j-1}. \tag{29}$$

By (28) and (22), identities (29) clearly hold for $r = 2$. Let us now suppose that (29) holds for some $r \in \{2, \dots, k+1\}$. Then, we have that

$$a_{i,k+r}^{(k+r)} / a_{i-1,k+r}^{(k+r)} = \frac{2k-i+3}{3k-i+3}. \tag{30}$$

Since $a_{i,j}^{(k+r+1)} = a_{i,j}^{(k+r)} - \left(a_{i,k+r}^{(k+r)} / a_{i-1,k+r}^{(k+r)} \right) a_{i-1,j}^{(k+r)}$, taking into account (29) and (30), we can write

$$\binom{2k+1}{j-1} a_{i,j}^{(k+r+1)} = \binom{3k-i+2}{2k-i+2} \binom{i-k-r}{i-j} - \frac{2k-i+3}{3k-i+3} \binom{3k-i+3}{2k-i+3} \binom{i-k-r-1}{i-j-1}.$$

Using the combinatorial identities $\frac{q}{p}\binom{p}{q} = \binom{p-1}{q-1}$ and $\binom{p}{q} - \binom{p-1}{q-1} = \binom{p-1}{q}$, for $p, q \in \mathbb{N}$, $p \geq q$, we have

$$\binom{2k+1}{j-1} a_{i,j}^{(k+r+1)} = \binom{3k-i+2}{2k-i+2} \left(\binom{i-k-r}{i-j} - \binom{i-k-r-1}{i-j-1} \right) = \binom{3k-i+2}{2k-i+2} \binom{i-k-r-1}{i-j},$$

confirming (29) for $r+1$.

By (2) and (29), we can easily deduce that the pivots $p_{i,j}$ of the NE of A satisfy

$$p_{i,j} = a_{i,j}^{(j)} = \binom{3k-i+2}{2k-i+2} \Big/ \binom{2k+1}{j-1}, \tag{31}$$

for $k+2 \leq j \leq 2k+2$ and $j \leq i$. Then, for the particular case $i = j$,

$$p_{i,i} = \binom{3k-i+2}{2k-i+2} \Big/ \binom{2k+1}{i-1}, \quad k+2 \leq i \leq 2k+2. \tag{32}$$

It can be easily checked that $p_{2k+2,2k+2} = 1$ and $p_{i,i}/p_{i+1,i+1} = \frac{3k-i+2}{i}$. So, equalities (27) hold for the diagonal pivots $p_{i,i}$, $k+2 \leq i \leq 2k+2$.

Finally, using (3) and (31), $m_{i,j} = p_{i,j}/p_{i-1,j} = \frac{2k-i+3}{3k-i+3}$, $k+2 \leq j < 2k+1$. Then, equalities (25) hold for the multipliers $m_{i,j}$, $k+2 \leq j \leq 2k+1$.

Now, for the computation of $\widetilde{m}_{i,j}$, let $\tilde{A}^{(1)} := (A^{(n+1)})^T$, and $\tilde{A}^{(r)} = (\tilde{a}_{ij}^{(r)})_{1 \leq i,j \leq n+1}$, $r = 2, \dots, n+1$, be the matrices obtained after $r-1$ steps of the NE process of $\tilde{A}^{(1)}$. Taking into account that $\tilde{a}_{i,j}^{(1)} = 0$, for $k+2 \leq i \leq 2k+2$, and $j < i$, we get $\widetilde{m}_{i,j} = 0$, for $k+2 \leq i \leq 2k+2$.

Using induction, we are going to see that, for $1 \leq j \leq i \leq k+1$, we have

$$\tilde{a}_{i,j}^{(r)} = \binom{j-1}{r-1} \binom{k+j-1}{j-1} \binom{k-j+1}{i-j} \Big/ \left(\binom{i-1}{r-1} \binom{2k+1}{i-1} \right). \tag{33}$$

Taking into account that $\tilde{A}^{(1)}[1, \dots, k+1] = (A[1, \dots, k+1])^T$, that is,

$$\tilde{a}_{i,j}^{(1)} = \binom{k+j-1}{j-1} \binom{k-j+1}{i-j} \Big/ \binom{2k+1}{i-1}, \quad 1 \leq j \leq i \leq k+1, \tag{34}$$

we can check that identities (33) clearly hold for $r = 1$. If (33) holds for $r \in \{2, \dots, k+1\}$, we have that

$$\tilde{a}_{i,r}^{(r)} / \tilde{a}_{i-1,r}^{(r)} = \frac{k-i+2}{2k-i+3}. \tag{35}$$

Since $\tilde{a}_{i,j}^{(r+1)} = \tilde{a}_{i,j}^{(r)} - \left(a_{i,r}^{(r)} / a_{i-1,r}^{(r)} \right) \tilde{a}_{i-1,j}^{(r)}$, taking into account (33), (35), and using the combinatorial identities $(p-q)\binom{p}{q} = (q+1)\binom{p}{q+1}$, $(p+1)\binom{p}{q} = (p-q+1)\binom{p+1}{q}$, for $p, q \in \mathbb{N}$, with $p \geq q$, we can deduce that

$$\binom{i-1}{r-1} \binom{2k+1}{i-1} \tilde{a}_{i,j}^{(r+1)} = \frac{j-r}{i-r} \binom{j-1}{r-1} \binom{k+j-1}{j-1} \binom{k-j+1}{i-j},$$

and so, (33) for $r+1$. Now, by (2) and (33), we deduce that the pivots $\tilde{p}_{i,j}$ satisfy

$$\tilde{p}_{i,j} = \tilde{a}_{i,j}^{(j)} = \binom{k+j-1}{j-1}\binom{k-j+1}{i-j} \Big/ \left(\binom{i-1}{j-1}\binom{2k+1}{i-1}\right), \tag{36}$$

for $1 \le i \le k+1$ and $j \le i$. Then, for the particular case $i = j$,

$$\widetilde{p}_{i,i} = \binom{k+i-1}{i-1} \Big/ \binom{2k+1}{i-1}, \quad 1 \le i \le k+1. \tag{37}$$

It can be easily checked that $\widetilde{p}_{1,1} = 1$, and $\widetilde{p}_{i+1,i+1}/\widetilde{p}_{i,i} = \frac{k+i}{2k-i+2}$, $i = 1, \dots, k$, and equalities (27) for the diagonal pivots $p_{i,i}$, $k+2 \le i \le 2k+2$, hold.

Finally, using (3) and (36), $\tilde{m}_{i,j} = \tilde{p}_{i,j}/\tilde{p}_{i-1,j} = \frac{k-i+2}{2k-i+3}$, $1 \le j < k+1$, and the equalities (25) hold for the multipliers $\tilde{m}_{i,j}$, $2 \le i \le k+1$.

In order to conclude the proof, let us observe that the pivots $p_{i,i}$ and multipliers $m_{i,j}$, $\widetilde{m}_{i,j}$ satisfying (27) and (25), respectively, can clearly be computed to HRA and so, the described bidiagonal factorization (4). ■

Let us remark that since the multipliers of the NE of the matrix A satisfy $m_{i,j} = 0$, $1 \le j \le k+1$, and $\widetilde{m}_{i,j} = 0$, $k+2 \le i \le 2k+2$, we have that $F_{2k+1} = F_{2k} = \cdots = F_{k+1} = I_{2k+2}$, and $G_1 = G_2 = \cdots = G_{k+1} = I_{2k+2}$, where I_{2k+2} denotes de $(2k+2) \times (2k+2)$ identity matrix. Then the bidiagonal factorization (4) reduces to

$$A = F_k \cdots F_1 D G_1 \cdots G_k,$$

with pivots $p_{i,i}$ and multipliers $m_{i,j}$, $\widetilde{m}_{i,j}$ satisfying (27) and (25), respectively.

Using Theorem 4, for an odd degree $n = 2k+1$, the bidiagonal factorization (4) of the change of basis matrix A described by (22) can be represented by means of the $(2k+2) \times (2k+2)$ matrix $BD(A) = (BD(A)_{i,j})_{1 \le i,j \le 2k+2}$ such that

$$BD(A)_{i,j} := \begin{cases} \binom{k+i-1}{i-1} \Big/ \binom{2k+1}{i-1}, & \text{if } 1 \le i = j \le k+1, \\ (k-j+2)/(2k-j+3), & \text{if } 1 \le i < j \le k+1, \\ \binom{3k-i+2}{2k-i+2} \Big/ \binom{2k+1}{i-1}, & \text{if } k+2 \le i = j \le 2k+2, \\ (2k-i+3)/(3k-i+3), & \text{if } k+2 \le j < i \le 2k+2, \\ 0, & \text{otherwise.} \end{cases} \tag{38}$$

The bidiagonal factorization (9) of the TP change of basis matrix $A = (a_{i,j})_{1 \le i,j \le n+1}$ for even $n = 2k$ can be obtained using a similar reasoning to that of Theorem 4 and taking into account that, in this case, A can be seen as a diagonal block matrix whose main-diagonal blocks are triangular matrices and the off-diagonal blocks are zero matrices. In particular, the first diagonal block $A[1, \dots, k] = (a_{i,j})_{1 \le i,j \le k}$ is an upper triangular matrix, $A[k+1] = a_{k+1,k+1} = 1$ and $A[k+2, \dots, 2k+1] = (a_{i,j})_{k+2 \le i,j \le 2k+1}$ is a lower triangular matrix.

Theorem 5. *For $n = 2k$, $k \in \mathbb{N}$, the change of basis matrix A satisfying (21) and entries described by (23) admits a factorization of the form (4) such that*

$$A = F_{2k}F_{2k-1} \cdots F_1 D G_1 \cdots G_{2k-1}G_{2k},$$

where F_i and G_i, $i = 1, \dots, 2k$, are the lower and upper triangular bidiagonal matrices given by (5) and $D = \operatorname{diag}(p_{1,1}, \dots, p_{2k+1,2k+1})$. This factorization can be computed to HRA. The off-diagonal entries $m_{i,j}$ and $\widetilde{m}_{i,j}$, $1 \le j < i \le 2k+1$, are given by

$$\begin{aligned} m_{i,j} &= 0, \quad 1 \le j \le k+1, \qquad m_{i,j} = \frac{2k-i+2}{3k-i+2}, \quad k+2 \le j \le 2k, \\ \widetilde{m}_{i,j} &= \frac{k-i+1}{2k-i+2}, \quad 2 \le i \le k, \qquad \widetilde{m}_{i,j} = 0, \quad k+1 \le i \le 2k+1. \end{aligned} \tag{39}$$

Moreover, the diagonal entries $p_{i,i}$, $1 \le i \le n+1$ are given by

$$p_{i,i} = \binom{k+i-1}{i-1} \Big/ \binom{2k}{i-1}, \ 1 \le i \le k, \quad p_{k+1,k+1} = 1, \quad p_{i,i} = \binom{3k-i+1}{2k-i+1} \Big/ \binom{2k}{i-1}, \quad k+2 \le i \le 2k+1, \tag{40}$$

and then satisfy

$$
\begin{aligned}
&p_{1,1} = 1, \quad p_{i+1,i+1} = \frac{k+i}{2k-i+1} p_{i,i}, \quad i = 1, \dots, k,\\
&p_{k+1,k+1} = 1,\\
&p_{2k+1,2k+1} = 1, \quad p_{i,i} = \frac{3k-i+1}{i} p_{i+1,i+1}, \quad i = 2k+1, \dots, k+2.
\end{aligned}
$$

It can also be deduced that for an even degree $n = 2k$, the bidiagonal factorization (4) reduces to

$$A = F_{k-1} \cdots F_1 D G_1 \cdots G_{k-1},$$

with pivots $p_{i,i}$ and multipliers $m_{i,j}$, $\widetilde{m}_{i,j}$ satisfying (40) and (39), respectively.

Using Theorem 5, for an even degree $n = 2k$, the bidiagonal factorization (4) of the change of basis matrix A described by (22) can be represented by means of the $(2k+1) \times (2k+1)$ matrix $BD(A) = (BD(A)_{i,j})_{1 \le i,j \le 2k+1}$ such that

$$
BD(A)_{i,j} := \begin{cases}
\binom{k+i-1}{i-1} / \binom{2k}{i-1}, & \text{if } 1 \le i = j \le k,\\
(k-j+1)/(2k-j+2), & \text{if } 1 \le i < j \le k,\\
1, & \text{if } i = j = k+1,\\
\binom{3k-i+1}{2k-i+1} / \binom{2k}{i-1}, & \text{if } k+2 \le i = j \le 2k+1,\\
(2k-i+2)/(3k-i+2), & \text{if } k+2 \le j < i \le 2k+1,\\
0, & \text{otherwise}.
\end{cases} \tag{41}
$$

As a direct consequence of Theorems 4 and 5, we can guarantee that the TP change of basis matrix A such that $(s_0^n, \dots, s_n^n)^T = A(B_0^n, \dots, B_n^n)^T$ can be computed to HRA. Then, taking into account that the Gram matrices M_B and M of the Bernstein and Said-Ball bases, respectively, are related by $M = AM_BA^T$, Corollary 1 guarantees that M is TP and provides the conditions so that it can be computed to HRA, as stated in the following result.

Algorithm 2. Computation of the bidiagonal decomposition of the matrix A in (22) to HRA

Require: odd degree n
Ensure: *BDAO* bidiagonal decomposition of A for an odd degree n to HRA

```
BDAO = zeros(n + 1)
k = (n − 1)/2
BDAO(1, 1) = 1
BDAO(2k + 2, 2k + 2) = 1
  for i = 2 : k + 1
    BDAO(i, i) = (k+i−1)/(2k−i+3) BDAO(i − 1, i − 1)
    BDAO(n + 2 − i, n + 2 − i) = BDAO(i, i)
  end
  for j = 2 : k + 1
    res = (k−j+2)/(2k−j+3)
    for i = 1 : min(k + 1, j − 1)
    BDAO(i, j) = res
    end
  end
  for i = k + 2 : 2k + 2
    res = (2k−i+3)/(3k−i+3)
    for j = k + 2 : i − 1
      BDAO(i, j) = res
    end
  end
```

Theorem 6. *The Gram matrix M of the Said-Ball basis of the space of polynomials $\mathbf{P}^n([0,1])$ with respect to the inner product (12) is TP. Furthermore, for any $\alpha, \beta > -1$ such that $\Gamma(\alpha+1)$, $\Gamma(\beta+1)$ and $\Gamma(\alpha+\beta+2)$ can be evaluated to HRA, M and its inverse can be computed to HRA.*

Now, we provide the pseudocode of Algorithms 2 and 3 for the computation to HRA of the bidiagonal decomposition of the matrix A such that $(s_0^n, \ldots, s_n^n)^T = A(B_0^n, \ldots, B_n^n)^T$ for an odd and even degree n, respectively. We can observe that both algorithms have a computational cost of $O(n)$ arithmetic operations.

Algorithm 3. Computation of the bidiagonal decomposition of the matrix A in (23) to HRA

Require: even degree n
Ensure: $BDAE$ bidiagonal decomposition of A for an even degree n to HRA
$BDAE = zeros(n+1)$
$k = n/2$
$BDAE(1,1) = 1$
$BDAE(k+1, k+1) = 1$
$BDAE(2k+1, 2k+1) = 1$
  for $i = 2 : k$
    $BDAE(i,i) = \frac{k+i-1}{2k-i+2} BDAE(i-1, i-1)$
    $BDAE(n+2-i, n+2-i) = BDAE(i,i)$
  end
  for $j = 2 : k+1$
    $res = \frac{k-j+1}{2k-j+2}$
    for $i = 1 : min(k+1, j-1)$
    $BDAE(i,j) = res$
    end
  end
  for $i = k+2 : 2k+1$
    $res = \frac{2k-i+2}{3k-i+2}$
    for $j = k+2 : i-1$
      $BDAE(i,j) = res$
    end
  end

We also present the pseudocode of Algorithm 4 for computing the bidiagonal decomposition (4) of a Said-Ball Gram matrix M. It requires the matrices computed by Algorithms 1–3, and calls the Matlab function `TNProduct` available in.[25] Let us recall that, given $A = BD(F)$ and $B = BD(G)$ to HRA, `TNProduct`(A, B) computes $BD(FG)$ to HRA. Its computational cost is $O(n^3)$.

Section 5 illustrates the accurate results obtained by using the proposed algorithms.

5 | NUMERICAL EXPERIMENTS

We have considered Gram matrices of Said-Ball bases with dimension $n+1 = 10, 11, \ldots, 24, 25$. We have taken values α, β satisfying conditions provided by Theorem 6 to guarantee that these matrices are TP and its bidiagonal decomposition (4) can be computed to HRA.

To test our algorithms, we have solved several fundamental problems in Numerical Linear Algebra. In order to analyze the accuracy of the results, all the solutions have also been calculated in Mathematica using a 100-digit arithmetic. In order to compute the relative errors in the computations, the values provided by Mathematica have been considered as the exact solution of the algebraic problem.

In Figure 1, the 2-norm condition number of the considered Gram matrices is depicted. For a matrix M, this conditioning has been obtained by means of the Mathematica command `Norm[M,2]` · `Norm[Inverse[M],2]`. We shall see that, due to the ill-conditioning of the matrices, traditional methods do not obtain accurate solutions. The reason is that these algorithms can suffer from inaccurate cancelations. On the contrary, the numerical results will illustrate the high

Algorithm 4. Computation of the bidiagonal decomposition of the Gram matrix M of the Said-Ball basis (19)

Require: α, β, n
Ensure: BDM bidiagonal decomposition of M
$a = 0$
$b = 1$
$BDMB = \texttt{zeros}(n+1)$
$BDAO = \texttt{zeros}(n+1)$
$BDAE = \texttt{zeros}(n+1)$
$B1 = \texttt{zeros}(n+1)$
$BDM = \texttt{zeros}(n+1)$
$BDMB = \texttt{BDMB}(\alpha, \beta, a, b, n)$
  If n is odd
    $BDAO = \texttt{BDAO}(n)$
    $B1 = \texttt{TNProduct}(BDAO, BDMB)$
    $BDM = \texttt{TNProduct}(B1, (BDAO)^T)$
  else
    $BDAE = \texttt{BDAE}(n)$
$B1 = \texttt{TNProduct}(BDAE, BDMB)$
$BDM = \texttt{TNProduct}(B1, (BDAE)^T)$

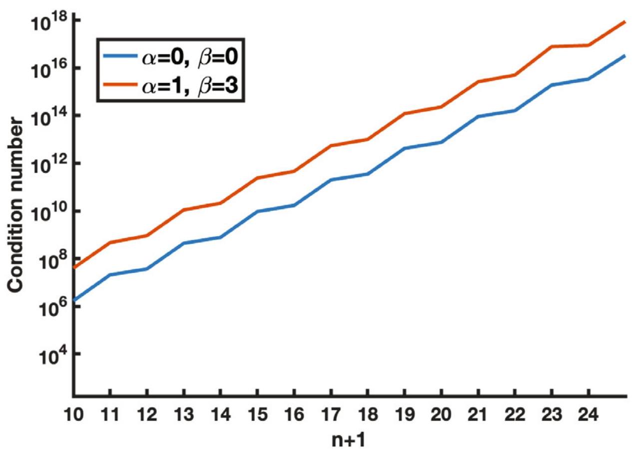

FIGURE 1 The 2-norm condition number of the Gram matrices M of the Said-Ball bases.

Algorithm 5. Computation of the singular values of M to HRA

Require: α, β, n
Ensure: $\mathbf{v} \in \mathbb{R}^{n+1}$ containing the singular values of M
$BDM = \texttt{zeros}(n+1)$
$BDM = \texttt{BDM}(\alpha, \beta, n)$
$\mathbf{v} = \texttt{TNSingularValues}(BDM)$

accuracy obtained when using the bidiagonal decomposition deduced in this article with the Matlab functions available in Reference 25.

Computation of singular values. Given $B = BD(A)$ to HRA, the Matlab function `TNSingularValues(B)` available in Reference 25 computes the singular values of a matrix A to HRA. Its computational cost is $O(n^3)$ (see Reference 24).

Algorithm 5 uses the bidiagonal decomposition provided by Algorithm 4 to compute the singular values of a Said-Ball Gram matrix M to HRA.

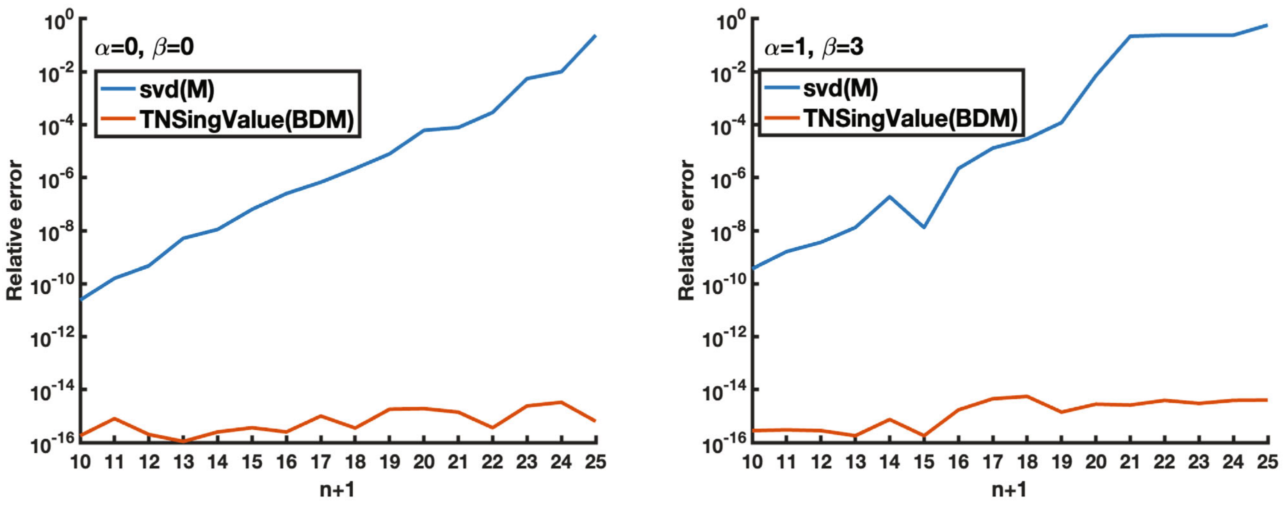

FIGURE 2 Relative error in the approximations to the lowest singular value of M.

Algorithm 6. Computation of the inverse of M to HRA

Require: α, β, n
Ensure: A matrix $MInv$ which is the inverse of M
$BDM = \texttt{zeros}(n+1)$
$MInv = \texttt{zeros}(n+1)$
$BDM = \texttt{BDM}(\alpha, \beta, n)$
$MInv = \texttt{TNInverseExpand}(BDM)$

For all the considered Gram matrices, we have compared the lowest singular value obtained using Algorithm 5 and the Matlab command `svd`. In this context, the values provided by Mathematica using 100-digit arithmetic have been considered as the exact solution of the algebraic problem and the relative error e of each approximation has been computed as $e := |a - \tilde{a}|/|a|$, where a denotes the singular value computed in Mathematica and $\tilde{a}$ the singular value computed in Matlab.

In Figure 2, the relative error of the approximations is shown. Looking at the results, we notice that our approach computes accurately the lowest singular value regardless the 2-norm condition number of the considered Gram matrices. In contrast, the Matlab command `svd` returns results that are not accurate at all.

Computation of the inverse matrix. Given $B = BD(A)$ to HRA, the Matlab function `TNInverseExpand`(B) available in Reference 25 returns A^{-1} to HRA, requiring $O(n^2)$ arithmetic operations (see Reference 33).

Algorithm 6 uses the bidiagonal decomposition provided by Algorithm 4 to compute the inverse of a Said-Ball Gram matrix M to HRA.

We have compared the inverses obtained using Algorithm 6 and the Matlab command `inv`. To look over the accuracy of these two methods we have compared both Matlab approximations with the inverse matrix A^{-1} computed by Mathematica using 100-digit arithmetic, taking into account the formula $e = \|A^{-1} - \widetilde{A}^{-1}\|_2/\|A^{-1}\|_2$ for the corresponding relative error, where $\widetilde{A}^{-1}$ denotes the inverse computed in Matlab.

The achieved relative errors are shown in Figure 3. We notice that our algorithm provides accurate results in contrast to the inaccurate results provided by the Matlab command `inv`.

Computation of solution of linear system $Mc = d$. Given $B = BD(A)$ to HRA and a vector d with alternating signs, the Matlab function `TNSolve`(B, d) available in Reference 25 returns the solution of the linear system $Ac = d$ to HRA. It requires $O(n^2)$ arithmetic operations (see Reference 25).

Algorithm 7 uses the bidiagonal decomposition of a Said-Ball Gram matrix M, provided by Algorithm 4, to compute the solution of the linear system $Mc = d$ to HRA.

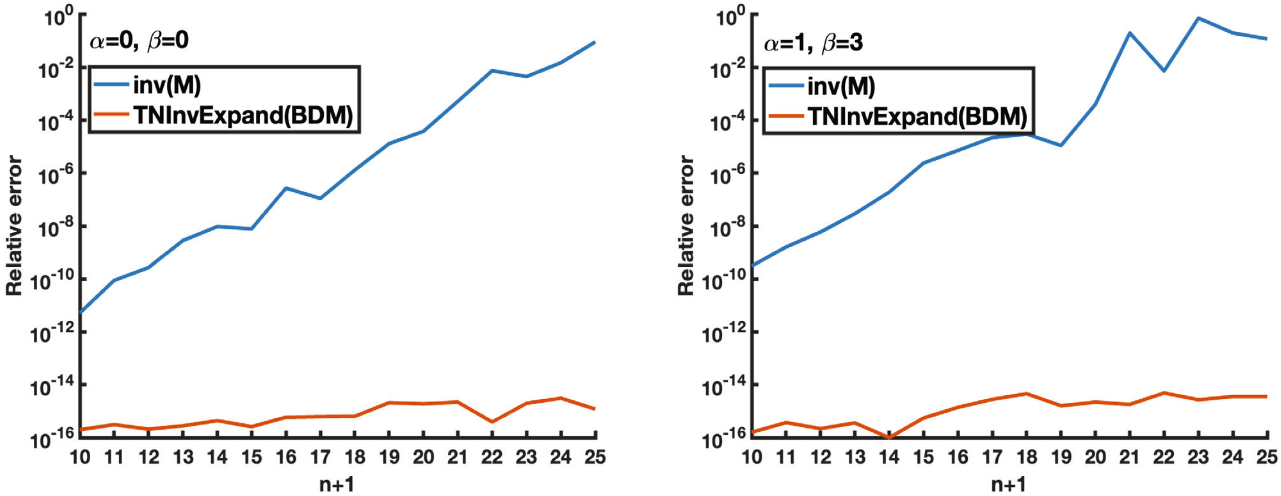

FIGURE 3 Relative error in the approximations to the inverse of M.

Algorithm 7. Computation of the solution of the linear system of $Mc = d$ to HRA

Require: $\alpha, \beta, n, d \in \mathbb{R}^{n+1}$
Ensure: $c \in \mathbb{R}^{n+1}$ containing the solution of the linear system $Mc = d$
$BDM = \texttt{zeros}(n+1)$
$c = \texttt{zeros}(n+1, 1)$
$BDM = \texttt{BDM}(\alpha, \beta, n)$
$c = \texttt{TNSolve}(BDM, d)$

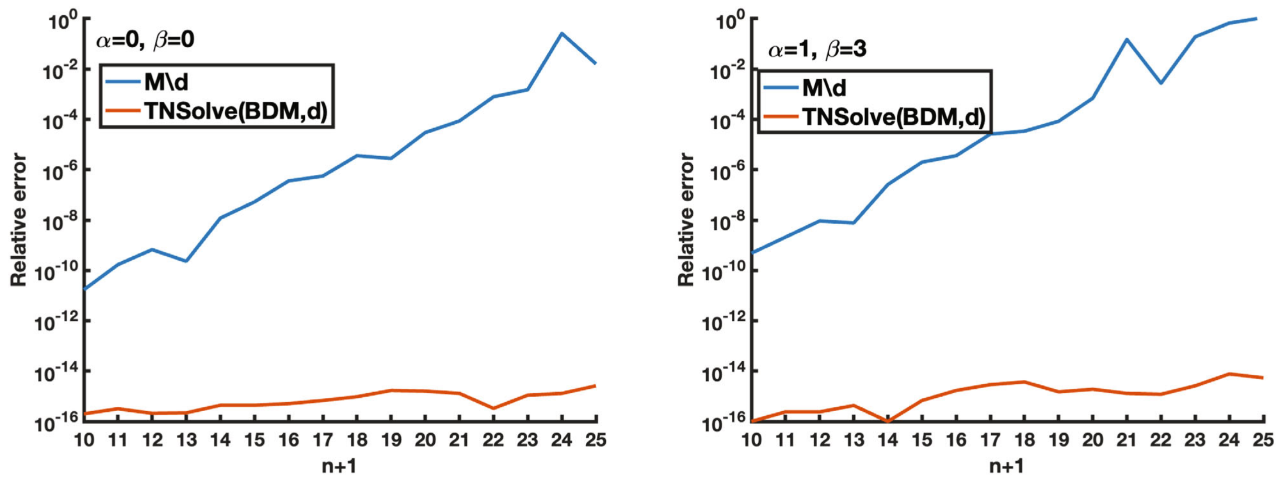

FIGURE 4 Relative error of the approximations to the solution of the linear systems $Mc = d$, where $d = ((-1)^{i+1} d_i)_{1 \le i \le n+1}$ and d_i, $i = 1, \ldots, n+1$, are random nonnegative integer values.

For all considered Gram matrices, we have compared the solutions of the different linear systems $Mc = d$ obtained using Algorithm 7 and the Matlab command `M\d`. The solution provided by Mathematica using 100-digit arithmetic has been considered as the exact solution c. Then, we have computed in Mathematica the relative error of the computed approximation with Matlab $\tilde{c}$, taking into account the formula $e = \|c - \tilde{c}\|_2 / \|c\|_2$.

In Figure 4 we show the relative errors when solving the aforementioned linear systems for different values of n. We notice that the proposed algorithm preserves the accuracy in spite of the dimension of the problem in contrast with the results obtained with the Matlab command \.

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CONFLICT OF INTEREST STATEMENT

This study does not have any conflicts to disclose.

DATA AVAILABILITY STATEMENT

The Matlab codes employed to run the numerical experiments are available upon request.

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