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New series expansions for the $\mathcal{H}$-function of communication theory

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ABSTRACT
The $\mathcal{H}$-function of communication theory plays an important role in the error rate analysis in digital communication with the presence of additive white Gaussian noise (AWGN) and generalized multipath fading conditions. In this paper we investigate several convergent and/or asymptotic expansions of $\mathcal{H}_p(z,b,\eta)$ for some limiting values of their variables and parameters: large values of z, large values of p, small values of η, and values of $b \to 1$. We provide explicit and/or recursive algorithms for the computation of the coefficients of the expansions. Some numerical examples illustrate the accuracy of the approximations.

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1. Introduction

The $\mathcal{H}$-function of communication theory is defined by means of the integral [1, Equation (4.12)]:

$$\mathcal{H}_p(z,b,\eta) := \frac{(1-b^2)^p}{2\pi}\int_0^\eta \frac{1}{1+x^2}\frac{1}{(1+b^2x^2)^p}\exp\left(-\frac{z^2}{2}\frac{1+x^2}{1+b^2x^2}\right)\mathrm{d}x, \qquad (1.1)$$

for $p \ge 0$, $z \ge 0$, $\eta \ge 0$, and $0 \le b^2 \le 1$. It may be equivalently written in the form [2, Equation (2.3)]:

$$\mathcal{H}_p(z,b,\eta) := \frac{(1-b^2)^p}{2\pi}\exp\left(-\frac{z^2}{2b^2}\right) \times \int_0^\eta \frac{1}{1+x^2}\frac{1}{(1+b^2x^2)^p}\exp\left(\frac{z^2}{2b^2}\frac{1-b^2}{1+b^2x^2}\right)\mathrm{d}x. \qquad (1.2)$$

This function arises in the error rate analysis in digital communication with the presence of additive white Gaussian noise (AWGN) and generalized multipath fading conditions (see [1,2]). Chapter 4 of monograph [1] is completely devoted to the study of this function. In [1, Section 4.3], the author provides basic relations and algebraic properties of the $\mathcal{H}$-function. For example, recurrence relations at $b \neq 1$ and recurrence relations at $z = 0$ and

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$p = n$ and $p = n + 1/2$ for $n \in \mathbb{N}$ are given. Expansions in terms of other special functions are also derived: i) a Fourier series in terms of the generalized Laguerre polynomials $L_n^p(z)$ [3],

$$\mathcal{H}_p(z,b,\varphi) = \frac{(1-b^2)^p}{4\pi} \exp\left(-\frac{z^2}{2}\right) \sum_{n=0}^{\infty} (1-b^2)^n L_n^{p-1}\left(\frac{z^2}{2}\right) B_\varphi\left(n+\frac{1}{2}, p+\frac{1}{2}\right),$$

where $\varphi := \arctan(\eta)$ and $B_z(a,b)$ is the incomplete beta function [4, Equation (8.17.1)]; ii) a series expansion in terms of the incomplete modified Bessel function $I_n(z,\psi)$ [1, p. 368] at $n \in \mathbb{N} \cup \{0\}$ and $b \neq 0$,

$$\mathcal{H}_p(z,b,\varphi) = \frac{(1-b^2)^n}{2^{2n+2}} \exp\left(-\frac{1+b^2}{4b^2} z^2\right) R_n(z,b,\varphi),$$

where

$$\begin{aligned} R_n(z,b,\varphi) := &\sum_{k=0}^{n-1} \binom{2n}{k} \sum_{m=0}^{\infty} \varepsilon_m \left(\frac{1-b}{1+b}\right)^m \left[I_{n-k-m}\left(\frac{1-b^2}{4b^2} z^2, 2\varphi\right) \right. \\ &\left. + I_{n-k+m}\left(\frac{1-b^2}{4b^2} z^2, 2\varphi\right)\right] + \binom{2n}{n} \sum_{m=0}^{\infty} \varepsilon_m \left(\frac{1-b}{1+b}\right)^m \\ &\times I_m\left(\frac{1-b^2}{4b^2} z^2, 2\varphi\right), \end{aligned}$$

with $\varphi := \arctan(b\eta)$, $\varepsilon_0 = 1$ and $\varepsilon_m = 2$ for $m > 0$. It is indicated in [1] that these series are the starting point to establish certain connections between the $\mathcal{H}$-function and other special functions, such as the generalized Q-function of Marcum, the Owen T-function, the Gaussian and Nicholson functions and the generalized circular function. The author also obtains limiting cases of some of the variables and parameters and their application in problems of calculation of error probability; infinite series containing $\mathcal{H}$-functions; upper and lower bounds, among others. In [2], new relations for this function are considered, including differentiation formulas with respect to z, η and b and integration formulas with respect to z; integral representations, recurrence relations or generating functions. With regard to series expansions, the authors provide the following expansion in powers of z [2, p. 4] (convenient for small z):

$$\begin{aligned} \mathcal{H}_p(z,b,\eta) = &\frac{(1-b^2)^p \eta}{2\pi\sqrt{1+b^2\eta^2}} \sum_{k=0}^{\infty} \frac{(-1/2)^k}{k!} F_1 \\ &\times \left(\frac{1}{2}; 1-k, \frac{1}{2}-p; \frac{3}{2}; \frac{(b^2-1)\eta^2}{1+b^2\eta^2}, \frac{b^2\eta^2}{1+b^2\eta^2}\right) z^{2k}, \end{aligned} \tag{1.3}$$

where $F_1(\alpha;\beta,\beta';\gamma;x,y)$ is the first Appell function [4, Equation (16.13.1)]. Expansion (1.3) is valid if $b^2 > (\eta^2-1)/(2\eta^2)$ and may be extended by using formula [5].

A representation in terms of the confluent Lauricella function is also given [2, Equation (4.1)],

$$\mathcal{H}_p(z,b,\eta) = \frac{(1-b^2)^p \eta\, e^{-z^2/2}}{2\pi\sqrt{b^2\eta^2+1}} \Phi_D^{(3)} \times \left(\frac{1}{2}, \frac{1}{2}-p, 1, \frac{3}{2}, \frac{b^2\eta^2}{b^2\eta^2+1}, \frac{(b^2-1)\eta^2}{b^2\eta^2+1}, \frac{(b^2-1)\eta^2 z^2}{2(b^2\eta^2+1)}\right), \tag{1.4}$$

where the function $\Phi_D^{(3)}$ is the confluent Lauricella function defined in [6]. Formula (1.4) is valid if $b^2 > (\eta^2-1)/(2\eta^2)$, but it may be extended by analytic continuation outside this region, region of interest in applications [2, p. 5].

In this paper, we investigate new asymptotic approximations of the $\mathcal{H}$-function, $\mathcal{H}_p(z,b,\eta)$, in certain regions of its variables and parameters. As these variables and parameters are related to the Rice-Nakagami probability density function (see [1, Chapter 4]) used in communication theory, in principle, it is of interest to approximate the $\mathcal{H}$-function for the range of values specified below formula (1.1), that is the range of values with a statistical meaning. In particular, we obtain new analytical expressions in broad regions of the parameters, and asymptotic approximations in certain limits: large values of z, large values of p, small values of η and values of $b \to 1$. These expansions, unlike those provided previously, are given in terms of simpler special functions, many of them, elementary functions. We provide explicit and/or recursive algorithms for the computation of the coefficients of the expansions. Some numerical examples illustrate the accuracy of the approximations.

2. Asymptotic expansion for large values of z

We seek for an asymptotic expansion of $\mathcal{H}_p(z,b,\eta)$ for large values of z and fixed p, b and η. Then, it is convenient to write (1.2) in the form of a Laplace-type integral,

$$\mathcal{H}_p(z,b,\eta) = \frac{(1-b^2)^p}{2\pi} \exp\left(-\frac{z^2}{2b^2}\right) \int_0^\eta e^{-z^2 f(x)} g(x)\, dx, \tag{2.1}$$

with

$$f(x) := \frac{b^2-1}{2b^2}\frac{1}{1+b^2x^2}, \quad g(x) := \frac{1}{1+x^2}\frac{1}{(1+b^2x^2)^p}.$$

The absolute minimum of the phase function $f(x)$ on the integration interval is located at $x=0$. Following the standard Laplace asymptotic method, we substitute $f(x)-f(0) = \frac{1}{2}f''(0)t^2$, with $\mathrm{sign}(t) = \mathrm{sign}(x)$, which leads to

$$x = \frac{t}{\sqrt{1-b^2t^2}}.$$

This substitution let us write (2.1) in the standard form

$$\mathcal{H}_p(z,b,\eta) = \frac{(1-b^2)^p}{2\pi} e^{-\frac{z^2}{2}} \int_0^{\frac{\eta}{\sqrt{1+b^2\eta^2}}} e^{-z^2\frac{1-b^2}{2}t^2} h(t)\, dt, \qquad h(t) := \frac{(1-b^2t^2)^{p-\frac{1}{2}}}{1+(1-b^2)t^2}. \tag{2.2}$$

Now, we consider the MacLaurin series expansion of $h(t)$,

$$h(t)=\sum_{n=0}^{\infty}(-1)^n(1-b^2)^n h_n(b,p)t^{2n},\quad h_n(b,p):=\sum_{k=0}^{n}\binom{p-\frac{1}{2}}{k}\left(\frac{b^2}{1-b^2}\right)^k. \tag{2.3}$$

Introducing (2.3) into (2.2) and interchanging sum and integral we obtain

$$\mathcal{H}_p(z,b,\eta)\sim\frac{(1-b^2)^{p-\frac{1}{2}}}{2\sqrt{2\pi}}\,\mathrm{e}^{-\frac{z^2}{2}}\sum_{n=0}^{\infty}\frac{(-2)^n}{z^{2n+1}}h_n(b,p)\gamma\left(n+\frac{1}{2},\frac{(1-b^2)\eta^2}{2(1+b^2\eta^2)}z^2\right), \tag{2.4}$$

where $\gamma(a,z)$ is an incomplete gamma function [4, Equation (8.2.1)]. From Laplace's method we know that the right hand side of (2.4) is an asymptotic expansion of $\mathcal{H}_p(z,b,\eta)$ for large z and fixed and moderate values of p, b and η. Moreover, using that [4, Equations (8.2.3), (8.11.2)], when $z\to\infty$,

$$\gamma\left(n+\frac{1}{2},\frac{(1-b^2)\eta^2}{2(1+b^2\eta^2)}z^2\right)\sim\Gamma\left(n+\frac{1}{2}\right)+\text{exponentiallysmallterms},$$

we derive the following Poincaré-type asymptotic expansion of $\mathcal{H}_p(z,b,\eta)$ for large z and fixed and moderate values of p, b and η given in terms of inverse powers of z:

$$\mathcal{H}_p(z,b,\eta)\sim\frac{(1-b^2)^{p-\frac{1}{2}}}{2\sqrt{2\pi}}\,\mathrm{e}^{-\frac{z^2}{2}}\sum_{n=0}^{\infty}(-2)^n h_n(b,p)\frac{\Gamma\left(n+\frac{1}{2}\right)}{z^{2n+1}}. \tag{2.5}$$

In particular, the first-order asymptotic approximation for large z is given by the following formula:

$$\mathcal{H}_p(z,b,\eta)\sim\frac{(1-b^2)^{p-\frac{1}{2}}}{2\sqrt{2\pi}}\frac{\mathrm{e}^{-\frac{z^2}{2}}}{z}. \tag{2.6}$$

The right hand side of (2.4) is an asymptotic expansion of $\mathcal{H}_p(z,b,\eta)$ for large z. But moreover, it is convergent, and then constitutes an analytic representation of $\mathcal{H}_p(z,b,\eta)$, when $\eta^2(1-2b^2)<1$. This can be proved as follows: the radius of the disk of convergence of the expansion of $h(t)$ in (2.2) at $t=0$ is $r=\min\{1/b,1/\sqrt{1-b^2}\}$. When $\eta^2(1-2b^2)<1$, this disk contains the integration interval $[0,\eta/\sqrt{1+b^2\eta^2}]$. Then, from uniform convergence, when we introduce (2.3) into (2.2) and interchange sum and integral, the equality in (2.2) remains valid.

In Table 1, we illustrate the accuracy of approximation (2.5) for some large values of z and moderate values of p, b and η.

Remark 2.1: As indicated in [1,2], the Owen T-function is a special case of the $\mathcal{H}$-function for $b=0$: $T(z,\eta)=\mathcal{H}_p(z,0,\eta)$. Then, as a particular case of (2.5), we obtain an asymptotic expansion of $T(z,\eta)$ for large values of z and moderate values of η by replacing $b=0$ into (2.5):

$$T(z,\eta)\sim\frac{\mathrm{e}^{-\frac{z^2}{2}}}{2\sqrt{2\pi}\,z}\sum_{n=0}^{\infty}(-2)^n\frac{\Gamma\left(n+\frac{1}{2}\right)}{z^{2n}},$$

that was previously obtained in [7, Equation (2.1)].

Table 1. Relative errors in the computation of $\mathcal{H}_p(z,b,\eta)$ for $p=3$, $b=0.45$, $\eta=1.3$ and several values of z by using expansion (2.5) with terms up to $n=4$.

	$z=5$	$z=10$	$z=20$	$z=40$
$n=0$	0.62e−01	0.16e−01	0.41e−02	0.10e−02
$n=1$	0.75e−02	0.51e−03	0.33e−04	0.21e−05
$n=2$	0.14e−02	0.25e−04	0.41e−06	0.64e−08
$n=3$	0.37e−03	0.17e−05	0.71e−08	0.28e−10
$n=4$	0.13e−03	0.15e−06	0.16e−09	0.16e−12

3. Asymptotic expansion for large values of p

3.1. Asymptotic expansion for large values of p in terms of Gauss hypergeometric functions

In order to approximate $\mathcal{H}_p(z,b,\eta)$ for large values of p and fixed z, b and η, we consider the integral representation (1.1) written in the form

$$\begin{aligned}\mathcal{H}_p(z,b,\eta) &= \frac{(1-b^2)^p}{2\pi}\int_0^\eta \frac{f(x)}{(1+b^2x^2)^p}\,\mathrm{d}x,\\ f(x) &:= \frac{1}{1+x^2}\exp\left(-\frac{z^2}{2}\frac{1+x^2}{1+b^2x^2}\right).\end{aligned} \tag{3.1}$$

The maximum of the factor $(1+b^2x^2)^{-p}=\exp\{-p\log(1+b^2x^2)\}$ on the integration interval in (3.1) is attained at $x=0$. Then, following the ideas of Laplace's method, we expect that an expansion of the other factor $f(x)$ at $x=0$ can provide an asymptotic expansion of $\mathcal{H}_p(z,b,\eta)$ for large p. Then, for $N=1,2,3,\ldots$, we consider the Taylor expansion of $f(x)$ at $x=0$ with the Lagrange form for the Taylor remainder,

$$f(x)=\sum_{n=0}^{N-1} c_n(z,b)x^{2n}+r_N(x,z,b),\qquad r_N(x,z,b)=\frac{f^{(2N)}(\xi,z,b)}{(2N)!}x^{2N},\ \xi\in(0,\eta), \tag{3.2}$$

with

$$c_n(z,b):=(-1)^n\sum_{k=0}^{n}(-1)^k a_k(z,b), \tag{3.3}$$

and the coefficients $a_k(z,b)$ satisfy, for $k=0,1,2,\ldots$, the recurrence relation

$$\begin{cases}2(k+2)a_{k+2}+[4b^2(k+1)+(1-b^2)z^2]a_{k+1}+2kb^4a_k=0,\\ a_0=\mathrm{e}^{-\frac{z^2}{2}},\quad a_1=-\dfrac{1-b^2}{2}z^2\,\mathrm{e}^{-\frac{z^2}{2}}.\end{cases}$$

This recurrence relation can been obtained from the differential equation $(1+b^2x^2)^2f_1'(x)+(1-b^2)z^2f_1(x)=0$ satisfied by the function $f_1(x):=\exp(-\frac{z^2}{2}\frac{1+x^2}{1+b^2x^2})$.

Introducing expansion (3.2) into (3.1) and interchanging sum and integral we find

$$\mathcal{H}_p(z,b,\eta)=\frac{(1-b^2)^p}{2\pi}\sum_{n=0}^{N-1}c_n(b,z)\frac{\eta^{2n+1}}{2n+1}\,{}_2F_1\times\left(n+\frac{1}{2},p;n+\frac{3}{2};-b^2\eta^2\right)+R_N(p,z,b,\eta), \tag{3.4}$$

wihere ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function [4, Equation (15.2.1)], and

$$R_N(p,z,b,\eta):=\frac{(1-b^2)^p}{2\pi(2N)!}\int_0^\eta f^{(2N)}(\xi,z,b)\frac{x^{2N}}{(1+b^2x^2)^p}\,dx. \tag{3.5}$$

Since $f\in\mathcal{C}^\infty([0,\eta])$, we have that $|f^{(2N)}(\xi,z,b)|\le\bar{M}_N(z,b,\eta)$ for $\xi\in(0,\eta)$ with $\bar{M}_N(z,b,\eta)>0$ independent of x (and of course of p). Then, we have that [4, Equation (15.6.1)]

$$|R_N(p,z,b,\eta)|\le\bar{M}_N(z,b,\eta)\frac{(1-b^2)^p\eta^{2N+1}}{2\pi(2N+1)!}\left|{}_2F_1\left(N+\frac{1}{2},p;N+\frac{3}{2};-b^2\eta^2\right)\right|.$$

From [8, Equation (12.0.6)] and [8, Equations (12.2.1), (12.2.20), (12.1.11)], when $p\to\infty$,

$${}_2F_1\left(n+\frac{1}{2},p;n+\frac{3}{2};-b^2\eta^2\right)\sim\frac{\Gamma\left(n+\frac{3}{2}\right)}{(\eta b)^{2n+1}p^{n+\frac{1}{2}}}. \tag{3.6}$$

This formula shows that the terms of the expansion (3.4) constitute an asymptotic sequence for large p. Moreover, it also shows that the remainder term $R_N(p,z,b,\eta)$ can be bounded in the form

$$|R_N(p,z,b,\eta)|\le\frac{M_N(z,b,\eta)}{p^{N+\frac{1}{2}}},$$

with $M_N(z,b,\eta)$ independent of p. Therefore, the right hand side of (3.4) is an asymptotic expansion of $\mathcal{H}_p(z,b,\eta)$ for large p and fixed and moderate values of z, b and η.

The right hand side of (3.4) is an asymptotic expansion of $\mathcal{H}_p(z,b,\eta)$ for large p. But moreover, it is convergent, and then constitutes an analytic representation of $\mathcal{H}_p(z,b,\eta)$, when $\eta<1$. This can be proved as follows: the radius of the disk of convergence of the expansion of $f(x)$ in (3.1) at $x=0$ is $r=\min\{1/b,1\}=1$. When $\eta<1$, this disk contains the integration interval $[0,\eta]$. Then, from uniform convergence, when we introduce (3.2) into (3.1) and interchange sum and integral, the equality in (3.1) remains valid.

In particular, the first-order asymptotic approximation of $\mathcal{H}_p(z,b,\eta)$ for large p is given by the following formula:

$$\mathcal{H}_p(z,b,\eta)\sim\frac{(1-b^2)^p\,e^{-\frac{z^2}{2}}}{4b\sqrt{\pi p}}. \tag{3.7}$$

Table 2 shows the accuracy of approximation (3.4) for different large values of p and moderate values of z, b and η.

Table 2. Relative errors in the computation of $\mathcal{H}_p(z,b,\eta)$ for $z = 1.5, b = 0.65, \eta = 2.6$ and several values of p by using expansion (3.4) with terms up to $N = 6$.

	$p = 50$	$p = 100$	$p = 500$	$p = 1000$
$N = 1$	0.38e−01	0.19e−01	0.39e−02	0.20e−02
$N = 2$	0.35e−02	0.89e−03	0.36e−04	0.90e−05
$N = 3$	0.50e−03	0.62e−04	0.49e−06	0.62e−07
$N = 4$	0.95e−04	0.57e−05	0.90e−08	0.56e−09
$N = 5$	0.23e−04	0.67e−06	0.20e−09	0.73e−11
$N = 6$	0.68e−05	0.94e−07	0.58e−11	0.96e−12

3.2. Asymptotic expansion for large values of p in terms of inverse powers of p

The expansion derived in the previous subsection is given in terms of hypergeometric functions. In this subsection we derive a different asymptotic expansion of $\mathcal{H}_p(z,b,\eta)$ for large values of p and fixed z, b and η, this time in terms of elementary functions. The key point is an appropriate change of the integration variable in the integral representation (1.1) that transforms this integral into a Laplace transform. We consider the change of variable $x \to t$ given by $1 + b^2x^2 = e^t$, obtaining

$$\mathcal{H}_p(z,b,\eta) = \frac{(1-b^2)^p b}{4\pi}\, e^{-\frac{z^2}{2b^2}} \int_0^{\log(1+b^2\eta^2)} \frac{e^{-pt}}{\sqrt{t}} h(t)\, dt, \tag{3.8}$$

with

$$h(t) := \frac{1}{1-(1-b^2)\,e^{-t}} \sqrt{\frac{t}{e^t - 1}} \exp\left(\frac{z^2}{2b^2}(1-b^2)\,e^{-t}\right). \tag{3.9}$$

Then, following Watson's lemma, when we are interested in an asymptotic expansion of this integral for large p, we must consider the power series expansion of $h(t)$ at $t = 0$:

$$h(t) = \sum_{n=0}^{\infty} e_n(b,z) t^n, \tag{3.10}$$

where the coefficients $e_n(b,z)$ are given by

$$e_n(b,z) := \sum_{j=0}^{n} \left(\sum_{k=0}^{j} a_k(b) c_{j-k} \right) d_{n-j}(b,z), \tag{3.11}$$

with the coefficients $a_n(b)$, c_n and $d_n(b,z)$ satisfying the following respective recurrence relations for $n = 1, 2, 3, \ldots,$

$$\begin{cases} (n+1)b^2 a_{n+1} + (n+1-b^2)a_n + \displaystyle\sum_{k=0}^{n-2} \frac{k+1}{(n-k)!} a_{k+1} = 0, \\ a_0 = \dfrac{1}{b^2}, \quad a_1 = \dfrac{1}{b^2} - \dfrac{1}{b^4}, \end{cases}$$

$$\begin{cases} c_n = -\dfrac{1}{2n} \displaystyle\sum_{k=0}^{n-1} \frac{n+k}{(n+1-k)!} c_k, \\ c_0 = 1, \end{cases} \qquad \begin{cases} d_n = -\dfrac{(1-b^2)z^2}{2b^2 n} \displaystyle\sum_{k=0}^{n-1} \frac{(-1)^{n-1-k}}{(n-1-k)!} d_k, \\ d_0 = e^{\frac{(1-b^2)z^2}{2b^2}}. \end{cases}$$

We have obtained these recurrence relations from the differential equations: $(\mathrm{e}^t - (1-b^2))h_1'(t) + (1-b^2)h_1(t) = 0$, with $h_1(t) := (1-(1-b^2)\,\mathrm{e}^{-t})^{-1}$; $2t(\mathrm{e}^t - 1)h_2'(t) - (\mathrm{e}^t - 1 - t\,\mathrm{e}^t)h_2(t) = 0$, with $h_2(t) := \sqrt{t/(\mathrm{e}^t - 1)}$; $h_3'(t) + \frac{z^2}{2b^2}(1-b^2)\,\mathrm{e}^{-t}h_3(t) = 0$, with $h_3(t) := \exp(z^2(1-b^2)\,\mathrm{e}^{-t}/(2b^2))$.

Introducing expansion (3.10) in (3.8) and interchanging summation and integration we obtain

$$\mathcal{H}_p(z,b,\eta) \sim \frac{(1-b^2)^p b}{4\pi}\,\mathrm{e}^{-\frac{z^2}{2b^2}} \sum_{n=0}^{\infty} e_n(b,z) \frac{\gamma\left(n+\frac{1}{2}, p\log(1+b^2\eta^2)\right)}{p^{n+1/2}}. \tag{3.12}$$

From Watson's lemma we know that the right hand side of (3.12) is an asymptotic expansion of $\mathcal{H}_p(z,b,\eta)$ for large p and fixed and moderate values of z, b and η. Moreover, from [4, Equations (8.2.3), (8.11.2)] we have that, when $p \to \infty$,

$$\gamma\left(n+\frac{1}{2}, p\log(1+b^2\eta^2)\right) \sim \Gamma\left(n+\frac{1}{2}\right) + \text{exponentiallysmallterms}.$$

Then we finally obtain the Poincaré-type asymptotic expansion

$$\mathcal{H}_p(z,b,\eta) \sim \frac{(1-b^2)^p b}{4\pi}\,\mathrm{e}^{-\frac{z^2}{2b^2}} \sum_{n=0}^{\infty} e_n(b,z) \frac{\Gamma\left(n+\frac{1}{2}\right)}{p^{n+1/2}}. \tag{3.13}$$

The right hand side of (3.13) is an asymptotic expansion of $\mathcal{H}_p(z,b,\eta)$ for large p. But moreover, it is convergent, and then constitutes an analytic representation of $\mathcal{H}_p(z,b,\eta)$, when $(1+b^2\eta^2)(1-b^2) < 1$. This can be proved as follows: the radius of the disk of convergence of the expansion of $h(t)$ in (3.9) at $t=0$ is $r = -\log(1-b^2)$. When $(1+b^2\eta^2)(1-b^2) < 1$, this disk contains the integration interval $[0, \log(1+b^2\eta^2)]$. Then, from uniform convergence, when we introduce (3.10) in (3.8) and interchange sum and integral, the equality in (3.8) remains valid.

The first-order asymptotic approximation provided by this formula is the same as the one provided by formula (3.6) given in (3.7).

Remark 3.1: The asymptotic sequence in expansion (3.13), inverse powers of p, is simpler than the asymptotic sequence (3.6), that consists of hypergeometric functions. As a counterpart, the computation of the coefficients $e_n(b,z)$ in expansion (3.13) is a little bit more involved than the computation of the coefficients $c_n(b,z)$ in expansion (3.7).

Table 3 illustrates the accuracy of approximation (3.13) for different large values of p and moderate values of z, b and η.

4. Asymptotic expansion for small values of η

For small values of η we consider the integral representation (1.2) of $\mathcal{H}_p(z,b,\eta)$ and just compute the MacLaurin series of $\mathcal{H}_p(z,b,\eta)$ at $\eta = 0$. Then, we obtain the following asymptotic expansion of $\mathcal{H}_p(z,b,\eta)$ for small values of η and fixed p, z and b in odd powers of η, that is convergent for $|\eta| < \min\{1, 1/b\} = 1$, and therefore constitute an analytic

Table 3. Relative errors in the computation of $\mathcal{H}_p(z,b,\eta)$ for $z = 1.5, b = 0.65, \eta = 2.6$ and several values of p by using expansion (3.13) with terms up to $n = 5$.

	$p = 50$	$p = 100$	$p = 500$	$p = 1000$
$n = 0$	0.30e−01	0.15e−01	0.31e−02	0.16e−02
$n = 1$	0.21e−02	0.53e−03	0.22e−04	0.55e−05
$n = 2$	0.20e−03	0.27e−04	0.22e−06	0.28e−07
$n = 3$	0.27e−04	0.18e−05	0.30e−08	0.19e−09
$n = 4$	0.43e−05	0.15e−06	0.50e−10	0.19e−11
$n = 5$	0.85e−06	0.14e−07	0.94e−12	0.31e−12

Table 4. Relative errors in the computation of $\mathcal{H}_p(z,b,\eta)$ for $p = 3, z = 5.6, b = 0.25$ and several values of η by using expansion (4.1) with terms up to $n = 3$.

	$\eta = 0.35$	$\eta = 0.1$	$\eta = 0.01$	$\eta = 0.005$
$n = 0$	0.53e−00	0.53e−01	0.53e−03	0.13e−03
$n = 1$	0.42e−00	0.26e−02	0.26e−06	0.16e−07
$n = 2$	0.21e−00	0.10e−03	0.10e−09	0.16e−11
$n = 3$	0.87e−01	0.33e−05	0.25e−013	0.72e−14

representation of $\mathcal{H}_p(z,b,\eta)$ for $|\eta| < 1$,

$$\mathcal{H}_p(z,b,\eta) = \frac{(1-b^2)^p}{2\pi}\,\mathrm{e}^{-\frac{z^2}{2}}\sum_{n=0}^{\infty} c_n(p,z,b)\frac{\eta^{2n+1}}{2n+1}, \tag{4.1}$$

where the coefficients $c_n(p,z,b)$ are given by

$$c_n(p,z,b) := \sum_{j=0}^{n}\left(\sum_{k=0}^{j}(-1)^{j-k}\binom{-p}{k}b^{2k}\right)a_{n-j}(b,z), \tag{4.2}$$

and $a_n(b,z)$ satisfy, for $n = 2,3,4,\ldots$, the recurrence relation

$$\begin{cases} 2na_n + \left(4b^2(n-1) + z^2(1-b^2)\right)a_{n-1} + 2b^4(n-2)a_{n-2} = 0, \\ a_0 = 1, \quad a_1 = -\dfrac{(1-b^2)z^2}{2}. \end{cases}$$

This recurrence relation can be obtained from the differential equation satisfied by the function $f(x) = \exp(\frac{z^2}{2b^2}\frac{1-b^2}{1+b^2x^2})$: $(1+b^2x^2)^2 f'(x) + z^2(1-b^2)xf(x) = 0$.

In Table 4 we illustrate the accuracy of approximation (4.1) for different small values of η and moderate values of p, z and b.

Remark 4.1: As in Remark 2.1, from (4.1) we can obtain a convergent expansion of the Owen T-function for $|\eta| < 1$. Replacing $b = 0$ into (4.1), we find

$$T(z,\eta) = \frac{\mathrm{e}^{-\frac{z^2}{2}}}{2}\sum_{n=0}^{\infty}\tilde{c}_n(z)\frac{\eta^{2n+1}}{2n+1}, \quad \tilde{c}_n(z) := \sum_{j=0}^{n}(-1)^j\tilde{a}_{n-j}(z), \tag{4.3}$$

where $\tilde{a}_n(z)$ satisfy, for $n = 2, 3, 4, \ldots$, the recurrence relation

$$\begin{cases} a_n = -\dfrac{z^2}{2n} a_{n-1}, \\ a_0 = 1, \quad a_1 = -\dfrac{z^2}{2}. \end{cases}$$

Using formulas [4, Equations (18.9.1), (18.9.2)] it is straightforward to show that $\tilde{c}_n(z) = L_n^{-n-1}(z^2/2)$, where $L_n^{\alpha}(x)$ is the Laguerre polynomial. Then, the asymptotic expansion (4.3) is just a reformulation of the one obtained in [7, Equation (5.3)] in terms of Laguerre polynomials.

5. Asymptotic expansion for $b \to 1$

In order to derive an asymptotic expansion of $\mathcal{H}_p(z, b, \eta)$ for values of b near 1, we consider the integral representation (1.2) and write

$$\mathcal{H}_p(z, b, \eta) = \frac{(1-b^2)^p}{2\pi} \exp\left(-\frac{z^2}{2b^2}\right) \int_0^{\eta} f(x, z, b) \frac{1}{1+x^2} \frac{1}{(1+b^2x^2)^p} \, \mathrm{d}x, \tag{5.1}$$

with

$$f(x, z, b) := \exp\left(\frac{z^2}{2b^2} \frac{1-b^2}{1+b^2x^2}\right). \tag{5.2}$$

For $b \to 1$, the argument of this exponencial function becomes small and then it seems reasonable to expand (5.2) in powers of the argument of the exponential function,

$$f(x, z, b) = \sum_{n=0}^{N-1} \frac{z^{2n}}{2^n b^{2n}} \frac{(1-b^2)^n}{(1+b^2x^2)^n n!} + r_N(x, z, b), \tag{5.3}$$

where $r_N(x, z, b)$ is the Taylor remainder that we write in the Lagrange form

$$r_N(x, z, b) := \frac{\mathrm{e}^{\xi}}{N!} \frac{z^{2N}}{2^N b^{2N}} \frac{(1-b^2)^N}{(1+b^2x^2)^N}, \qquad \frac{z^2}{2b^2} \frac{1-b^2}{1+b^2\eta^2} < \xi < \frac{z^2}{2b^2}(1-b^2). \tag{5.4}$$

Introducing expansion (5.3) into (5.1) and interchanging sum and integral we obtain

$$\begin{aligned} \mathcal{H}_p(z, b, \eta) = \frac{(1-b^2)^p}{2\pi} \exp\left(-\frac{z^2}{2b^2}\right) \Bigg[\sum_{n=0}^{N-1} \frac{z^{2n}\eta}{2^n b^{2n}} \frac{(1-b^2)^n}{n!} \\ \times F_1\left(\frac{1}{2}; n+p, 1; \frac{3}{2}; -b^2\eta^2, -\eta^2\right) + R_N(z, b) \Bigg], \end{aligned} \tag{5.5}$$

where $F_1(a; b_1, b_2; c; z_1, z_2)$ is the first Appell function [4, Equation (16.13.1)], and

$$R_N(z, b) := \int_0^{\eta} r_N(x, z, b) \frac{1}{1+x^2} \frac{1}{(1+b^2x^2)^p} \mathrm{d}x. \tag{5.6}$$

Table 5. Relative errors in the computation of $\mathcal{H}_p(z,b,\eta)$ for $p = 1$, $z = 2.1$, $\eta = 1.5$ and small values of $1-b$ by using expansion (5.5) for $n = 0$, 1, 2, 3.

	$b = 0.9$	$b = 0.99$	$b = 0.999$
$n = 0$	0.35e−00	0.35e−01	0.35e−02
$n = 1$	0.72e−01	0.66e−03	0.66e−05
$n = 2$	0.11e−02	0.87e−05	0.85e−08
$n = 3$	0.12e−03	0.88e−07	0.85e−11

Fix an arbitrary value b_0 with $0 < b_0 < 1$. As the function $(1-b^2)/b^2$ is a decreasing function of b, we have from (5.4) that, for $b \in [b_0, 1]$,

$$0 < \xi < \xi_0 := \frac{z^2}{2b_0^2}(1-b_0^2).$$

Then, for $0 < b_0 < b < 1$, we can bound $e^{\xi} \leq M_1(z)$ in (5.4), with $M_1(z)$ independent of b, and we find

$$|R_N(z,b)| \leq M_1(z)\frac{z^{2N}\eta}{2^N b^{2N}}\frac{(1+b)^N}{N!}\left|F_1\left(\frac{1}{2}; N+p, 1; \frac{3}{2}; -b^2\eta^2, -\eta^2\right)\right|(1-b)^N. \quad (5.7)$$

Then, for $0 < b_0 < b < 1$,

$$|R_N(z,b)| \leq M_N(z,\eta,p,b_0)(1-b)^N, \quad (5.8)$$

with $M_N(z,\eta,p,b_0) > 0$ independent of b. On the other hand, it is clear that the terms of the expansion (5.5) constitute and asymptotic sequence for $b \to 1$. Therefore, the right hand side of (5.5) is an asymptotic expansion of $\mathcal{H}_p(z,b,\eta)$ when $b \to 1$.

In Table 5, we illustrate the accuracy of approximation (5.5) for different small values of $1-b$ and moderate values of p, z and η.

6. Conclusions

In this paper, we add some more information to the study of the $\mathcal{H}$-function of communication theory $\mathcal{H}_p(z,b,\eta)$ given in [1,2], introducing new asymptotic expansions in certain regions of its variables and parameters: large values of z, large values of p, small values of η and values of $b \to 1$. These expansions are given in terms of simpler special functions, many of them, elementary functions. Approximations for small p and b are still unknown and are subject of further investigation.

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