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Accurate and fast computations with Green matrices[☆]

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ABSTRACT

This paper provides a linear time complexity method to obtain the bidiagonal decomposition of Green matrices with high relative accuracy. In addition, when the Green matrix is nonsingular and totally positive, this bidiagonal decomposition can be used to compute the eigenvalues, the inverse and the solution of some linear system of equations with high relative accuracy. A numerical example illustrates the advantages of this method.

1. Introduction

Let us recall that an algorithm computes to high relative accuracy (HRA) when it only uses products, quotients, additions of numbers with the same sign or subtractions of initial data (cf. [1]). In other words, the only forbidden operation is the subtraction of numbers (which are not initial data) with the same sign. Finding an adequate parameterization of the matrix is the first step to derive algorithms with HRA. Among the classes of matrices for which algorithms to HRA have been constructed, we can mention some subclasses of nonsingular totally positive matrices (see, for instance, [2–4]). Let us recall that a matrix is *totally positive* (TP) if all its minors are nonnegative and it is *strictly totally positive* (STP) if they are positive (see [5,6]). As shown in [7], for a nonsingular TP matrix A, if we know its bidiagonal factorization $\mathcal{BD}(A)$, then we can perform many algebraic computations with HRA with the software of [8]. For instance, its eigenvalues, its singular values, its inverse and the solution of linear systems $Ax = b$, where b has alternating signs. In this paper, we provide a method of $\mathcal{O}(n)$ elementary operations to obtain bidiagonal factorizations of Green matrices with HRA. Recall that the bidiagonal factorization $\mathcal{BD}(A)$ arises naturally in the process of Neville elimination (see [9]). This process is an elimination procedure alternative to Gaussian elimination, which, roughly speaking, makes zeros in a column by adding to each row an adequate multiple of the previous one.

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Green matrices (see [5,6,10]) can be considered as discrete version of Green functions (see p. 237 of [10]). These functions arise in the Sturm–Liouville boundary-value problem. They have important applications (see [10]). A subclass of Green matrices is given by the Schoenmakers–Coffey matrices, which have important financial applications (see [11,12]). For Schoenmakers–Coffey matrices, a parameterization of n parameters leading to HRA computations was presented in [3]. We now present a parameterization of $2n$ parameters leading to HRA computations for Green matrices. It is known that the matrix entries do not provide an adequate parameterization for accurate computations with TP matrices, in contrast to the entries of $\mathcal{BD}(A)$. In Section 2, we first present in Theorem 2.1 the HRA bidiagonal factorization of Green matrices and its applications to other HRA computations when A is nonsingular TP. Section 3 includes a numerical experiment confirming the advantages of using our HRA methods.

2. Main result

Given two sequences of nonzero real numbers $(u_i)_{1\le i\le n}$, $(v_i)_{1\le i\le n}$, a *Green matrix* $A=(a_{ij})_{1\le i,j\le n}$ is the symmetric matrix given by $a_{ij}=u_iv_j$ if $i\le j$.

Let us recall the characterization of TP Green matrices (cf. Theorem 4.2 of [6] or p. 214 of [5]): a Green matrix is TP if and only if the sequences $(u_i)_{1\le i\le n}$, $(v_i)_{1\le i\le n}$ are formed by numbers of the same strict sign and

$$(0<)\frac{u_1}{v_1}\le\cdots\le\frac{u_n}{v_n}. \tag{1}$$

Taking into account (1), the initial parameters that we shall use to get HRA algorithms will be:

$$v_i, \quad r_i:=\frac{u_i}{v_i}, \qquad i=1,\ldots,n. \tag{2}$$

Observe, for instance, that the entries of the Green matrix $A=(a_{ij})_{1\le i,j\le n}$ can be written as $a_{ij}=r_iv_iv_j$ if $i\le j$ and that (1) is transformed into $(0<)r_1\le\cdots\le r_n$.

Let us now obtain the bidiagonal factorization of a Green matrix A with HRA. Therefore, when A is also nonsingular TP, we can also calculate with HRA its eigenvalues (which coincide with its singular values by the symmetry of the matrix), its inverse and the solution of some linear systems.

Theorem 2.1. *If A is a Green matrix and we know the parameters (2), then we can compute the bidiagonal factorization of A with HRA. If, in addition, A is nonsingular TP, then we can also calculate its eigenvalues, A^{-1} and the solution of linear systems $Ax=b$ (where b has alternating signs) with HRA.*

Proof. We start by performing Neville elimination. In general, Neville elimination does not provide HRA algorithms because it uses subtractions. However, in this case it will allow us to obtain $\mathcal{BD}(A)$ to HRA. We first subtract v_n/v_{n-1} times the last but one row to the last one in order to produce a zero at entry (n,n) of A. We can observe that this elementary operation also produces zeros in the remaining off-diagonal entries of the last row. The (n,n) entry of the last row transforms into

$$d_n:=u_nv_n-\frac{v_n}{v_{n-1}}u_{n-1}v_n=u_nv_n\frac{r_n-r_{n-1}}{r_n}=v_n^2(r_n-r_{n-1}). \tag{3}$$

Analogously (by symmetry), if we now subtract v_n/v_{n-1} times the last but one column to the last one, we obtain a matrix M with zeros in the last column up to place (n,n), where d_n remains.

Let us denote by $E_i(\alpha)$ $(2\le i\le n)$ the $n\times n$ elementary matrix that has unit diagonal, α in place $(i,i-1)$ and 0 elsewhere. Then the matrix form of the previous step can be written as $E_n\left(\frac{-v_n}{v_{n-1}}\right)AE_n\left(\frac{-v_n}{v_{n-1}}\right)^T=M$.

Continuing this procedure to make zeros in entries $(n-1,1),(1,n-1),\dots,(2,1),(1,2)$, we get the factorization

$$E_2\left(\frac{-v_2}{v_1}\right)\cdots E_n\left(\frac{-v_n}{v_{n-1}}\right)AE_n\left(\frac{-v_n}{v_{n-1}}\right)^T\cdots E_2\left(\frac{-v_2}{v_1}\right)^T=D, \tag{4}$$

where D is the diagonal matrix with bidiagonal entries

$$d_1=u_1v_1,\qquad d_i=u_iv_i\frac{r_i-r_{i-1}}{r_i}=v_i^2(r_i-r_{i-1}),\qquad i>1. \tag{5}$$

Taking into account that $E_i(\alpha)^{-1}=E_i(-\alpha)$, we get the following bidiagonal factorization $\mathcal{BD}(A)$ of A:

$$A=E_n\left(\frac{v_n}{v_{n-1}}\right)\cdots E_2\left(\frac{v_2}{v_1}\right)DE_2\left(\frac{v_2}{v_1}\right)^T\cdots E_n\left(\frac{v_n}{v_{n-1}}\right)^T. \tag{6}$$

Observe that all entries of the bidiagonal factorization (6) can be obtained with HRA.

If A is nonsingular TP, we can use the algorithms of [7] to derive the announced algebraic computations with HRA from the bidiagonal factorization of A. □

Remark 2.2. The computational cost for the given bidiagonal factorization of the proof of Theorem 2.1 for an $n\times n$ Green matrix is $\mathcal{O}(n)$ elementary operations. Taking into account the computational costs of the algorithms given in [7,13] for the corresponding algebraic computations of these matrices (with the particular structure of their bidiagonal factorizations), we can conclude that, using the bidiagonal factorization of a nonsingular Green matrix, the computational cost to compute the eigenvalues is $\mathcal{O}(n^2)$ elementary operations, whereas the computational cost to compute its inverse or solve a linear system of equations is $\mathcal{O}(n)$ elementary operations.

3. Numerical tests

If the bidiagonal decomposition $\mathcal{BD}(A)$ of a nonsingular TP matrix A is known to HRA, Koev [7] devised algorithms to compute the eigenvalues, the singular values of A and the solution of linear systems of equations $Ax=b$, where b has an alternating sign pattern. Koev implemented these algorithms in order to be used with Matlab and Octave in the software library TNTool available for download in [8]: functions `TNEigenvalues`, `TNSingularValues` and `TNSolve`. In addition, the inverse A^{-1} can be computed by using the algorithm presented in [13]: function `TNInverseExpand` in [8].

The bidiagonal decomposition of a Green matrix deduced in the proof of Theorem 2.1 has been implemented in the function `TNBDGreen` to be used in Matlab and Octave.

Now we will illustrate the results presented in Section 2 with numerical examples. In our numerical tests, we have considered the Green matrix A of order 20 whose bidiagonal decomposition is given by the parameters $v=(v_i)_{i=1}^{20}$ and $r=(r_i)_{i=1}^{20}$ defined by

$$v_i=i \quad\text{and}\quad r_i=1+\frac{1}{2^{30-i}} \quad \text{for } i=1,\dots,20.$$

It corresponds to the Green matrix with parameters given by the sequence v and the sequence $u=(u_i)_{i=1}^{20}$ defined by $u_i=i\left(1+\frac{1}{2^{30-i}}\right)$ for $i=1,\dots,20$. Since it satisfies (1), the matrix A is TP. Then, by Theorem 2.1, $\mathcal{BD}(A)$ can be computed to HRA and, in addition, using this bidiagonal decomposition and the software library [8] the eigenvalues of A, A^{-1} and the solution of linear systems $Ax=b$ (where b has alternating signs) can also be computed to HRA. Observe that this matrix A is very ill-conditioned (its condition number is $\kappa(A)=1.97e+12$), as happens with many other TP matrices. This is the reason why it is necessary to design adapted algorithms, since standard algorithms will give low accuracy.

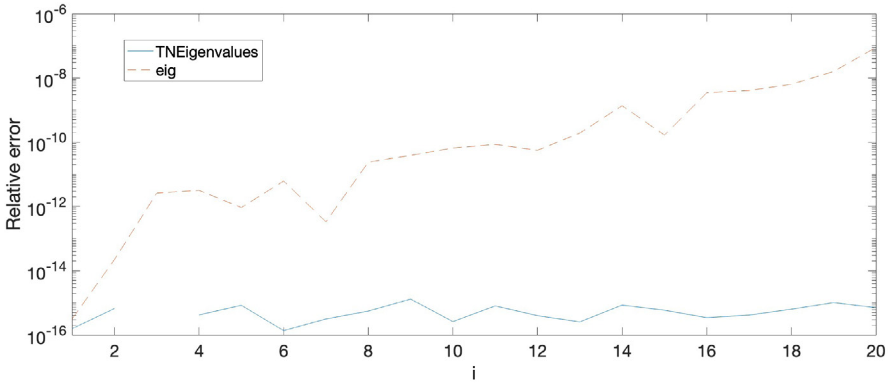

Fig. 1. Relative errors when computing the eigenvalues λ_i, $i = 1, \ldots, 20$, with Matlab.

First we have computed the eigenvalues of matrix A with Mathematica using a 100-digit precision. In addition, we have also computed the eigenvalues in Matlab with the usual function `eig` and with the function `TNEigenvalues` using $\mathcal{BD}(A)$ to HRA obtained with `TNBDGreen`. Then we have calculated the relative errors of the approximation to the eigenvalues obtained by both algorithms in Matlab considering the eigenvalues obtained with Mathematica as exact. In order to show the relative errors we have considered the eigenvalues ordered in decreasing order: $\lambda_1 > \lambda_2 > \cdots > \lambda_{20} > 0$. In Fig. 1, we can see the relative errors of the approximation to the eigenvalues λ_i, $i = 1, \ldots, 20$, of matrix A by both methods. We can observe that the bidiagonal decomposition $\mathcal{BD}(A)$ to HRA joint with Plamen Koev software library provide more accurate results than these obtained by using the Matlab command `eig`.

Data availability

No data was used for the research described in the article.

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