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Equivalence Classes of Fundamental Sequences of Rational Numbers

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~~Sister Marlene Greatens, O.S.F.~~ has submitted an

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I hereby accept and approve this essay.

Signed: ______________________

Department Chairman

EQUIVALENCE CLASSES OF FUNDAMENTAL SEQUENCES
OF
RATIONAL NUMBERS

by

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INTRODUCTION

Several methods have been devised for obtaining the real numbers from rational numbers. The purpose of this paper is to demonstrate how a complete, ordered field can be established through equivalence classes of fundamental sequences of rational numbers. It is the method used by G. Cantor for constructing the real number system.

There are sequences of rational numbers which converge to a limit but whose limit is not a rational number. Such sequences provide a method for obtaining the real numbers. The equivalence classes of fundamental sequences of rational numbers form a field which contains a subfield isomorphic to the field of rational numbers. This field of equivalence classes is actually the real number system. It is a complete, ordered field since every fundamental sequence in the field has a limit in the field. This is essentially what this paper establishes.

An example here will clarify the discussion. Suppose we take the square root of two by the usual algorithm and carry the extraction to a large number of decimal places. The successive approximations to this square root yields the sequence of numbers, 1, 1.4, 1.41, 1.412,.... The resultant is a convergent sequence of numbers, all of which are rational. Each is a closer approximation to the so-called irrational number, $\sqrt{2}$. The square root of two, then, is a real number which has been obtained from a sequence of rational numbers. This sequence has no limit among the rationals but does have limit in the reals.

The first chapter presents basic definitions and theorems needed to discuss equivalence classes and fundamental sequences. Chapter two outlines the field properties and shows that the equivalence classes of fundamental sequences form a field. In chapter three an ordered field is established from the set of equivalence classes of fundamental sequences of rational numbers. Finally, chapter four shows that the ordered field of equivalence classes of fundamental sequences of rational numbers possesses completeness.

CHAPTER I

EQUIVALENCE CLASSES AND FUNDAMENTAL SEQUENCES

PART I

Since this discussion is concerned particularly with equivalence classes, we begin by defining an equivalence classes, we begin by defining an equivalence relation.

DEFINITION 1.1. A <u>relation,</u> $\sim$, on S is a subset of S X S.

For a,b ε S, we write a $\sim$ b if (a,b) is a member of the subset, $\sim$.

DEFINITION 1.2. A relation is an <u>equivalence</u> <u>relation</u> for a set S if, for every a,b,c ε S:

(1) a $\sim$ a;

(2) if a $\sim$ b, then b $\sim$ a; and

(3) if a $\sim$ b and b $\sim$ c, then a $\sim$ c.

The following important theorem is an immediate consequence of these definitions.

THEOREM 1.1. Let $\sim$ be an equivalence relation on a set S.

For $a, b \in S$, let $S_a = \{x \in S | x \sim a\}$ and $S_b = \{x \in S | x \sim b\}$. Then (i) $S_a = S_b$ if and only if $b \sim a$, and

(ii) if $S_a \cap S_b \neq \emptyset$, then $S_a = S_b$.

Proof: (i) Since $\sim$ is an equivalence relation, $a \sim a$. Thus $a \in S_a$ and $S_a \neq \phi$. If $b \in S_a$, $b \sim a$ and hence $a \sim b$ so that $a \in S_b$. Suppose $x \in S_b$. Then $x \sim b$ and $b \sim a$ implies $x \sim a$. Thus $S_b \subseteq S_a$. Suppose $y \in S_a$. Then $y \sim a$ and $a \sim b$ implies $y \sim b$. Then $y \in S_b$ and $S_a \subseteq S_b$. Therefore, $S_a = S_b$ if $a \sim b$.

(ii) If $S_a \cap S_b \neq \phi$, then there exists $c \in S$ such that $c \in S_a$ and $c \in S_b$. Then $c \sim a$ and $c \sim b$. $c \sim a$ implies $a \sim c$ so that $a \sim b$ and $S_a = S_b$.

DEFINITION 1.3. Let $S = \bigcup_{i=1}^{n} S_i$. If $S_i \cap S_j = \phi$ for $i \neq j$, then this set of subsets, $\{S_i\}$, is called a <u>partition</u> of S.

By Theorem 1.1 an equivalence relation defines a partition. Each S_a, S_b, as defined in Theorem 1.1, is called an equivalence class.

THEOREM 1.2. A partition P of a set S defines an equivalence relation in the following way: $a \sim b$ if and only if there is a set $R \in P$ such that $a \in R$ and $b \in R$.

Proof: (1) $\sim$ is reflexive since $a \in R$ implies $a \sim a$.

(2) $\sim$ is symmetric for, if $a \sim b$ then there exists $R \in P$ such that if $a, b \in R$, we have $b \sim a$.

(3) $\sim$ is transitive for, if $m \sim n$ and $n \sim p$ then there exists $R \in P$ such that $m,n \in R$ and there exists $Q \in P$ such that $n,p \in Q$. Since P is a partition and $n \in Q \cap R$, $Q = R$ implies $m,p \in R$ and $m \sim p$.

The following will serve as an example of an equivalence relation. We define an equivalence relation on S where the members of S are pairs of positive integers.

Let $(a,b) \sim (c,d)$ if and only if $a + d = b + c$. This is an equivalence relation since it can easily be shown that the reflexive, symmetric, and transitive properties hold. $(a,b) \sim (a,b)$ since $a + b = b + a$ in the set of integers. If $(a,b) \sim (c,d)$, then $(c,d) \sim (a,b)$ for if $a + d = b + c$, then clearly $c + b = d + a$. Suppose, next, that $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$. Then $a + d = b + c$ and $c + f = d + e$. This implies that $(a + d) + (c + f) = (b + c) + (d + e)$ and $a + f = b + e$. Therefore, $(a,b) \sim (e,f)$.

It is possible to define addition and multiplication on this set so that the system becomes a ring. That is, let $D = S \times S = \{(a,b) | a,b \in S\}$. Let $[a,b] = \{(a',b') \in D | (a',b') \sim (a,b)\}$ and let I be the set of all equivalence classes, $[a,b]$. Let the operations be defined in the following way:

$$+ : [a,b] + [c,d] = [a + c, b + d]$$

$$\cdot : [a,b] \cdot [c,d] = [ac + bd, ad + bc].$$

Then $\{I; +, \cdot\}$ is a commutative ring with identity.

PART II

Since this discussion is concerned with equivalence classes of fundamental sequences of rational numbers, it is necessary now to insert a few remarks on sequences and, in particular, fundamental sequences.

DEFINITION 1.4. A sequence of rational numbers is a mapping from the set of positive integers into the set of rational numbers.

DEFINITION 1.5. A sequence, $\{r_n\}$, of rational numbers is fundamental, or Cauchy, if, for every rational number $\varepsilon > 0$, there exists a positive integer N such that if $n,m > N$, then $|r_n - r_m| < \varepsilon$.

We shall define a relation for the set of Cauchy sequences of rational numbers, and then show that this is, in fact, an equivalence relation.

DEFINITION 1.6. Let $\{r_n\}$ and $\{s_n\}$ be fundamental sequences. Let $\sim$ denote the relation such that $\{r_n\} \sim \{s_n\}$ if, for every rational number $\varepsilon > 0$, there is an integer N such that if $n > N$, then $|r_n - s_n| < \varepsilon$.

LEMMA 1.1. Suppose that $\{r_n\}$ has limit r. If $\{s_n\}$ is a fundamental sequence, then the following two statements are equivalent:

(1) $\{r_n\} \sim \{s_n\}$

(2) $\{s_n\}$ has limit r.

Proof: Suppose $\varepsilon > 0$.

(1) => (2). Suppose $\{r_n\} \sim \{s_n\}$. Then there is an M_1 such that if $n > M_1$, then $|r - r_n| < \varepsilon/2$ and an M_2 such that if $n > M_2$, $|r_n - s_n| < \varepsilon/2$. Now let $M = \max(M_1, M_2)$. Then we have

$$|r - s_n| \leq |r - r_n| + |r_n - s_n|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \text{ if } n > M.$$

It follows that r is the limit of $\{s_n\}$.

(2) => (1). If $\{s_n\}$ has limit r, there exists an N_1 such that if $n > N_1$, $|r - s_n| < \varepsilon/2$. Also, since $\{r_n\}$ has limit r, there exists N_2 such that if $n > N_2$, $|r - r_n| < \varepsilon/2$. Let $N = \max(N_1, N_2)$. Then if $n > N$,

$$|r_n - s_n| = |(r_n - r) + (r - s_n)|$$
$$\leq |r - r_n| + |r - s_n|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence, $\{r_n\} \sim \{s_n\}$.

LEMMA 1.2. The relation, $\sim$, of Definition 1.6 is an equivalence relation.

Proof: (1) $\{r_n\} \sim \{r_n\}$ since for every $\varepsilon > 0$, and for all n, $|r_n - r_n| = 0 < \varepsilon$.

(2) Suppose $\{r_n\} \sim \{s_n\}$. For each $\varepsilon > 0$, there exists $N > 0$ such that $|r_n - s_n| < \varepsilon$ and $|s_n - r_n| < \varepsilon$ if $n > N$. This implies $\{s_n\} \sim \{r_n\}$.

(3) Suppose $\{r_n\} \sim \{s_n\}$ and $\{s_n\} \sim \{t_n\}$. Then, if $\varepsilon > 0$, there is an $M > 0$ such that $|r_n - s_n| < \varepsilon/2$ and $|s_n - t_n| < \varepsilon/2$ if $n > M$. It follows that $|r_n - t_n| \leq |r_n - s_n| + |s_n - t_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence, $\{r_n\} \sim \{t_n\}$.

CHAPTER II

THE FIELD PROPERTIES

PART I

Since we wish to discuss fundamental sequences of rational numbers, we shall assume that the rational number system is given as an ordered field. Below we list the properties of an ordered field.

DEFINITION 2.1. The statement that F is a <u>field</u> means that the following statements are true.

1. There are binary operations, "+", and "•", on F which are called addition and multiplication respectively.

2. For a,b,c ε F,

   (a) $(a + b) + c = a + (b + c)$;

   (b) $a + b = b + a$;

   (c) For every a,b ε F there is an x ε F such that $a + x = b$;

   (d) $a(bc) = (ab)c$

   (e) $a(b + c) = ab + ac$, $(b + c)a = ba + ca$;

   (f) There is an element, say e, in F such that for every a ε F, $a + e = e + a + a$ and $e \cdot a = e$;

   (g) If a ε F, $a \neq 0$, and b ε F, there is an x ε F such that $ax = b$;

(h) $ab = ba$;

(i) F has at least two elements;

(j) There is an element, $e \in F$, such that for every $a \in F$, $ae = ea = a$.

DEFINITION 2.2. The statement that the field F is an ordered field means that the following statements are true.

(1) There is a relation defined on F, called greater than, and denoted ">" such that:

(a) for every $a,b \in F$, $a \neq b$, either $a > b$ or $b > a$;

(b) if $a > b$, then it is not true that $b > a$;

(c) if $a > b$ and $b > c$, then $a > c$.

(2) If $a > 0$ and $b > 0$, then $ab > 0$ and $a + b > 0$.

We note without proof that the rational number system R has no proper ordered subfield[1] and every ordered field contains a subfield isomorphic with the ordered field of rational numbers. This means that if F is an ordered field which contains a proper ordered subfield F_1, F_1 is isomorphic with the ordered field of rational numbers.

Thus, for every $x_1, y_1 \in F_1$,

$f(x_1) + f(y_1) = f(x_1 + y_1)$;

$f(x_1) \cdot f(y_1) = f(x_1 y_1)$; and if

$x_1 > y_1$, then $f(x_1) > f(y_1)$.

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[1]Casper Goffman, Real Functions (New York: Holt, Rinehart, and Winston, 1963), p. 29.

R is unique in regard to this. It is the only system exhibiting this special property.

PART II

In order to provide a suitable definition for the sum and product of two equivalence classes, we establish the following lemmas. The equivalence classes are referred to as ρ, σ, τ, etc.

LEMMA 2.1. Suppose that $\{r_n\}$ and $\{s_n\}$ are fundamental sequences of rational numbers. Then $\{r_n + s_n\}$ is a fundamental sequence.

Proof: Let $\varepsilon > 0$. Since $\{r_n\}$ is a fundamental sequence, there is an N_1 such that if $n,m > N_1$, then $|r_n - r_m| < \varepsilon/2$; and there exists N_2 such that if $n,m > N_2$, then $|s_n - s_m| < \varepsilon/2$. Let $N = \max(N_1,N_2)$. Then

$$|(r_n + s_n) - (r_m + s_m)| \leq |r_n - r_m| + |s_n - s_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence, $\{r_n + s_n\}$ is a fundamental sequence.

LEMMA 2.2. If $\{r_n\}$ is a fundamental sequence of rational numbers, there is a rational number $M > 0$ such that $|r_n| \leq M$ for every positive integer n.

Proof: Choose N such that for every $n > N$, $|r_n - r_N| < 1$. Let $M = \max(|r_1|, |r_2|, \ldots, |r_N|) + 1$. Then for every $n \leq N$, $|r_n| < M$. For $n > N$, $|r_n| < |r_N| + 1 \leq M$.

LEMMA 2.3. If $\{r_n\}$ and $\{s_n\}$ are fundamental sequences of rational numbers, then $\{r_n s_n\}$ is a fundamental sequence.

Proof: Let $\varepsilon > 0$. There exist rational numbers M_1, M_2 such that $|r_n| < M_1$ and $|s_n| < M_2$ for every n. Also, there is an N_1 such that for $n,m > N_1$, $|r_n - r_m| < \frac{\varepsilon}{2M_2}$ and an N_2 such that for $n,m > N_2$, $|s_n - s_m| < \frac{\varepsilon}{2M_1}$. Let $N = \max(N_1,N_2)$.

If $n,m > N$,

$$\begin{aligned} &|r_n s_n - r_m s_m| \\ &\quad = |r_n s_n - r_n s_m + r_n s_m - r_m s_m| \\ &\quad \leqq |r_n s_n - r_n s_m| + |r_n s_m - r_m s_m| \\ &\quad \leqq |r_n| \cdot |s_n - s_m| + |s_m| \cdot |r_n - r_m| \\ &\quad < M_1 \cdot |s_n - s_m| + M_2 \cdot |r_n - r_m| \\ &\quad < M_1 \cdot \frac{\varepsilon}{2M_1} + M_2 \cdot \frac{\varepsilon}{2M_2} = \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence, there is an N such that if $n,m > N$, we have $|r_n s_n - r_m s_m| < \varepsilon$. Therefore, $\{r_n s_n\}$ is a fundamental sequence.

We are now able to establish the following theorem.

THEOREM 2.1. If $\{r_n\} \sim \{r_n'\}$ and $\{s_n\} \sim \{s_n'\}$, where $\{r_n\}$, $\{r_n'\}$, $\{s_n\}$ and $\{s_n'\}$ are fundamental sequences of rational numbers, then $\{r_n + s_n\} \sim \{r_n' + s_n'\}$, and $\{r_n s_n\} \sim \{r_n' s_n'\}$.

Proof: For $\varepsilon > 0$, there exists an N_1 such that if $n > N_1$, $|r_n - r_n'| < \varepsilon/2$ and an N_2 such that if $n > N_2$, $|s_n - s_n'| < \varepsilon/2$. Take $N = \max(N_1,N_2)$. Then, if $n > N$, $|(r_n + s_n) - (r_n' + s_n')| \leqq |r_n - r_n'| + |s_n - s_n'| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Hence, $\{r_n + s_n\} \sim \{r_n' + s_n'\}$.

To show $\{r_n s_n\} \sim \{r_n' s_n'\}$, let $\varepsilon > 0$. There exists $M_1 > 0$, $M_2 > 0$, $M_3 > 0$, and $M_4 > 0$ such that $|r_n| < M_1$, $|r_n'| < M_2$, $|s_n| < M_3$ and $|s_n'| < M_4$ for every n. Let $M = \max(M_1, M_2, M_3, M_4)$. We have $\{r_n\} \sim \{r_n'\}$ and $\{s_n\} \sim \{s_n'\}$ and, therefore, there exists N_1 such that if $n > N_1$, $|r_n - r_n'| < \frac{\varepsilon}{2M}$ and an N_2 so that if $n > N_2$, $|s_n - s_n'| < \frac{\varepsilon}{2M}$.

Let $N = \max(N_1, N_2)$. Then for $n > N$,

$$\begin{aligned} |r_n s_n - r_n' s_n'| &= |r_n s_n - r_n' s_n + r_n' s_n - r_n' s_n'| \\ &\leqq |r_n s_n - r_n' s_n| + |r_n' s_n - r_n' s_n'| \\ &\leqq |s_n| \cdot |r_n - r_n'| + |r_n'| \cdot |s_n - s_n'| \\ &< M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence, for every $\varepsilon > 0$, there is an N such that if $n > N$, $|r_n s_n - r_n' s_n'| < \varepsilon$; so $\{r_n s_n\} \sim \{r_n' s_n'\}$.

DEFINITION 2.3. If ρ and σ are equivalence classes of fundamental sequences of rational numbers, their <u>sum</u>, $\rho + \sigma$, is the set to which $\{r_n + s_n\}$ belongs and their <u>product</u>, $\rho\sigma$, is the set to which $\{r_n s_n\}$ belongs where $\{r_n\}$ is any element of ρ and $\{s_n\}$ is any element of σ.

Before proving that the set of equivalence classes of fundamental sequences of rational numbers is a field, we establish four lemmas.

LEMMA 2.4. If $\{r_n\}$ and $\{s_n\}$ are fundamental sequences of rational numbers, then $\{s_n - r_n\}$ is a fundamental sequence.

THEOREM 2.2. The set of equivalence classes of fundamental sequences of rational numbers, along with the operations defined in Definition 2.3, is a field.

Proof: Let F^* be the set of equivalence classes of fundamental sequences of rational numbers. To show that F^* is a field we show that the properties listed in Definition 2.1 are satisfied. Suppose that ρ, σ, τ are the equivalence classes of F^*. Suppose also that $\{r_n\} \epsilon \rho$, $\{s_n\} \epsilon \sigma$, and $\{t_n\} \epsilon \tau$.

(a) $\{r_n + (s_n + t_n)\}$ is a member of $\rho + (\sigma + \tau)$ and is equal to $\{(r_n + s_n) + t_n\}$ and hence is also a member of $(\rho+\sigma) + \tau$. Therefore, $\rho + (\sigma + \tau) = (\rho + \sigma) + \tau$. This follows from Theorem 1.1.

(b) $\{r_n + s_n\} = \{s_n + r_n\}$ and hence is a member of both $\rho + \sigma$ and $\sigma + \rho$. Therefore, $\rho + \sigma = \sigma + \rho$.

(c) Let $x_n = r_n - s_n$ for $n = 1,2,3,....$ Then $\{s_n + x_n\} = \{r_n\}$. If χ is the equivalence class to which $\{x_n\}$ belongs, then $\sigma + \chi = \rho$.

(d) $\{r_n(s_n t_n)\} = \{(r_n s_n) t_n\}$ and is a member of $\rho(\sigma\tau)$ and $(\rho\sigma)\tau$. This implies $\rho(\sigma\tau) = (\rho\sigma)\tau$.

(e) $\{r_n(s_n + t_n)\} = \{r_n s_n + r_n t_n\}$. Again, by Theorem 1.1, this implies $\rho(\sigma + \tau) = \rho\sigma + \rho\tau$. Also, $\{(r_n + s_n)t_n\} = \{r_n t_n + s_n t_n\}$ implies $(\rho + \sigma)\tau = \rho\tau + \sigma\tau$.

CHAPTER III

THE ORDER RELATION

We now impose order on this field and show that we have an ordered field.

LEMMA 3.1. If $\{r_n\}$ and $\{s_n\}$ are fundamental sequences of rational numbers, then either (1) $\{r_n\} \sim \{s_n\}$; or (2) there exists $k > 0$ and an N, such that for $n > N$, $r_n > s_n + k$ or $s_n > r_n + k$.

Proof: Suppose $\{r_n\}$ is not equivalent to $\{s_n\}$. Then there is a $k > 0$ such that for every positive integer, i, there is a $j > i$ so that $|r_j - s_j| > 2k$. Since $\{r_n\}$ and $\{s_n\}$ are fundamental sequences, there is an N_1 such that if $n,m > N_1$, $|r_n - r_m| < k/2$. There is an N_2 such that if $n,m > N_2$, $|s_n - s_m| < k/2$. Choose $N \geq \max(N_1,N_2)$ so that $|r_N - s_N| > 2k$. Then it follows that $r_N > s_N$ or $s_N > r_N$. Without loss of generality, suppose $r_N > s_N$. It follows that $r_N > s_N + 2k$. Also, for every $n > N$, $|r_N - r_n| < k/2$ and $|s_N - s_n| < k/2$. Hence $r_n > r_N - k/2 > s_N + 2k - k/2 = s_N + 3k/2$. Since $s_N > s_n - k/2$ we have $s_N + 3k/2 > s_n - k/2 + 3k/2$. This implies $s_N + 3k/2 > s_n + k$ and so $r_n > s_n + k$. Therefore, if $s_N > r_N$ and $n > N$, we have $s_n > r_n + k$.

LEMMA 3.2. If $\{r_n\}$, $\{r_n'\}$, $\{s_n\}$ and $\{s_n'\}$ are fundamental sequences of rational numbers with $\{r_n\} \sim \{r_n'\}$ and $\{s_n\} \sim \{s_n'\}$, the same relationship holds between $\{r_n'\}$ and $\{s_n'\}$, as exists between $\{r_n\}$ and $\{s_n\}$.

Proof: Suppose $\{r_n\} \sim \{s_n\}$. Then $\{r_n'\} \sim \{r_n\} \sim \{s_n\} \sim \{s_n'\}$ implies $\{r_n'\} \sim \{s_n'\}$.

Suppose now that $\{r_n\}$ is not equivalent to $\{s_n\}$. Suppose, also, without loss of generality that there is a $k > 0$ and an N_1 such that if $n > N_1$, $r_n > s_n + k$. Because of the equivalence relation there exist integers N_2 and N_3 such that if $n > N_2$, then $|r_n - r_n'| < k/3$ and if $n > N_3$ then $|s_n - s_n'| < k/3$. This implies that for every $n > N_2$, $r_n' > r_n - k/3$ and for $n > N_3$, $s_n > s_n' - k/3$. Let $N = \max(N_1, N_2, N_3)$. For every $n > N$, we have $r_n' > r_n - k/3 > s_n + k - k/3 > s_n + k - k/3 - k/3 = s_n + k/3$. Hence, $r_n' > s_n' + k/3$ so that there is an $l > 0$ ($l = k/3$) and an N such that for every $n > N$, $r_n' > s_n' + l$. This completes the proof.

We now define the order relation between the equivalence classes.

DEFINITION 3.1. Suppose that ρ and σ are equivalence classes of fundamental sequences of rational numbers. The statement that $\rho > \sigma$ means that if $\{r_n\} \in \rho$ and $\{s_n\} \in \sigma$, then there exists $k > 0$ such that if N is a positive integer, there is an integer $n > N$ such that $r_n > s_n + k$.

THEOREM 3.1. If ρ, σ, τ are equivalence classes of fundamental sequences of rational numbers, then

(1) for every ρ and σ, $\rho \neq \sigma$, either $\rho > \sigma$, or $\sigma > \rho$;

(2) if $\rho > \sigma$, then it is not true that $\sigma > \rho$;

(3) if $\rho > \sigma$ and $\sigma > \tau$, then $\rho > \tau$.

The first two assertions of the theorem follow immediately from these lemmas and the definition. We now show that the transitive property is also true.

Suppose $\{r_n\} \in \rho$, $\{s_n\} \in \sigma$, and $\{t_n\} \in \tau$. If $\rho > \sigma$, there exist $k > 0$ and an N_1 such that if $n > N_1$, $r_n > s_n + k$. If $\sigma > \tau$ and $l > 0$, there exists an N_2 such that if $n > N_2$, $s_n > t_n + l$. Thus there exists $m > 0$ ($m = l + k$) and an N ($N = \max(N_1,N_2)$) such that $r_n > t_n + m$ whenever $n > N$. Hence, $\rho > \tau$.

This completes the proof of the theorem. It leads to the following important proposition.

THEOREM 3.2. The equivalence classes of fundamental sequences of rational numbers form an ordered field.

Proof: Suppose that $\rho > \theta$ and $\sigma > \theta$. Select $\{r_n\} \in \rho$ and $\{s_n\} \in \sigma$. There exist rational numbers $k_1 > 0$ and $k_2 > 0$ and integers N_1 and N_2 so that if $n > N_1$, $r_n > k$, and if $n > N_2$, $s_n > k_2$. If $N = \max(N_1,N_2)$, then for $n > N$ $r_n + s_n > k_1 + k_2 > 0$ and $r_n s_n > k_1 k_2 > 0$. This implies that $\rho + \sigma > \theta$ and $\rho\sigma > \theta$.

We have shown that the set of all equivalence classes of fundamental sequences of rational numbers form an ordered field. We are aware that the rational number system itself is an ordered field. In investigating that field, we noted that every ordered field has a subfield the sequence 1, 1, 1,... . Hence, for every rational number, r, there exists exactly one equivalence class, ρ , which has the constant sequence, {r, r, r,...}.

Suppose now that ρ is an equivalence class and that the sequence $\{r_n\}$ is a member of ρ and $\{r_n\}$ has limit r, a rational number. Then the sequence {r, r, r,...} is a member of ρ; so ρ is in $F_1{}^*$ and the isomorphism maps into r. Hence, every sequence that has a rational sequential limit is a member of some equivalence class in $F_1{}^*$. This characterizes the members of $F_1{}^*$.

CHAPTER IV

COMPLETENESS

In investigating the field of rational numbers, we find that it does not possess completeness; that is, not every fundamental sequence has a limit in the rational number system. The following question naturally arises. Does this ordered field of fundamental sequences of rational numbers have the completeness property?

The purpose of this paper is to show that the field of all equivalence classes is complete. To do this, we use the notion of a fundamental sequence of fundamental sequences. We define it and then explain the notation.

DEFINITION 4.1. A sequence $\{r_{1,n}\}$, $\{r_{2,n}\}$, ..., $\{r_{m,n}\}$,..., of fundamental sequences is called a <u>fundamental sequence of fundamental sequences</u> if, for every $\varepsilon > 0$, there is an N such that if k, l > N, then there is a $\nu(k, l)$ such that if $\mu > \nu(k,l)$ then $|r_{k,\mu} - r_{l,\mu}| < \varepsilon$.

NOTATION: For $\varepsilon > 0$, there is an N such that for every $k, l > N$ the sequences $r_{k,1}, r_{k,2}, r_{k,3}, \ldots, r_{k,n} \ldots$, and $r_{\ell,1}, r_{\ell,2}, \ldots, r_{\ell,n}, \ldots$ are related in the following way. There is a positive integer $\nu(k,l)$, which may be different for other choices of k and l, such that for every $\mu > \nu(k,l)$, $|r_{k,\mu} - r_{\ell,\mu}| < \varepsilon$.

It is in the equivalence classes that the completeness property is satisfied. The following lemma allows us to define a fundamental sequence of equivalence classes.

LEMMA 4.1. Suppose $\{s_k\}$ and $\{r_l\}$ are sequences of fundamental sequences of rational numbers and $s_k \sim r_k$ for each k. Then if $\{r_{1,n}\}, \{r_{2,n}\}, \ldots$ is fundamental, then $\{s_k\}$ is also a fundamental sequence of fundamental sequences of rational numbers.

Proof: Suppose $\varepsilon > 0$. There is an N such that if $k,l > N$, then there is a $\nu(k,l)$ such that if $\mu > \nu(k,l)$, then $|r_{k,\mu} - r_{\ell,\mu}| < \varepsilon/3$. But, for every m, there is a $\nu(m)$ such that if $\mu > \nu(m)$ then $|r_{m,\mu} - s_{m,\mu}| < \varepsilon/3$. For every pair k,l of positive integers with $k,l > N$, let $\mu(k,l) = \max(\nu(k,l), \nu(k), \nu(l))$. For every $\mu > \mu(k,l)$ where $k,l > N$ we have

$$|s_{k,\mu} - s_{\ell,\mu}| \leq |s_{k,\mu} - r_{k,\mu}| + |r_{k,\mu} - r_{\ell,\mu}| + |r_{\ell,\mu} - s_{\ell,\mu}|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

It follows that $\{s_{1,n}\}$, $\{s_{2,n}\}, \ldots$ is a fundamental sequence of fundamental sequences of rational numbers.

DEFINITION 4.2. The sequence $\rho_1, \rho_2, \ldots$ of <u>equivalence classes</u> of fundamental sequences of rational numbers is <u>fundamental</u> if every sequence $\{r_{1,n}\}$, $\{r_{2,n}\}, \ldots$ is fundamental, where $\{r_{m,n}\} \in \rho_m$, $m = 1,2,\ldots.$

DEFINITION 4.3. If $\{r_{1,n}\}$, $\{r_{2,n}\}, \ldots$ is a fundamental sequence of fundamental sequences of rational numbers, a fundamental sequence $\{r_n\}$ is its limit if, for every $\varepsilon > 0$, there is an N such that for every $k > N$, there is a $\nu(k)$ such that if $n > \nu(k)$ then $|r_n - r_{k,n}| < \varepsilon$.

We can proceed now to the most basic part of this discussion. We prove the following theorem.

THEOREM 4.1. Every fundamental sequence of fundamental sequences of rational numbers has a limit.

Proof: Suppose that $\{r_{1,n}\}$, $\{r_{2,n}\}, \ldots, \{r_{m,n}\}, \ldots$ is a fundamental sequence of fundamental sequences of rational numbers. Let $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \ldots + \varepsilon_p + \ldots$ be a convergent series of positive rational terms.

(1) There are integers m_1 and n_1 such that

(a) if $n > n_1$, then $|r_{m_1,n_1} - r_{m_1,n}| < \varepsilon_1$

and

(b) if $m > m_1$, then there is a $\nu_1(m,m_1)$ such that if $\nu > \nu_1(m,m_1)$ then $|r_{m_1,\nu} - r_{m,\nu}| < \varepsilon_1$.

(2) There are integers k_2 and l_2 such that

(a) if $l > l_2$, then $|r_{k_2,\ell_2} - r_{k_2,\ell}| < 1/4\varepsilon_2$

and

(b) if $k > k_2$, then there is a $\nu(k,k_2)$ such that if $\nu > \nu(k,k_2)$, then

$$|r_{k,\nu} - r_{k_2,\nu}| < 1/4\varepsilon_2.$$

Select $m_2 > \max(k_2,m_1)$. Select $n_2 >$ $\max(l_2,n_1,\nu_1(m_2,m_1),\nu(m_2,k_2))$.

Then

(i) $|r_{m_1,n_1} - r_{m_2,n_2}|$

$$\leq |r_{m_1,n_1} - r_{m_1,n_2}| + |r_{m_1,n_2} - r_{m_2,n_2}|$$

$$< \varepsilon_1 + \varepsilon_1 = 2\varepsilon_1;$$

(ii) if $n > n_2$, then $n > k_2$, $n > l_2$, and $n > \nu(m_2,k_2)$, so

$$|r_{m_2,n_2} - r_{m_2,n}| \leq |r_{m_2,n_2} - r_{k_2,n_2}| + |r_{k_2,n_2} - r_{k_2,n}| + |r_{k_2,n} - r_{m_2,n}|$$

$$< 1/4\varepsilon_2 + |r_{k_2,n_2} - r_{k_2,\ell_2}| + |r_{k_2,\ell_2} - r_{k_2,n}| + 1/4\varepsilon_2$$

$$< \varepsilon_2;$$

(iii) for each $m > m_2$ let $\nu_2(m,m_2) =$

$\max(\nu(m,k_2),\nu(m_2,k_2))$.

If $m > m_2$ and $\nu > \nu_2(m,m_2)$, then $m > k_2$,

$m_2 > k_2$, $\nu > \nu(m,k_2)$ and $\nu > \nu(m_2,k_2)$ so

$$|r_{m,\nu} - r_{m_2,\nu}|$$

$$\leq |r_{m,\nu} - r_{k_2,\nu}| + |r_{k_2,\nu} - r_{m_2,\nu}|$$

$$< 1/4\varepsilon_2 + 1/4\varepsilon_2 < \varepsilon_2.$$

We can continue in a similar manner for each integer. The following argument completes the induction procedure.

(p) Suppose that p is an integer greater than 2. Suppose also that for each integer $q = 1,2,\ldots, p - 1$, there are numbers m_q and n_q such that the following statements are true:

(q - i) $|r_{m_q,n_q} - r_{m_i,n_i}| < 2\varepsilon_i$ for $i = q - 1$;

(q - ii) if $n > n_q$, then $|r_{m_q,n_q} - r_{m_q,n}| < \varepsilon_q$;

and

(q - iii) for each $m > m_q$, there is a number $\nu_q(m,m_q)$ such that if $\nu > \nu_q(m,m_q)$, then

$$|r_{m_q\nu} - r_{m,\nu}| < \varepsilon_q.$$

There are integers k_q and l_q such that

(a) if $l > l_q$, then $|r_{k_q,\ell_q} - r_{k_q,\ell}| < 1/4\varepsilon_q$;

and

(b) if $k > k_q$, then there is a $\nu(k,k_q)$ such that if $\nu > \nu(k,k_q)$, then

$$|r_{k,\nu} - r_{k_q,\nu}| < 1/4\varepsilon_q.$$

There are integers k_p and l_p such that

(a) if $l > l_p$, then $|r_{k_p,\ell_p} - r_{k_p,\ell}| < 1/4\varepsilon_p$;

and

(b) $k > k_p$, then there is a $\nu(k,k_p)$ such that if $\nu > \nu(k,k_p)$, then

$$|r_{k,\nu} - r_{k_p,\nu}| < 1/4\varepsilon_p.$$

Select $m_p > \max(k_p,m_q)$. Select $n_p > \max(l_p, n_q, \nu_q(m_p,m_q), \nu(m_p,k_p))$.

Then

(i) $|r_{m_q,n_q} - r_{m_p,n_p}|$

$$\leq |r_{m_q,n_q} - r_{m_q,n_p}| + |r_{m_q,n_p} - r_{m_p,n_p}|$$

$$< \varepsilon_q + \varepsilon_q = 2\varepsilon_q;$$

(ii) if $n > n_p$, then $n > k_p$, $n > l_p$ and $n > \nu(m_p,k_p)$,

so $|r_{m_p,n_p} - r_{m_p,n}| \leq |r_{m_p,n_q} - r_{k_p,n_q}| +$

$$|r_{k_p,n_p} - r_{k_p,n}| + |r_{k_p,n} - r_{m_p,n}|$$

$$< 1/4\varepsilon_p + |r_{k_p,n_p} - r_{k_p,\ell_p}| + |r_{k_p,\ell_p} - r_{k_p,n}| + 1/4\varepsilon_p < \varepsilon_p.$$

(iii) for each $m > m_p$, let $\nu_p(m,m_p) = \max(\nu(m,k_p),\nu(m_p,k_p))$.

If $m > m_p$ and $\nu > \nu_p(m,m_p)$, then $m > k_p$, $m_p > k_p$, $\nu > \nu(m,k_p)$ and $\nu > \nu(m_p,k_p)$ so

$$|r_{m,\nu} - r_{m_p,\nu}| \leq |r_{m,\nu} - r_{k_p,\nu}| + |r_{k_p,\nu} - r_{m_p,\nu}|$$

$$< 1/4\, \varepsilon_p + 1/4 \varepsilon_p < \varepsilon_p.$$

Let $\varepsilon > 0$. There is a positive integer p such that if $k > p$, $j > 0$ then $\varepsilon_k + \varepsilon_{k+1} + \ldots + \varepsilon_{k+j-1} < \varepsilon/2$; thus, for every $k > p$ and $j > 0$ we have

$$|r_{m_k,n_k} - r_{m_{k+j},n_{k+j}}| \leq |r_{m_k,n_m} - r_{m_{k+1},n_{k+1}}| + |r_{m_{k+1},n_{k+1}} - r_{m_{k+2},n_{k+2}}| + \ldots + |r_{m_{k+j-1},n_{k+j-1}} - r_{m_{k+j},n_{k+j}}|$$

$$< 2\varepsilon_k + 2\varepsilon_{k+1} + \ldots + 2\varepsilon_{k+j-1} < \varepsilon .$$

Hence, $\{r_{m_p,n_p}\}$ is a fundamental sequence.

Again take $\varepsilon > 0$. There is a p such that for $j > 0$, $\sum_{k=p}^{p+j} \varepsilon_k < \varepsilon/4$. Let $m > m_p$. Consider the fundamental sequence $r_{m_1}, r_{m_2}, \ldots$. There is an N such that if $\mu, \nu > N$ then $|r_{m,\mu} - r_{m,\nu}| < \varepsilon/4$. There is a $\mu > N$ such that $|r_{m,\mu} - r_{m_p,n_p}| < \varepsilon_p$. For every $\nu > \max(p,N)$ we have

$$|r_{m_p,n_p} - r_{m,\nu}| \leq |r_{m_p,n_p} - r_{m_p,n_p}| +$$

$$|r_{m_p,n_p} - r_{m,\mu}| + |r_{m,\mu} - r_{m,\nu}|$$

$$< \varepsilon/4 + 2\varepsilon/4 + \varepsilon/4 = \varepsilon.$$

This means that the fundamental sequence of fundamental sequences has a limit and the limit is itself a fundamental sequence.

This proof is complemented by the following theorem.

THEOREM 4.2. If $\{r_{1,n}\}$, $\{r_{2,n}\}$, ... is a fundamental sequence of fundamental sequences and if $\{r_{1,n}\} \sim \{s_{1,n}\}$, $\{r_{2,n}\} \sim \{s_{2,n}\}$, ..., then $\{s_{1,n}\}$, $\{s_{2,n}\}$, ... is a fundamental sequence of fundamental sequences and if $\{r_n\}$ and $\{s_n\}$ are their limits, then $\{r_n\} \sim \{s_n\}$.

Proof: From Lemma 4.1 we know that $\{s_{1,n}\}$, $\{s_{2,n}\}$, ... is necessarily a fundamental sequence. It remains only to show that their respective limits are equivalent. This follows from the definitions of equivalence relation and the limit of a fundamental sequence.

Let $\varepsilon > 0$. There exists N_1 such that if $m > N_1$ there is a $\nu(m)$ such that if $n > \nu(m)$ then $|r_{m,n} - r_n| < \varepsilon/3$. There exists N_2 such that if $m > N_2$ there is a $\nu(m)$ such that if $n > \nu(m)$ then $|s_{m,n} - s_n| < \varepsilon/3$. For every m there is a $\nu(m)$ such that if $n > \nu(m)$ then $|r_{m,n} - s_{m,n}| < \varepsilon/3$. For every m let $N = \max(N_1,N_2,\nu(m))$. Then for $n,m > N$, we have

$$|r_n - s_n| = |r_n - r_{m,n} + r_{m,n} - s_{m,n} + s_{m,n} - s_n|$$
$$\leq |r_{m,n} - r_n| + |r_{m,n} - s_{m,n}| + |s_{m,n} - s_n|$$
$$\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Hence, $\{r_n\} \sim \{s_n\}$. This completes the proof.

Suppose now that $\rho_1, \rho_2, \ldots$ is a sequence of equivalence classes of fundamental sequences. Suppose also that we select two sequences $\{r_{1,n}\}, \{r_{2,n}\}, \ldots$ and $\{s_{1,n}\}, \{s_{2,n}\}, \ldots$ such that $\{s_{k,n}\} \in \rho_k$ and $\{r_{k,n}\} \in \rho_k$ for each integer k. It follows from Theorem 4.1 and Theorem 4.2 that these two sequences have limits which are equivalent fundamental sequences and which belong to some equivalence class ρ. It follows then that every fundamental sequence of equivalence classes has a limit in the sense of the following definition.

DEFINITION 4.4. If $\rho_1, \rho_2, \ldots, \rho_m, \ldots$ is a fundamental sequence of equivalence classes, the limit is the equivalence class to which $\{r_n\}$ belongs, where $\{r_n\}$ is the limit of $\{r_{1,n}\}, \{r_{2,n}\}, \ldots$ and $\{r_{1,n}\} \in \rho_1, \{r_{2,n}\} \in \rho_2, \ldots, \{r_{m,n}\} \in \rho_m, \ldots.$

CONCLUSION

We have thus displayed Cantor's method for obtaining the real numbers from the rationals. Further discussion would show that the real numbers are a field, and that the real number system is complete. Essentially, this is what has been accomplished with the equivalence classes of rational numbers which give rise to the real numbers. The following theorem summarizes the notion of completeness of the real numbers.

THEOREM 5.1. Suppose that $\rho_1, \rho_2, \ldots$ is a sequence of equivalence classes. The following two statements are equivalent.

(i) If σ is an equivalence class, $\sigma > \theta$, then there is an integer, N, such that if $n > N$ and $m > N$, then $-\sigma < \rho_m - \rho_n < \sigma$.

(ii) There is an equivalence class, ρ, such that if $\sigma > \theta$, then there is an integer N such that if $n > N$, then $-\sigma < \rho - \rho_n < \sigma$.

Proof: We prove this by showing that the statements (i) and (ii) are equivalent to Definitions 4.2 and 4.3 respectively.

Definition 4.2 implies (i).

Let $\{r_{m,\mu}\} \in \rho_m$ and $\{r_{n,\mu}\} \in \rho_n$. $\rho_1, \rho_2, \ldots$

is a fundamental sequence of equivalence classes means that $\{r_{m,\mu}\}$ is fundamental for m = 1,2,3,... and $\{r_{n,\mu}\}$ is fundamental for n = 1,2,3,.... $\{r_{m,\mu} - r_{n,\mu}\}$ is a member of $\rho_m - \rho_n$.

LEMMA 5.1. If $\sigma > \theta$, then there is a rational number $\varepsilon > 0$ such that if τ_ε is the equivalence class containing $\{\varepsilon,\varepsilon,\varepsilon,...\}$, then $\theta < \tau_\varepsilon < \sigma$.

Proof: Suppose not. If k is a positive integer, let $\tau_{1/k}$ denote the equivalence class containing {1/k, 1/k,...}. By assumption $\theta < \sigma < \tau_{1/2k} < \tau_{1/k}$ for each k. If k is an integer, then there is a member, $\{s_n\}$, of σ and an integer N such that $|s_n - 0| < 1/k$ for each n > N. Hence $\{s_n\}$ has limit 0 and $\sigma = \theta$, contrary to hypothesis.

Suppose now that $\sigma > \theta$ and ε is a positive rational number such that $\tau_\varepsilon < \sigma$. There exists an N such that if m,n > N, then there is a $\nu(m,n)$ such that if $\mu > \nu(k,l)$, then $|r_{m,\mu} - r_{n,\mu}| < \varepsilon$. Then if $\tau_\varepsilon > \theta$ is the equivalence class to which $\{\varepsilon,\varepsilon,\varepsilon,...\}$ belongs, we have $-\sigma < -\tau_\varepsilon < \rho_m - \rho_n < \tau_\varepsilon < \sigma$.

(i) implies Definition 4.2.

Suppose $\varepsilon > 0$. Then there exists an equivalence class, σ, containing $\{\varepsilon,\varepsilon,\varepsilon,...\}$. There is an integer N such that if m,n > N, then $-\sigma < \rho_m - \rho_n < \sigma$. Let $\{r_{m,\mu}\} \in \rho_m$ and $\{r_{n,\mu}\} \in \rho_n$. It follows that the sequence $\{r_{m,\mu} - r_{n,\mu}\}$

belongs to the equivalence class $\rho_m - \rho_n$. Then we have from Definition 3.1 that for some $N > 0$ $|r_{m,\mu} - r_{n,\mu}| < \varepsilon$ if $M > N$. This implies that $\{r_{1,n}\}$, $\{r_{2,n}\}$,... is a fundamental sequence of fundamental sequences.

A similar argument can be used to show that (ii) implies Definition 4.3, and Definition 4.3 implies (ii). This proves the theorem.

As stated previously, our particular interest was in the technique itself, especially since most authors of elementary analysis texts emphasize the Dedekind method. It is interesting to see that the two approaches are essentially equivalent. In fact, the Dedekind cuts form an ordered field isomorphic with the field of equivalence classes of fundamental sequences of rational numbers.

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