On the Extendibility of a D(4)-Pair of Pell Numbers

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Abstract

A Diophantine m-tuple with property D(ℓ) is a set of m integers such that the product of any two integers plus ℓ results in a perfect square. This thesis establishes that a particular family of D(4) pairs of Pell numbers can be extended to a D(4) triple by exactly one Pell number. A similar result has been found for the Diophantine triples of Fibonacci numbers, a discussion of which is included in the first chapter of this thesis. This chapter finishes with a statement of the main result of my thesis, and the subsequent chapters discuss several topics in number theory which were used to prove the main result in chapter 5. Specifically, results about continued fractions, Pell-type equations, and linear forms in logarithms were used. These topics are the subjects of chapters 2, 3 and 4, which contain some history and discussions of the important results. The conclusion of this thesis discusses some possible generalizations.

Key words and phrases: Linear forms in logarithms; Diophantine triples; Pellian equations; Pell numbers.

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1 Introduction

1.1 The Property of Diophantus

It has been noticed by Diophantus of Alexandria that the set $\left\{\frac{1}{16}, \frac{33}{16}, \frac{68}{16}, \frac{105}{16}\right\}$ has the property that taking the product of any two numbers and adding one results in a rational square. Sets of integers have been found with similar properties. For example, consider the set $\{1, 3, 8\}$ and observe:

$$1 \times 3 + 1 = 2^2, \qquad 1 \times 8 + 1 = 3^2, \qquad 3 \times 8 + 1 = 5^2.$$

The set $\{1, 3, 8\}$ is an example of what's called a Diophantine triple with property $D(1)$. More generally, a Diophantine m-tuple with property $D(\ell)$ is a set $\{a_1, a_2, ..., a_m\}$, usually consisting of integers, such that $a_i a_j + \ell$ is a perfect square for any $i \neq j$. The first set of four integers with this property was found by Fermat, who noticed that $\{1, 3, 8, 120\}$ is a $D(1)$-quadruple. It was proved by Baker and Davenport in [1] that 120 is the only integer that can extend $\{1, 3, 8\}$ to a $D(1)$-quadruple. Their work uses results about lower bounds for linear forms in logarithms, which are also used to solve the main problem of my thesis in chapter 5. A discussion about linear forms in logarithms is included in chapter 4. Their paper is also the first appearance of a method called Baker-Davenport reduction, which is commonly used when studying Diophantine m-tuples.

1.2 The Fibonacci Sequence

It is well-known that Diophantine triples can be constructed from the Fibonacci sequence. The Fibonacci sequence is defined as

$$F_{n+1} = F_n + F_{n-1}, \quad F_0 = 0,\ F_1 = 1.$$

The sequence has been frequently studied in recent years in the context of Diophantine triples. It satisfies the well-known Catalan's identity:

$$F_n^2 - F_{n-r} F_{n+r} = (-1)^{n-r} F_r^2.$$

A simple family of $D(1)$-triples can be constructed by performing the following substitutions into Catalan's identity:

$$\begin{aligned}
(n, r) = (2n+1, 1) \quad &\Rightarrow \quad F_{2n} F_{2n+2} + F_1^2 = F_{2n+1}^2 \\
(n, r) = (2n+3, 1) \quad &\Rightarrow \quad F_{2n+2} F_{2n+4} + F_1^2 = F_{2n+3}^2 \\
(n, r) = (2n+2, 2) \quad &\Rightarrow \quad F_{2n} F_{2n+4} + F_2^2 = F_{2n+2}^2
\end{aligned}$$

Since $F_1 = F_2 = 1$, this shows that $\{F_{2n}, F_{2n+2}, F_{2n+4}\}$ is a $D(1)$-triple. It is possible to construct families of $D(\ell)$ triples for values of $\ell > 1$. The Fibonacci sequence can be solved to obtain the formula for the nth term $F_n = \frac{\varphi^n - \bar{\varphi}^n}{\varphi - \bar{\varphi}}$, where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\bar{\varphi} = \frac{1-\sqrt{5}}{2}$. This is called Binet's Formula. Define the sequence $L_n := \varphi^n + \bar{\varphi}^n$ (this sequence is known as the Lucas Numbers), and observe that $L_n F_n = (\varphi^n + \bar{\varphi}^n) \cdot \frac{\varphi^n - \bar{\varphi}^n}{\varphi - \bar{\varphi}} = \frac{\varphi^{2n} - \bar{\varphi}^{2n}}{\varphi - \bar{\varphi}} = F_{2n}$. Similar to before, we can perform substitutions into Catalan's identity. There will be an additional step of multiplying two of the equations by L_r^2 and applying the identity $L_r F_r = F_{2r}$.

$$\begin{aligned} F_{2n}F_{2n+2r} + F_r^2 &= F_{2n+r}^2 \\ F_{2n+2r}F_{2n+4r} + F_r^2 &= F_{2n+3r}^2 \\ F_{2n}F_{2n+4r} + F_{2r}^2 &= F_{2n+2r}^2 \end{aligned} \xrightarrow[\text{equations by } L_r^2]{\text{multiply first two}} \begin{aligned} L_r^2 F_{2n}F_{2n+2r} + F_{2r}^2 &= L_r^2 F_{2n+r}^2 \\ L_r^2 F_{2n+2r}F_{2n+4r} + F_{2r}^2 &= L_r^2 F_{2n+3r}^2 \\ F_{2n}F_{2n+4r} + F_{2r}^2 &= F_{2n+2r}^2 \end{aligned}$$

This shows that $\left\{F_{2n}, L_r^2 F_{2n+2r}, F_{2n+4r}\right\}$ is a $D\left(F_{2r}^2\right)$-triple. Taking $r = 2, 3$ gives the $D(9)$-triple $\{F_{2n}, 9F_{2n+4}, F_{2n+8}\}$ and the $D(64)$-triple $\{F_{2n}, 16F_{2n+6}, F_{2n+12}\}$, respectively. The cases of $r = 2, 3$ were studied in [4], where the authors were concerned with whether it is possible to extend $\left\{L_r^2 F_{2n+2r}, F_{2n+4r}\right\}$ to a $D\left(F_{2r}^2\right)$-triple by another Fibonacci number. In the case of $r = 2$, they showed for $n > 1$ that the only Fibonacci number which can extend $\left\{L_r^2 F_{2n+2r}, F_{2n+4r}\right\}$ to a $D(9)$-triple is F_{2n}. In the case of $r = 3$, they proved the same result under the assumption that $3|n$.

1.3 Binary Recurrence Relations

While the Fibonacci sequence has been commonly studied in the context of Diophantine triples, it is possible to construct Diophantine triples using any binary recurrence relation of the form

$$S_{n+1} = a \cdot S_n + b \cdot S_{n-1}, \quad S_0 = 0,\ S_1 \neq 0, \quad a^2 \neq -4b.$$

Just like the Fibonacci sequence, this more general recurrence can be solved for the n^{th} term to obtain the Binet-like formula $S_n = S_1 \frac{\alpha^n - \bar{\alpha}^n}{\alpha - \bar{\alpha}}$, where $\alpha,\ \bar{\alpha}$ are distinct roots of the polynomial $\lambda^2 - a\lambda - b$. I prove this fact below.

Theorem 1.1. $S_n = S_1 \frac{\alpha^n - \bar{\alpha}^n}{\alpha - \bar{\alpha}}$

Proof. The recurrence relation can be expressed using matrices:

$$\begin{bmatrix} S_n \\ S_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix} \begin{bmatrix} S_{n-1} \\ S_n \end{bmatrix} \xrightarrow{\text{apply recursively}} \begin{bmatrix} S_n \\ S_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}^n \begin{bmatrix} S_0 \\ S_1 \end{bmatrix}$$

The characteristic polynomial of $\begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}$ is $\lambda^2 - a\lambda - b$. Since $a^2 \neq -4b$, the eigenvalues $\alpha, \bar{\alpha}$ are distinct. The corresponding eigenvectors are $\begin{pmatrix} 1 \\ \alpha \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \bar{\alpha} \end{pmatrix}$, respectively. Therefore, $\begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}$ has the following diagonalization:

$$\begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix} = \frac{1}{\alpha - \bar{\alpha}} \begin{bmatrix} 1 & 1 \\ \alpha & \bar{\alpha} \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{bmatrix} \begin{bmatrix} -\bar{\alpha} & 1 \\ \alpha & -1 \end{bmatrix}$$

With this, we perform matrix multiplication to see that see that

$$\begin{pmatrix} S_n \\ S_{n+1} \end{pmatrix} = \frac{1}{\alpha - \bar{\alpha}} \begin{bmatrix} 1 & 1 \\ \alpha & \bar{\alpha} \end{bmatrix} \begin{bmatrix} \alpha^n & 0 \\ 0 & \bar{\alpha}^n \end{bmatrix} \begin{bmatrix} -\bar{\alpha} & 1 \\ \alpha & -1 \end{bmatrix} \begin{pmatrix} 0 \\ S_1 \end{pmatrix} = \frac{S_1}{\alpha - \bar{\alpha}} \begin{pmatrix} \alpha^n - \bar{\alpha}^n \\ \alpha^{n-1} - \bar{\alpha}^{n-1} \end{pmatrix}$$

And so we have the formula for S_n: $S_n = S_1 \frac{\alpha^n - \bar{\alpha}^n}{\alpha - \bar{\alpha}}$ □

Using this formula, it is possible to prove a variant of Catalan's identity:

$$S_n^2 - S_{n-r}S_{n+r} = (-b)^{n-r} S_r^2.$$

Define the sequence $C_n := \alpha^n + \bar{\alpha}^n$, so that $C_n S_n = S_{2n}$. Combining these two facts, we find that

$$\begin{aligned} C_r^2 S_{2n} S_{2n+2r} + b S_{2r}^2 &= C_r^2 S_{2n+r}^2 \\ C_r^2 S_{2n+2r} S_{2n+4r} + b S_{2r}^2 &= C_r^2 S_{2n+3r}^2 \\ S_{2n} S_{2n+4r} + b S_{2r}^2 &= S_{2n+2r}^2 \end{aligned}$$

and we therefore notice that $\left\{S_{2n}, C_r^2 S_{2n+2r}, S_{2n+4r}\right\}$ is a $D\left(bS_{2r}^2\right)$-triple.

1.4 The Pell Sequence

The problem of my thesis is similar to what was done in [4] and [11], where my work pertains to the Pell sequence as opposed to the Fibonacci sequence. The Pell numbers are historically noteworthy for being involved in the approximations of $\sqrt{2}$ by rational numbers. They were known as early as 130 C.E. by Theon of Smyrna [6], who used the term "side and diameter numbers" to describe the integer solutions to the equation

$$x^2 - 2y^2 = \pm 1, \quad x, y \in \mathbb{Z}$$

This equation is a particular case of Pell's equation, which is discussed in a later chapter in more detail. There is a straightforward procedure to solve this equation for relatively small x and y. We can simply substitute values of y one-by-one into the expression $2y^2 \pm 1$, and take note of whether or not the resulting number is a square. If it is a square, then that particular pair (x, y) is a solution. By this process we can find the first few solutions:

n	1	2	3	4	5	6	7	8	...
x_n	1	3	7	17	41	99	239	577	...
y_n	1	2	5	12	29	70	169	408	...

The sequence y_n is called the Pell sequence. It seems to follow the recurrence relation $y_{n+1} = 2y_n + y_{n-1}$, and the sequence x_n appears to follow $x_n = y_n + y_{n-1}$. Note that if (x_n, y_n) is a solution to $x^2 - 2y^2 = \pm 1$, then $\left(\frac{x_n}{y_n}\right)^2 - 2 = \pm\frac{1}{y_n^2}$. Since the right-hand side approaches zero as the denominators grow larger, we find that $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right)^2 = 2$. This means that solutions to $x^2 - 2y^2 = \pm 1$ can be used to produce a rational approximation $\frac{x}{y}$ of $\sqrt{2}$, and the approximations become more accurate as the size of the denominator increases.

Indeed, it can be verified that the Pell sequence, defined as

$$P_{n+1} = 2P_n + P_{n-1}, \quad P_0 = 0,\ P_1 = 1$$

can be used to generate solutions $(x, y) = (P_{n-1} + P_n,\ P_n)$ to the equation $x^2 - 2y^2 = \pm 1$. This fact was mentioned by Theon of Smyrna, although he did not provide a proof, but rather verified the first few cases. The proof I will give requires the Catalan-like identity:

$$P_n^2 = P_{n-1}P_{n+1} + (-1)^{n+1}$$

Since irrational numbers were contentious at this point in history, I would like to give a proof without reference to irrational numbers. First, using a similar recursive process to theorem 1.1, we can establish that

$$\begin{pmatrix} P_n \\ P_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} P_{n-1} \\ P_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

From these, we obtain the following matrix formula for the Pell-numbers:

$$\begin{pmatrix} P_{n-1} & P_n \\ P_n & P_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}^n$$

The Catalan-like identity results from taking the determinant of both sides. Now, we substitute

$(x, y) = (P_{n-1} + P_n,\ P_n)$ into the expression $x^2 - 2y^2$:

$$
\begin{aligned}
(P_{n-1} + P_n)^2 - 2P_n^2 &= P_{n-1}^2 + 2P_nP_{n-1} - P_n^2 \\
&= P_nP_{n-2} + 2P_nP_{n-1} - P_n^2 + (-1)^n && \bullet \text{ by the Catalan - like identity} \\
&= P_n\left(P_{n-2} + 2P_{n-1}\right) - P_n^2 + (-1)^n \\
&= P_n^2 - P_n^2 + (-1)^n && \bullet \text{ by definition of Pell numbers} \\
&= (-1)^n
\end{aligned}
$$

Thus it has been established that $(P_{n-1} + P_n,\ P_n)$ is a solution to $x^2 - 2y^2 = \pm 1$. This means that numbers of the form $\frac{P_{n-1}+P_n}{P_n}$ give rational approximations of $\sqrt{2}$, with the approximations becoming more accurate as we go deeper into the sequence.

Now I will state the problem which I have been asked to solve. From section 1.3, we can see that $\{P_{2n}, 4P_{2n+2}, P_{2n+4}\}$ is a $D(4)$-triple. The main problem of this thesis is to show that that the only Pell number which can extend the set $\{4P_{2n+2}, P_{2n+4}\}$ to a $D(4)$-triple is P_{2n}. That is, I prove the following theorem.

Theorem 1.2. *The set $\{P_k, 4P_{2n+2}, P_{2n+4}\}$ is a $D(4)$-triple if and only if $k = 2n$.*

The plan of attack to prove this result is as follows:

- Set up a Pellian equation for the triple $\{P_k, 4P_{2n+2}, P_{2n+4}\}$ and use a lemma from [4] to classify the full solution set.
- Use bounds on linear forms in logarithms to narrow down the possible solutions to a finite list.
- Use results about continued fractions, as well as a method called Baker-Davenport reduction, to reduce the number of solutions to a more computationally manageable size.
- Test the remaining possibilities one-by-one.

The solution to this problem uses results about continued fractions, pellian equations, and linear forms in logarithms. As such, the next three chapters discuss the important background theory and results about these topics.

2 Continued Fractions

2.1 Huygens' Planetarium

In 1680, the Dutch mathematician Christiaan Huygens had set out to construct a planetarium to model the solar system using interconnected gears [12]. He had encountered the problem of determining the number of teeth he ought to use in his interconnected gears, so that the planets in his model reflected the orbital periods of the planets in the solar system.

Using data on the orbital periods of the planets derived from Johannes Kepler, he was able to determine that it takes Mercury $\frac{25335}{105190}$ years to rotate once around the sun – that is, for every 105190 rotations Mercury completes around the sun, the Earth rotates around the sun 25335 times. Thus, in an ideal model, he would have one gear with 25335 teeth, and the other with 105190 teeth. For practical reasons, it was not possible for him to have gears with such large numbers of teeth. As such, he was confronted with another problem of approximating the ratio $\frac{25335}{105190}$ with a number sufficiently close to it, but with the numerator and denominator not exceeding a certain size.

To address this problem, Huygens was able to come up with the following representation of $\frac{25335}{105190}$:

$$\frac{25335}{105190} = \cfrac{1}{4 + \cfrac{1}{6 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{7 + \cfrac{1}{1 + \cfrac{1}{2}}}}}}}}}}}$$

The next section explains how to calculate such a representation. By truncating this expression after 5 divisions, he obtained the number

$$\cfrac{1}{4 + \cfrac{1}{6 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}}} = \frac{33}{137}$$

which is remarkably close to $\frac{25335}{105190}$, and with a much smaller numerator and denominator. The error of the approximation is

$$\left|\frac{25335}{105190} - \frac{33}{137}\right| = 0.00002602173...$$

With this result, Mercury could be modelled in the planetarium using one gear with 33 teeth, and another with 137 teeth. The representation Huygens found for $\frac{25335}{105190}$ is now known as a continued fraction. The fact that truncating the continued fraction after a certain number of terms results in a close approximation of $\frac{25335}{105190}$ is not a coincidence. Continued fractions can be used to generate rational approximations for real numbers, while having restrictions on the size of the denominator. Huygens' work on the planetarium is the first to demonstrate this application.

2.2 Representation of Real Numbers by Continued Fractions

A finite simple continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

It is possible to represent any rational number by a finite continued fraction with integer $a_0, a_1, ..., a_n$, using a method which parallels the Euclidean algorithm. To do this for a reduced fraction $\frac{p}{q} > 0$, we can perform computations according to the following recurrence relation:

$$\frac{r_k}{r_{k+1}} = \left\lfloor \frac{r_k}{r_{k+1}} \right\rfloor + \frac{r_{k+2}}{r_{k+1}}, \quad r_0 = p,\ r_1 = q.$$

Since the r_k are strictly decreasing, we will eventually find that $r_n = 0$. If we denote $a_k = \left\lfloor \frac{r_k}{r_{k+1}} \right\rfloor$, then by noting that $\frac{r_{k+1}}{r_k} = \frac{1}{a_k + r_{k+2}/r_{k+1}}$, a series of substitutions reveals that

$$\frac{r_0}{r_1} = a_0 + \frac{r_2}{r_1} = a_0 + \cfrac{1}{a_1 + \cfrac{r_3}{r_2}} = ... = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-2}}}}$$

It is obvious that a finite continued fraction is rational, so this establishes that a number is rational if and only if it can be represented by a finite simple continued fraction.

The process described above for finding a continued fraction representation of a rational number can be generalized to irrational real numbers as well. For a positive real number α define the recurrence relation

$$\alpha_{k+1} = (\alpha_k - \lfloor \alpha_k \rfloor)^{-1}, \quad \alpha_0 = \alpha.$$

One can rearrange this to see that $\alpha_k = \lfloor \alpha_k \rfloor + \frac{1}{\alpha_{k+1}}$. Therefore,

$$\alpha = \lfloor \alpha_0 \rfloor + (\alpha_0 - \lfloor \alpha_0 \rfloor) = \lfloor \alpha_0 \rfloor + \frac{1}{\alpha_1} = \lfloor \alpha_0 \rfloor + \cfrac{1}{\lfloor \alpha_1 \rfloor + \cfrac{1}{\alpha_2}} = \lfloor \alpha_0 \rfloor + \cfrac{1}{\lfloor \alpha_1 \rfloor + \cfrac{1}{\lfloor \alpha_2 \rfloor + \cfrac{1}{\ddots}}}$$

If we denote $a_k = \lfloor \alpha_k \rfloor$, then it appears as though we can represent α by the infinite continued fraction

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}, \quad a_0, a_1, a_2, ... \in \mathbb{Z}^+$$

Here the $a_0, a_1, a_2, ...$ are called partial quotients. As with all infinite processes, there still remains the question of convergence. It is a fundamental fact in the theory of continued fractions that every continued fraction with positive integer partial quotients converges to an irrational real number, and that every irrational real number can be represented in a unique way as an infinite continued fraction whose partial quotients are positive integers, except for the first partial quotient which may be any integer. (See Theorem 1.2.13 in [2]).

2.3 Some Definitions and Basic Results

The theory of continued fractions is used in each of the upcoming chapters. For further discussion, the introduction of some common terminology is in order.

Definition 2.1 (Infinite Simple Continued Fraction)**.** *Let $a_0, a_1, a_2, ...$ be real numbers. Define the expression*

$$\zeta := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

This is called an infinite simple continued fraction, and is denoted more compactly by $[a_0, a_1, a_2, a_3, ...]$.

- *The number a_j is called the jth partial quotient of ζ.*

- *The following expression, which we denote by $[a_0, a_1, a_2, ...a_n]$ is called the nth convergent of ζ:*

$$[a_0, a_1, a_2, ...a_n] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{... + a_{n-1} + \cfrac{1}{a_n}}}}$$

- *A continued fraction with a repeating block of partial quotients is called a periodic continued fraction, and is denoted by*
- $[a_0, a_1, a_2, ...a_k, \overline{a_{k+1}, a_{k+2}, ...a_{k+\ell}}]$
- *where $[a_0, a_1, a_2, ...a_k]$ are the initial block of partial quotients, which is followed by the block $[a_{k+1}, a_{k+2}, ...a_{k+\ell}]$ of partial quotients which repeats indefinitely. Here the length of the repeating block, ℓ, is called the period of ζ.*

The convergents of a continued fraction are given by the following recurrence relations:

Theorem 2.1. *Let $\frac{p_k}{q_k} = [a_0, a_1, a_2, ..., a_k]$. Then for $n \geq 0$,*

$$p_n = a_n p_{n-1} + p_{n-2} \quad \textit{and} \quad q_n = a_n q_{n-1} + q_{n-2}$$

where it's defined that $p_{-2} := 0,\ p_{-1} := 1,\ q_{-2} := 1,\ q_{-1} := 0$.

The convergents satisfy the following identity which is frequently useful:

Theorem 2.2. *For $n \geq 0$,*

$$q_n p_{n-2} - p_n q_{n-1} = (-1)^n$$

Quadratic irrational numbers can be characterized in terms of periodic continued fractions.

Theorem 2.3. *A real number α is a quadratic irrational if and only if its simple continued fraction representation is eventually periodic.*

Theorem 2.4. *If $d \in \mathbb{Z}^+$ is not a perfect square, then*

$$\sqrt{d} = \left[a_0, \overline{a_1, ..., a_{n-1}, 2a_0}\right]$$

where $a_0 = \left\lfloor \sqrt{d} \right\rfloor$, and $a_{n-1} = a_1, a_{n-2} = a_2, ...$ are positive integers.

These theorems are theorem 1.3.8 and 1.3.9 in [2]. The latter theorem is useful in solving Pell's equation, which is discussed in the next chapter.

The upcoming lemmas 2.5 and 2.6 are used to prove lemma 2.11, which is used in section 5.4. They are also used to prove theorem 2.7, which establishes that successive convergents are closer rational approximations of an irrational number.

Lemma 2.5. *If $a_i \in \mathbb{Z}^+$ for each $i \in \mathbb{N}$, then $a_0 < [a_0, a_1, a_2, ...] < a_0 + 1$.*

Proof. It's evident that $[a_0, a_1, a_2, ...] = a_0 + \frac{1}{[a_1, a_2, a_3, ...]} > a_0$. Now suppose, to the contrary, that the second inequality doesn't hold. Then

$$[a_0, a_1, a_2, ...] = a_0 + \frac{1}{[a_1, a_2, a_3, ...]} \geq a_0 + 1 \quad \Rightarrow \quad 1 \geq [a_1, a_2, a_3, ...] \quad \Rightarrow \quad 1 \geq a_1 + \frac{1}{[a_2, a_3, a_4, ...]} > a_1$$

which is absurd. □

Lemma 2.6. *Let $\alpha = [a_0, a_1, a_2, ...]$ for integers $a_0, a_1, a_2, ...$. Define $x_i = [a_i, a_{i+1}, ...]$. Then*

$$\alpha = \frac{(-1)^k}{q_k \left(x_{k+1} q_k + q_{k-1}\right)}$$

Proof. By theorem 2.1 we have

$$a = [a_0, a_1, ..., a_k, x_{k+1}] = \frac{x_{k+1} p_k + p_{k-1}}{x_{k+1} q_k + q_{k-1}}$$

Pairing this result with theorem 1.2,

$$a - \frac{p_k}{q_k} = \frac{p_{k-1} q_k - q_{k-1} p_k}{q_k \left(x_{k+1} q_k + q_{k-1}\right)} = \frac{(-1)^k}{q_k \left(x_{k+1} q_k + q_{k-1}\right)}$$

□

2.4 Approximation of Irrational Real Numbers using Rational Numbers

We have seen that an irrational real number α can be represented by a convergent infinite continued fraction. Each successive convergent is closer approximations of α. This is a corollary of the following theorem:

Theorem 2.7. *Let α be an irrational number and let $\frac{p_k}{q_k}$ denote convergents of its simple continued fraction. Then*

$$q_n \left| \alpha - \frac{p_n}{q_n} \right| > q_{n+1} \left| \alpha - \frac{p_{n+1}}{q_{n+1}} \right|$$

Proof. By lemma 2.6,

$$q_k \left| \alpha - \frac{p_k}{q_k} \right| = \frac{1}{x_{k+1} q_k + q_{k-1}} \quad \text{where} \quad x_{k+1} = [a_{k+1}, a_{k+2}, a_{k+3}, ...]$$

By lemma 2.5, we know that

$$a_{k+1} < x_{k+1} < a_{k+1} + 1.$$

From this and theorem 2.1, it follows that

- $x_{k+1}q_k + q_{k-1} \geq a_{k+1}q_k + q_{k-1} = q_{k+1}$
- $x_{k+1}q_k + q_{k-1} = (x_{k+1} - a_{k+1})\, q_k + q_{k+1} < q_k + q_{k+1}$

Therefore

$$\frac{1}{q_k + q_{k+1}} < q_k \left|\alpha - \frac{p_k}{q_k}\right| \leq \frac{1}{q_{k+1}}$$

Note also the inequality $q_k + q_{k+1} \leq q_k + a_{k+2}q_{k+1} = q_{k+2}$. Therefore,

$$q_{k+1}\left|\alpha - \frac{p_{k+1}}{q_{k+1}}\right| \leq \frac{1}{q_{k+2}} \leq \frac{1}{q_k + q_{k+1}} < q_k \left|\alpha - \frac{p_k}{q_k}\right|$$

□

An important application of continued fractions is to approximate irrational real numbers by rational numbers, with restrictions on the size of the denominator. If $\frac{p_n}{q_n}$ is a convergent of a real number α, then $\frac{p_n}{q_n}$ is closer to α than any other rational number with lesser or equal denominator. That is to say that

$$\left|\alpha - \frac{p_n}{q_n}\right| < \left|\alpha - \frac{p'}{q'}\right| \quad \text{for any } p'/q' \neq p_n/q_n \text{ where } 0 < q' \leq q_n.$$

In this sense, convergents of continued fractions are said to give the best approximations of α. This is a corollary of the following theorem.

Theorem 2.8. *Let $\alpha > 0$ be a real number. If $\frac{p_n}{q_n}$, $n \geq 2$ is a convergent of α, then*

$$|q_n\alpha - p_n| < |q\alpha - p|$$

for any p/q which is not a convergent of α, where $0 < q \leq q_n$.

Proof. If $q = q_n$, then it means $p \neq p_n$. Note that

$$\left|\frac{p}{q} - \frac{p_n}{q_n}\right| = \frac{|p - p_n|}{q_n} \geq \frac{1}{q_n} \quad \text{and} \quad \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}} < \frac{1}{2q_n}$$

Where the second inequality is true since $q_{n+1} > 2$ for $n \geq 2$. Therefore,

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{2q_n} = \frac{1}{q_n} - \frac{1}{2q_n} < \left|\frac{p}{q} - \frac{p_n}{q_n}\right| - \left|\alpha - \frac{p_n}{q_n}\right| \leq \left|\alpha - \frac{p}{q}\right|$$

Multiplying the inequality by $q = q_n$ gives $|q_n\alpha - p_n| < |q\alpha - p|$.

Now suppose $0 < q < q_n$. Consider the linear system

$$q_n x + q_{n-1} y = q$$
$$p_n x + p_{n-1} y = p$$

By theorem 2.2 we know that

$$\det \begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix} = q_n p_{n-1} - q_{n-1} p_n = (-1)^n$$

This means that the system solvable for integer values of x and y. Note that x and y are nonzero. If one of them was 0, it would make $\frac{p}{q}$ a convergent of α. Since $q_n x + q_{n-1} y = q$ and $0 < q < q_n$, it means x and y are of opposite signs. By lemma 2.6, we also know that $q_n\alpha - p_n$ and $q_{n-1}\alpha - p_{n-1}$ are of opposite signs. Therefore,

$$|q\alpha - p| = |q_n\alpha - p_n|\,|x| + |q_{n-1}\alpha - p_{n-1}|\,|y| > |q_{n-1}\alpha - p_{n-1}| > |q_n\alpha - p_n|$$

□

The converse of this theorem is also true.

Theorem 2.9. *If $|q'\alpha - p'| < |q\alpha - p|$ for all $p/q \neq p'/q'$ with $1 < q \leq q'$, then $\frac{p'}{q'}$ is a convergent of the continued fraction of α.*

Proof. Suppose, to the contrary, that $\frac{p'}{q'}$ is not a convergent of α. For any $q' > 1$ there are convergents q_{k-1} and q_k such that $q_{k-1} < q' \leq q_k$. In the proof of the previous theorem, we have established in this case that $|q_{k-1}\alpha - p_{k-1}| < |q'\alpha - p'|$. Since $q_{k-1} < q'$, this means $\frac{p'}{q'}$ can not satisfy the premise of the theorem. Thus, $\frac{p'}{q'}$ is a convergent. □

The next theorem is a criterion due to Legendre, and is used in section 5.4. It gives a condition for when a rational number is a convergent of a continued fraction representation of an irrational number.

Lemma 2.10. *Let α be an irrational number. Let $p, q \in \mathbb{Z}$ with $q > 1$ such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$. Then $\frac{p}{q}$ is a convergent of α.*

Proof. Suppose $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$. Let $\frac{p'}{q'} \neq \frac{p}{q}$ satisfy $|q'\alpha - p'| \leq |q\alpha - p| = q\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q}$. Then

$$\frac{1}{qq'} \leq \frac{|pq' - qp'|}{qq'} = \left|\frac{p}{q} - \frac{p'}{q'}\right| \leq \left|\alpha - \frac{p}{q}\right| + \left|\alpha - \frac{p'}{q'}\right| < \frac{1}{2q^2} + \frac{1}{2qq'}$$

Rearranging this inequality reveals that $q < q'$. By the previous theorem, this implies that $\frac{p}{q}$ is a convergent of α. □

The next result is used 5.4 and gives bounds for how well a convergent approximates an irrational number in terms of the subsequent partial quotient in its continued fraction representation.

Lemma 2.11. *Let x be an irrational number, let $\frac{p_k}{q_k}$ be the kth convergent of its continued fraction representation, and let a_{k+1} be its $(k+1)^{st}$ partial quotient. The following inequality holds:*

$$\frac{1}{q_k^2\left(a_{k+1}+2\right)} < \left|x - \frac{p_k}{q_k}\right| \leq \frac{1}{q_k^2 a_{k+1}}$$

Proof. By lemma 2.6,

$$\left|x - \frac{p_k}{q_k}\right| = \frac{1}{q_k\left(x_{k+1}q_k + q_{k-1}\right)} \quad \text{where} \quad x_{k+1} = [a_{k+1}, a_{k+2}, a_{k+3}, ...]$$

By lemma 2.5, we know that $a_{k+1} < x_{k+1} < a_{k+1} + 1$. Since $q_{k-1} < q_k$, it follows that

$$\left|x - \frac{p_k}{q_k}\right| = \frac{1}{q_k\left(x_{k+1}q_k + q_{k-1}\right)} > \frac{1}{q_k\left(\left(a_{k+1}+1\right)q_k + q_{k-1}\right)} > \frac{1}{q_k\left(\left(a_{k+1}+1\right)q_k + q_k\right)} = \frac{1}{q_k^2\left(a_{k+1}+2\right)}$$

We can also see that

$$\left|x - \frac{p_k}{q_k}\right| = \frac{1}{q_k\left(x_{k+1}q_k + q_{k-1}\right)} < \frac{1}{q_k\left(a_{k+1}q_k + q_{k-1}\right)} < \frac{1}{q_k^2 a_{k+1}}$$

□

3 Pellian Equations

3.1 Pell's Equation

Let d be a positive integer which is not a perfect square. The Diophantine equation $x^2 - dy^2 = 1$ is called Pell's equation. As this is a Diophantine equation, we are interested in its integer solutions. A solution (x_0, y_0) in which x_0 and y_0 have their smallest positive values is called the fundamental solution. Note that it is in fact enough to say that x_0 is minimized, as this will imply that y_0 is minimized also. To see this, suppose that

$$x_1^2 - dy_1^2 = 1, \quad x_2^2 - dy_2^2 = 1, \quad 0 < x_1 < x_2$$

Then

$$1 = x_1^2 - dy_1^2 < x_2^2 - dy_1^2 = x_2^2 - dy_2^2 + d\left(y_2^2 - y_1^2\right) = 1 + d\left(y_2^2 - y_1^2\right)$$

If we assume y_1 and y_2 are positive, this implies that $y_2 > y_1$. We could similarly show that the minimal x_0 is uniquely determined by the minimal y_0.

The fundamental solution to Pell's equation can be found with the use of continued fractions, as described by the next theorem:

Theorem 3.1. *Let n be the period of the continued fraction of $\sqrt{d}$. Then the fundamental solution to Pell's equation is*

$$(x_0, y_0) = \begin{cases} (p_{k-1}, q_{k-1}) & \text{if } k \text{ is even} \\ (p_{2k-1}, q_{2k-1}) & \text{if } k \text{ is odd} \end{cases}$$

Where $\frac{p_k}{q_k}$ is the kth convergent of $\sqrt{d}$.

The proof of this result uses the fact that $\sqrt{d}$ has a periodic continued fraction representation. More information on solving Pell's equation using continued fractions can be found in [6].

The general solution to Pell's equation can be expressed in terms of the fundamental solution, in accordance with the following result, which is theorem 104 in [10].

Theorem 3.2. *Pell's equation has infinitely many solutions. Moreover, all of the solutions with positive x and y are of the form*

$$x_n + y_n\sqrt{d} = \left(x_0 + y_0\sqrt{d}\right)^n, \quad n = 0, 1, 2, 3, ...$$

Where (x_0, y_0) is the fundamental solution.

3.2 Generalized Pell's Equation

There is also the generalized Pell's equation, which is the Diophantine equation

$$u^2 - dv^2 = N, \quad N \in \mathbb{Z}^+$$

We are interested in classifying the entire solution set of this equation. To do so, we will use the related Pell's equation $x^2 - dy^2 = 1$. If $u' + v'\sqrt{d}$ is a particular solution to $u^2 - dv^2 = N$, and $x_0 + y_0\sqrt{d}$ is the fundamental solution to $x^2 - dy^2 = 1$, then for any integer n,

$$u + v\sqrt{d} = \left(u' + v'\sqrt{d}\right)\left(x_0 + y_0\sqrt{d}\right)^n$$

is also a solution to $u^2 - dv^2 = N$. The set $\left\{u + v\sqrt{d} = \left(u' + v'\sqrt{d}\right)\left(x_0 + y_0\sqrt{d}\right)^n : n \in \mathbb{Z}\right\}$ forms what is called a class of solutions of $u^2 - dv^2 = N$. The generalized Pell's equation may have multiple classes of solutions. For example, consider the equation

$$u^2 - 5y^2 = 20$$

It can be checked that $u_1 + v_1\sqrt{d} = 5 + \sqrt{5}$ and $u_2 + v_2\sqrt{d} = 10 + 4\sqrt{5}$ are solutions to this equation. For these two solutions to be in the same class, there would need to be an integer n such that

$$10 + 4\sqrt{5} = \left(5 + \sqrt{5}\right)\left(9 + 4\sqrt{5}\right)^n \quad \Rightarrow \quad \left(9 + 4\sqrt{5}\right)^n = \tfrac{10+4\sqrt{5}}{5+\sqrt{5}} = \tfrac{3}{2} + \tfrac{1}{2}\sqrt{5}$$

But the coefficients from expanding $\left(9 + 4\sqrt{5}\right)^n$ would be integers, so these two solutions cannot be of the same class. This particular example can be generalized to classify whether two solutions are of the same class. Note that for two solutions $u + v\sqrt{d}$ and $u' + v'\sqrt{d}$ to be of the same class, there must be an integer n such that

$$u + d\sqrt{v} = \left(u' + v'\sqrt{d}\right)\left(x_0 + y_0\sqrt{d}\right)^n \quad \Rightarrow \quad \frac{u + d\sqrt{v}}{u' + v'\sqrt{d}} = \left(x_0 + y_0\sqrt{d}\right)^n$$

It is possible to write $\frac{u+v\sqrt{d}}{u'+v'\sqrt{d}} = s + d\sqrt{t}$. If s and t are not both integers, then the two solutions cannot be of the same class since expanding $\left(x_0 + y_0\sqrt{d}\right)^n$ results in integer coefficients. If s and t are both integers, then $s + d\sqrt{t}$ is a solution to the Pell's equation $x^2 - dy^2 = 1$, so $s + t\sqrt{d} = \left(x_0 + y_0\sqrt{d}\right)^n$ for some $n \in \mathbb{Z}$. Therefore $u + d\sqrt{v} = \left(u' + v'\sqrt{d}\right)\left(x_0 + y_0\sqrt{d}\right)^n$, meaning the solutions are of the same class.

Although a generalized Pell's equation can have multiple solution classes, it can be shown that there are finitely many. In [10], Nagell defines the fundamental solution of a particular class as the solution (u_0, v_0) with the smallest nonnegative v occurring in that class. This restriction also implies $|u|$ is the smallest in the class. Section 58 of [10] establishes bounds which narrow down the possible fundamental solutions to a finite list. Since there are finitely many fundamental solutions, and each class is entirely determined by one of its solutions, it means there are finitely many classes of solutions to a generalized Pell's equation.

3.3 Lemma on Pellian Equations

The following result gives a general solution to a certain class of Pellian equations. It is a generalization of a result from [11], and is proven and applied in [4]. This lemma will be of use in chapter 5 of this thesis.

Lemma 3.3. *Let $\{a, b, c\}$ be a $D\left(l^2\right)$-triple – that is, there exist positive integers r, s, t such that*

$$ab + l^2 = r^2, \quad ac + l^2 = s^2, \quad \text{and} \quad bc + l^2 = t^2$$

Suppose that $a < b < a\left(4 + \frac{4}{l^2}\right)$. If one of the following conditions holds:

i) $l = 2$

ii) *l is an odd prime and $l|ab$, or*

iii) *$l^2|a$ or $l^2|b$*

Then all solutions of the equation

$$at^2 - bs^2 = l^2\left(a - b\right)$$

are of the form

$$t\sqrt{a} + s\sqrt{b} = \left(\pm l\sqrt{a} + l\sqrt{b}\right)\left(\frac{r + \sqrt{ab}}{l}\right)^{\nu} \quad \nu \in \mathbb{Z}^+$$

4 Linear Forms in Logarithms

4.1 Hilbert's Seventh Problem

In the year 1900, David Hilbert published a list of 23 unsolved problems which he believed would have a major impact on mathematics. Among those problems, the seventh problem inquired on the transcendence of α^β where $\alpha \neq 0, 1$ is algebraic and β is an algebraic irrational. This problem was solved by Gelfond and Schneider in 1935, where they obtained the following result:

If α and β are nonzero algebraic numbers with $\log \alpha$ and $\log \beta$ linearly independent over $\mathbb{Q}$, then $\log \alpha$ and $\log \beta$ are linearly independent over the algebraic numbers.

The answer to Hilbert's seventh problem is a consequence of this result. Let $\alpha \neq 0, 1$ be algebraic and $\beta \in \mathbb{Q}$ be irrational. Suppose that α^β is algebraic. This would make $\log\left(\alpha^\beta\right)$ and $\log \alpha$ linearly dependent over the algebraic numbers:

$$\log\left(\alpha^\beta\right) - \beta \log \alpha = 0$$

By Gelfond and Schneider's theorem, this would imply that $\log\left(\alpha^\beta\right)$ and $\log \alpha$ are linearly dependent over $\mathbb{Q}$ - that is, there is a rational number b such that

$$\log\left(\alpha^\beta\right) - b \log \alpha = 0$$

After rearranging we find that $\beta = b$, which contradicts the irrationality of β. Thus, it must be the case that α^β is transcendental.

In 1966, Baker was able to generalize the theorem of Gelfond and Schneider to an arbitrary number of logarithms:

If $\alpha_1, \alpha_2, ..., \alpha_m$ are nonzero algebraic numbers with $\log \alpha_1, ..., \log \alpha_m$ linearly independent over $\mathbb{Q}$, then $\beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 + ... + \beta_m \log \alpha_m \neq 0$ for any algebraic numbers $\beta_1, ..., \beta_m$ not all zero.

Furthermore, Baker was interested in resolving the question of how far away from zero a linear combination of logarithms of algebraic numbers is. He was successful, and the results had far-reaching applications to other areas of number theory.

4.2 Theorems of Matveev and Laurent

Definition 4.1. *(Linear Form in Logarithms) Let $\alpha_1, \alpha_2, ..., \alpha_n$ and $\beta_0, \beta_1, \beta_2, ..., \beta_n$ be complex algebraic numbers. A linear form in logarithms is an expression of the form*

$$\Lambda = \beta_0 + \beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 + ... + \beta_n \log \alpha_n$$

Where log denotes any determination of the logarithm.

Since Baker's work, there have been several results which give lower bounds for linear forms in logarithms. By using these lower bounds, linear forms in logarithms have applications in solving Diophantine equations. A typical strategy is as follows:

- Rewrite the Diophantine equation into an exponential equation with variables in the exponents.
- Associate the exponential equation to a linear form in logarithms, Λ.
- Using standard algebraic manipulations, find an upper bound for Λ .
- Using results about linear forms in logarithms, obtain a lower bound for Λ .
- Compare the upper and lower bounds to narrow down the possible solutions to a finite list.
- Reduce the size of this list to a more manageable size using various methods such as Baker-Davenport reduction.
- Test the remaining possible solutions one-by-one.

For examples of applications, see [2].

Two results for lower bounds are stated below. The first is due to Matveev, and the second is due to Laurent. Both of these theorems require the definition of absolute logarithmic height, which is defined as follows:

Definition 4.2. *(absolute logarithmic height) Let γ be an algebraic number with minimal polynomial over $\mathbb{Z}$ is a $\prod_{j=1}^{d} \left(x - \gamma^{(j)}\right)$, where $\gamma^{(1)}, \gamma^{(2)}, ..., \gamma^{(d)}$ are the conjugates of γ, including γ. Define*

$$h(\gamma) = \frac{1}{d}\left(\log a + \sum_{j=1}^{d} \log \max\left(1, \left|\gamma^{(j)}\right|\right)\right)$$

The following theorem due to Matveev is from [9].

Theorem 4.1. *Let Λ be a linear form in logarithms of multiplicatively independent, totally real algebraic numbers $\alpha_1, \alpha_2, ..., \alpha_N$ with nonzero rational integer coefficients $b_1, b_2, ..., b_N$. Let $h(\alpha_j)$ denote the absolute logarithmic height of α_j. Let D be the degree of $\mathbb{Q}(\alpha_1, \alpha_2, ..., \alpha_N)$ as a field extension over $\mathbb{Q}$. Define the numbers A_j and E so that*

$$A_j \geq \max\left\{Dh(\alpha_j), |\log \alpha_j|\right\}, \quad 1 \leq j \leq N \quad \textit{and} \quad E = \max\left\{1, \max\left\{|b_j|\, \tfrac{A_j}{A_N};\ 1 \leq j \leq N\right\}\right\}$$

Then

$$\log|\Lambda| > -C(N)\, C_0 W_o D^2 \Omega$$

where

$$C(N) = \tfrac{8}{(N-1)!}(N+2)(2N+3)(4e(N+1))^{N+1} \qquad C_0 = \log\left(e^{4.4N+7}N^{5.5}D^2\log(eD)\right)$$
$$W_0 = \log(1.5eED\log(eD)) \qquad \Omega = A_1 \cdot A_2 ... \cdot A_N$$

The following theorem on linear forms in two logarithms is due to Laurent [7].

Theorem 4.2. *Let $\gamma_1 > 1$ and $\gamma_2 > 1$ be two real multiplicatively independent algebraic numbers, $b_1, b_2 \in \mathbb{Z}$ not both 0, and $\Lambda = b_2 \log\gamma_2 - b_1 \log\gamma_1$. Let*

$$D = [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}].$$

Let

$$h_i \geq \max\left\{h(\gamma_i), \frac{|\log\gamma_i|}{D}, \frac{1}{D}\right\} \quad \text{for } i = 1, 2 \qquad b' \geq \frac{|b_1|}{Dh_2} + \frac{|b_2|}{Dh_1}$$

Then $\log|\Lambda| \geq -17.9 \cdot D^4\left(\max\left\{\log b' + 0.38, \frac{30}{D}, 1\right\}\right)^2 h_1 h_2$.

4.3 Baker-Davenport Reduction

After applying the previous two theorems, the bounds may still be too large to practically check the remaining possibilities. The following result due to Dujella [3] is a variation of a method first used by Baker and Davenport in [1] to reduce the bounds to a more manageable size. It is part (a) of lemma 5 in [3].

Remark. Dujella's original lemma in [3] had the requirement that $q > 6M$. His lemma had two parts, and this requirement is only necessary for part (b) of his lemma, which is not needed in this thesis. As such, I have omitted this requirement in the theorem below. I have included a proof to show that there is no need to assume that $q > 6M$.

Lemma 4.3. *Assume κ and μ are real numbers and M is a positive integer. Let p/q be the convergent of the continued fraction expansion of κ such that $q > 1$, and let*

$$\varepsilon = \|\mu q\| - M \cdot \|\kappa q\|$$

where $\|\cdot\|$ denotes the distance to the nearest integer. If $\varepsilon > 0$, then there is no solution to the inequality

$$0 < j\kappa - k + \mu < AB^{-j}$$

In integers j and k with

$$\frac{\log(Aq/\varepsilon)}{\log B} \leq j \leq M$$

.

Proof. Since p/q is a convergent of κ, theorem 2.9 implies that

$$\|\kappa q\| = |\kappa q - p|\,.$$

Suppose $0 \leq j \leq M$ and that $0 < j\kappa - k + \mu < AB^{-j}$. Applying the reverse triangle inequality, we find that

$$\begin{aligned} q\,(j\kappa - k + \mu) &= \mu q + jp - kq + j\,(\kappa q - p) \\ &= (\mu q + jp - kq) \pm j\,\|\kappa q\| \\ &\geq |\mu q - (kq - jp)| - j\,\|\kappa q\| \\ &\geq \|\mu q\| - M\,\|\kappa q\| \\ &= \varepsilon \end{aligned}$$

The above inequality implies that $\varepsilon < qAB^{-j}$, which further implies that

$$j < \frac{\log\left(Aq/\varepsilon\right)}{\log B}$$

□

This reduction process can be straightforwardly implemented in Maple™. A code for doing so is found in section 5.5.

5 On the Extendibility of a D(4)-Pair of Pell Numbers

5.1 The D(4)-triple

The Pell numbers are defined by the recurrence relation

$$P_n = 2P_{n-1} + P_{n-2} \quad \text{with initial conditions} \quad P_0 = 0,\ P_1 = 1.$$

The n^{th} term is given by

$$P_n = \tfrac{1}{2\sqrt{2}}\left(\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n\right)$$

Herein we will denote $\alpha := 1+\sqrt{2}$, which gives the following representations:

$$P_k = \frac{\alpha^k - \bar{\alpha}^k}{\alpha - \bar{\alpha}} \quad \text{and} \quad P_k = \frac{\alpha^k - (-1)^k \alpha^{-k}}{2\sqrt{2}} \tag{5.1}$$

Where $\bar{\alpha} = 1-\sqrt{2}$. The Pell numbers satisfy an analogue of Catalan's identity:

$$P_n^2 - P_{n-r}P_{n+r} = (-1)^{n-r} P_r^2$$

The Pell numbers also satisfy the following bounds in terms of α:

Lemma 5.1. $\alpha^{n-2} + 1 \le P_n < \alpha^{n-1}$ *for* $n \ge 2$. *The first inequality is strict for* $n > 2$.

Proof. For $P_n < \alpha^{n-1}$, start with $P_n = \frac{\alpha^n - \bar{\alpha}^n}{\alpha - \bar{\alpha}}$, multiply top and bottom by α^n and use $(\alpha\bar{\alpha})^n = (-1)^n$.

$$P_n = \frac{\alpha^n - \bar{\alpha}^n}{\alpha - \bar{\alpha}} = \frac{\alpha^{2n} - (-1)^n}{\alpha^{n+1} + \alpha^{n-1}} = \alpha^{n-1}\frac{\alpha^{2n} - (-1)^n}{\alpha^{2n} + \alpha^{2n-2}} < \alpha^{n-1}\frac{\alpha^{2n}+1}{\alpha^{2n}+1} = \alpha^{n-1}$$

For $\alpha^{n-2} + 1 \le P_n$, equality holds when $n = 2$. For $n > 2$, use induction. The inequality is strict when $n = 3$ and $n = 4$. For some $k > 4$, assume that $\alpha^{k-2} + 1 < P_k$ and that $\alpha^{k-3} + 1 < P_{k-1}$. Then

$$P_{k+1} - 1 = 2P_k + P_{k-1} - 1 > 2\left(P_k - 1\right) + \left(P_{k-1} - 1\right) > 2\alpha^{k-2} + \alpha^{k-3} = \left(2\alpha^{-1} + \alpha^{-2}\right)\alpha^{k-1} = \alpha^{k-1}$$

□

The following theorem, first stated in Chapter 1, is the main result of this chapter:

Theorem 5.2. *The set* $\{P_{2n+4}, 4P_{2n+2}, P_k\}$ *is a* $D\,(4)$*-triple if and only if* $k = 2n$.

It can be shown using Catalan's identity that $\{P_{2n+4}, 4P_{2n+2}, P_{2n}\}$ is a $D\,(4)$-triple, so it remains to show that $k = 2n$ is the only solution which makes $\{P_{2n+4}, 4P_{2n+2}, P_k\}$ a $D\,(4)$-triple. If

$\{P_{2n+4}, 4P_{2n+2}, P_k\}$ is a $D(4)$-triple, then for some integers X and Y we have

$$P_{2n+4}P_k + 4 = X^2 \quad \text{and} \quad 4P_{2n+2}P_k + 4 = Y^2$$

We can eliminate P_k to obtain the Pellian equation

$$4P_{2n+2}X^2 - P_{2n+4}Y^2 = 4\left(4P_{2n+2} - P_{2n+4}\right)$$

We now apply lemma 3.3 with the quantities $a = 4P_{2n+2}$, $b = P_{2n+4}$ and $l = 2$. Since $4P_{2n+2} < P_{2n+4} < 20P_{2n+2}$, this equation has general solution

$$Y\sqrt{P_{2n+4}} + 2X\sqrt{P_{2n+2}} = \left(2\sqrt{P_{2n+4}} \pm 4\sqrt{P_{2n+2}}\right)\left(P_{2n+3} + \sqrt{P_{2n+2}P_{2n+4}}\right)^{\nu} \quad \nu \in \mathbb{Z}^+ \tag{5.2}$$

Define the sequences V_j, U_j by

$$V_j + U_j\sqrt{P_{2n+2}P_{2n+4}} := \left(P_{2n+3} + \sqrt{P_{2n+2}P_{2n+4}}\right)^j \tag{5.3}$$

This results in

$$\begin{aligned} Y\sqrt{P_{2n+4}} + 2X\sqrt{P_{2n+2}} &= \left(2\sqrt{P_{2n+4}} \pm 4\sqrt{P_{2n+2}}\right)\left(V_j + U_j\sqrt{P_{2n+2}P_{2n+4}}\right) \\ &= \underbrace{(2V_j \pm 4P_{2n+2}U_j)}_{Y}\sqrt{P_{2n+4}} + \underbrace{(2U_jP_{2n+4} \pm 4V_j)}_{2X}\sqrt{P_{2n+2}} \end{aligned}$$

Which gives the expressions for X_j and Y_j:

$$X = X_j = \pm 2V_j + U_jP_{2n+4} \quad \text{and} \quad Y = Y_j = \pm 4P_{2n+2}U_j + 2V_j \tag{5.4}$$

Thus,

$$\begin{aligned} P_{2n+4}P_k + 4 = X^2 = (\pm 2V_j + U_jP_{2n+4})^2 \\ 4P_{2n+2}P_k + 4 = Y^2 = (\pm 4P_{2n+2}U_j + 2V_j)^2 \end{aligned} \xrightarrow{\text{rearrange for } P_k} \begin{aligned} P_k &= \tfrac{4V_j^2 - 4}{P_{2n+4}} + P_{2n+4}U_j^2 \pm 4U_jV_j \\ P_k &= \tfrac{V_j^2 - 1}{P_{2n+2}} + 4P_{2n+2}U_j^2 \pm 4U_jV_j \end{aligned}$$

By eliminating the first term on the right-hand side in both of the above equations for P_k, we get

$$\left(\tfrac{1}{P_{2n+2}} - \tfrac{4}{P_{2n+4}}\right)P_k = \left(\tfrac{P_{2n+4}}{P_{2n+2}} - \tfrac{16P_{2n+2}}{P_{2n+4}}\right)U_j^2 + \left(\pm\tfrac{1}{P_{2n+2}} \mp \tfrac{4}{P_{2n+4}}\right)4U_jV_j$$

Dividing both sides by $\left(\tfrac{1}{P_{2n+2}} - \tfrac{4}{P_{2n+4}}\right)$ gives the equation

$$P_k = \left(P_{2n+4} + 4P_{2n+2}\right)U_j^2 \pm 4U_jV_j \tag{5.5}$$

We call the resulting expression $C_j^{\pm}$:

$$C_j^{\pm} := \pm 4U_jV_j + (P_{2n+4} + 4P_{2n+2})\, U_j^2.$$

The goal is to find the values of j such that $C_j^{\pm}$ results in a Pell number – that is, to find a pair of integers (j,k) such that $P_k = C_j^{\pm}$. Note that when $j = 1$ we have:

$$\begin{aligned} C_1^- &= -4P_{2n+3} + P_{2n+4} + 4P_{2n+2} \\ &= P_{2n+2} - 2P_{2n+1} - 4P_{2n+3} + P_{2n+4} + 3P_{2n+2} + 2P_{2n+1} \\ &= P_{2n+2} - 2P_{2n+1} - 2P_{2n+3} + 4P_{2n+2} + 2P_{2n+1} \\ &= P_{2n+2} - 2P_{2n+1} - 2P_{2n+3} + 2P_{2n+3} \\ &= P_{2n} \end{aligned}$$

and also $P_{2n+6} < C_1^+ < P_{2n+7}$, which follows from:

$$\begin{aligned} C_1^+ &= 4P_{2n+3} + P_{2n+4} + 4P_{2n+2} \\ &= P_{2n+6} + 4P_{2n+3} - (P_{2n+6} - P_{2n+4}) + 4P_{2n+2} \\ &= P_{2n+6} + 4P_{2n+3} - 2P_{2n+5} + 4P_{2n+2} \\ &= P_{2n+6} + 2P_{2n+3} + 4\,(P_{2n+2} - P_{2n+4}) \\ &= P_{2n+6} + 10P_{2n+3} \\ &< P_{2n+6} + 3P_{2n+3} + 7P_{2n+4} \\ &= P_{2n+6} + 3P_{2n+5} + P_{2n+4} \\ &= 2P_{2n+6} + P_{2n+5} \\ &= P_{2n+7} \end{aligned}$$

So C_1^- is the already-known solution of $k = 2n$ and C_1^+ cannot be a Pell number.

From equation (5.2) we can obtain recursive forms for X_j and Y_j:

$$\begin{aligned} Y_{j+1}\sqrt{P_{2n+4}} + 2X_{j+1}\sqrt{P_{2n+2}} &= \left(Y_j\sqrt{P_{2n+4}} + 2X_j\sqrt{P_{2n+2}}\right)\left(P_{2n+3} + \sqrt{P_{2n+2}P_{2n+4}}\right) \\ &= (Y_jP_{2n+3} + 2X_jP_{2n+2})\sqrt{P_{2n+4}} + (Y_jP_{2n+4} + 2X_jP_{2n+3})\sqrt{P_{2n+2}} \end{aligned}$$

Which means $2X_{j+1} = 2X_jP_{2n+3} + Y_jP_{2n+4}$. Therefore,

$$\begin{aligned}
X_{j+1}^{\pm} &= X_jP_{2n+3} + \tfrac{1}{2}Y_jP_{2n+4} \\
&= \left(\pm 2V_j + P_{2n+4}U_j\right)P_{2n+3} + \left(\pm 2P_{2n+2}U_j + V_j\right)P_{2n+4} && \bullet \text{ by (5.4)} \\
&= \pm 2V_jP_{2n+3} + P_{2n+4}P_{2n+3}U_j \pm 2P_{2n+2}P_{2n+4}U_j + V_jP_{2n+4} \\
&= \left(\pm 2P_{2n+3} + P_{2n+4}\right)V_j + \left(P_{2n+3} \pm 2P_{2n+2}\right)P_{2n+4}U_j \\
&> \pm 2V_j + P_{2n+4}U_j \\
&= X_j^{\pm} && \bullet \text{ by (5.4)}
\end{aligned}$$

So we have that $X_{j+1}^{\pm} > X_j^{\pm}$ and $X_1^{+} > X_1^{-} > 0$. We are interested in solutions (j,k) that satisfy $P_{2n+4}P_k + 4 = \left(X_j^{\pm}\right)^2$. When $j = 1$ we have $P_{2n+4}P_{2n} + 4 = \left(X_1^{-}\right)^2$. For any solution (j,k) with $j > 1$,

$$P_{2n+4}P_k + 4 = \left(X_j^{\pm}\right)^2 > \left(X_1^{-}\right)^2 = P_{2n+4}P_{2n} + 4$$

which implies that $P_k > P_{2n}$. Based on this, we conclude that any solution different from $(j,k) = (1,2n)$ would have $j \geq 2$ and $k > 2n$. Define $\beta_n := P_{2n+3} + \sqrt{P_{2n+3}^2 - 1}$. Then $\beta_n^{-1} = P_{2n+3} - \sqrt{P_{2n+3}^2 - 1}$, so from (5.3),

$$\begin{aligned}
V_j + U_j\sqrt{P_{2n+2}P_{2n+4}} &= \beta_n^{j} \\
V_j - U_j\sqrt{P_{2n+2}P_{2n+4}} &= \beta_n^{-j}
\end{aligned} \quad \Rightarrow \quad V_j = \tfrac{\beta_n^j + \beta_n^{-j}}{2} \quad \text{and} \quad U_j = \tfrac{\beta_n^j - \beta_n^{-j}}{2\sqrt{P_{2n+2}P_{2n+4}}}$$

So $C_j^{\pm}$ can be rewritten as

$$\begin{aligned}
C_j^{\pm} &:= \pm\frac{\beta_n^{2j} - \beta_n^{-2j}}{\sqrt{P_{2n+2}P_{2n+4}}} + \left(P_{2n+4} + 4P_{2n+2}\right)\frac{\beta_n^{2j} + \beta_n^{-2j} - 2}{4P_{2n+2}P_{2n+4}} \\
&= \pm\frac{\beta_n^{2j}}{\sqrt{P_{2n+2}P_{2n+4}}} + \frac{\left(P_{2n+4} + 4P_{2n+2}\right)\beta_n^{2j}}{4P_{2n+2}P_{2n+4}} - \frac{\pm\beta_n^{-2j}}{\sqrt{P_{2n+2}P_{2n+4}}} + \frac{\left(P_{2n+4} + 4P_{2n+2}\right)\beta_n^{-2j}}{4P_{2n+2}P_{2n+4}} - \frac{P_{2n+4} + 4P_{2n+2}}{2P_{2n+2}P_{2n+4}} \\
&= \underbrace{\left(\frac{\pm 1}{\sqrt{P_{2n+2}P_{2n+4}}} + \frac{P_{2n+4} + 4P_{2n+2}}{4P_{2n+2}P_{2n+4}}\right)}_{\gamma_n^{\pm}}\beta_n^{2j} - \frac{P_{2n+4} + 4P_{2n+2}}{2P_{2n+2}P_{2n+4}} + \underbrace{\left(\frac{\mp 1}{\sqrt{P_{2n+2}P_{2n+4}}} + \frac{P_{2n+4} + 4P_{2n+2}}{4P_{2n+2}P_{2n+4}}\right)}_{\gamma_n^{\mp}}\beta_n^{-2j}
\end{aligned}$$

Define $\gamma_n^{\pm} := \frac{\pm 1}{\sqrt{P_{2n+2}P_{2n+4}}} + \frac{P_{2n+4} + 4P_{2n+2}}{4P_{2n+2}P_{2n+4}}$ so that the problem may be expressed as finding $j \geq 2$ and $k > 2n$ which satisfy the equation

$$\gamma_n^{\pm}\beta_n^{2j} - \frac{P_{2n+4} + 4P_{2n+2}}{2P_{2n+2}P_{2n+4}} + \gamma_n^{\mp}\beta_n^{-2j} = \frac{\alpha^k - \bar{\alpha}^k}{2\sqrt{2}} \quad \text{where} \quad \alpha = 1 + \sqrt{2}. \tag{5.6}$$

5.2 A Linear Form in Three Logarithms

From (5.6) we find that

$$1-\frac{1}{\gamma_n^{\pm}\beta_n^{2j}}\frac{\alpha^k}{2\sqrt{2}}=\frac{1}{\gamma_n^{\pm}\beta_n^{2j}}\left(\frac{P_{2n+4}+4P_{2n+2}}{2P_{2n+2}P_{2n+4}}-\frac{\bar{\alpha}^k}{2\sqrt{2}}\right)-\frac{\gamma_n^{\mp}\beta_n^{-2j}}{\gamma_n^{\pm}\beta_n^{2j}} \tag{5.7}$$

Observe that the left-hand side above is $1-e^{-\Lambda}$, where Λ is the following linear form in three logarithms:

$$\Lambda := 2j\log\beta_n - k\log\alpha + \log\left(2\sqrt{2}\gamma_n^{\pm}\right)$$

It will be established in 5.6 that Λ is positive. We will be able to get an upper bound for the left-hand side of equation (5.7), which will give us an upper bound for Λ using the following lemma.

Lemma 5.3. *For $0<x<\frac{4}{3}$, the inequality $2\left(1-e^{-x}\right)>x$ is true.*

Proof. Using the Taylor series of the logarithm, we have

$$-\log\left(1-\frac{x}{2}\right)=\frac{x}{2}+\sum_{k=2}^{\infty}\frac{x^k}{2^k k}<\frac{x}{2}+\sum_{k=2}^{\infty}\frac{x^k}{2^{k+1}}=\frac{x}{2}+\frac{x^2}{4\left(2-x\right)}<x$$

which is equivalent to $2\left(1-e^{-x}\right)>x$. □

Using theorem 4.1, we will be able to get a lower bound for Λ. Once we have an upper and lower bound for Λ, we will be able to compare these two bounds to obtain the following bound for n and j:

Proposition 5.4. *If equation (5.5) has a positive integer solution (j,k) with $j>1$, then*

$$j<1.9241\times10^{12}\left(4n+7\right)\log\left(39j\left(4n+7\right)\right).$$

To get an upper bound for $1-e^{-\Lambda}$, we will start by finding bounds on $\gamma_n^{\pm}$. Using the identity $\pm\frac{1}{xy}+\frac{y^2+4x^2}{4x^2y^2}=\left(\frac{1}{y}\pm\frac{1}{2x}\right)^2$ we have

$$\sqrt{\gamma_n^{\pm}}=\pm\frac{1}{\sqrt{P_{2n+4}}}+\frac{1}{2\sqrt{P_{2n+2}}}$$

Lemma 5.5. *$\gamma_n^{\pm}$ satisfy the following:*

$$0.02081\alpha^{-2n-2}<\gamma_n^{-}<0.02093\alpha^{-2n-2}\quad\text{and}\quad 2.36395\alpha^{-2n-2}<\gamma_n^{+}<2.36514\alpha^{-2n-2}$$

Proof. We have

$$\begin{aligned}\sqrt{\gamma_n^\pm} &= \pm\frac{1}{\sqrt{P_{2n+4}}} + \frac{1}{2\sqrt{P_{2n+2}}}\\ &= \pm\frac{1}{\sqrt{\left(\alpha^{2n+4}-\alpha^{-2n-4}\right)/\left(2\sqrt{2}\right)}} + \frac{1}{2\sqrt{\left(\alpha^{2n+2}-\alpha^{-2n-2}\right)/\left(2\sqrt{2}\right)}}\\ &= 2^{3/4}\alpha^{-n-1}\left(\pm\frac{1}{\alpha\sqrt{1-\alpha^{-4n-8}}} + \frac{1}{2\sqrt{1-\alpha^{-4n-4}}}\right)\end{aligned}$$

From the Taylor series of $(1-x)^{-1/2}$, for $0 < x < 1$ we have

$$1 + \frac{1}{2}x < \frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + ... < 1 + \frac{x}{2}\left(\frac{1}{1-x}\right)$$

So

$$\frac{1}{\alpha}\left(1+\frac{1}{2}\alpha^{-4n-8}\right) < \frac{1}{\alpha\sqrt{1-\alpha^{-4n-8}}} < \frac{1}{\alpha}\left(1 + \frac{\alpha^{-4n-8}}{2\left(1-\alpha^{-4n-8}\right)}\right)$$

Hence,

$$0.41421356 < \frac{1}{\alpha} < \frac{1}{\alpha\sqrt{1-\alpha^{-4n-8}}} < \frac{1}{\alpha}\left(1+\frac{1}{2}\frac{1}{\alpha^{4n+8}-1}\right) < 0.414218846$$

Similarly,

$$0.5 < \frac{1}{2}\left(1+\frac{1}{2}\alpha^{-4n-4}\right) < \frac{1}{2\sqrt{1-\alpha^{-4n-4}}} < \frac{1}{2}\left(1+\frac{\alpha^{-4n-4}}{2\left(1-\alpha^{-4n-4}\right)}\right) < 0.50021665$$

We then obtain bounds for $\sqrt{\gamma_n^\pm}$

$$0.085781154 < 2^{-3/4}\alpha^{n+1}\sqrt{\gamma_n^-} = -\frac{1}{\alpha\sqrt{1-\alpha^{-4n-8}}} + \frac{1}{2\sqrt{1-\alpha^{-4n-4}}} < 0.08600309$$

$$0.91421356 < 2^{-3/4}\alpha^{n+1}\sqrt{\gamma_n^+} = \frac{1}{\alpha\sqrt{1-\alpha^{-4n-8}}} + \frac{1}{2\sqrt{1-\alpha^{-4n-4}}} < 0.914435496$$

So we get the following bounds:

$$0.02081\alpha^{-2n-2} < \gamma_n^- < 0.02093\alpha^{-2n-2} \quad \text{and} \quad 2.36395\alpha^{-2n-2} < \gamma_n^+ < 2.36514\alpha^{-2n-2}$$

□

With this we can show that Λ is positive and obtain an upper bound for it.

Lemma 5.6. *$0 < \Lambda < 4046\beta_n^{-2j}$ for $j \geq 2$.*

Proof. First, we show that $\Lambda > 0$. $\Lambda = \log 2\sqrt{2}\gamma_n^\pm\beta_n^{2j}\alpha^{-k} > 0$ if and only if $2\sqrt{2}\gamma_n^\pm\beta_n^{2j}\alpha^{-k} > 1$. For an argument by contradiction, suppose this is not the case. This implies that

$$\gamma_n^{\pm}\beta_n^{2j} \le \frac{\alpha^k}{2\sqrt{2}} \quad \text{and} \quad \frac{2\sqrt{2}}{\alpha^k} \le \frac{\beta_n^{-2j}}{\gamma_n^{\pm}} \le \frac{\beta_n^{-2j}}{\gamma_n^{-}} \tag{5.8}$$

This gives

$$\begin{aligned}
\tfrac{1}{P_{2n+2}} &< \tfrac{14}{11}\left(\tfrac{1}{2P_{2n+2}} + \tfrac{2}{P_{2n+4}}\right) && \bullet \text{ because } P_{2n+4} < 7P_{2n+2} \\
&= \tfrac{14}{11}\left(\tfrac{P_{2n+4}+4P_{2n+2}}{2P_{2n+2}P_{2n+4}}\right) \\
&= \tfrac{14}{11}\left(\tfrac{\bar{\alpha}^k}{2\sqrt{2}} + \gamma_n^{\mp}\beta_n^{-2j} + \left(\gamma_n^{\pm}\beta_n^{2j} - \tfrac{\alpha^k}{2\sqrt{2}}\right)\right) && \bullet \text{ by equation (5.6)} \\
&\le \tfrac{14}{11}\left(\gamma_n^{\mp}\beta_n^{-2j} + \tfrac{\bar{\alpha}^k}{2\sqrt{2}}\right) && \bullet \text{ by (5.8)} \\
&\le \tfrac{14}{11}\left(\gamma_n^{\mp}\beta_n^{-2j} + \tfrac{2\sqrt{2}}{8\alpha^k}\right) && \bullet \text{ since } \bar{\alpha}^k \le \alpha^{-k} \\
&\le \tfrac{14}{11}\beta_n^{-2j}\left(\gamma_n^{\mp} + \tfrac{1}{8\gamma_n^{-}}\right) && \bullet \text{ by (5.8)}
\end{aligned}$$

Which we apply below, along with the bounds on $\gamma_n^{\pm}$ from lemma 5.5:

$$\begin{aligned}
P_{2n+2}^{j}P_{2n+4}^{j} &= \left(P_{2n+3}^2 - 1\right)^j && \bullet \text{ by Catalan's identity} \\
&< \beta_n^{2j} \\
&< \tfrac{14}{11}P_{2n+2}\left(\gamma_n^{\mp} + \tfrac{1}{8\gamma_n^{-}}\right) && \bullet \text{ by the previous result} \\
&< P_{2n+2}\left(3.02\alpha^{-2n-2} + 7.61\alpha^{2n+2}\right) && \bullet \text{ by lemma 5.5} \\
&< P_{2n+2}\cdot 20.65\left(\tfrac{\alpha^{2n+3}+\alpha^{-2n-3}}{2\sqrt{2}}\right) \\
&= 20.65P_{2n+2}P_{2n+3} && \bullet \text{ by (5.1)} \\
&< 20.65P_{2n+2}P_{2n+4}
\end{aligned}$$

Which implies that

$$20.65 > P_{2n+2}^{j-1}P_{2n+4}^{j-1} \ge P_4 \cdot P_6 = 12 \cdot 70,$$

which is a contradiction. We now establish the upper bound for Λ, first by finding the upper bound for $1 - e^{-\Lambda} = 1 - \frac{\alpha^k}{2\sqrt{2}}\frac{1}{\gamma_n^{\pm}\beta_n^{2j}}$.

$$\begin{aligned}
0 &< 1 - \tfrac{\alpha^k}{2\sqrt{2}}\tfrac{1}{\gamma_n^{\pm}\beta_n^{2j}} && \bullet \text{ since } \Lambda > 0 \\
&= \tfrac{1}{\gamma_n^{\pm}\beta_n^{2j}}\left(\tfrac{P_{2n+4}+4P_{2n+2}}{2P_{2n+2}P_{2n+4}} - \tfrac{\bar{\alpha}^k}{2\sqrt{2}}\right) - \tfrac{\gamma_n^{\mp}\beta_n^{-2j}}{\gamma_n^{\pm}\beta_n^{2j}} && \bullet \text{ by equation (5.7)} \\
&\le \tfrac{1}{\gamma_n^{\pm}\beta_n^{2j}}\left(\tfrac{P_{2n+4}+4P_{2n+2}}{2P_{2n+2}P_{2n+4}} + \tfrac{1}{2\sqrt{2}\alpha^k}\right) && \bullet \text{ since } (\bar{\alpha}\alpha)^k = (-1)^k \\
&< \tfrac{1}{\beta_n^{2j}\gamma_n^{\pm}}\left(\tfrac{5}{2P_{2n+2}} + \tfrac{1}{2\sqrt{2}\alpha^{2n+1}}\right) && \bullet \text{ since } P_{2n+2} < P_{2n+4} \\
&= \tfrac{1}{\beta_n^{2j}\times 0.02081\alpha^{-2n-2}}\left(\tfrac{5}{2P_{2n+2}} + \tfrac{1}{2\sqrt{2}\alpha^{2n+1}}\right) && \bullet \text{ by lemma 5.5} \\
&< \tfrac{2023}{\beta_n^{2j}} \\
&< \tfrac{1}{2}
\end{aligned}$$

Note that $1-e^{-x}<\frac{1}{2}$ implies that $x<\ln 2$, so by lemma 5.3 we must have $\Lambda<2\left(1-e^{-\Lambda}\right)<4046\beta_n^{-2j}$. □

We will now work towards using theorem 4.1 to get a lower bound for Λ. To apply this theorem, we will take $(\alpha_1,\alpha_2,\alpha_3)=\left(\beta_n,\alpha,2\sqrt{2}\gamma_n^{\pm}\right)$. We will need to find the degree D of $\mathbb{Q}(\alpha_1,\alpha_2,\alpha_3)$ as a field extension over $\mathbb{Q}$, and to establish that $\alpha_1,\alpha_2,\alpha_3$ are multiplicatively independent – that is, for $p,q,r\in\mathbb{Z}$, that $\alpha_1^p\alpha_2^q\alpha_3^r=1$ if and only if $p=q=r=0$. In order to do this, we will need to establish the following result:

Lemma 5.7. *$P_{2n+2}P_{2n+4}$ is neither a square nor 2 times a square.*

Proof. The fact that $P_{2n+2}P_{2n+4}$ is not a square follows simply from Catalan's identity:

$$P_{2n+2}P_{2n+4}=P_{2n+3}^2-1$$

As we know that consecutive integers will not both be squares. For the sake of contradiction, suppose that $P_{2n+2}P_{2n+4}=2Y^2$ for some integer Y. Catalan's identity gives:

$$X^2-2Y^2=1 \quad \text{where } X=P_{2n+3} \text{ for some } n\in\mathbb{Z}^+ \tag{5.9}$$

We find the fundamental solution $(X,Y)=(3,2)$, and with this obtain the general solution $X_j+Y_j\sqrt{2}=\left(3+2\sqrt{2}\right)^j$, $j\in\mathbb{Z}^+$. With this we can obtain the general solution for X_j:

$$\begin{array}{l} X+Y\sqrt{2}=\left(3+2\sqrt{2}\right)^j \\ X-Y\sqrt{2}=\left(3-2\sqrt{2}\right)^j \end{array} \xrightarrow{\text{eliminate } Y} \quad X=\frac{\left(3+2\sqrt{2}\right)^j+\left(3-2\sqrt{2}\right)^{-j}}{2}$$

Noting that $3+2\sqrt{2}=\alpha^2$, we have $X_j=\frac{\alpha^{2j}+\alpha^{-2j}}{2}$. Thus we have

$$X_j=P_{2n+3} \quad \Rightarrow \quad \frac{\alpha^{2j}+\alpha^{-2j}}{2}=\frac{\alpha^{2n+3}+\alpha^{-2n-3}}{2\sqrt{2}}$$

Following from this result, we obtain the inequalities:

$$\begin{aligned}
\frac{\alpha^{2n+2}+\alpha^{-2n-2}}{2} &= \frac{\alpha^{-1}\alpha^{2n+3}+\alpha^{-2n-3}\alpha}{2} \\
&< \frac{\frac{1}{\sqrt{2}}\alpha^{2n+3}+\frac{1}{\sqrt{2}}\alpha^{-2n-3}+\left(\alpha^{-1}-\frac{1}{\sqrt{2}}\right)\alpha^{2n+3}+\left(\alpha-\frac{1}{\sqrt{2}}\right)\alpha^{-2n-3}}{2} \\
&< \frac{\frac{1}{\sqrt{2}}\alpha^{2n+3}+\frac{1}{\sqrt{2}}\alpha^{-2n-3}-0.29\alpha^{2n+3}+1.71\alpha^{-2n-3}}{2} \\
&< \frac{\alpha^{2n+3}+\alpha^{-2n-3}}{2\sqrt{2}} \\
&= \frac{a^{2j}+a^{2j}}{2}
\end{aligned}$$

$$
\begin{aligned}
\frac{\alpha^{2n+4}+\alpha^{-2n-4}}{2} &= \frac{\alpha^{2n+3}\alpha+\alpha^{-2n-3}\alpha^{-1}}{2} \\
&= \frac{\frac{1}{\sqrt{2}}\alpha^{2n+3}+\frac{1}{\sqrt{2}}\alpha^{-2n-3}+\left(\alpha-\frac{1}{\sqrt{2}}\right)\alpha^{2n+3}+\left(\alpha^{-1}-\frac{1}{\sqrt{2}}\right)\alpha^{-2n-3}}{2} \\
&> \frac{\frac{1}{\sqrt{2}}\alpha^{2n+3}+\frac{1}{\sqrt{2}}\alpha^{-2n-3}+1.7\alpha^{2n+3}-0.3\alpha^{-2n-3}}{2} \\
&> \frac{\alpha^{2n+3}+\alpha^{-2n-3}}{2\sqrt{2}} \\
&= \frac{\alpha^{2j}+\alpha^{-2j}}{2}
\end{aligned}
$$

The above inequalities show that $2n+2 < 2j$ and that $2j < 2n+4$, respectively. Since $n+1 < j < n+2$, it means that j cannot be an integer, so equation (5.9) does not have a solution. With this we conclude that $P_{2n+2}P_{2n+4}$ is neither a square nor 2 times a square. □

With this fact established, we can take $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) = \mathbb{Q}\left(\sqrt{d}, \sqrt{2}\right)$ where d is the squarefree part of $P_{2n+2}P_{2n+4}$. This field extension has basis $\left\{1, \sqrt{2}, \sqrt{d}, \sqrt{2d}\right\}$ as a vector space over $\mathbb{Q}$, so its degree D is 4.

Proposition 5.8. *$\alpha_1, \alpha_2, \alpha_3$ are multiplicatively independent.*

Proof. Suppose, to the contrary, that there exist $p, q, r \in \mathbb{Z}$, not all zero, such that $\alpha_1^p\alpha_2^q\alpha_3^r = 1$. By lemma 5.7 it follows that $\mathbb{Q}\left(\sqrt{d}\right)$ is a quadratic field different from $\mathbb{Q}\left(\sqrt{2}\right)$. Since $\gamma_n^{\pm}, \beta_n \in \mathbb{Q}\left(\sqrt{d}\right)$ for all n, by closure under multiplication we have

$$\beta_n^p\left(\gamma_n^{\pm}\right)^r = \left(2\sqrt{2}\right)^{-r}\alpha_1^p\alpha_3^r \in \mathbb{Q}\left(\sqrt{d}\right)$$

However, by rearranging $\alpha_1^p\alpha_2^q\alpha_3^r = 1$ we find that $\left(2\sqrt{2}\right)^{-r}\alpha_1^p\alpha_3^r = \left(2\sqrt{2}\right)^{-r}\alpha_2^{-q}$, and the right-hand side of this equation is always irrational in $\mathbb{Q}\left(\sqrt{2}\right)$ unless $q = 0$ and r is even (in this case it is rational). Thus, letting $q = 0$ and $r = 2k$ we have that $\alpha_1^p\alpha_3^{2k} = 1$, or moreover that $\alpha_1^{-p} = \alpha_3^{2k}$.

Note that α_1 and α_1^{-1} are both algebraic integers, so a_1^{-p} is an algebraic integer, and thus α_3^{2k} must also be. However, the minimal polynomial of α_3^{2k} has constant term

$$\left(8\gamma_n^+\gamma_n^-\right)^{2k} = \left(\frac{P_{2n+4} - 4P_{2n+2}}{\sqrt{2}P_{2n+2}P_{2n+4}}\right)^{4k} < 1 \quad \text{for all } n$$

The constant term is never an integer, so α_3^{2k} is not an algebraic integer – a contradiction. To see that the constant term is always less than 1, the inequality $0 < \frac{P_{2n+4}-4P_{2n+2}}{\sqrt{2}P_{2n+2}P_{2n+4}} < 1$ can be verified by noting that $P_{2n+4} - 4P_{2n+2} = 2P_{2n+1} + P_{2n+2} > 0$ and

$$\begin{aligned} 4P_{2n+2} + \sqrt{2}P_{2n+2}P_{2n+4} - P_{2n+4} &= 4P_{2n+2} + \sqrt{2}P_{2n+2}P_{2n+4} - 2P_{2n+3} - P_{2n+2} \\ &= 3P_{2n+2} + \sqrt{2}P_{2n+3}^2 - 2P_{2n+3} - \sqrt{2} \\ &= 3P_{2n+2} + \sqrt{2}\Big(P_{2n+3} - \tfrac{\sqrt{2}}{2}\Big)^2 - \tfrac{3\sqrt{2}}{2} \\ &> 0 \end{aligned}$$

Rearrange this and raise to the power $4k$ to obtain $\left(\frac{P_{2n+4}-4P_{2n+2}}{\sqrt{2}P_{2n+2}P_{2n+4}}\right)^{4k} < 1$. □

From lemma 5.1 we know that $\alpha^{\lambda-2} + 1 < P_\lambda < \alpha^{\lambda-1}$. With this fact we find that

$$\begin{aligned} \beta_n &= P_{2n+3} + \sqrt{P_{2n+3}^2 - 1} < 2P_{2n+3} < 2\alpha^{2n+2} < \alpha^{2n+3}, \quad \text{and} \\ \beta_n &= P_{2n+3} + \sqrt{P_{2n+3}^2 - 1} > 2P_{2n+3} - 1 > 2\alpha^{2n+1} \end{aligned} \tag{5.10}$$

With this we can prove the following fact which will be useful when applying theorem 4.1.

Lemma 5.9. $k \leq j\,(4n+7)$

Proof. First note that $\frac{1}{9} > \frac{1}{16} + \left(\frac{1}{12}\right)^2 \geq \frac{1}{16} + \left(\frac{1}{P_{2n+2}}\right)^2$. Multiplying by $(P_{2n+2}P_{2n+4})^2$ shows that

$$\tfrac{1}{9}(P_{2n+2}P_{2n+4})^2 > \tfrac{1}{16}(P_{2n+2}P_{2n+4})^2 + P_{2n+4}^2$$

Also note that $24 \times 16 < 12 \times 70 = P_4P_6 \leq P_{2n+2}P_{2n+4}$ implies that $24P_{2n+2}P_{2n+4} < \left(\frac{1}{4}P_{2n+2}P_{2n+4}\right)^2$. Therefore,

$$\begin{aligned} &(P_{2n+4} + 4P_{2n+2})^2 = 16P_{2n+2}^2 + 8P_{2n+2}P_{2n+4} + P_{2n+4}^2 \\ &< 16P_{2n+2}P_{2n+4} + 8P_{2n+2}P_{2n+4} + P_{2n+4}^2 \\ &= 24P_{2n+2}P_{2n+4} + P_{2n+4}^2 \\ &< \left(\tfrac{1}{4}P_{2n+2}P_{2n+4}\right)^2 + P_{2n+4}^2 \\ &< \tfrac{1}{9}(P_{2n+2}P_{2n+4})^2 \end{aligned}$$

Which means that $P_{2n+4} + 4P_{2n+2} > \frac{1}{3}P_{2n+2}P_{2n+4}$. Using this we establish the result:

$$\begin{array}{rcll}
\alpha^{k-1} & < & 3\alpha^{k-2} & \\
 & < & 3P_k & \bullet \text{ by lemma 5.1} \\
 & = & \pm 12U_jV_j + 3U_j^2\left(P_{2n+4} + 4P_{2n+2}\right) & \bullet \text{ by equation (5.5)} \\
 & < & 12U_jV_j + U_j^2 P_{2n+2}P_{2n+4} & \bullet \text{ by the previous result} \\
 & < & \left(V_j + U_j\sqrt{P_{2n+2}P_{2n+4}}\right)^2 - V_j^2 & \bullet \text{ because } 12 < 2\sqrt{P_4P_6} \leq 2\sqrt{P_{2n+2}P_{2n+4}} \\
 & < & \left(V_j + U_j\sqrt{P_{2n+2}P_{2n+4}}\right)^2 & \\
 & = & \beta_n^{2j} & \bullet \text{ by equation (5.3) and definition of } \beta_n \\
 & < & \alpha^{2j(2n+3)} & \bullet \text{ by (5.10)}
\end{array}$$

Therefore $k \leq j\,(4n+7)$. □

We have everything needed to apply Matveev's theorem 4.1 to get an upper bound for $-\log|\Lambda|$. From lemma 5.6 we have the lower bound $2j\log\beta_n - \log 4046 < -\log|\Lambda|$. Combining these two bounds will allow us to prove proposition 5.4.

Proof. (proof of proposition 5.4) To prove this, we will apply theorem 4.1 to the linear form in logarithms

$$\Lambda = 2j\log\beta_n - k\log\alpha + \log\left(2\sqrt{2}\gamma_n^{\pm}\right)$$

with the following quantities as specified by theorem 4.1:

$$N = 3 \qquad D = 4 \qquad b_1 = 2j \qquad b_2 = -k \qquad b_3 = 1$$
$$\alpha_1 = \beta_n \qquad \alpha_2 = \alpha \qquad \alpha_3 = 2\sqrt{2}\gamma_n^{\pm}$$

We have already established that $D = 4$ and $\alpha_1, \alpha_2, \alpha_3$ are multiplicatively independent. Since α_1 and α_2 are both algebraic integers with degree 2 and their conjugates are less than 1, their absolute logarithmic heights are

$$h\left(\alpha_1\right) = \tfrac{1}{2}\log\beta_n \quad \text{and} \quad h\left(\alpha_2\right) = \tfrac{1}{2}\log\alpha$$

For α_3, note that γ_n^+ and γ_n^- are roots of the polynomial

$$\left(x - \gamma_n^+\right)\left(x - \gamma_n^-\right) = x^2 + 2\left(\frac{P_{2n+4} + 4P_{2n+2}}{4P_{2n+2}P_{2n+4}}\right)x + \left(\frac{P_{2n+4} - 4P_{2n+2}}{4P_{2n+2}P_{2n+4}}\right)^2$$

Clearing the denominators, we find that the minimal polynomial of $\gamma_n^{\pm}$ has leading coefficient $16P_{2n+2}^2P_{2n+4}^2$. Since $|\gamma_n^{\pm}| < 1$ for all n, and $P_\lambda < \alpha^\lambda/2^{3/2}$ for positive even λ, we have

$$h\left(\gamma_n^{\pm}\right) = \tfrac{1}{2}\log\left(16P_{2n+2}^2P_{2n+4}^2\right) = \log\left(4P_{2n+2}P_{2n+4}\right) < (4n+6)\log\left(\alpha\right) + \log\left(\tfrac{1}{2}\right)$$

Thus we can take

$$\begin{aligned}
h(\alpha_3) &= h\left(2\sqrt{2}\gamma_n^{\pm}\right) \\
&< h\left(2\sqrt{2}\right) + h\left(\gamma_n^{\pm}\right) \\
&< \tfrac{3}{2}\log 2 + (4n+6)\log(\alpha) + \log\left(\tfrac{1}{2}\right) \\
&< \tfrac{3}{2}\log 2 + \log\left(\tfrac{1}{2}\right) + (4n+6)\log(\alpha) \\
&< \log\alpha + (4n+6)\log(\alpha) \\
&= (4n+7)\log(\alpha)
\end{aligned}$$

Since we need $A_i \geq \max\{Dh(\alpha_i), |\log\alpha_i|\}$, where $D = 4$, we take

$$A_1 = 2\log\beta_n \qquad A_2 = 2\log\alpha \qquad A_3 = 4(4n+7)\log\alpha$$

Note that the requirement that $A_3 > |\log\alpha_3|$ is met:

$$\begin{aligned}
\left|\log\left(2\sqrt{2}\gamma_n^{\pm}\right)\right| &\leq \left|\log\left(2\sqrt{2}\times 0.02081\alpha^{-2n-2}\right)\right| \\
&< \left|\log\left(2\sqrt{2}\times 0.02081\alpha^{-4(4n+4)}\right)\right| \\
&< \left|\log\left(\alpha^{-4(4n+7)}\right)\right| \\
&= 4(4n+7)\log\alpha
\end{aligned}$$

For E we have

$$\begin{aligned}
E &= \max\left\{1, \max\left\{|b_j|\tfrac{A_j}{A_3} : 1 \leq j \leq 3\right\}\right\} \\
&= \max\left\{1, \max\left\{\tfrac{|b_1|A_1}{A_3}, \tfrac{|b_2|A_2}{A_3}, \tfrac{|b_3|A_3}{A_3}\right\}\right\} \\
&= \max\left\{\tfrac{j\log\beta_n}{(4n+7)\log\alpha}, \tfrac{k}{2(4n+7)}, 1\right\} \\
&\leq \max\left\{\tfrac{j}{2}, \tfrac{j}{2}, 1\right\} && \bullet \text{ by (5.10) and lemma 5.9} \\
&< j(4n+7)
\end{aligned}$$

To apply lemma 4.1, take the quantities

$$C(3) = \tfrac{8}{2}(5)(9)(16e)^4 < 6.45\times 10^8 \qquad C_0 = \log e^{20.2}3^{5.5}(16)\log(4e) < 30$$

$$W_0 = \log(1.5eE\cdot 4\log 4e) < \log(39j(4n+7)) \qquad \Omega = (2\log\beta_n)(2\log\alpha)(4(4n+7)\log\alpha)$$

Therefore,

$$\begin{aligned}
2j\log\beta_n - \log 4046 &< -\log|\Lambda| && \bullet \text{ by lemma 5.6} \\
&< C(3)C_0W_0D^2\Omega && \bullet \text{ by theorem 4.1} \\
&< 3.8481\times 10^{12}(4n+7)(\log\beta_n)\log(39j(4n+7))
\end{aligned}$$

which implies

$$j < 1.9241 \times 10^{12} \left(4n+7\right) \log\left(39j\left(4n+7\right)\right)$$

$\square$

5.3 Linear Form in Two Logarithms

Firstly, define a new linear form in three logarithms, Λ_0, by substituting $(j,k) = (1,2n)$ into Λ:

$$\Lambda_0 := 2\log\beta_n - 2n\log\alpha + \log\left(2\sqrt{2}\gamma_n^{\pm}\right)$$

By the easily verifiable identity $x+\sqrt{x^2-1} = 2x\left(1 - \frac{1}{2x\left(x+\sqrt{x^2-1}\right)}\right)$, and that $P_{2n+3} = \frac{\alpha^{2n+3}}{2\sqrt{2}}\left(1+\frac{1}{\alpha^{4n+6}}\right)$, we find that

$$\beta_n = P_{2n+3} + \sqrt{P_{2n+3}^2 - 1} = 2P_{2n+3}\left(1 - \frac{1}{2P_{2n+3}\left(P_{2n+3}+\sqrt{P_{2n+3}^2-1}\right)}\right) = \frac{1}{\sqrt{2}}\alpha^{2n+3}\underbrace{\left(1+\frac{1}{\alpha^{4n+6}}\right)\left(1-\frac{1}{2P_{2n+3}\beta_n}\right)}_{\delta_n}$$

Let's define $\delta_n = \left(1+\frac{1}{\alpha^{4n+6}}\right)\left(1-\frac{1}{2P_{2n+3}\beta_n}\right)$. We then obtain

$$\begin{aligned}
\Lambda - \Lambda_0 &= \left(2j\log\beta_n - k\log\alpha + \log\left(2\sqrt{2}\gamma_n^{\pm}\right)\right) - \left(2\log\beta_n - 2n\log\alpha + \log\left(2\sqrt{2}\gamma_n^{\pm}\right)\right) \\
&= (2j-2)\log\beta_n - (k-2n)\log\alpha \\
&= (2j-2)\left(\log\left(\frac{1}{\sqrt{2}}\right) + (2n+3)\log\alpha + \log\delta_n\right) - (k-2n)\log\alpha \\
&= (2j-2)\log\left(\frac{1}{\sqrt{2}}\right) + \underbrace{\left[(2j-2)(2n+3) - (k-2n)\right]}_{K}\log\alpha + (2j-2)\log\delta_n
\end{aligned}$$

If we define the linear form in two logarithms:

$$\Lambda_1 := K\log\alpha - (j-1)\log(2) \quad \text{where} \quad K = (2j-1)(2n+3) - k - 3,$$

this means that

$$\Lambda_1 = \Lambda - \Lambda_0 - (2j-2)\log\delta_n$$

which, by the triangle inequality, implies that $|\Lambda_1| \leq |\Lambda| + |\Lambda_0| + (2j-2)\left|\log\delta_n\right|$.

In this section the task is to find an upper bound for $|\Lambda_1|$, and then Laurent's theorem 4.2 gives a lower bound. Combining these bounds with the result from proposition 5.4 will allow us to get the following bounds for n and j:

Proposition 5.10. *If equation (5.5) has a positive integer solution (j, k) with $j > 1$, then*

$$j < 9.19 \times 10^{18} \quad \textit{and} \quad n < 20358$$

We already have an upper bound for Λ from lemma 5.6. In order to get an upper bound for $|\Lambda_1|$, it remains to find an upper bound for $|\Lambda_0|$ and for $|\log \delta_n|$. We will begin with $|\Lambda_0|$.

Lemma 5.11. $|\Lambda_0| < 15911\beta_n^{-2}$

Proof. For now assume $n \geq 2$. Substituting $(j, k) = (1, 2n)$ into (5.7) we obtain

$$1 - \frac{1}{\gamma_n^{\pm}\beta_n^2}\frac{\alpha^{2n}}{2\sqrt{2}} = \frac{1}{\gamma_n^{\pm}\beta_n^2}\frac{P_{2n+4} + 4P_{2n+2}}{2P_{2n+2}P_{2n+4}} - \frac{1}{\gamma_n^{\pm}\beta_n^2}\frac{\bar{\alpha}^{2n}}{2\sqrt{2}} - \frac{1}{\gamma_n^{\pm}\beta_n^2}\gamma_n^{\mp}\beta_n^{-2} \tag{5.11}$$

Observe that $1 - e^{-\Lambda_0} = 1 - e^{-\log\left(\beta_n^2\alpha^{-2n}2\sqrt{2}\gamma_n^{\pm}\right)} = 1 - \frac{1}{\gamma_n^{\pm}\beta_n^2}\frac{\alpha^{2n}}{2\sqrt{2}}$, the left-hand side of the above equation. This part of the proof is split into two cases:

$$(1) \quad 1 - e^{-\Lambda_0} \leq 0 \qquad \text{and} \qquad (2) \quad 1 - e^{-\Lambda_0} > 0$$

If $1 - e^{-\Lambda_0} \leq 0$, then

$$\begin{aligned}
0 &\leq \frac{1}{\gamma_n^{\pm}\beta_n^2}\frac{\alpha^{2n}}{2\sqrt{2}} - 1 && \bullet \ \text{because } 1 - e^{-\Lambda_0} \leq 0 \\
&= \frac{\frac{\bar{\alpha}^{2n}}{2\sqrt{2}} + \gamma_n^{\mp}\beta_n^{-2}}{\gamma_n^{\pm}\beta_n^2} - \frac{1}{\gamma_n^{\pm}\beta_n^2}\frac{P_{2n+4}+4P_{2n+2}}{2P_{2n+2}P_{2n+4}} && \bullet \ \text{by equation 5.11} \\
&< \frac{\frac{\beta_n^2\alpha^{-2n}}{2\sqrt{2}} + \gamma_n^{\mp}}{\gamma_n^{\pm}\beta_n^4} \\
&< \frac{\frac{1}{8\gamma_n^{\pm}} + \gamma_n^{\mp}}{\gamma_n^{\pm}\beta_n^4} && \bullet \ \text{because } 1 - e^{-\Lambda_0} \leq 0 \text{ implies } \frac{\alpha^{-2n}}{2\sqrt{2}} \leq \frac{1}{8\gamma_n^{\pm}\beta_n^2} \\
&< \frac{\frac{1}{8\times 0.02081\alpha^{-2n-2}} + 2.36514\alpha^{-2n-2}}{\beta_n^4 0.02081\alpha^{-2n-2}} && \bullet \ \text{by lemma 5.5} \\
&< \frac{113.654 + 288.646\alpha^{4n+4}}{\beta_n^4} \\
&< \frac{113.654 + 288.646\alpha^{4n+4}}{4\beta_n^2\alpha^{4n}} && \bullet \ \text{by (5.10)} \\
&< 2480\beta_n^{-2}
\end{aligned}$$

Note that $e^{-x} - 1 \geq 0$ implies $x \leq 0$. It follows in this case that

$$|\Lambda_0| = -\Lambda_0 < e^{-\Lambda_0} - 1 < 2480\beta_n^{-2}.$$

If $1 - e^{-\Lambda_0} > 0$, then

$$\begin{array}{rcll} 0 & < & 1 - \frac{1}{\gamma_n^{\pm}\beta_n^2}\frac{\alpha^{2n}}{2\sqrt{2}} & \bullet \text{ because } 1 - e^{-\Lambda_0} > 0 \\ & = & \frac{1}{\gamma_n^{\pm}\beta_n^2}\frac{P_{2n+4}+4P_{2n+2}}{2P_{2n+2}P_{2n+4}} - \frac{\frac{\bar{\alpha}^{2n}}{2\sqrt{2}}+\gamma_n^{\mp}\beta_n^{-2}}{\gamma_n^{\pm}\beta_n^2} & \bullet \text{ by equation (5.11)} \\ & < & \frac{1}{\gamma_n^{\pm}\beta_n^2}\left(\frac{1}{2P_{2n+2}} + \frac{2}{P_{2n+4}}\right) & \\ & < & \frac{1}{\gamma_n^{\pm}\beta_n^2}\left(\frac{5}{2P_{2n+2}}\right) & \bullet \text{ because } P_{2n+2} < P_{2n+4} \\ & < & 701\beta_n^{-2} & \\ & < & \frac{1}{2} & \end{array}$$

Note that $0 < 1 - e^{-x} < \frac{1}{2}$ implies that $0 < x < \ln 2$, so by lemma 5.3 we must have

$$|\Lambda_0| = \Lambda_0 < 2\left(1 - e^{-\Lambda_0}\right) < 1402\beta_n^{-2}.$$

If $n = 1$, then

$$|\Lambda_0| = 2\log\left(29 + \sqrt{29^2 - 1}\right) - 2\log\left(1 + \sqrt{2}\right) + \log\left(2\sqrt{2}\left(\pm\frac{1}{\sqrt{70}} + \frac{1}{2\sqrt{12}}\right)^2\right) < 4.7326 < 15911\beta_1^{-2}$$

In any case, we have $|\Lambda_0| < 15911\beta_n^{-2}$. □

To find a bound for $|\log\delta_n|$, note that for $0 < x < \frac{1}{2}$ and $0 < y$ we have the following:

$$-\log(1 - x) < 2x \quad \text{and} \quad \log(1 + y) < y$$

Using this and the bounds from lemma 5.1 and (5.10), we obtain the bound for $|\log\delta_n|$:

$$|\log\delta_n| \leq \left|\log\left(1 - \frac{1}{2P_{2n+3}\beta_n}\right)\right| + \left|\log\left(1 + \frac{1}{\alpha^{4n+6}}\right)\right| < \frac{1}{P_{2n+3}\beta_n} + \frac{1}{\alpha^{4n+6}} < \frac{1}{2\alpha^{4n+2}} + \frac{1}{\alpha^{4n+6}} < \frac{18}{\alpha^{4n+6}}$$

Using the bounds for $|\Lambda_0|$ and $|\log\delta_n|$, and the bound for $|\Lambda|$, we prove the following result:

Lemma 5.12. $|\Lambda_1| < \frac{7j+29074}{\alpha^{4n+4}}$

Proof. Bringing together $|\Lambda| < 4046\beta_n^{-2j}$, $|\Lambda_0| < 15911\beta_1^{-2}$ and $|\log\delta_n| < \frac{18}{\alpha^{4n+6}}$, we have that

$$\begin{aligned} |\Lambda_1| &\leq |\Lambda| + |\Lambda_0| + (2j-2)\,|\log\delta_n| \\ &< 4046\beta_n^{-2j} + 15911\beta_1^{-2} + (2j-2)\,\frac{18}{\alpha^{4n+6}} \\ &< \frac{4046}{\left(2\alpha^{2n+1}\right)^{2j}} + \frac{15911}{\left(2\alpha^{2n+1}\right)^2} + (2j-2)\,\frac{18}{\alpha^{4n+6}} \\ &< \frac{7j+29074}{\alpha^{4n+4}} \end{aligned}$$

□

Now, using the theorem 4.2 due to Laurent on linear forms in two logarithms, we can prove proposition 5.10.

Proof. (proof of proposition 5.10) We will apply theorem 4.2 on $\Lambda_1 := K\log\alpha - (j-1)\log(2)$. We have

$$D = 2 \qquad \gamma_1 = 2 \qquad \gamma_2 = \alpha \qquad b_1 = j-1 \qquad b_2 = K$$

Also we take h_1 and h_2 as shown below:

$$h_1 = \log 2 \geq \max\left\{h(\gamma_1), \tfrac{|\log\gamma_1|}{D}, \tfrac{1}{D}\right\} = \max\left\{\log 2, \tfrac{\log 2}{4}, \tfrac{1}{4}\right\} = \log 2$$
$$h_2 = \tfrac{1}{2} \geq \max\left\{h(\gamma_2), \tfrac{|\log\gamma_2|}{D}, \tfrac{1}{D}\right\} = \max\left\{\tfrac{1}{2}\log\alpha, \tfrac{1}{4}\log\alpha, \tfrac{1}{4}\right\}$$

By lemma 5.12,

$$K < \frac{(j-1)\log(2) + (7j+29074)\,\alpha^{-4n-4}}{\log\alpha} < 0.794j + 27.799$$

And because

$$\frac{|b_1|}{Dh_2} + \frac{|b_2|}{Dh_1} = (j-1) + \frac{|K|}{2\log 2} < 1.573j + 19.0523 =: b'$$

Applying theorem 4.2 we obtain the bound

$$\log|\Lambda_1| > -17.9 \cdot 8\log 2 \cdot \left(\max\{\log(1.573j + 19.0523) + 0.38,\ 15\}\right)^2$$

And from lemma 5.12 we have the bound

$$\log|\Lambda_1| < \log(7j + 29074) - (4n+4)\log\alpha$$

Combining these two bounds yields

$$n < 10.279(\max\{\log(1.573j + 19.0523) + 0.38, 15\})^2 + 0.104\log(7j + 29074)$$

If $\log(1.573j + 19.0523) + 0.38 < 15$, then $j < 2.81554 \times 10^6$. Otherwise,

$$n < 10.279(\log(1.573j + 19.0523) + 0.38)^2 + 0.104\log(7j + 29074)$$

In proposition 5.4 we found the bound $j < 1.9241 \times 10^{12}(4n+7)\log(39j(4n+7))$. Bringing these two results together, we have

$$j < 1.9241 \times 10^{12} \left(4\left(10.279(\log(1.573j + 19.0523) + 0.38)^2 + 0.104\log(7j + 29074)\right) + 7\right)$$
$$\times \log\left(39j\left(4\left(10.279(\log(1.573j + 19.0523) + 0.38)^2 + 0.104\log(7j + 29074)\right) + 7\right)\right)$$

Which implies $j < 9.19 \times 10^{18}$ and therefore $n < 20358$. □

5.4 Refining the Bounds

In this section, the bounds on n and j are improved before Baker-Davenport reduction is applied in the next section.

Lemma 5.12 gives

$$|K\log\alpha - (j-1)\log 2| < \frac{7j + 29074}{\alpha^{4n+4}} \xrightarrow{\text{divide by } j-1} \left|\frac{\log 2}{\log\alpha} - \frac{K}{j-1}\right| < \frac{7j + 29074}{(j-1)\,\alpha^{4n+4}\log\alpha}$$

Assume that

$$\frac{7j + 29074}{(j-1)\,\alpha^{4n+12}\log\alpha} < \frac{1}{2(j-1)^2}$$

Then by the inequality above,

$$\left|\frac{\log 2}{\log\alpha} - \frac{K}{j-1}\right| < \frac{1}{2(j-1)^2}$$

By lemma 2.10, $\frac{K}{j-1}$ is a convergent of the continued fraction of $\frac{\log 2}{\log\alpha}$. The 38th convergent of continued fraction of $\frac{\log 2}{\log\alpha}$ is

$$\frac{7486685157270191075}{9519719241472897252}$$

Its denominator is larger than the upper bound of 9.19×10^{18} established for j, so $\frac{K}{j-1}$ cannot be equal to the 38th convergent, nor any convergent that follows it. Therefore $\frac{K}{j-1}$ is a convergent that occurs among the first 37 convergents of $\frac{\log 2}{\log\alpha}$. By theorem 2.7, we can use the denominator of the 37th convergent

$$\frac{5063552340916761513}{6438576704834547937}$$

to obtain the lower bound:

$$\left|\frac{\log 2}{\log\alpha} - \frac{K}{j-1}\right| \geq \left|\frac{\log 2}{\log\alpha} - \frac{5063552340916761513}{6438576704834547937}\right| > 1.00 \times 10^{-38}$$

Combining these bounds we obtain

$$10^{-38} < \frac{7j + 29074}{(j-1)\,\alpha^{4n+4}\log\alpha} < \frac{29200}{\alpha^{4n+4}\log\alpha}$$

Which implies that $n < 27$. We now apply lemma 2.11 to deduce that

$$\left|\frac{\log 2}{\log \alpha} - \frac{p_r}{q_r}\right| \geq \frac{1}{(a_{r+1}+2)\, q_r^2}$$

where $\frac{p_r}{q_r}$ is the r^{th} convergent of $\frac{\log 2}{\log \alpha}$, and a_{r+1} is the $(r+1)^{\text{st}}$ partial quotient of $\frac{\log 2}{\log \alpha}$. Therefore, since $\frac{K}{j-1}$ is among the first 37 convergents of $\frac{\log 2}{\log \alpha}$, we have for $2 \leq r \leq 37$ that

$$\min_{2\leq r\leq 37}\left\{\frac{1}{(a_{r+1}+2)\,(j-1)^2}\right\} < \left|\frac{\log 2}{\log \alpha} - \frac{K}{j-1}\right| < \frac{7j+29074}{(j-1)\,\alpha^{4n+4}\log \alpha}$$

Since $\max\{a_{r+1} : 2 \leq r \leq 37\} = a_{27} = 100$,

$$\alpha^{4n+4} < 102\,(j-1)\,(7j+29074)\,(\log \alpha)^{-1}$$

All of this was under the assumption that $\frac{7j+29074}{(j-1)\alpha^{4n+4}\log \alpha} < \frac{1}{2(j-1)^2}$. If this is not the case, then

$$\alpha^{4n+4} \leq 2\,(j-1)\,(7j+29074)\,(\log \alpha)^{-1}$$

In either case, $\alpha^{4n+4} < 9 \times 10^5 j^2$. This leads to the following result.

Proposition 5.13. *If equation (5.5) has a positive integer solution (j,k) with $j > 1$, then*

$$n < 0.568 \log j + 3.889$$

Combining this result with the bound for j found in proposition 5.4, we get

$$j < 1.9241 \times 10^{12}\,(4\,(0.568 \log j + 3.889) + 7) \log (39j\,(4\,(0.568 \log j + 3.889) + 7))$$

Which implies $j < 9.21 \times 10^{15}$ and $n < 25$.

5.5 Baker-Davenport Reduction

We will apply the method of Baker-Davenport Reduction described in lemma 4.3. We know from lemma 5.6 that

$$0 < 2j \log \beta_n - k \log \alpha + \log\left(2\sqrt{2}\gamma_n^{\pm}\right) < 4046\beta_n^{-2j}$$

So we can apply lemma 4.3 with the quantities

$$\kappa = \frac{2\log \beta_n}{\log \alpha}, \quad \mu = \frac{\log\left(2\sqrt{2}\gamma_n^{\pm}\right)}{\log \alpha}, \quad A = \frac{4046}{\log \alpha}, \quad B = \beta_n^2, \quad M = 9.21 \times 10^{15}, \quad 1 \leq n \leq 24$$

The following code written by myself in Maple™ carries out the reduction for each $1 \leq n \leq 24$. Note that there is a "±" symbol in the definition of `G(n)`. This needs to be specified as "+" or "-" to differentiate between γ_n^+ and γ_n^-, respectively.

```
> with(NumberTheory):
  P := n -> 1/2*((1 + sqrt(2))^n - (1 - sqrt(2))^n)/sqrt(2):
  n := 1:
  while n < 25 do
      B(n) := P(2*n + 3) + sqrt(P(2*n + 3)^2 - 1);
      G(n) := (±1/sqrt(P(2*n + 4)) + 1/(2*sqrt(P(2*n + 2))))^2;
      Digits := 10000;
      i := 0;
      t := 2*ln(B(n))/ln(1 + sqrt(2));
      cf := ContinuedFraction(t);
      M := 9.21*10^15;
      u := ln(2*sqrt(2)*G(n))/ln(1 + sqrt(2));
      epsilon := -1;
      while epsilon <= 0 do
          q := Denominator(cf, i);
          epsilon := evalf(abs(u*q - round(u*q)) - M*abs(t*q -    round(t*q)));
          i := i + 1;
      end do:
      q := Denominator(cf, i);
      epsilon := evalf(abs(u*q - round(u*q)) - M*abs(t*q -    round(t*q)));
      nBound[n] := floor(evalf(log(4046*q/(ln(1 + sqrt(2))*epsilon))/log(B(n)^2)));
      n := n + 1;
  end do:
  for n to 24 do
      nBound[n];
  end do;
```

In each case we find that $j \leq 6$, which implies $n \leq 4$.

Proposition 5.14. *If equation (5.5) has solutions* (j, k) *with* $j > 1$*, then* $j \leq 6$ *and* $n \leq 4$.

Applying this result to the equation $P_k = C_j^{\pm}$, we can prove theorem 5.2.

Proof. (proof of theorem 5.2) By testing each case one-by-one, we find that no combination of n and j with $1 < j \leq 6$ and $1 \leq n \leq 4$ results in $C_j^{\pm}$ being a Pell number. When $j = 1$, we have already seen that ${C_1}^- = P_{2n}$ and $P_{2n+6} < C_1^+ < P_{2n+7}$. □

6 Conclusion

It may be possible to generalize this result to other binary recurrences. If S_n is defined by the recurrence relation $S_{n+1} = c \cdot S_n + S_{n-1}$ with initial conditions $S_0 = 0$ and $S_1 = 1$ for a positive integer c, then it was shown in section 1.3 that $\left\{S_{2n}, c^2 S_{2n+2}, S_{2n+4}\right\}$ is a $D\left(c^2\right)$-triple. The case of $c = 2$ was covered in this paper. It may be possible to use the method in this paper to prove a similar result for other values of c. However, this method on its own will come short when trying to make a general argument for all values of c. The theorem of Matveev requires that $\alpha_1, \alpha_2, \alpha_3$ are multiplicatively independent. For general c we have that $\alpha_2 = \frac{1}{2}\left(c + \sqrt{c^2+4}\right)$. To establish multiplicative independence, we would need to prove that the squarefree part of $S_{2n+2}S_{2n+4}$ is different from the squarefree part of $c^2 + 4$. In the case of $c = 2$, this result was proved in lemma 5.7. If we leave c unspecified, we will not have a general expression for the squarefree part of $c^2 + 4$, so we will not be able to find the fundamental solution of the Pellian equation that comes up in lemma 5.7. We can, however, make the restriction that $c^2 + 4$ is squarefree. It has been shown by Estermann [5] that there are infinitely many squarefree numbers of this form. In this case, the relevant Pellian equation becomes $X^2 - \left(c^2+4\right)Y^2 = 1$. I was able to find the continued fraction representation $\sqrt{c^2+4} = \begin{cases} \left[c; \overline{\frac{c}{2}, 2c}\right] & \text{if } c \text{ is even} \\ \left[c; \overline{\frac{c-1}{2}, 1, 1, \frac{c-1}{2}, 2c}\right] & \text{if } c \text{ is odd} \end{cases}$. With this, the fundamental solution to $X^2 - \left(c^2+4\right)Y^2 = 1$ can be found and it can be proved by a similar argument as lemma 5.7 that $S_{2n+2}S_{2n+4}$ is neither a square nor $c^2 + 4$ times a square. After this point, the bounds found from applying theorems 4.1 and 4.2 would have an additional variable c in them. It may be worthwhile to carry out a similar procedure and see if a general argument can be made which works for all c such that $c^2 + 4$ is squarefree, or at the very least show a similar result for other particular values of c.

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