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The focus of this dissertation is to study positive solutions for classes of nonlinear steady state reaction diffusion equations and systems. In particular, we consider four focuses. In Focus 1, we establish sufficient conditions on the reaction term for which the bifurcation diagram for positive solutions for a nonlinear reaction diffusion equation is $\Sigma-$shaped. In Focus 2, we extend the study in Focus 1 for classes of coupled reaction diffusion equations. In Focus 3, we analyze the classes of diffusive Lotka-Volterra competition models in fragmented patches. Finally, in Focus 4, we use the finite element method for the numerical computation of bifurcation diagrams in dimension $N = 2$ for examples in Focus 1 and Focus 3.

We establish analytical results in any dimension, namely, we establish existence, nonexistence, multiplicity, and uniqueness results. Our existence and multiplicity results are achieved by the method of sub-supersolutions. Via computational methods we also obtain approximate bifurcation diagrams describing the structure of the steady states. Namely, we obtain these bifurcation diagrams via a quadrature method and Mathematica computations in the one-dimensional case, and via the use of finite element methods and nonlinear solvers in Matlab in the two-dimensional case.

This dissertation aims to significantly enrich the mathematical and computational analysis of steady states to classes of reaction diffusion equations and systems.

ANALYSIS OF STEADY STATES TO CLASSES OF REACTION DIFFUSION EQUATIONS AND SYSTEMS

by

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Approved by

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Ratnasingham Shivaji
Committee Chair

To my parents.

APPROVAL PAGE

This dissertation written by Ananta Acharya has been approved by the following committee of the Faculty of The Graduate School at The University of North Carolina Greensboro.

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CHAPTER I

INTRODUCTION

The study of steady state reaction diffusion equations is of great importance in many applications such as population dynamics, combustion theory, nonlinear heat generation and chemical reactor theory (see [Ari69], [BIS81], [CC03], [CL70], [Fif79], [FK69], [KC67], [KJD+79], [Mur03], [OL01], [Par61], [Par74], [Sat75], [Sem35], [Ske51], [Tam79], [Tur52] and [ZBLM85]). The time dependent models that arise are of the form:

$$\begin{cases} u_t = d\Delta u + f(u);\ x \in \Omega_0,\ t > 0, \\ u(x,0) = u_0(x);\ x \in \Omega_0, \\ Bu \equiv u = 0;\ x \in \partial\Omega_0,\ t > 0 \ \text{ or } \ Bu \equiv \frac{\partial u}{\partial \eta} + \gamma u = 0; \gamma > 0,\ x \in \partial\Omega_0,\ t > 0, \end{cases} \tag{1.A}$$

where $\Delta u := \text{div}(\nabla u)$ is the Laplacian operator of u, $d > 0$ is the diffusion coefficient, $\Omega_0 \subset \mathbb{R}^N$ with $N > 1$, is a bounded domain with smooth boundary $\partial\Omega_0$ or $\Omega_0 = (0,1)$, $f : [0,\infty) \to \mathbb{R}$ is the reaction term, and $\frac{\partial u}{\partial \eta}$ is the outward normal derivative of u. In the applications mentioned above, u describes a population density, a mass concentration or a temperature distribution, and in these cases, only non-negative solutions ($u \geq 0$ in $\overline{\Omega}_0$) are relevant. The steady states of (1.A) (if they exist) are needed to understand the dynamics of the solutions of (1.A). For the case when $u = 0$; $x \in \partial\Omega_0$ (Dirichlet or hostile boundary condition), mathematicians have developed a rich literature, namely, for nonlinear elliptic partial differential equations of the form:

$$\begin{cases} -\Delta u = \lambda f(u);\ x \in \Omega_0, \\ u = 0;\ x \in \partial\Omega_0. \end{cases} \tag{1.B}$$

In recent history there has been a lot of interest in models where a parameter influences the equation as well as the boundary conditions, namely of the form:

$$\begin{cases} -\Delta u = \lambda f(u);\ x \in \Omega_0, \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda} u = 0;\ x \in \partial\Omega_0. \end{cases} \tag{1.C}$$

In particular, see [AFS21], [CGS19], [FSSS19], [FMS20] and [GMRS18]. Also, see Focus 3 where it is shown how the steady state equations for an ecological model take this form. In [GMRS18], authors obtain the exact bifurcation diagram (see Figure 1) for the case when $f(s) = s(1-s)$. In [FSSS19], for classes of $f(s)$, the authors establish S-shaped bifurcation diagrams. An example, satisfying their hypothesis is $f(s) = e^{\frac{\beta s}{\beta + s}}; \beta \gg 1$ (see Figure 2).

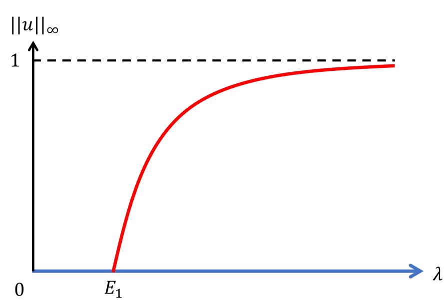


Figure 1. Exact bifurcation diagram of (1.C) when $f(s) = s(1-s)$.

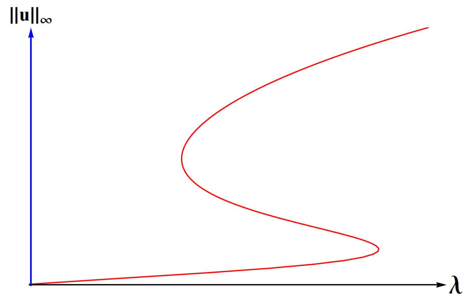


Figure 2. Bifurcation diagram of (1.C) when $f(s) = e^{\frac{\beta s}{\beta + s}}; \beta \gg 1$.

In [AFS21], authors extend the study in [FSSS19] to classes of systems of the form:

$$\begin{cases} -\Delta u = \lambda f(v);\ \Omega \\ -\Delta v = \lambda g(u);\ \Omega \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda} u = 0;\ \partial\Omega \\ \frac{\partial v}{\partial \eta} + \sqrt{\lambda} v = 0;\ \partial\Omega, \end{cases} \tag{1.D}$$

and establish an S-shaped bifurcation diagram when f and g satisfy certain hypotheses. An example satisfying these hypotheses is:

$$f = f_{\alpha,k}(s) = \begin{cases} e^{\frac{s}{s+1}} - 1; s < k \\ [e^{\frac{\alpha s}{\alpha + s}} - e^{\frac{\alpha k}{\alpha + k}}] + [e^{\frac{k}{k+1}} - 1]; s \geq k \end{cases}$$

$$g = g_k(s) = \begin{cases} 2(1+s)^{\frac{1}{2}} - 2; s < k \\ [\frac{1}{2}(1+s)^2 - \frac{1}{2}(1+k)^2] + [2(1+k)^{\frac{1}{2}} - 2]; s \geq k, \end{cases}$$

when the parameters α and k are large.

In this research we enrich the literature on multiplicity results for (1.C) and (1.D) and study an ecological model (system) where a parameter (related to the patch size) arises in the reaction term and the boundary condition. Namely, the dissertation has the following focuses:

Focus 1: Establish sufficient conditions on f for which the bifurcation diagram for positive solutions to (1.C) is $\Sigma-$shaped.

Focus 2: Extend the study in Focus 1 for classes of coupled reaction diffusion equations.

Focus 3: Analysis of classes of diffusive Lotka-Volterra competition models in fragmented patches.

Focus 4: Numerical computation of bifurcation diagrams in dimension $N = 2$ for examples in Focus 1 and Focus 3.

1.1 Focus 1: Establish sufficient conditions on f for which the bifurcation diagram for positive solutions to (1.C) is $\Sigma-$shaped

We study positive solutions to the steady state reaction diffusion equation of the form:

$$\begin{cases} -\Delta u = \lambda f(u);\ \Omega \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda} u = 0;\ \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a positive parameter, Ω is a bounded domain in $\mathbb{R}^N$ when $N > 1$ (with smooth boundary $\partial\Omega$) or $\Omega = (0, 1)$, and $\frac{\partial u}{\partial \eta}$ is the outward normal derivative of u. Here $f(s) = ms + g(s)$ where $m \geq 0$ (constant) and $g \in C^2[0, r) \cap C[0, \infty)$ for some

$r > 0$.

Part I: Motivational example in the dimension $N = 1$ case.

First, we consider the 1-dimensional form of (1.1) with $\Omega = (0, 1)$:

$$\begin{cases} -u'' = \lambda f(u); \ (0,1) \\ -u'(0) + \sqrt{\lambda}u(0) = 0 \\ u'(1) + \sqrt{\lambda}u(1) = 0, \end{cases} \tag{1.2}$$

and we take the function f as follows:

$f(s) = ms + g(s)$ where

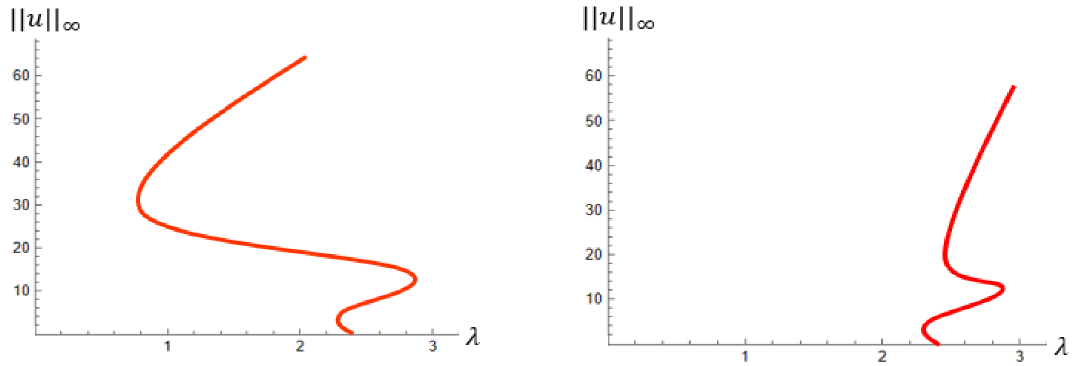
$$g(s) = g_{\alpha,k}(s) = \begin{cases} e^{\frac{cs}{c+s}} - 1; s \leq k \\ [e^{\frac{\alpha s}{\alpha+s}} - e^{\frac{\alpha k}{\alpha+k}}] + [e^{\frac{ck}{c+k}} - 1]; s > k. \end{cases} \tag{1.3}$$

Here $c > 2$ is a fixed number, $m \geq 0$, $\alpha > 0$ and $k > 0$ are parameters. We use the Quadrature method (discussed in Chapter II) to obtain the bifurcation diagrams. We obtained the following approximate Σ-shaped bifurcation diagrams for the positive solutions of (1.2) for certain combinations of α and k (see Figures 3, 4).

Remark: See also [LSS12], where for an ecological model involving logistic growth, grazing, constant yield harvesting, and Dirichlet boundary condition, it was established that the bifurcation diagram for positive solutions is at least Σ-shaped.

Part II: Analytical results for the general domain case.

Here, we consider (1.1) and analyze the positive solutions in any dimension $N \geq 1$. In particular, we discuss the existence of multiple positive solutions for certain ranges



(a) $\alpha = 5.5, k = 5, c = 2.5$ (b) $\alpha = 3.1, k = 3, c = 2.5$

Figure 3. Approximate bifurcation diagrams for (1.2) when $m = 0$.

of λ leading to the occurrence of Σ-shaped bifurcation diagrams. We establish our existence and multiplicity results via the method of sub-supersolutions.

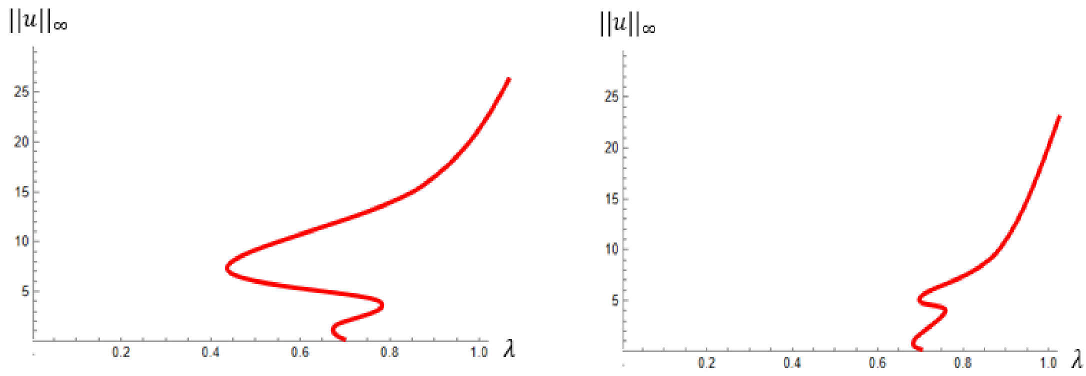
We first introduce some hypotheses that we use.

(H_1) $g(0) = 0,\ g'(0) = 1,\ g''(0) > 0,\ g'(s) > 0; s \geq 0$, and $\lim_{s\to\infty} \frac{g(s)}{s} = 0$.

First, we recall $E_1(m,k)$ from [GMRS18]. Namely, $E_1(m,k)$ is the principal eigenvalue of:

$$\begin{cases} -\Delta z = Emz;\ \Omega \\ \frac{\partial z}{\partial \eta} + k\sqrt{E}z = 0;\ \partial\Omega \end{cases} \tag{1.4}$$

for $m \in (0,\infty)$ and $k \in [0,\infty)$. Note that $E_1(m,k)$ is increasing in k and decreasing in m.



(a) $\alpha = 5.5, k = 5, c = 2.5$ (b) $\alpha = 3.1, k = 3, c = 2.5$

Figure 4. Approximate bifurcation diagrams for (1.2) when $m = 1$.

Let $A_m = E_1(m, 1)$. Then A_m is a strictly decreasing function of m with:

$$\lim_{m \to 0} A_m = \infty. \tag{1.5}$$

Further, for a fixed $\lambda > 0$, let $\sigma_{\lambda,m}$ be the principal eigenvalue and $\theta_{\lambda,m} > 0$ on $\overline{\Omega}$ be the corresponding normalized eigenfunction of:

$$\begin{cases} -\Delta\theta = (\sigma + \lambda) m\theta; \ \Omega \\ \frac{\partial\theta}{\partial\eta} + \sqrt{\lambda}\theta = 0; \ \partial\Omega. \end{cases} \tag{1.6}$$

We note that $\sigma_{\lambda,m} > 0$ when $\lambda < A_m$, $\sigma_{\lambda,m} < 0$ when $\lambda > A_m$, and $\sigma_{\lambda,m} \to 0$ as $\lambda \to A_m$.

Next, let v be the unique solution of:

$$\begin{cases} -\Delta v = 1; \ \Omega \\ \frac{\partial v}{\partial\eta} + v = 0; \ \partial\Omega, \end{cases} \tag{1.7}$$

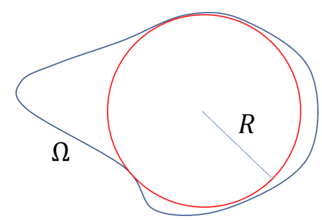
and let w be the unique solution of:

$$\begin{cases} -\Delta w = 1; \ \Omega \\ \frac{\partial w}{\partial \eta} + \sqrt{\frac{A_1}{2}} w = 0; \ \partial\Omega. \end{cases} \tag{1.8}$$

Now, we introduce additional hypotheses (H_2) and (H_3):

(H_2) There exist $a_1 > 0$, $b_1 > 0$ such that $a_1 < b_1$ and $\min\{A_m, \frac{\mathbf{a_1}}{\mathbf{f(a_1)}} \frac{1}{\|v\|_\infty}\} > \max\{\frac{\mathbf{b_1}}{\mathbf{f(b_1)}} \frac{2NC_N}{R^2}, A_{m+1}, 1\}$.

(H_3) There exist $a_2 > 0$, $b_2 > 0$ such that $a_2 < b_2$ and $\frac{\mathbf{a_2}}{\mathbf{f(a_2)}} \frac{1}{\|w\|_\infty} \geq A_{m+1} > \max\{\frac{\mathbf{b_2}}{\mathbf{f(b_2)}} \frac{2NC_N}{R^2}, \frac{A_1}{2}\}$, where $C_N = \frac{(N+1)^{N+1}}{2N^N}$ and R is the radius of the largest inscribed ball in Ω.



We believe a typical f which is likely to produce such a $\Sigma-$shaped bifurcation diagram is as follows:

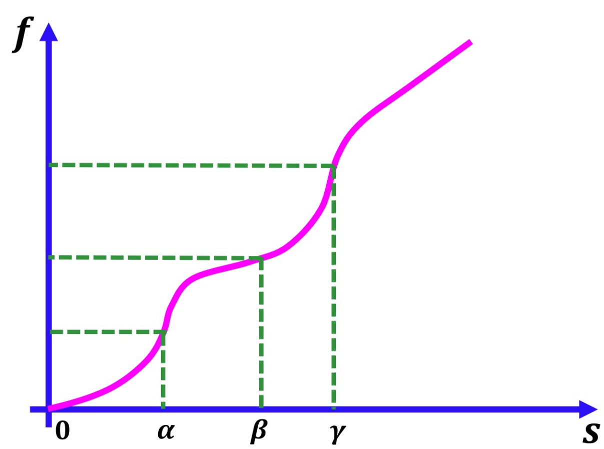


Figure 5. Shape of f producing multiplicity.

Convex on $(0, \alpha)$ for some $\alpha > 0$ driving the bifurcation curve initially to the left, a strong concavity on (α, β) with $\beta > \alpha$ making the bifurcation curve go back to the right, a strong convexity on (β, γ) with $\gamma > \beta$ driving the bifurcation curve back again to the left, and then a strong concavity on (γ, ∞) bringing the curve eventually to the right (see Figure 5).

In this case we expect the shape of $\frac{s}{f(s)}$ to be of the form in Figure 6, and when $\frac{\ell_1}{\ell_2} \gg 1$ our hypotheses are satisfied.

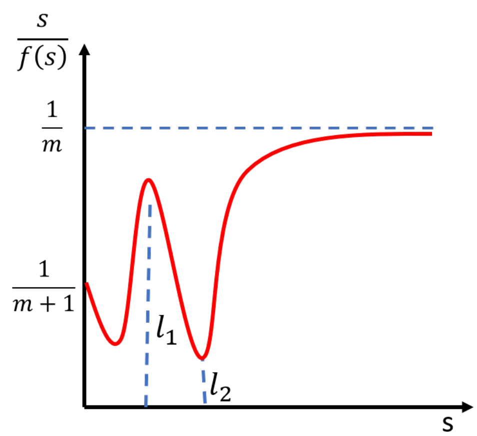


Figure 6. Shape of $\frac{s}{f(s)}$.

We establish the following results:

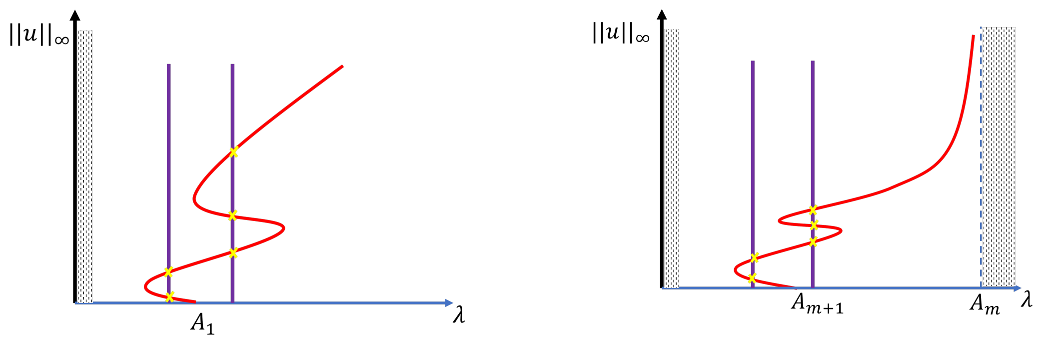
Existence and Multiplicity Results:

Theorem 1.1.

a) Let (H_1) *hold. Then* (1.1) *has a positive solution for* $\lambda \in [A_{m+1}, A_m)$*. In particular,* (1.1) *has a positive solution* u_λ *for* $\lambda < A_m$ *and* $\lambda \approx A_m$ *such that* $u_\lambda \to \infty$ *as* $\lambda \to A_m^-$*. Further, there exists* $\overline{\lambda} < A_{m+1}$ *such that* (1.1) *has at least two positive solutions for*

$\lambda \in [\overline{\lambda}, A_{m+1})$. *(Here, by $\lambda \approx A_m$, we mean λ is close to A_m.)*

b) Let (H_1) and (H_2) hold. Then (1.1) has at least three positive solutions for $\lambda \in \Big(\max\{\frac{b_1}{f(b_1)}\frac{2NC_N}{R^2}, A_{m+1}, 1\}, \min\{A_m, \frac{a_1}{f(a_1)}\frac{1}{\|v\|_\infty}\}\Big)$.



(a) $m = 0$ (b) $m > 0$

Figure 7. Expected bifurcation diagrams for (1.1) when the hypotheses of Theorem 1.1(b) are satisfied.

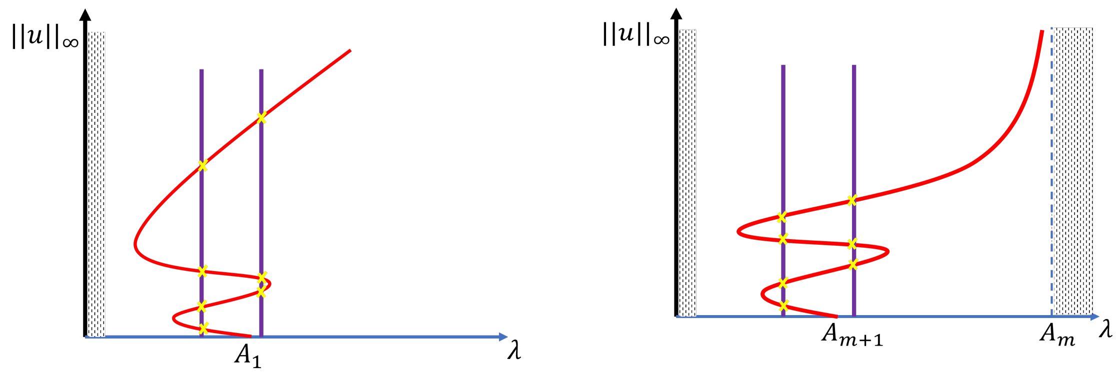
Theorem 1.2. *Let (H_1) and (H_3) hold. Then there exists $\lambda^* \in \Big(\max\{\frac{b_2}{f(b_2)}\frac{2NC_N}{R^2}, \frac{A_1}{2}\}, A_{m+1}\Big)$ such that (1.1) has at least four positive solutions for $\lambda \in [\lambda^*, A_{m+1})$.*

Corollary 1.3. *Let (H_1) - (H_3) hold. Then there exists λ^* such that (1.1) has a positive solution for $\lambda \in [\lambda^*, A_m)$, a positive solution u_λ for $\lambda < A_m$ and $\lambda \approx A_m$ such that $u_\lambda \to \infty$ as $\lambda \to A_m^-$, at least four positive solutions for $\lambda \in [\lambda^*, A_{m+1})$ and at least three positive solutions for $\lambda \in \Big(\max\{\frac{b_1}{f(b_1)}\frac{2NC_N}{R^2}, A_{m+1}, 1\}, \min\{A_m, \frac{a_1}{f(a_1)}\frac{1}{\|v\|_\infty}\}\Big)$.*

<u>Nonexistence Results:</u>

We also prove the following:

Theorem 1.4. (1.1) *has no positive solutions for $\lambda \approx 0$ and when $m > 0$ for $\lambda > A_m$.*



(a) $m = 0$ (b) $m > 0$

Figure 8. Expected bifurcation diagrams for (1.1) when the hypotheses of Corollary 1.3 are satisfied.

Remark: Focus 1 results are now published in [AFQS21].

1.2 Focus 2: Extend the study in Focus 1 for classes of coupled reaction diffusion equations

We study positive solutions to classes of steady state reaction diffusion systems of the form:

$$\begin{cases} -\Delta u = \lambda f(v);\ \Omega \\ -\Delta v = \lambda g(u);\ \Omega \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda} u = 0;\ \partial\Omega \\ \frac{\partial v}{\partial \eta} + \sqrt{\lambda} v = 0;\ \partial\Omega, \end{cases} \tag{1.9}$$

where $\lambda > 0$ is a positive parameter, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ for $N > 1$ or $\Omega = (0, 1)$, and $\frac{\partial z}{\partial \eta}$ is the outward normal derivative of z. Here $f, g \in C^2[0, r) \cap C[0, \infty)$ for some $r > 0$. Further, we assume that f and g are increasing functions such that $f(0) = 0 = g(0)$, $f'(0) = g'(0) = 1$, $f''(0) > 0, g''(0) > 0$, and $\lim_{s\to\infty} \frac{f(Mg(s))}{s} = 0$ for all $M > 0$. Under certain additional assumptions on f and g we prove that the bifurcation diagram for positive solutions of this system is at least $\Sigma-$shaped.

This study extends the results of Focus 1. In this study, our focus is to show that in the case of a system like (1.9), both the reaction terms f and g do not have to exhibit similar alternating convexity concavity properties to produce a $\Sigma-$shaped bifurcation curve, and, in fact, they both do not have to be sub-linear at infinity. We establish that $\Sigma-$shaped bifurcation curves occur when f and g satisfy a combined sublinear condition at ∞ ($\lim_{s\to\infty} \frac{f(Mg(s))}{s} = 0; \forall M > 0$). In particular, one of the nonlinearities can be superlinear at ∞ (see Figure 9).

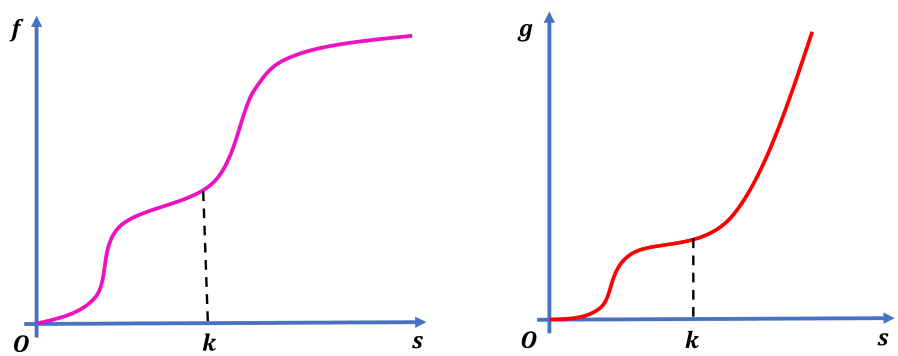


Figure 9. Prototypical shapes of f and g producing a $\Sigma-$shaped bifurcation diagram.

Recall (1.4) and let $A_1 = E_1(1, 1)$. Next, recall (1.6) for a fixed $\lambda > 0$, let $\sigma_\lambda = \sigma_{\lambda,1}$ be the principal eigenvalue and $\theta_\lambda = \theta_{\lambda,1}$ on $\overline{\Omega}$ be the corresponding eigenfunction such that $\|\theta_\lambda\|_\infty = 1$. We note that $\sigma_\lambda > 0$ when $\lambda < A_1$, $\sigma_\lambda < 0$ when $\lambda > A_1$, and

$\sigma_\lambda \to 0$ as $\lambda \to A_1$.

Next, recall v from (1.7) and w from (1.8). Now, we introduce our hypotheses $(H_4)-(H_7)$ which we use to establish our results. Assume that f, g are increasing and satisfy:

(H_4) $f(0) = g(0) = 0,\ f'(0) = g'(0) = 1,\ f''(0) > 0, g''(0) > 0.$

(H_5) $\lim\limits_{s\to\infty} \frac{f(Mg(s))}{s} = 0$ for all $M > 0$.

(H_6) There exist $a_1 > 0$, $b_1 > 0$ such that $a_1 < b_1$ and $Q_1(a_1)\frac{1}{\|v\|_\infty} > \max\left\{Q_2(b_1)\frac{2NC_N}{R^2}, A_1, 1\right\}$, where $C_N = \frac{(N+1)^{N+1}}{2N^N}$ and R is the radius of the largest inscribed ball in Ω.

Here, for $0 < a < b$,

$$Q_1(a) := \min\left\{\frac{a}{f(a)}, \frac{a}{g(a)}\right\} \tag{1.10}$$

and

$$Q_2(b) := \max\left\{\frac{b}{f(b)}, \frac{b}{g(b)}\right\}. \tag{1.11}$$

(H_7) There exist $a_2 > 0$, $b_2 > 0$ such that $a_2 < b_2$ and $Q_1(a_2)\frac{1}{\|w\|_\infty} \geq A_1 > Q_2(b_2)\frac{2NC_N}{R^2}$.

Then we establish the following results:

Theorem 1.5.

a) Let $(H_4)-(H_5)$ *hold. Then there exists* $\overline{\lambda} < A_1$ *such that* (1.9) *has a positive solution for* $\lambda \geq \overline{\lambda}$, *at least two positive solutions for* $\lambda \in [\overline{\lambda}, A_1)$, *and a positive* *solution* (u_λ, v_λ) *for* $\lambda \gg 1$ *such that* $\|u_\lambda\|_\infty$, $\|v_\lambda\|_\infty \to \infty$ *as* $\lambda \to \infty$.

b) Let $(H_4) - (H_6)$ *hold. Then* (1.9) *has at least three positive solutions for*

$$\lambda \in \left(\max\{A_1, Q_2(b_1)\frac{2NC_N}{R^2}, 1\}, \frac{Q_1(a_1)}{\|v\|_\infty}\right) := I.$$

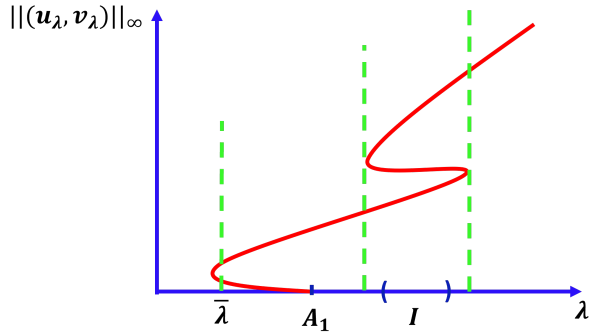


Figure 10. Bifurcation diagram for (1.9) when the hypotheses of Theorem 1.5(b) $(H_4 - H_6)$ hold.

Theorem 1.6. *Let* $(H_4) - (H_5)$, *and* (H_7) *hold. Then there exists* $\lambda^* \in \left(\max\{Q_2(b_2)\frac{2NC_N}{R^2}, \frac{A_1}{2}\}, A_1\right)$ *such that* (1.9) *has at least four positive solutions for* $\lambda \in [\lambda^*, A_1)$.

Corollary 1.7. *Let* $(H_4) - (H_7)$ *hold. Then there exist* $\overline{\lambda}(< A_1)$ *and* $\lambda^*(< A_1)$ *such that* (1.9) *has a positive solution for* $\lambda \geq \overline{\lambda}$, *a positive solution* (u_λ, v_λ) *for* $\lambda \gg 1$ *such that* $\|u_\lambda\|_\infty, \|v_\lambda\|_\infty \to \infty$ *as* $\lambda \to \infty$, *at least four positive solutions for* $\lambda \in [\lambda^*, A_1)$, *and at least three positive solutions for* $\lambda \in \left(\max\{A_1, Q_2(b_1)\frac{2NC_N}{R^2}, 1\}, \frac{Q_1(a_1)}{\|v\|_\infty}\right)$.

Remark: Focus 2 results are now published in [ASF22].

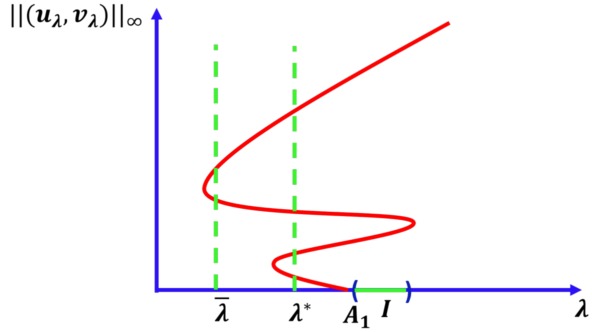


Figure 11. Bifurcation diagram for (1.9) when the hypotheses of Corollary 1.3 ($H_4 - H_7$) hold.

1.3 Focus 3: Analysis of classes of diffusive Lotka-Volterra competition models in fragmented patches.

We study the diffusive Lotka-Volterra (L-V) two species competition model coupled with boundary conditions that allow for the study of the effects of habitat fragmentation on the system. The model is built upon the reaction diffusion framework which has seen tremendous success in the study of spatially structured systems in the literature, see [CC03], [Fif79], [HLV94], [Lev74], [Lev81], [Mur03], [Oku81] and references therein for a detailed history of the framework. We assume that two species are dwelling in a single focal patch $\Omega_0 = \{lx \mid x \in \Omega\}$ with patch size $l > 0$ and $\Omega = (0, 1)$ or $\Omega \subset \mathbb{R}^N$ having unit measure (e.g. if $N = 2$ then the area of Ω is one) and smooth boundary with $N = 2, 3$, that is surrounded by a hostile matrix, denoted by $\Omega_0^c = \mathbb{R}^N \backslash \overline{\Omega}_0$, where it is assumed that organisms experience exponential decay at fixed rate, say $S_0 > 0$ (see Figure 12).

We also denote the boundary of Ω_0 by $\partial\Omega_0$. The two organisms follow an unbiased random walk inside both the patch and matrix, while on the patch/matrix interface a

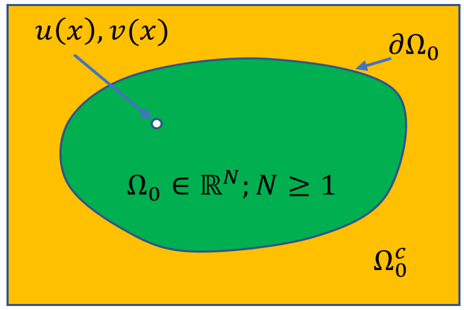


Figure 12. Habitat Ω_0 and the exterior matrix Ω_0^c

discontinuity between the density in the patch and matrix is allowed at the interface (via a biased random walk), while maintaining continuity in the flux (see e.g. [ML86], [Ova04], [OC03]). Here organisms recognize the patch/matrix interface and modify their random walk movement probability (i.e. probability of an organism moving at a given time step in the random walk process), random walk step length (i.e. distance that an organism moves during a given time step), and/or probability of remaining in the patch (say α). In this patch-level setting, we equate dispersal from the patch to organisms reaching the patch/matrix interface, leaving the patch with probability $1 - \alpha$ (taken to be constant), and entering the matrix, where they still have the opportunity to re-enter the patch at the interface. Following the derivation given in [CGS19], the diffusive competitive Lotka-Volterra system becomes:

$$\begin{cases} u_t = D_1 \Delta u + r_1 u(1 - \frac{1}{K_1} u - \frac{a_1}{K_1} v);\ t > 0, x \in \Omega_0 \\ v_t = D_2 \Delta v + r_2 v(1 - \frac{1}{K_2} v - \frac{a_2}{K_2} u);\ t > 0, x \in \Omega_0 \\ u(0,x) = u_0(x); x \in \Omega_0 \\ v(0,x) = v_0(x); x \in \Omega_0 \\ D_1 \alpha_1 \frac{\partial u}{\partial \eta} + S_1^*[1 - \alpha_1] u = 0;\ t > 0, x \in \partial\Omega_0 \\ D_2 \alpha_2 \frac{\partial v}{\partial \eta} + S_2^*[1 - \alpha_2] v = 0;\ t > 0, x \in \partial\Omega_0, \end{cases} \tag{1.12}$$

and it will exactly model the study system in the case of a one-dimensional patch in the sense that steady states of (1.12) and their stability properties will be exactly the same as those of the study system (see [CGS19] and references therein) while providing a reasonable approximation of the study system in the case of a simply connected, convex patch in two- or three-dimensions. In this model, $D_i > 0$ represents the patch diffusion rate, $r_i > 0$ represents the patch intrinsic growth rate, $K_i > 0$ represents the patch carrying capacity, $a_i > 0$ represents the scale of competitive effect from the other competitor, $u_0(x), v_0(x)$ represent the initial population density distributions in the patch, and α_i represents the probability of an individual remaining in the patch upon reaching the boundary ($i = 1$ for u and $i = 2$ for v). The term $\frac{\partial}{\partial \eta}$ denotes the outward normal derivative operator. Note that the parameter $S_i^* \geq 0$ represents the matrix hostility towards an organism, has units of length by time, and can assume different forms depending upon the patch/matrix interface assumptions (see [CGS19]). The boundary is absorbing, i.e., all individuals that reach the boundary will emigrate, when $\alpha_i \equiv 0$, whereas the boundary is reflecting, i.e. the emigration rate is zero, when $\alpha_i \equiv 1$.

We now introduce a standard scaling:

$$\tilde{x} = \frac{x}{l} \ \& \ \tilde{t} = r_1 t.$$

After applying this scaling and dropping the tilde, (1.12) becomes:

$$\begin{cases} u_t = \frac{1}{\lambda}\Delta u + u(1-u-b_1 v); \ t>0, x\in\Omega \\ v_t = \frac{1}{\lambda}\Delta v + r_0 v(1-v-b_2 u); \ t>0, x\in\Omega \\ u(0,x) = u_0(x); \ x\in\Omega \\ v(0,x) = v_0(x); \ x\in\Omega \\ \frac{\partial u}{\partial\eta} + \sqrt{\lambda}\gamma_1 u = 0; \ t>0, x\in\partial\Omega \\ \frac{\partial v}{\partial\eta} + \sqrt{\lambda}\gamma_2 v = 0; \ t>0, x\in\partial\Omega \end{cases} \tag{1.13}$$

with the corresponding steady state equation:

$$\begin{cases} -\Delta u = \lambda u(1-u-b_1 v); \ \Omega \\ -\Delta v = \lambda r v(1-v-b_2 u); \ \Omega \\ \frac{\partial u}{\partial\eta} + \sqrt{\lambda}\gamma_1 u = 0; \ \partial\Omega \\ \frac{\partial v}{\partial\eta} + \sqrt{\lambda}\gamma_2 v = 0; \ \partial\Omega, \end{cases} \tag{1.14}$$

where $\lambda = \frac{r_1 l^2}{D_1}$, $r_0 = \frac{r_2}{r_1}$, $D_0 = \frac{D_2}{D_1}$, $r = \frac{r_0}{D_0}$, $b_i = \frac{a_i}{K_i}; i = 1,2$, $\gamma_1 = \frac{S_1^*}{\sqrt{r_1 D_1}}\frac{1-\alpha_1}{\alpha_1}$, and $\gamma_2 = \frac{S_2^*}{\sqrt{r_1 D_1 D_0}}\frac{1-\alpha_2}{\alpha_2}$ are all unitless. Also, recall that Ω has a length, area, or volume of one. Hence, for fixed r_1, r_2, D_1, D_2, the composite parameter λ is proportional to the patch size squared, γ_1 is proportional to the effective matrix hostility towards u, and

γ_2 is proportional to the effective matrix hostility towards v. The composite parameter b_i denotes the scale of the competitive effect of one organism onto the other, e.g., b_1 measures the competitive effect of v on u. We will denote $b_1, b_2 \in (0,1)$ as weak competition, $b_1 = 1 = b_2$ as neutral competition, either $0 < b_1 \leq b_2$ or $0 < b_2 \leq b_1$ as semistrong competition, and $b_1, b_2 \in [1, \infty)$ as strong competition.

In the case that $\gamma_1 = 0 = \gamma_2$, (1.13) becomes the classical diffusive homogeneous Lotka-Volterra competition model whose dynamics have been studied extensively (see [Bro80], [Has78] and [HN16]).

Part I: Study in the dimension $N = 1$ the case and when $b_1 = 0$.

Here, we consider a case when the species u does not have a competitor and consider the case when $\Omega = (0,1)$. Namely, we analyze positive solutions for:

$$\begin{cases} -v'' = \lambda r v(1 - v - b_2 u); \ (0,1) \\ -v'(0) + \sqrt{\lambda}\gamma_2 v(0) = 0 \\ v'(1) + \sqrt{\lambda}\gamma_2 v(1) = 0, \end{cases} \tag{1.15}$$

where u is the positive solution of:

$$\begin{cases} -u'' = \lambda u(1 - u); \ (0,1) \\ -u'(0) + \sqrt{\lambda}\gamma_1 u(0) = 0 \\ u'(1) + \sqrt{\lambda}\gamma_1 u(1) = 0. \end{cases} \tag{1.16}$$

First, using the Quadrature method (which will be discussed in Chapter II) and Mathematica computations, we numerically approximate the unique positive solution u of (1.16). Then using this approximation, we employ the Shooting method (will be discussed in Chapter II) to numerically approximate positive solutions of (1.15) and generate bifurcation diagrams of the positive solutions of (1.15). We choose values of r, γ_1, γ_2 to obtain results for three different cases: $E_1(r,\gamma_2) < E_1(1,\gamma_1), E_1(r,\gamma_2) > E_1(1,\gamma_1)$ and $E_1(r,\gamma_2) = E_1(1,\gamma_1)$, where $E_1(m,k)$ is as in (1.4). Further, the problem:

$$\begin{cases} -\Delta z = \lambda m z(1-z);\ \Omega \\ \frac{\partial z}{\partial \eta} + \sqrt{\lambda} k z = 0;\ \partial\Omega \end{cases} \tag{1.17}$$

has a unique positive solution for $\lambda > E_1(m,k)$ and has the exact bifurcation diagram for positive solutions given in Figure 13 (see [GMRS18]).

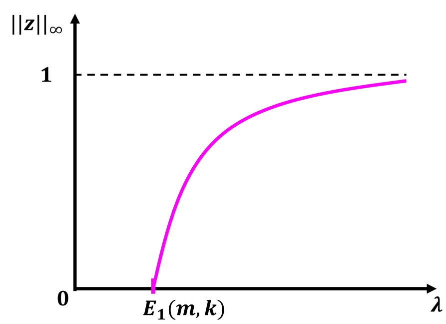
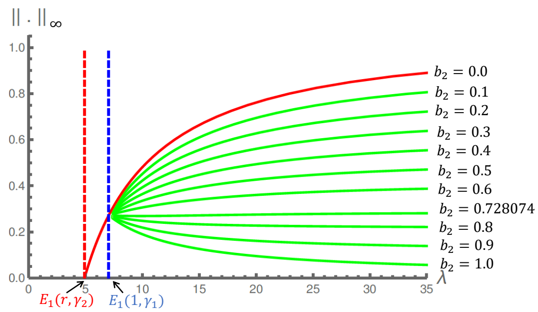
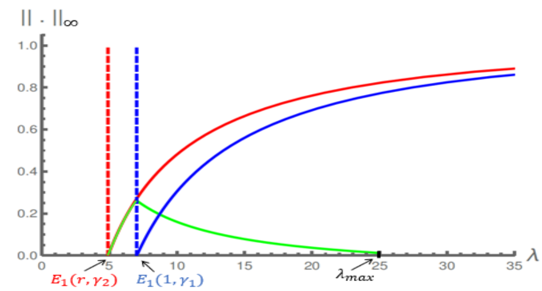


Figure 13. Exact bifurcation diagram for positive solutions of model (1.17).

Here we provide the bifurcation curves we obtained for the positive solution v of (1.15). The blue and red curves represent the bifurcation curves corresponding to the independent solutions u and v respectively, and the green curves represent the bifurcation curves for the solution v when it is affected by u with competition strength b_2.



(a) Approximate bifurcation diagrams for different values of $b_2 \leq 1$.



(b) Approximate bifurcation diagrams when $b_2 = 1.1$.

Figure 14. Approximate bifurcation diagrams for (1.15) when $E_1(r, \gamma_2) < E_1(1, \gamma_1)$ and b_2 varies.

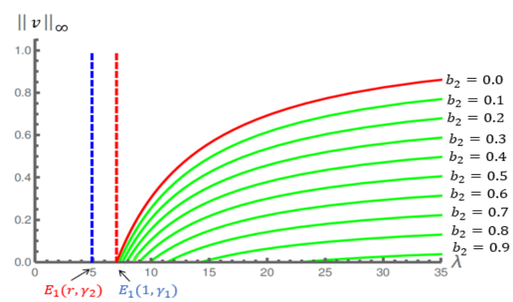
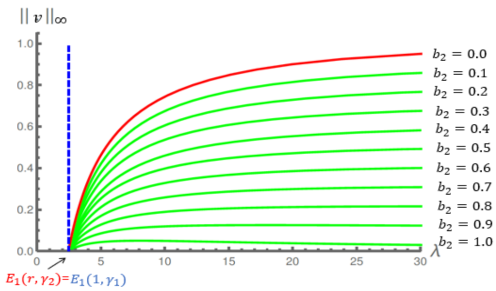


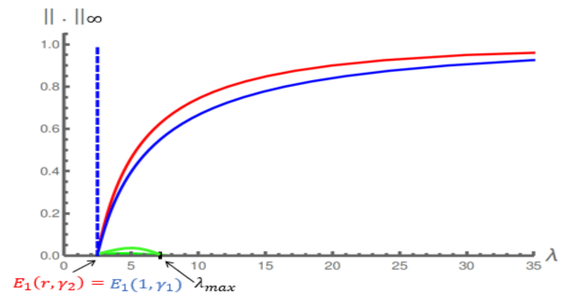
Figure 15. Approximate bifurcation diagrams for (1.15) when $E_1(1, \gamma_1) < E_1(r, \gamma_2)$ and b_2 varies.

Part II: Dimension $N \geq 1$ case when both $b_1, b_2 \neq 0$.

Motivated by our study in Part I, here we consider the case when both of the species have competition inside the domain in any dimension ($N \geq 1$). Namely, we consider



(a) Approximate bifurcation diagrams for different values of $b_2 \leq 1$.



(b) Approximate bifurcation diagram when $b_2 = 1.1$.

Figure 16. Approximate bifurcation diagrams for (1.15) when $E_1(r,\gamma_2) = E_1(1,\gamma_1)$ and b_2 varies.

the following problem:

$$\begin{cases} -\Delta u = \lambda u(1-u-b_1 v);\ \Omega \\ -\Delta v = \lambda r v(1-v-b_2 u);\ \Omega \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda}\gamma_1 u = 0;\ \partial\Omega \\ \frac{\partial v}{\partial \eta} + \sqrt{\lambda}\gamma_2 v = 0;\ \partial\Omega. \end{cases} \tag{1.18}$$

Now, we recall the dynamics of the following single species model and discuss some important eigenvalue problems for which our coexistence results will be built upon:

$$\begin{cases} W_t = \frac{1}{\lambda R}\Delta W + W(1-b-W);\ t>0, x\in\Omega \\ W(0,x) = W_0(x);\ x\in\Omega \\ \frac{\partial W}{\partial \eta} + \sqrt{\lambda}\gamma W = 0;\ t>0, x\in\partial\Omega \end{cases} \tag{1.19}$$

with corresponding steady state equation:

$$\begin{cases} -\Delta W = \lambda RW(1-b-W); \Omega \\ \frac{\partial W}{\partial \eta} + \sqrt{\lambda}\gamma W = 0; \partial\Omega, \end{cases} \tag{1.20}$$

where $R > 0, b, \gamma \geq 0$, and W_0 is a smooth non-negative function. From [GMRS18], the complete dynamics of (1.19) can be determined via the sign of the principal eigenvalue $\sigma_0 = \sigma_0(\lambda, R, b, \gamma)$ of:

$$\begin{cases} -\Delta\phi_0 - \lambda R(1-b)\phi_0 = \sigma_0\phi_0; \Omega \\ \frac{\partial \phi_0}{\partial \eta} + \sqrt{\lambda}\gamma\phi_0 = 0; \partial\Omega \end{cases} \tag{1.21}$$

with corresponding eigenfunction ϕ_0 which can be chosen such that $\phi_0 > 0;\ \partial\Omega$. Also, we recall from [GMRS18] the eigenvalue problem:

$$\begin{cases} -\Delta\phi = R(1-b)E\phi; \Omega \\ \frac{\partial \phi}{\partial \eta} + \sqrt{\lambda}\gamma\phi = 0; \partial\Omega. \end{cases} \tag{1.22}$$

For fixed R, b, and γ, let $E_1(R, b, \gamma)$ denote the principal eigenvalue of (1.22) with corresponding eigenfunction ϕ which can be chosen such that $\phi > 0;\ \overline{\Omega}$.

See Figure 17 for an exact bifurcation curve of positive solutions of (1.20). Note that we will denote $W_{R,\gamma,0}$ as $W_{R,\gamma}$ or simply W_R and $E_1(R, 0, \gamma)$ as $E_1(R, \gamma)$ when there is no confusion regarding the context.

We establish Theorems 1.8-1.11 stated below.

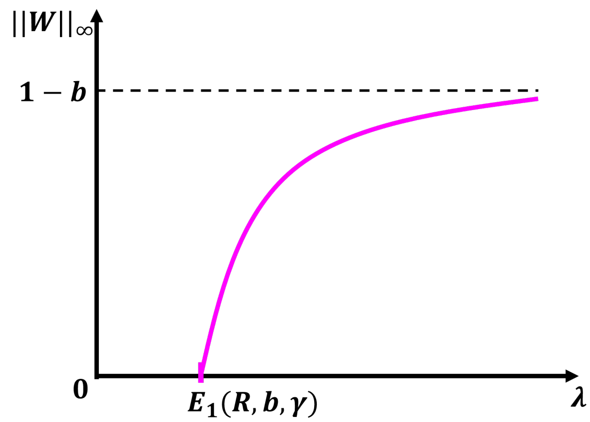


Figure 17. Exact bifurcation diagram for positive solutions of (1.20).

Theorem 1.8. *(Nonexistence). For $r > 0$, $b_1, b_2 \geq 0$ and $\gamma_1, \gamma_2 \geq 0$, if any of the following hold then* (1.18) *has no positive solution.*

(A) $\lambda \leq \max\{E_1(1, \gamma_1), E_1(r, \gamma_2)\}$*;*

(B) $\gamma_1 = \gamma_2$*, and either of the following hold:*

(i) $b_2 \leq 1 \leq b_1$ *and* $1 \leq r \leq \frac{b_1}{b_2}$*, with at least one inequality being strict;*

(ii) $b_1 \leq 1 \leq b_2$ *and* $\frac{b_1}{b_2} \leq r \leq 1$*, with at least one inequality being strict;*

(C) $\gamma_1 > \gamma_2, b_2 \leq 1 \leq b_1$*, and* $1 \leq r \leq \frac{b_1}{b_2}$*;*

(D) $\gamma_1 < \gamma_2, b_1 \leq 1 \leq b_2$*, and* $\frac{b_1}{b_2} \leq r \leq 1$*;*

(E) $b_1 > 1, b_2 < \frac{b_1 - 1}{b_1}$ *and* $\lambda \gg 1$*;*

(F) $b_2 > 1, b_1 < \frac{b_2 - 1}{b_2}$ *and* $\lambda \gg 1$*;*

(G) $E_1(1, \gamma_1) < E_1(r, \gamma_2), b_2 > 0$ *and* $\lambda < E_1(r, \gamma_2) + \delta(b_2)$*, for* $\delta(b_2) > 0$*;*

(H) $E_1(1, \gamma_1) > E_1(r, \gamma_2), b_1 > 0$ *and* $\lambda < E_1(1, \gamma_1) + \delta(b_1)$*, for* $\delta(b_1) > 0$.

Theorem 1.9. *(Existence). Let* $r^* = \frac{E_1(1,\gamma_2)}{E_1(1,\gamma_1)}$*. For* $r > 0, b_1, b_2 \geq 0$*, and* $\gamma_1, \gamma_2 \geq 0$ *the following hold:*

(A) If $b_1, b_2 < 1$*, then* (1.18) *has at least one positive solution,* (u, v)*, for* $\lambda > \max\left\{\frac{E_1(1,\gamma_1)}{1-b_1}, \frac{E_1(r,\gamma_2)}{1-b_2}\right\}$*. Furthermore, every positive solution* (u, v) *of* (1.18) *will satisfy:*

(i) for $\lambda > \max\{E_1(1, \gamma_1), E_1(r, \gamma_2)\}$,

$$0 < u(x, \lambda) \leq W_{1,\gamma_1,0}(x, \lambda); \overline{\Omega},$$

$$0 < v(x, \lambda) \leq W_{r,\gamma_2,0}(x, \lambda); \overline{\Omega}.$$

(ii) for $\lambda > \max\left\{\frac{E_1(1,\gamma_1)}{1-b_1}, \frac{E_1(r,\gamma_2)}{1-b_2}\right\}$

$$W_{1,\gamma_1,b_1}(x, \lambda) < u(x, \lambda) \leq W_{1,\gamma_1,0}(x, \lambda); \overline{\Omega},$$

$$W_{r,\gamma_2,b_2}(x, \lambda) < v(x, \lambda) \leq W_{r,\gamma_2,0}(x, \lambda); \overline{\Omega}.$$

(iii) if $r = 1$ *and* $\gamma_1 = \gamma_2$ *(implying that* $E_1(1, \gamma_1) = E_1(r, \gamma_2)$*) then for* $\lambda > E_1(1, \gamma_1)$,

$$u(x, \lambda) = \frac{1 - b_1}{1 - b_1 b_2} W_{1,\gamma_1,0}(x, \lambda); \overline{\Omega},$$

$$v(x,\lambda) = \frac{1-b_2}{1-b_1 b_2} W_{1,\gamma_1,0}(x,\lambda); \overline{\Omega}.$$

(B) *If* $b_1 = b_2 = 1$*,* $\gamma_1 = \gamma_2$*, and* $r = 1$ *(implying that* $E_1(1,\gamma_1) = E_1(r,\gamma_2)$*), then* (1.18) *has infinitely many solutions for* $\lambda > E_1(1,\gamma_1)$ *of the form:*

$$(u(x,\lambda), v(x,\lambda)) = (sW_{1,\gamma_1,0}(x,\lambda), (1-s)W_{1,\gamma_1,0}(x,\lambda)); \ \overline{\Omega}, s \in (0,1).$$

(C) *If* $b_1 < 1 \leq b_2$*,* $\gamma_1 > 0$*, and* $r > r^*$ *(implying* $E_1(r,\gamma_2) < E_1(1,\gamma_1)$*), then for* $b_1 \approx 0$ *there exist* $\lambda_1(r,b_1,b_2,\gamma_1,\gamma_2), \lambda_2(r,b_2,\gamma_1,\gamma_2) > E_1(1,\gamma_1)$ *such that* (1.18) *has at least one positive solution,* (u,v)*, for* $\lambda \in (\lambda_1,\lambda_2)$*. Furthermore,* (u,v) *will satisfy:*

$$W_{1,\gamma_1,b_1}(x,\lambda) < u(x,\lambda) < W_{1,\gamma_1,0}(x,\lambda); \ \overline{\Omega}$$

$$0 < v(x,\lambda) < W_{r,\gamma_2,0}(x,\lambda); \ \overline{\Omega}.$$

(D) *If* $b_2 < 1 \leq b_1$*,* $\gamma_2 > 0$*, and* $r < r^*$ *(implying* $E_1(r,\gamma_2) > E_1(1,\gamma_1)$*), then for* $b_2 \approx 0$ *there exist* $\lambda_1(r,b_1,b_2,\gamma_1,\gamma_2), \lambda_2(r,b_2,\gamma_1,\gamma_2) > E_1(r,\gamma_2)$ *such that* (1.18) *has at least one positive solution,* (u,v)*, for* $\lambda \in (\lambda_1,\lambda_2)$*. Furthermore,* (u,v) *will satisfy:*

$$0 < u(x,\lambda) < W_{1,\gamma_1,0}(x,\lambda); \ \overline{\Omega},$$

$$W_{r,\gamma_2,b_2}(x,\lambda) < v(x,\lambda) < W_{r,\gamma_2,0}(x,\lambda);\ \overline{\Omega}.$$

(E) If $b_1, b_2 > 1, \gamma_1 = \gamma_2$, and $r = 1$ (implying that $E_1(r,\gamma_2) = E_1(1,\gamma_1)$), then (1.18) *has at least one positive solution for $\lambda > E_1(1,\gamma_1)$, given by:*

$$(u(x,\lambda), v(x,\lambda)) = \left(\frac{1-b_1}{1-b_1b_2}W_{1,\gamma_1,0}(x,\lambda), \frac{1-b_2}{1-b_1b_2}W_{1,\gamma_1,0}(x,\lambda)\right);\overline{\Omega}.$$

Theorem 1.10. *(Uniqueness). For $r > 0, b_1, b_2 < 1$, and $\gamma_1, \gamma_2 \geq 0$ the following hold:*

(A) If $b_1, b_2 < 1, r = 1$, and $\gamma_1 = \gamma_2$, then (1.18) *has at most one positive solution for any $\lambda > 0$.*

(B) For $\lambda > \max\{E_1, 1, \gamma_1), E_1(r,\gamma_2)\}$ if

$$4 > \frac{b_1^2}{r}\sup_{\Omega}\left\{\frac{W_{1,\gamma_1}(x,\lambda)}{W_{r,\gamma_2}(x,\lambda)}\right\} + 2b_1b_2 + rb_2^2\sup_{\Omega}\left\{\frac{W_{r,\gamma_2}(x,\lambda)}{W_{1,\gamma_1}(x,\lambda)}\right\}, \tag{1.23}$$

then (1.18) *has at most one positive solution. In particular, if $b_1, b_2 \approx 0$, then* (1.23) *holds and* (1.18) *has a unique positive solution for* $\lambda > \max\left\{\frac{E_1(1,\gamma_1)}{1-b_1}, \frac{E_1(r,\gamma_2)}{1-b_2}\right\}$.

Theorem 1.11. *(Stability). Suppose that $r > 0, b_1, b_2 \geq 0$, $\gamma_1, \gamma_2 \geq 0$, and $\lambda > 0$ are such that $\sigma_1, \sigma_2 < 0$. The following hold:*

(A) If $\sigma_3 > 0$ or $\sigma_4 > 0$, then $(W_{1,\gamma_1}, 0)$ or $(0, W_{r,\gamma_2})$ is asymptotically stable, respectively.

(B) If $\sigma_3 < 0$ or $\sigma_4 < 0$, then $(W_{1,\gamma_1}, 0)$ or $(0, W_{r,\gamma_2})$ is unstable, respectively.

(C) If $\sigma_3, \sigma_4 < 0$, then there exist a max-min $(\overline{u}, \underline{v})$ and a min-max $(\underline{u}, \overline{v})$ positive solution of (1.18) *with $0 \le \underline{u} \le \overline{u} \le W_{1,\gamma_1}$ and $0 \le \underline{v} \le \overline{v} \le W_{r,\gamma_2}$ on $\overline{\Omega}$ such that:*

(i) if $\overline{u}(x) \le u(0,x) \le W_{1,\gamma_1}(x); \Omega$ and $0 < v(0,x) \le \underline{v}(x); \Omega$, then the unique positive solution of (1.13)*, $(u(t,x), v(t,x))$, converges to $(\overline{u}, \underline{v})$ as $t \to \infty$.*

(ii) if $0 < u(0,x) \le \underline{u}(x); \Omega$ and $\overline{v}(x) \le v(0,x) \le W_{r,\gamma_2}(x); \Omega$, then the unique positive solution of (1.13)*, $(u(t,x), v(t,x))$, converges to $(\underline{u}, \overline{v})$ as $t \to \infty$.*

(iii) $(\overline{u}, \underline{v}) = (\underline{u}, \overline{v})$ if and only if there is a unique positive solution of (1.18)*. Moreover, this coexistence state is globally asymptotically stable.*

(iv) There does not exist an asymptotically stable positive solution of (1.18) *arbitrarily close to $(W_{1,\gamma_1}, 0)$ or $(0, W_{r,\gamma_2})$.*

Remark: Focus 3 results are now published in [ABC$^+$23].

1.4 Focus 4: Numerical computation of bifurcation diagrams in dimension $N = 2$ for examples in Focus 1 and Focus 3.

1.4.1 Study in dimension $N = 2$ of an elliptic boundary value problem where a parameter influences the differential equation as well as the boundary

Here we obtain the bifurcation diagrams using the finite element method for positive solutions (numerically) to:

$$\begin{cases} -\Delta u = \lambda f(u); \ x \in \Omega, \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda} u = 0; \ x \in \partial\Omega, \end{cases} \tag{1.24}$$

where $\Omega = (0,1) \times (0,1)$ and the reaction term f is given as:

$$f(s) = f_{\alpha,k}(s) = \begin{cases} e^{\frac{cs}{c+s}} - 1; s \leq k \\ [e^{\frac{\alpha s}{\alpha+s}} - e^{\frac{\alpha k}{\alpha+k}}] + [e^{\frac{ck}{c+k}} - 1]; s > k. \end{cases} \tag{1.25}$$

Here $c > 2$ is a fixed number, $\alpha > 0$ and $k > 0$ are parameters. We obtain approximate bifurcation diagrams, as in the $N = 1$ case, which are Σ-shaped when $\alpha, k \gg 1$.

1.4.2 Study in dimension $N = 2$ of an ecological problem

Here we obtain the numerical bifurcation diagrams using the finite element method ([LB13]) to the following problem:

$$\begin{cases} -\Delta u = \lambda u(1 - u - b_1 v); \ \Omega \\ -\Delta v = \lambda r v(1 - v - b_2 u); \ \Omega \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda}\gamma_1 u = 0; \ \partial\Omega \\ \frac{\partial v}{\partial \eta} + \sqrt{\lambda}\gamma_2 v = 0; \ \partial\Omega, \end{cases} \tag{1.26}$$

where $\lambda, \gamma_1, \gamma_2 > 0$, $b_1, b_2 \geq 0$, and $\Omega = (0,1) \times (0,1)$.

We obtain bifurcation diagrams for positive solutions to (1.26) and explore how they evolve as b_1, b_2 vary.

We now describe the plan for the rest of this dissertation. In Chapter II, we state some preliminaries that we use in the proofs of our results. In Chapter III, we provide the proofs of results stated in Focus 1. Namely, we provide proofs of Theorems 1.1 - 1.2, Theorem 1.4, and Corollary 1.3 in Chapter III. Chapter IV is devoted to the proofs of results stated in Focus 2. Namely, we provide proofs of Theorem 1.5 - 1.6

and Corollary 1.7. In Chapter V, we provide proofs of Theorems 1.8- 1.11 stated in Focus 3. Chapter VI is dedicated to computational results for examples in Focus 4. Finally, in Chapter VII, we provide conclusions and future directions.

CHAPTER II

PRELIMINARIES

2.1 Method of Sub and Supersolutions

Consider the boundary value problem:

$$\begin{cases} -\Delta u = \lambda f(u);\ \Omega \\ \frac{\partial u}{\partial \eta} + \gamma\sqrt{\lambda}u = 0;\ \partial\Omega \end{cases} \tag{2.1}$$

where λ, γ are positive parameters and f is a smooth function. We first introduce definitions of a (strict) subsoultion and a (strict) supersolution of (2.1), and state a sub-supersolution theorem and a three solution theorem that are used to prove existence and multiplicity results for position solutions. By a subsolution of (2.1), we mean a function $\psi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ that satisfies:

$$\begin{cases} -\Delta \psi \leq \lambda f(\psi);\ \Omega \\ \frac{\partial \psi}{\partial \eta} + \gamma\sqrt{\lambda}\psi \leq 0;\ \partial\Omega. \end{cases}$$

By a supersolution of (2.1), we mean a function $z \in C^2(\Omega) \cap C^1(\overline{\Omega})$ that satisfies:

$$\begin{cases} -\Delta z \geq \lambda f(z);\ \Omega \\ \frac{\partial z}{\partial \eta} + \gamma\sqrt{\lambda}z \geq 0;\ \partial\Omega. \end{cases}$$

By a strict subsolution (supersolution) of (2.1) we mean a subsolution (supersolution) which is not a solution. Then the following results hold:

Lemma 2.1. *(see [Sat72], [Ama72]) Let ψ and z be a subsolution and a supersolution of* (2.1) *respectively such that $\psi \leq z$. Then* (2.1) *has a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that $\psi \leq u \leq z$.*

Lemma 2.2. *(see [Ama72], [Shi87]) Let $\underline{u}_1$ and $\overline{u}_2$ be a subsolution and a supersolution of* (2.1) *respectively such that $\underline{u}_1 \leq \overline{u}_2$ in Ω. Let $\underline{u}_2$ and $\overline{u}_1$ be a strict subsolution and a strict supersolution of* (2.1) *respectively such that $\underline{u}_2$, $\overline{u}_1 \in [\underline{u}_1, \overline{u}_2]$ and $\underline{u}_2 \not\leq \overline{u}_1$. Then* (2.1) *has at least three solutions u_1, u_2 and u_3 where $u_i \in [\underline{u}_i, \overline{u}_i]$ for $i = 1, 2$ and $u_3 \in [\underline{u}_1, \overline{u}_2] \backslash ([\underline{u}_1, \overline{u}_1] \cup [\underline{u}_2, \overline{u}_2])$.*

Similarly, by a subsolution of (1.9) we mean $(\psi, \overline{\psi}) \in [C^2(\Omega) \cap C^1(\bar{\Omega})] \times [C^2(\Omega) \cap C^1(\bar{\Omega})]$ that satisfies:

$$\begin{cases} -\Delta\psi \leq \lambda f(\overline{\psi});\ \Omega \\ -\Delta\overline{\psi} \leq \lambda g(\psi);\ \Omega \\ \frac{\partial\psi}{\partial\eta} + \sqrt{\lambda}\psi \leq 0;\ \partial\Omega \\ \frac{\partial\overline{\psi}}{\partial\eta} + \sqrt{\lambda}\ \overline{\psi} \leq 0;\ \partial\Omega, \end{cases}$$

and by a supersolution of (1.9) we mean $(Z, \overline{Z}) \in [C^2(\Omega) \cap C^1(\bar{\Omega})] \times [C^2(\Omega) \cap C^1(\bar{\Omega})]$ that satisfies:

$$\begin{cases} -\Delta Z \geq \lambda f(\overline{Z});\ \Omega \\ -\Delta\overline{Z} \geq \lambda g(Z);\ \Omega \\ \frac{\partial Z}{\partial\eta} + \sqrt{\lambda}Z \geq 0;\ \partial\Omega \\ \frac{\partial\overline{Z}}{\partial\eta} + \sqrt{\lambda}\ \overline{Z} \geq 0;\ \partial\Omega. \end{cases}$$

By a strict subsolution (strict supersolution) of (1.9) we mean a subsolution (supersolution) which is not a solution.

Now we state two results that we will use later.

Lemma 2.3. *Let* $(\psi, \overline{\psi})$ *and* $(Z, \overline{Z})$ *be a subsolution and a supersolution of* (1.9) *respectively such that* $(\psi, \overline{\psi}) \leq (Z, \overline{Z})$. *Then* (1.9) *has a solution* $(u, v) \in [C^2(\Omega) \cap C^1(\bar{\Omega})] \times [C^2(\Omega) \cap C^1(\bar{\Omega})]$ *such that* $(u, v) \in [(\psi, \overline{\psi}), (Z, \overline{Z})]$.

Lemma 2.4. *Let* $(\psi_1, \overline{\psi}_1)$ *be a subsolution,* $(\phi_2, \overline{\phi}_2)$ *a strict supersolution,* $(\psi_2, \overline{\psi}_2)$ *a strict subsolution, and* $(\phi_1, \overline{\phi}_1)$ *a supersolution for* (1.9) *such that* $(\psi_1, \overline{\psi}_1) \leq (\psi_2, \overline{\psi}_2) \leq (\phi_1, \overline{\phi}_1)$, $(\psi_1, \overline{\psi}_1) \leq (\phi_2, \overline{\phi}_2) \leq (\phi_1, \overline{\phi}_1)$, *and* $(\psi_2, \overline{\psi}_2) \not\leq (\phi_2, \overline{\phi}_2)$. *Then* (1.9) *has at least three positive solutions* $(u_i, v_i), i = 1, 2, 3$, *such that* $(u_1, v_1) \in [(\psi_1, \overline{\psi}_1), (\phi_2, \overline{\phi}_2)]$, $(u_2, v_2) \in [(\psi_2, \overline{\psi}_2), (\phi_1, \overline{\phi}_1)]$, *and* $(u_3, v_3) \in [(\psi_1, \overline{\psi}_1), (\phi_1, \overline{\phi}_1)] \backslash ([(\psi_1, \overline{\psi}_1), (\phi_2, \overline{\phi}_2)] \cup [(\psi_2, \overline{\psi}_2), (\phi_1, \overline{\phi}_1)])$.

2.2 Quadrature method

Let us consider the problem:

$$\begin{cases} -u'' = \lambda f(u); \; (0, 1) \\ -u'(0) + \sqrt{\lambda}\gamma u(0) = 0 \\ u'(1) + \sqrt{\lambda}\gamma u(1) = 0, \end{cases} \tag{2.2}$$

where λ, γ are positive parameters and $f \in C^1[0, r); r > 0$ is non-negative. We use the Quadrature method used in [GMRS18] which was first introduced for Dirichlet boundary conditions in [Lae71]. Let $u(x)$ be a positive solution to (2.2). Since (2.2) is autonomous, any positive solution u of (2.2) must be symmetric about $x = \frac{1}{2}$, increasing on $(0, \frac{1}{2})$, and decreasing on $(\frac{1}{2}, 1)$. Let $u(0) = u(1) = q$ and $\|u\|_\infty = u(\frac{1}{2}) = \rho$. Note that $u'(\frac{1}{2}) = 0$.

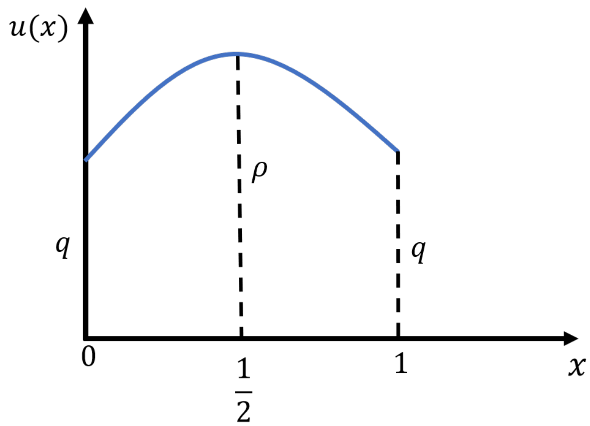


Figure 18. Shape of a positive solution to (1.16).

Then multiplying the differential equation (2.2) by u' we get

$$-u''u' = \lambda f(u)u'. \tag{2.3}$$

By integrating both sides, we obtain

$$-\frac{[u'(x)]^2}{2} = \lambda F(u(x)) + C, \tag{2.4}$$

where $F(s) = \int_0^s f(t)dt$. Now, applying $u'(\frac{1}{2}) = 0$ and $u(\frac{1}{2}) = \rho$, we get $C = -\lambda F(\rho)$. Thus

$$u'(x) = \sqrt{2\lambda(F(\rho) - F(u(x)))}; \quad x \in \left[0, \frac{1}{2}\right].$$

Further integration from 0 to x; $x \in [0, \frac{1}{2})$, yields

$$\int_0^x \frac{u'(x)ds}{\sqrt{F(\rho) - F(u(s))}} = \sqrt{2\lambda}x. \tag{2.5}$$

Through a change of variables and using the fact that $u(0) = q$ we have

$$\int_q^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda}x; \quad x \in \Big[0, \frac{1}{2}\Big). \tag{2.6}$$

Now, letting $x \to \frac{1}{2}$, we get

$$\sqrt{2}\int_q^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{\lambda}. \tag{2.7}$$

For the improper integral in (2.7) to exist, we must have $f(\rho) > 0$ and $F(\rho) > F(s); s \in [0, \rho)$. Using the boundary conditions we note that ρ and q must satisfy

$$F(\rho) = \frac{2F(q) + \gamma^2 q^2}{2}. \tag{2.8}$$

It is easy to verify that given $\rho \in (0, r)$, there exists a unique $q = q(\rho) \in (0, \rho)$ satisfying (2.8). Also,

$$G(\rho) = \sqrt{2}\int_{q(\rho)}^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}}$$

is well defined and continuous on (0, 1). Further, if λ, ρ and $q(\rho)$ satisfy

$$\sqrt{\lambda} = G(\rho) = \sqrt{2}\int_{q(\rho)}^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}}, \tag{2.9}$$

it can be proven that for each $x \in [0, \frac{1}{2})$ there is a unique $u(x) \in [0, \rho)$ that satisfies the equation

$$\int_{q(\rho)}^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda}x. \tag{2.10}$$

Now defining $u(\frac{1}{2}) = \rho$, and, for $x \in (\frac{1}{2}, 1]$ defining $u(x) = u(1 - x)$, it can be shown that $u \in C^2[0, 1]$ and satisfies (1.16).

Hence (2.9), namely, $S = \{(\lambda, \rho) | \rho \in (0, r), G(\rho) = \sqrt{\lambda}\}$ describes the bifurcation diagram for positive solutions of (1.16). For given λ, ρ and q satisfying (2.8) and (2.9), we use (2.10) with the Mathematica nonlinear solver to approximate $u = u_\lambda$.

2.3 Shooting method

To find the solutions of (1.15), we use the Shooting method. Recall that in the Quadrature method we discussed how to approximate the positive solution $u (= u_\lambda)$ of (1.16). Now we discuss a numerical Shooting method which is employed to approximate the positive solution v of (1.15) in the asymmetric competition case when $b_1 = 0$. Namely we approximate the solution v of:

$$\begin{cases} -v'' = \lambda r v[1 - v - b u_\lambda]; (0, 1) \\ \quad -v'(0) + \sqrt{\lambda}\gamma_2 v(0) = 0 \\ \quad v'(1) + \sqrt{\lambda}\gamma_2 v(1) = 0. \end{cases} \tag{2.11}$$

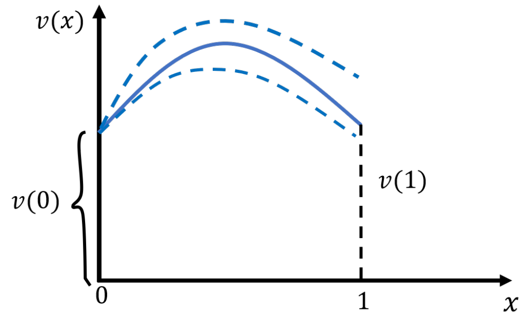


Figure 19. Shooting from $x = 0$ to $x = 1$.

Let $v(0) = \delta$ and $v' = z$. Then we obtain the following system of ordinary differential equations:

$$\begin{cases} v' = z; \ (0,1) \\ -z' = \lambda r v(1 - v - bu_\lambda); \ (0,1) \\ z(1) = -\sqrt{\lambda}\gamma_2 v(1) \\ v(0) = \delta, z(0) = \sqrt{\lambda}\gamma_2\delta. \end{cases} \tag{2.12}$$

For a given value of $\delta > 0$, we use the `ParametricNDSolve` method in Mathematica to approximate solutions of (2.12). This process can be explained as a shooting from $x = 0$ (where $v(0) = \delta$ and $z(0) = \sqrt{\lambda}\gamma_2\delta$) and checking at $x = 1$ to see if $z(1) = -\sqrt{\lambda}\gamma_1 v(1)$.

2.4 Finite element method

Shooting and Quadrature methods do not work for $N \geq 2$ in general. Here, we discuss the variational formulation and a finite element method (see [LB13]) that we will be using to obtain the numerical solutions of (1.24) and (1.26) when $\Omega = (0,1) \times (0,1)$

in $\mathbb{R}^2$. For the discussion, let us consider the following problem:

$$\begin{cases} -\Delta u = \lambda f(u); \ \Omega \\ \frac{\partial u}{\partial \eta} + \gamma\sqrt{\lambda}u = 0; \ \partial\Omega, \end{cases} \tag{2.13}$$

where $\lambda, \gamma > 0$, $\Omega = (0,1) \times (0,1)$, and f is continuous.

2.4.1 Variational Formulation

Let

$$V := H^1(\Omega) = \{v \in L_2(\Omega) | \nabla v \in L_2(\Omega)\},$$

where $\Omega = (0,1) \times (0,1) \in \mathbb{R}^2$. Then we take $v \in V$ and multiply equation (2.13) by v to obtain:

$$(-\Delta u)v = \lambda f(u)v.$$

Using integration by parts, we obtain:

$$\int_\Omega \nabla u \cdot \nabla v dx - \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v ds = \lambda \int_\Omega f(u) v dx.$$

Now, using the boundary condition, we obtain:

$$\int_\Omega \nabla u \cdot \nabla v dx + \gamma\sqrt{\lambda} \int_{\partial\Omega} uv ds = \lambda \int_\Omega f(u) v dx. \tag{2.14}$$

In general, the solution of (2.14) is not known and the numerical solution is important to analyze. Here, we take $\Omega = (0,1) \times (0,1)$ in $\mathbb{R}^2$, and for a given triangulation of Ω

(see Figure 20), we find a finite dimensional approximation for u by using the finite element method.

2.4.2 Finite Element Method Formulation

Let

$$V_h := \{v \in C^0(\overline{\Omega}) : v|_K \in P_1(K) \ \ \forall K \in \mathcal{K}_h\},$$

where $\mathcal{K}_h$ is a shape-regular triangulation of Ω with mesh size parameter h (see Figure 20).

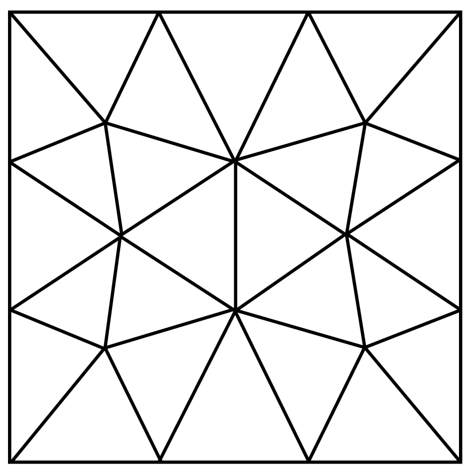


Figure 20. Triangulation ($\mathcal{K}_h$) of the domain.

Note that V_h is conforming in the sense that $V_h \subset V$. The finite element method for (2.13) is to find $u_h \in V_h$ such that

$$\int_\Omega \nabla u_h \nabla v_h dx - \lambda \int_\Omega f(u_h) v_h dx + \int_{\partial\Omega} \gamma\sqrt{\lambda} u_h v_h ds = 0 \quad \text{for all} \quad v_h \in V_h. \qquad (2.15)$$

Then we derive nonlinear system of equations by the following way:

Let $n_h := dim(V_h)$ such that $V_h = Span\{\varphi_i\}_{i=1}^{n_h}$ with the following property:

$$\varphi_j(x_i) = \delta_{ij} = \begin{cases} 1; \ i = j \\ 0; \ i \neq j. \end{cases}$$

Now, using this basis, we note that the finite element formulation (2.15) is equivalent to

$$\int_\Omega \nabla u_h \nabla \varphi_i dx - \lambda \int_\Omega f(u_h)\varphi_i dx + \int_{\partial\Omega} \gamma\sqrt{\lambda} u_h \varphi_i ds = 0; \quad i = 1, 2, ..., n_h.$$

Since $u_h \in V_h$, we can write u_h as the linear combination of φ_j $(j = 1, 2, ...n_h)$. That is,

$$u_h = \sum_{j=1}^{n_h} \xi_j \varphi_j.$$

Now, (2.15) can be written as

$$\begin{aligned} &\int_\Omega \nabla \left(\sum_{j=1}^{n_h} \xi_j \varphi_j \right) \nabla \varphi_i dx - \lambda \int_\Omega f(\sum_{j=1}^{n_h} \xi_j \varphi_j)\varphi_i dx + \gamma\sqrt{\lambda} \int_{\partial\Omega} \sum_{j=1}^{n_h} \xi_j \varphi_j \varphi_i ds = 0 \\ \implies &\sum_{j=1}^{n_h} \xi_j \int_\Omega \nabla \varphi_j \nabla \varphi_i dx - \lambda \int_\Omega f(\sum_{j=1}^{n_h} \xi_j \varphi_j)\varphi_i dx + \gamma\sqrt{\lambda} \sum_{j=1}^{n_h} \xi_j \int_{\partial\Omega} \varphi_j \varphi_i ds = 0 \end{aligned} \tag{2.16}$$

for all $i = 1, 2, ... n_h$, which leads to a system of nonlinear equations of the form $F(u) = 0$, where F is a nonlinear function and u is the solution vector which represents the coefficients of the expansion of u_h in terms of basis functions. The nonlinear system can be solved by Newton's method.

CHAPTER III

PROOFS OF THEOREMS 1.1 - 1.2, THEOREM 1.4, AND COROLLARY 1.3 STATED IN FOCUS 1

First we construct sub-super solutions for certain λ ranges. Recall $\theta_{\lambda,m}$ and $\sigma_{\lambda,m}$ (see (1.6)).

Construction of a small strict subsolution ψ_1 for $\lambda < A_{m+1}$ and $\lambda \approx A_{m+1}$ when (H_1) is satisfied.

We first note that $f''(s) > 0$ for $s \approx 0$ since $g''(0) > 0$. Hence there exists $A^* > 0$ and $s_1 > 0$ such that $f''(s) > A^*$ for $s < s_1$. Let $\psi_1 = \delta_\lambda \theta_{\lambda,m+1}$ where $\delta_\lambda = \frac{2(m+1)\sigma_{\lambda,m+1}}{\lambda A^* \min_{\overline{\Omega}} \theta_{\lambda,m+1}}$. We note that $\sigma_{\lambda,m+1} > 0$, $\sigma_{\lambda,m+1} \to 0$ as $\lambda \to A_{m+1}^-$, and $\min_{\overline{\Omega}} \theta_{\lambda,m+1} \not\to 0$ as $\lambda \to A_{m+1}^-$. Thus $\delta_\lambda \to 0^+$ as $\lambda \to A_{m+1}^-$. Now by Taylor's Theorem, we have $f(\psi_1) = f(0) + f'(0)\psi_1 + \frac{f''(\zeta)}{2}{\psi_1}^2 = (m+1)\psi_1 + \frac{f''(\zeta)}{2}{\psi_1}^2$ for some $\zeta \in [0, \psi_1]$. Then we have

$-\Delta\psi_1 - \lambda f(\psi_1)$

$$\begin{aligned} &= \delta_\lambda(\sigma_{\lambda,m+1} + \lambda)(m+1)\theta_{\lambda,m+1} - \lambda\Big[(m+1)\delta_\lambda\theta_{\lambda,m+1} + \frac{f''(\zeta)}{2}(\delta_\lambda\theta_{\lambda,m+1})^2\Big] \\ &< \delta_\lambda\theta_{\lambda,m+1}\Big[(m+1)\sigma_{\lambda,m+1} - \frac{\lambda A^*}{2}\delta_\lambda \min_{\overline{\Omega}} \theta_{\lambda,m+1}\Big] = 0;\ \Omega \end{aligned}$$

by our choice of δ_λ, for $\lambda < A_{m+1}$ and $\lambda \approx A_{m+1}$ such that $\psi_1 < s_1$. Also, $\frac{\partial\psi_1}{\partial\eta} + \sqrt{\lambda}\psi_1 = 0$ on $\partial\Omega$ since $\theta_{\lambda,m+1}$ satisfies this boundary condition. Thus, there exists $\overline{\lambda} < A_{m+1}$ such that ψ_1 is a strict subsolution of (1.1) for $\lambda \in [\overline{\lambda}, A_{m+1})$.

Construction of a small subsolution ψ_2 for $\lambda \in [A_{m+1}, A_m)$ when and (H_1) is satisfied.

We note that $f'(0) = m+1$, $\sigma_{\lambda,m+1} \leq 0$ for $\lambda \in [A_{m+1}, A_m)$ and $\sigma_{\lambda,m+1} \to 0$ as $\lambda \to A_{m+1}$. Let $\psi_2 = n_\lambda \theta_{\lambda,m+1}$ with $n_\lambda > 0$. Now, consider $H(s) = (\sigma_{\lambda,m+1} + \lambda)(m+1)s - \lambda f(s)$. Then we have $H(0) = 0$, $H'(0) = \sigma_{\lambda,m+1}(m+1) \leq 0$ and $H''(0) = -\lambda f''(0) < 0$ since $f''(0) > 0$. This implies that $-\Delta\psi_2 = n_\lambda(\sigma_{\lambda,m+1}+\lambda)(m+1)\theta_{\lambda,m+1} < \lambda f(n_\lambda\theta_{\lambda,m+1}) = \lambda f(\psi_2)$ in Ω for $n_\lambda \approx 0$. We also have $\frac{\partial\psi_2}{\partial\eta} + \sqrt{\lambda}\psi_2 = 0$ on $\partial\Omega$ since $\theta_{\lambda,m+1}$ satisfies this boundary condition. Thus ψ_2 is a subsolution of (1.1) for $n_\lambda \approx 0$ when $\lambda \in [A_{m+1}, A_m)$.

Construction of a subsolution ψ_3 for $\lambda < A_m$ and $\lambda \approx A_m$ such that $\|\psi_3\|_\infty \to \infty$ as $\lambda \to A_m^-$ when (H_1) is satisfied

Let $m > 0$ and $\psi_3 = \epsilon_\lambda\theta_{\lambda,m}$ where $\epsilon_\lambda = \frac{\lambda g\left(\min_{\overline{\Omega}}\theta_{\lambda,m}\right)}{m\sigma_{\lambda,m}\|\theta_{\lambda,m}\|_\infty}$. We note that $\epsilon_\lambda > 0$ since $\sigma_{\lambda,m} > 0$ for $\lambda < A_m$. Further, $\epsilon_\lambda \to \infty$ as $\lambda \to A_m^-$ since $\sigma_{\lambda,m} \to 0^+$ as $\lambda \to A_m^-$ and $\min_{\overline{\Omega}}\theta_{\lambda,m} \not\to 0$. This implies that $\|\psi_3\|_\infty \to \infty$ as $\lambda \to A_m^-$. Now we have

$$\begin{aligned}
-\Delta\psi_3 - \lambda f(\psi_3) &= \epsilon_\lambda[(\lambda + \sigma_{\lambda,m})m\theta_{\lambda,m}] - \lambda[m\epsilon_\lambda\theta_{\lambda,m} + g(\epsilon_\lambda\theta_{\lambda,m})] \\
&= \epsilon_\lambda m\sigma_{\lambda,m}\theta_{\lambda,m} - \lambda g(\epsilon_\lambda\theta_{\lambda,m}) \\
&\leq \epsilon_\lambda m\sigma_{\lambda,m}\|\theta_{\lambda,m}\|_\infty - \lambda g(\epsilon_\lambda\theta_{\lambda,m}) \\
&= \lambda[g\big(\min_{\overline{\Omega}}\theta_{\lambda,m}\big) - g(\epsilon_\lambda\theta_{\lambda,m})] \\
&\leq 0;\ \Omega
\end{aligned}$$

for $\lambda \approx A_m$, since $\epsilon_\lambda > 1$ for $\lambda \approx A_m$ and g is increasing. Hence, we have $-\Delta\psi_3 \leq \lambda f(\psi_3)$ in Ω. Also, on the boundary we have $\frac{\partial\psi_3}{\partial\eta} + \sqrt{\lambda}\psi_3 = 0$ since $\theta_{\lambda,m}$ satisfies this boundary condition. Consequently ψ_3 is a subsolution of (1.1) such that $\|\psi_3\|_\infty \to \infty$ as $\lambda \to A_m^-$.

Next, let $m = 0$. Here we can show (1.1) has a subsolution ψ_3 such that $\|\psi_3\|_\infty \to \infty$ as $\lambda \to \infty$ by using a well known result in [CGS93] for semipositone problems. Namely, define $h \in C^2([0,\infty))$ such that $h(0) < 0$, $h(s) \leq f(s)$ for $s \in (0,\infty)$ and $\lim_{s\to\infty} h(s) > 0$. Then the boundary value problem

$$\begin{cases} -\Delta w = \lambda h(w); \quad \Omega, \\ \quad w = 0; \quad \partial\Omega, \end{cases}$$

has a solution $\overline{w}_\lambda > 0$ for $\lambda \gg 1$ such that $\|\overline{w}_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$. Since by the Hopf maximum principle $\frac{\partial \overline{w}_\lambda}{\partial \eta} < 0$ on $\partial\Omega$, it is easy to show that $\psi_3 = \overline{w}_\lambda$ is a subsolution of (1.1) for $\lambda \gg 1$ such that $\|\psi_3\|_\infty \to \infty$ as $\lambda \to \infty$.

Construction of a strict subsolution ψ_4 for $\lambda > \frac{b}{f(b)}\frac{2NC_N}{R^2}$ where $b = b_1$ when (H_2) is satisfied and $b = b_2$ when (H_3) is satisfied

Here we construct a strict subsolution ψ_4 for $\lambda > \frac{b}{f(b)}\frac{2NC_N}{R^2}$ using the iteration of a subsolution $\tilde{\psi}$ created originally in [RS04] and later also used in [LSS11]. Namely, the authors in [LSS11] take ψ to be the solution of:

$$\begin{cases} -\psi''(r) - \frac{N-1}{r}\psi'(r) = \lambda f(w(r)); \ r \in (0,R) \\ \qquad \psi'(0) = 0 = \psi(R), \end{cases} \tag{3.1}$$

where R is the radius of the largest inscribed ball, B_R, in Ω (see Figure 21) and

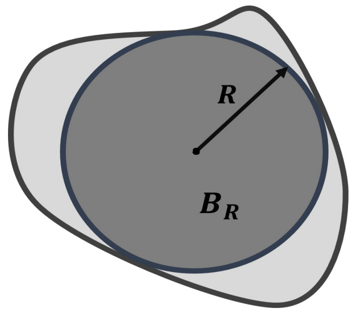


Figure 21. Largest inscribed ball in Ω.

$w(r) = b\rho(r)$ with

$$\rho(r) = \begin{cases} 1; & r \in [0, \epsilon] \\ 1 - \left[1 - \left(\frac{R-r}{R-\epsilon}\right)^{\beta}\right]^{\alpha}; & r \in (\epsilon, R], \alpha, \beta > 1. \end{cases}$$

When $\lambda > \frac{b}{f(b)} \frac{2NC_N}{R^2}$ for certain choices of $\alpha > 1$, $\beta > 1$, and $\epsilon \in (0, 1)$, it was proven that (see [RS04] for details) $\psi \geq w$ on $[0, R]$ and, hence, is a subsolution of (3.1) since f is increasing. Now since $f(0) = 0$ it follows that

$$\tilde{\psi} = \begin{cases} \psi; & B_R \\ 0; & \Omega \backslash B_R \end{cases}$$

is a strict subsolution of:

$$\begin{cases} -\Delta u = \lambda f(u); & \Omega \\ u = 0; & \partial\Omega \end{cases}$$

for $\lambda > \frac{b}{f(b)} \frac{2NC_N}{R^2}$ such that $\|\tilde{\psi}\|_{\infty} \geq b$.

Now let ψ_4 be the first iteration of $\tilde{\psi}$, namely, ψ_4 be the solution to the problem:

$$\begin{cases} -\Delta\psi_4 = \lambda f(\tilde{\psi});\ \Omega \\ \frac{\partial\psi_4}{\partial\eta} + \sqrt{\lambda}\psi_4 = 0;\ \partial\Omega. \end{cases}$$

Then we have $-\Delta(\psi_4 - \tilde{\psi}) \geq 0$ and $\frac{\partial(\psi_4 - \tilde{\psi})}{\partial\eta} + \sqrt{\lambda}(\psi_4 - \tilde{\psi}) = -\frac{\partial\tilde{\psi}}{\partial\eta} > 0$ by the Hopf maximum principle. This implies that $\psi_4 > \tilde{\psi}$ in Ω. Hence, ψ_4 is a strict subsolution of (1.1) for $\lambda > \frac{b}{f(b)}\frac{2NC_N}{R^2}$.

Construction of a large supersolution Z_1 for $\lambda < A_m$ when (H_1) is satisfied

Let $m > 0$. Choose $Z_1 = M\theta_{\lambda,m}$ for $M > 0$. Then $-\Delta Z_1 - \lambda f(Z_1) = M(\sigma_{\lambda,m} + \lambda)m\theta_{\lambda,m} - \lambda[mM\theta_{\lambda,m} + g(M\theta_{\lambda,m})] = mM\theta_{\lambda,m}\left[\sigma_{\lambda,m} - \frac{\lambda g(M\theta_{\lambda,m})}{mM\theta_{\lambda,m}}\right] > 0$ in Ω for $M \gg 1$ since $\sigma_{\lambda,m} > 0$ for $\lambda < A_m$ and $\frac{g(s)}{s} \to 0$ as $s \to \infty$. Further, $\frac{\partial Z_1}{\partial\eta} + \sqrt{\lambda}Z_1 = 0$ on $\partial\Omega$ since $\theta_{\lambda,m}$ satisfies this boundary condition. Hence, Z_1 is a supersolution of (1.1) for $M \gg 1$.

Next, let $m = 0$. Here we choose $Z_1 = Me_\lambda$, where e_λ is the unique solution of $-\Delta e = 1$ in Ω and $\frac{\partial e}{\partial\eta} + \sqrt{\lambda}e = 0$ on $\partial\Omega$. Note $e_\lambda > 0$ on $\overline{\Omega}$. Then $-\Delta Z_1 - \lambda f(Z_1) = M - \lambda g(Me_\lambda) \geq M\left[1 - \lambda\frac{g(M\|e_\lambda\|_\infty)}{M\|e_\lambda\|_\infty}\|e_\lambda\|_\infty\right] > 0$ for $M \gg 1$ since g is increasing and $\frac{g(s)}{s} \to 0$ as $s \to \infty$. Also, $\frac{\partial Z_1}{\partial\eta} + \sqrt{\lambda}Z_1 = 0$ on $\partial\Omega$ since e_λ satisfies this boundary condition. Hence, Z_1 is a supersolution of (1.1) for $M \gg 1$.

Construction of a strict supersolution Z_2 for $\lambda < A_{m+1}$ when (H_1) is satisfied

Let $Z_2 = m_\lambda \theta_{\lambda,m+1}$ and $l(s) = (\sigma_{\lambda,m+1} + \lambda)(m+1)s - \lambda f(s)$. We note that $\sigma_{\lambda,m+1} > 0$ for $\lambda < A_{m+1}$. Then we have $l(0) = 0$ and $l'(0) = (\sigma_{\lambda,m+1} + \lambda)(m+1) - \lambda f'(0) = \sigma_{\lambda,m+1}(m+1) > 0$ since $f'(0) = m+1$. This implies that $-\Delta Z_2 = m_\lambda(\sigma_{\lambda,m+1} + \lambda)(m+1)\theta_{\lambda,m+1} > \lambda f(m_\lambda \theta_{\lambda,m+1}) = \lambda f(Z_2)$ in Ω for $m_\lambda \approx 0$. On the boundary, we have $\frac{\partial Z_2}{\partial \eta} + \sqrt{\lambda} Z_2 = 0$ since $\theta_{\lambda,m+1}$ satisfies this boundary condition. Thus Z_2 with $m_\lambda \approx 0$ is a strict supersolution of (1.1) for $\lambda < A_{m+1}$.

Construction of a strict supersolution Z_3 for $\lambda \in \left(1, \frac{a_1}{f(a_1)} \frac{1}{\|v\|_\infty}\right)$ when (H_2) is satisfied

Let $Z_3 = \frac{a_1 v}{\|v\|_\infty}$ where v is as in (1.7). Then $-\Delta Z_3 = \frac{a_1}{\|v\|_\infty} > \lambda f(a_1) \geq \lambda f(Z_3)$ since $\lambda < \frac{a_1}{f(a_1)} \frac{1}{\|v\|_\infty}$ and f is increasing. Further, Z_3 satisfies $\frac{\partial Z_3}{\partial \eta} + \sqrt{\lambda} Z_3 = \frac{a_1}{\|v\|_\infty} \frac{\partial v}{\partial \eta} + \sqrt{\lambda} \frac{a_1 v}{\|v\|_\infty} > \frac{a_1}{\|v\|_\infty}[\frac{\partial v}{\partial \eta} + v] = 0$ on $\partial\Omega$ since $\lambda > 1$. Thus Z_3 is a strict supersolution of (1.1) for $\lambda \in \left(1, \frac{a_1}{f(a_1)} \frac{1}{\|v\|_\infty}\right)$.

Construction of a strict supersolution Z_4 for $\lambda \in \left(\frac{A_1}{2}, \frac{a_2}{f(a_2)} \frac{1}{\|w\|_\infty}\right)$ when (H_3) is satisfied

Let $Z_4 = \frac{a_2 w}{\|w\|_\infty}$ where w is as in (1.8). Then $-\Delta Z_4 = \frac{a_2}{\|w\|_\infty} > \lambda f(a_2) \geq \lambda f(Z_4)$ since $\lambda < \frac{a_2}{f(a_2)} \frac{1}{\|w\|_\infty}$ and f is increasing. Further, Z_4 satisfies $\frac{\partial Z_4}{\partial \eta} + \sqrt{\lambda} Z_4 = \frac{a_2}{\|w\|_\infty} \frac{\partial w}{\partial \eta} + \sqrt{\lambda} \frac{a_2 w}{\|w\|_\infty} > \frac{a_2}{\|w\|_\infty}[\frac{\partial w}{\partial \eta} + \sqrt{\frac{A_1}{2}} w] = 0$ on $\partial\Omega$ since $\lambda > \frac{A_1}{2}$. Thus Z_4 is a strict supersolution of (1.1) for $\lambda \in \left(\frac{A_1}{2}, \frac{a_2}{f(a_2)} \frac{1}{\|w\|_\infty}\right)$.

Now we prove Theorems 1.1-1.2, Corollary 1.3 and Theorem 1.4.

3.1 Proof of Theorem 1.1

a) Let M be as in the construction of the supersolution Z_1 and n_λ be as in the construction of the subsolution ψ_2. We choose $M \gg 1$ and $n_\lambda \approx 0$ such that $Z_1 \geq \psi_2$. By Lemma 2.1, (1.1) has a positive solution $u_\lambda \in [\psi_2, Z_1]$ for $\lambda \in [A_{m+1}, A_m)$.

Recall the subsolution ψ_3 of (1.1). Now we choose $M \gg 1$ such that $\psi_3 \leq Z_1$. Hence, recalling that $\|\psi_3\|_\infty \to \infty$ as $\lambda \to A_m^-$, by Lemma 2.1, (1.1) has a positive solution $u_\lambda \in [\psi_3, Z_1]$ such that $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to A_m^-$.

Next, let $\lambda \in [\overline{\lambda}, A_{m+1})$ where $\overline{\lambda}$ is as in the construction of the strict subsolution ψ_1. We note that $\psi_0 = 0$ is a solution and hence a subsolution of (1.1). Recall the strict supersolution Z_2 of (1.1). Now we choose m_λ small enough such that $\|Z_2\|_\infty < \|\psi_1\|_\infty$. Next, we choose $M \gg 1$ such that $\psi_1 \leq Z_1$ and $Z_2 \leq Z_1$ (see Figure 22). By Lemma 2.2, (1.1) has at least two positive solutions $u_1 \in [\psi_1, Z_1]$ and

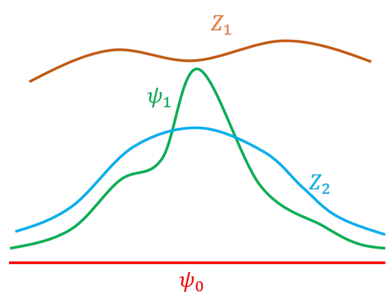


Figure 22. Subsolutions ψ_0, ψ_1 and supersolutions Z_1, Z_2.

$u_2 \in [\psi_0, Z_1] \backslash ([\psi_0, Z_2] \cup [\psi_1, Z_1])$ for $\lambda \in [\overline{\lambda}, A_{m+1})$.

b) Recall the strict subsolution ψ_4 when $b = b_1$ and the strict supersolution Z_3 of (1.1). Now we choose n_λ small enough such that $\psi_2 \leq \psi_4$ and $\psi_2 \leq Z_3$. Next we choose $M \gg 1$ such that $\psi_4 \leq Z_1$ and $Z_3 \leq Z_1$ (see Figure 23). We note that $\|\psi_4\|_\infty \geq b_1 > a_1 = \|Z_3\|_\infty$. By Lemma 2.2, (1.1) has at least three positive solutions for $\lambda \in \left(\max\{\frac{b_1}{f(b_1)}\frac{2NC_N}{R^2}, A_{m+1}, 1\}, \min\{A_m, \frac{a_1}{f(a_1)}\frac{1}{\|v\|_\infty}\}\right)$. We note that in the construction of ψ_2, ψ_4, Z_1, and Z_3, the intersection of intervals of λ is $\left(\max\{\frac{b_1}{f(b_1)}\frac{2NC_N}{R^2}, A_{m+1}, 1\}, \min\{A_m, \frac{a_1}{f(a_1)}\frac{1}{\|v\|_\infty}\}\right)$. This completes the proof. □

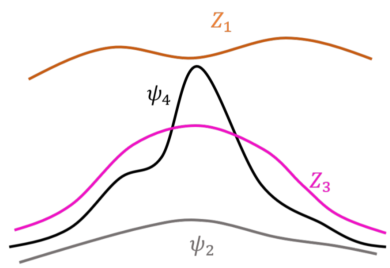


Figure 23. Subsolutions ψ_2, ψ_4 and supersolutions Z_1, Z_3.

3.2 Proof of Theorem 1.2

Let $\lambda^* = \overline{\lambda}$ and ψ_0 be as in the proof of Theorem 1.1. Recall the strict supersolution Z_4 and the strict subsolution ψ_4 when $b = b_2$. First we choose $\lambda^* > \max\{\frac{b_2}{f(b_2)}\frac{2NC_N}{R^2}, \frac{A_1}{2}\}$, $\lambda^* < A_{m+1}$, and $\lambda^* \approx A_{m+1}$ (making $\delta_\lambda \approx 0$) such that $\psi_1 < \psi_4$ and $\psi_1 < Z_4$ for $\lambda \in [\lambda^*, A_{m+1})$. Next, we choose m_λ small enough such that $\|Z_2\|_\infty < \|\psi_1\|_\infty$. Further, we can choose $M \gg 1$ such that $\psi_1 \leq Z_1$ and $Z_2 \leq Z_1$ (see Figure (24)). By Lemma 2.1, (1.1) has a positive solution $u_1 \in [\psi_0, Z_1] \backslash ([\psi_0, Z_2] \cup [\psi_1, Z_1])$ for $\lambda \in [\lambda^*, A_{m+1})$. We also have $\psi_4 \leq Z_1$, $Z_4 \leq Z_1$ for $M \gg 1$ and $\|\psi_4\|_\infty \geq b_2 > a_2 = \|Z_4\|_\infty$ (see Figure 24). Again, by Lemma 2.2, (1.1) has at least three positive solutions $u_2 \in [\psi_1, Z_4]$,

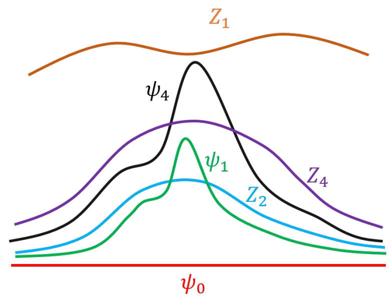


Figure 24. Subsolutions ψ_0, ψ_1, ψ_4 and supersolutions Z_1, Z_2, Z_4.

$u_3 \in [\psi_4, Z_1]$, and $u_4 \in [\psi_1, Z_1] \backslash ([\psi_1, Z_4] \cup [\psi_4, Z_1])$ for $\lambda \in [\lambda^*, A_{m+1})$. Hence (1.1) has at least four positive solutions for $\lambda \in [\lambda^*, A_{m+1})$. This completes the proof. □

3.3 Proof of Corollary 1.3

We note that the proof of Corollary 1.3 is an immediate consequence of the proof of Theorem 1.1 and Theorem 1.2. □

3.4 Proof of Theorem 1.4

First, we show the non-existence of positive solutions for $\lambda \approx 0$. Let u be a positive solution of (1.1). Then by the Green's second identity we obtain:

$$\begin{aligned} 0 &= \int_\Omega [\theta_{\lambda,m+1}\Delta u - u\Delta\theta_{\lambda,m+1}]dx \\ &= \int_\Omega [-\lambda f(u) + u(\sigma_{\lambda,m+1} + \lambda)(m+1)]\theta_{\lambda,m+1}dx \\ &\geq \int_\Omega [-\lambda M u + u(\sigma_{\lambda,m+1} + \lambda)(m+1)]\theta_{\lambda,m+1}dx \\ &= \int_\Omega \lambda \left\{ \frac{(m+1)\sigma_{\lambda,m+1}}{\lambda} - [M - (m+1)] \right\} u\theta_{\lambda,m+1}dx, \end{aligned} \tag{3.2}$$

where $M > (m+1)$ is such that $f(s) \leq Ms$ for all $s \in [0, \infty)$. Now for $\lambda < A_{m+1}$, $\sigma_{\lambda,m+1} > 0$, and $\lim_{\lambda \to 0} \frac{\sigma_{\lambda,m+1}}{\lambda} = \infty$ (see [FMSS]). This contradicts (3.2) for $\lambda \approx 0$ and hence (1.1) has no positive solution for $\lambda \approx 0$.

Next, when $m > 0$, if u is a positive solution of (1.1), then again by the Green's second identity we obtain:

$$\begin{aligned} 0 &= \int_\Omega [\theta_{\lambda,m} \Delta u - u \Delta \theta_{\lambda,m}] dx \\ &= \int_\Omega [-\lambda f(u) + u(\sigma_{\lambda,m} + \lambda) m] \theta_{\lambda,m} dx \\ &\leq \int_\Omega [-\lambda m u + u(\sigma_{\lambda,m} + \lambda) m] \theta_{\lambda,m} dx \\ &= \int_\Omega m \sigma_{\lambda,m} u \theta_{\lambda,m} dx \end{aligned} \tag{3.3}$$

since $f(s) \geq ms$ on $[0, \infty)$. Now if $\lambda > A_m$ then $\sigma_{\lambda,m} < 0$ which contradicts (3.3). Hence (1.1) has no positive solution for $\lambda > A_m$. $\square$

CHAPTER IV

PROOFS THEOREMS 1.5 - 1.6 AND COROLLARY 1.7 STATED IN FOCUS 2

First we construct sub-super solutions for certain λ ranges. Here we extend the ideas used in Chapter III appropriately for the systems case to construct sub-super solutions for (1.9) when f and g satisfy a combined sub-linear condition at infinity. Recall θ_λ and σ_λ (see (1.6)).

Construction of a small strict subsolution $(\psi_1, \overline{\psi}_1)$ for $\lambda < A_1$ and $\lambda \approx A_1$ when (H_4) is satisfied. In particular, here we will construct this subsolution so that $\overline{\psi}_1 = \psi_1$

We first note that $f''(s) > 0$ and $g''(s) > 0$ for $s \approx 0$. Hence there exists $A^* > 0$ and $s_1 > 0$ such that $f''(s), g''(s) > A^*$ for $s < s_1$. Let $(\psi_1, \psi_1) = (\delta_\lambda\theta_\lambda, \delta_\lambda\theta_\lambda)$ where $\delta_\lambda = \frac{2\sigma_\lambda}{\lambda A^* \min_{\overline{\Omega}} \theta_\lambda}$. We note that $\sigma_\lambda > 0$; $\lambda < A_1$, $\sigma_\lambda \to 0$ as $\lambda \to A_1^-$, and $\min_{\overline{\Omega}} \theta_\lambda \not\to 0$ as $\lambda \to A_1^-$. Thus $\delta_\lambda \to 0$ as $\lambda \to A_1^-$ since $\frac{1}{2}\delta_\lambda \lambda A^* \min_{\overline{\Omega}} \theta_\lambda = \sigma_\lambda$. Now by Taylor's Theorem, we have $f(\psi_1) = f(0) + f'(0)\psi_1 + \frac{f''(\zeta)}{2}\psi_1^2 = \psi_1 + \frac{f''(\zeta)}{2}{\psi_1}^2$ for some $\zeta \in [0, \psi_1]$. Then we have:

$$\begin{aligned}
-\Delta\psi_1 - \lambda f(\psi_1) &= \delta_\lambda(\sigma_\lambda + \lambda)\theta_\lambda - \lambda\Big[\delta_\lambda\theta_\lambda + \frac{f''(\zeta)}{2}(\delta_\lambda\theta_\lambda)^2\Big] \\
&< \delta_\lambda\theta_\lambda\Big[\sigma_\lambda - \frac{\lambda A^*}{2}\delta_\lambda \min_{\overline{\Omega}} \theta_\lambda\Big] = 0;\ \Omega
\end{aligned}$$

for $\lambda < A_1$ and $\lambda \approx A_1$ (so that $\delta_\lambda \approx 0$ and hence $\psi_1 < s_1$). Similarly, $-\Delta\psi_1 < \lambda g(\psi_1)$ for $\lambda < A_1$ and $\lambda \approx A_1$. Also, $\frac{\partial \psi_1}{\partial \eta} + \sqrt{\lambda}\psi_1 = 0$ on $\partial\Omega$ since θ_λ satisfies this boundary condition. Thus, there exists $\overline{\lambda} < A_1$ such that (ψ_1, ψ_1) is a strict subsolution of (1.9) for $\lambda \in [\overline{\lambda}, A_1)$.

Construction of a small subsolution $(\psi_2, \overline{\psi}_2)$ for $\lambda \geq A_1$ when (H_4) is satisfied. In particular, here we will construct this subsolution so that $\overline{\psi}_2 = \psi_2$

We have $f'(0) = g'(0) = 1$, and $\sigma_\lambda \leq 0$ for $\lambda \geq A_1$. Let $(\psi_2, \psi_2) = (n_\lambda\theta_\lambda, n_\lambda\theta_\lambda)$ with $n_\lambda > 0$. Now, consider $H(s) = (\sigma_\lambda + \lambda)s - \lambda f(s)$. Then we have $H(0) = 0$, $H'(0) = \sigma_\lambda \leq 0$ and $H''(0) = -\lambda f''(0) < 0$ since $f''(0) > 0$. This implies that $-\Delta\psi_2 = n_\lambda(\sigma_\lambda + \lambda)\theta_\lambda < \lambda f(n_\lambda\theta_\lambda) = \lambda f(\psi_2)$ in Ω for $n_\lambda \approx 0$. Similarly, $-\Delta\psi_2 < \lambda g(\psi_2)$ for $\lambda \geq A_1$ and $n_\lambda \approx 0$. We also have $\frac{\partial\psi_2}{\partial\eta} + \sqrt{\lambda}\psi_2 = 0$ on $\partial\Omega$ since θ_λ satisfies this boundary condition. Thus (ψ_2, ψ_2) is a subsolution of (1.9) for $n_\lambda \approx 0$ when $\lambda \geq A_1$.

Construction of a subsolution $(\psi_3, \overline{\psi}_3)$ for $\lambda \gg 1$ such that $\|\psi_3\|_\infty \to \infty$ as $\lambda \to \infty$ when (H_4) is satisfied. In particular, here we will construct this subsolution so that $\overline{\psi}_3 = \psi_3$

Noting that $f(0) = g(0) = 0$ and both f, g are increasing, define $h \in C^2([0,\infty))$ such that $h(0) < 0$, $h(s) \leq f(s)$, and $h(s) \leq g(s)$ for $s \in (0,\infty)$ and $\lim_{s\to\infty} h(s) = \gamma$ for some $\gamma > 0$. Then the Dirichlet boundary value problem:

$$\begin{cases} -\Delta w = \lambda h(w); & \Omega \\ w = 0; & \partial\Omega, \end{cases}$$

has a positive solution $\overline{w}_\lambda$ for $\lambda \gg 1$ such that $\|\overline{w}_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$ (see [CGS93]). It is easy to see that $(\overline{w}_\lambda, \overline{w}_\lambda)$ is a subsolution of (1.9) for $\lambda \gg 1$ since $h \leq f, h \leq g$ and $\frac{\partial\overline{w}_\lambda}{\partial\eta} < 0; \partial\Omega$. Consequently $(\psi_3, \psi_3) = (\overline{w}_\lambda, \overline{w}_\lambda)$ is a subsolution of (1.9) for $\lambda \gg 1$

such that $\|\psi_3\|_\infty \to \infty$ as $\lambda \to \infty$.

Construction of a strict subsolution $(\psi_4, \overline{\psi}_4)$ for $\lambda > Q_2(b)\frac{2NC_N}{R^2}$ where $b = b_1$ when $(H_4) - (H_6)$ are satisfied and $b = b_2$ when $(H_4), (H_5),$ and (H_7) are satisfied

Here we construct a strict subsolution $(\psi_4, \overline{\psi}_4)$ for $\lambda > Q_2(b)\frac{2NC_N}{R^2}$. We note that in [ASR06], the authors showed that the boundary value problem:

$$\begin{cases} -\Delta u = \lambda f(v);\ \Omega \\ -\Delta v = \lambda g(u);\ \Omega \\ \quad u = 0;\ \partial\Omega \\ \quad v = 0;\ \partial\Omega, \end{cases}$$

has a strict subsolution $(\overline{u}_0, \overline{v}_0)$ for $\lambda \geq Q_2(b)\frac{2NC_N}{R^2}$ such that $||\overline{u}_0||_\infty \geq b$ and $||\overline{v}_0||_\infty \geq b$. Let $(\psi_4, \overline{\psi}_4)$ be the first iteration of $(\overline{u}_0, \overline{v}_0)$, namely, $(\psi_4, \overline{\psi}_4)$ be the solution to the problem:

$$\begin{cases} -\Delta \psi_4 = \lambda f(\overline{v}_0);\ \Omega \\ -\Delta \overline{\psi}_4 = \lambda g(\overline{u}_0);\ \Omega \\ \frac{\partial \psi_4}{\partial \eta} + \sqrt{\lambda}\psi_4 = 0;\ \partial\Omega \\ \frac{\partial \overline{\psi}_4}{\partial \eta} + \sqrt{\lambda}\ \overline{\psi}_4 = 0;\ \partial\Omega. \end{cases}$$

Then by the comparison principle $(\psi_4, \overline{\psi}_4) > (\overline{u}_0, \overline{v}_0);\ \bar{\Omega}$. Hence $(\psi_4, \overline{\psi}_4)$ is a strict subsolution of (1.9) such that $||\psi_4||_\infty \geq b$ and $||\overline{\psi}_4||_\infty \geq b$.

Construction of a large supersolution $(Z_1, \overline{Z}_1)$ for any $\lambda > 0$ when $(H_4) - (H_5)$ are satisfied

Let e_λ be the positive solution of:

$$\begin{cases} -\Delta e = 1; \ \Omega \\ \frac{\partial e}{\partial \eta} + \sqrt{\lambda} e = 0; \ \partial\Omega. \end{cases} \tag{4.1}$$

We consider three different cases.

Case I: Assume both f and g are bounded. Let $\lambda > 0$. Take $(Z_1, \overline{Z}_1) = (\lambda M_\lambda \frac{e_\lambda}{\|e_\lambda\|_\infty}, \lambda M_\lambda \frac{e_\lambda}{\|e_\lambda\|_\infty})$ and choose M_λ large such that $\frac{M_\lambda \lambda}{\|e_\lambda\|_\infty} \geq \lambda f(\frac{\lambda M_\lambda e_\lambda}{\|e_\lambda\|_\infty})$. This implies $-\Delta Z_1 - \lambda f(\overline{Z}_1) \geq 0$ for $M_\lambda \gg 1$, and, by a similar argument, we see that $-\Delta \overline{Z}_1 - \lambda g(Z_1) \geq 0$ for $M_\lambda \gg 1$. Also on the boundary we have $\frac{\partial Z_1}{\partial \eta} + \sqrt{\lambda} Z_1 = 0$ and $\frac{\partial \overline{Z}_1}{\partial \eta} + \sqrt{\lambda}\ \overline{Z}_1 = 0$. Hence $(Z_1, \overline{Z}_1)$ is a supersolution for $M_\lambda \gg 1$.

Case II: Assume $g(s) \to \infty$ as $s \to \infty$. Let $\lambda > 0$. Take $(Z_1, \overline{Z}_1) = (M_\lambda e_\lambda, \lambda g(M_\lambda \|e_\lambda\|_\infty) e_\lambda)$ with $M_\lambda > 0$. Then by choosing M_λ large we obtain

$$\frac{1}{\lambda \|e_\lambda\|_\infty} \geq \frac{f(\lambda \|e_\lambda\|_\infty g(M_\lambda \|e_\lambda\|_\infty))}{M_\lambda \|e_\lambda\|_\infty}$$

which implies that $M_\lambda - \lambda f(\lambda g(M_\lambda \|e_\lambda\|_\infty) e_\lambda) \geq 0$ since f is increasing. Hence $-\Delta Z_1 - \lambda f(\overline{Z}_1) \geq 0$. We also have $\lambda g(M_\lambda \|e_\lambda\|_\infty) - \lambda g(M_\lambda e_\lambda) \geq 0$ since g is increasing. This implies that $-\Delta \overline{Z}_1 - \lambda g(Z_1) \geq 0$. Further, on the boundary we have $\frac{\partial Z_1}{\partial \eta} + \sqrt{\lambda} Z_1 = \frac{\partial \overline{Z}_1}{\partial \eta} + \sqrt{\lambda}\ \overline{Z}_1 = 0$ since e_λ satisfies this boundary condition. Hence $(Z_1, \overline{Z}_1)$ is a supersolution of (1.9) for $M_\lambda \gg 1$.

Case III: Assume $f(s) \to \infty$ as $s \to \infty$ and g is bounded. Let $\lambda > 0$. Take $(Z_1, \overline{Z}_1) = (\lambda f(M_\lambda \|e_\lambda\|_\infty) e_\lambda, M_\lambda e_\lambda)$ with $M_\lambda > 0$. Then since f is increasing,

$\lambda f(M_\lambda \|e_\lambda\|_\infty) - \lambda f(M_\lambda e_\lambda) \geq 0$ which implies that $-\Delta Z_1 - \lambda f(\overline{Z}_1) \geq 0$. Also we have $M_\lambda \geq \lambda g(\lambda f(M_\lambda \|e_\lambda\|_\infty) e_\lambda)$ for $M_\lambda \gg 1$. This implies that $-\Delta \overline{Z}_1 - \lambda g(Z_1) \geq 0$. Further, on the boundary we have $\frac{\partial Z_1}{\partial \eta} + \sqrt{\lambda} Z_1 = \frac{\partial \overline{Z}_1}{\partial \eta} + \sqrt{\lambda}\ \overline{Z}_1 = 0$ since e_λ satisfies this boundary condition. Hence $(Z_1, \overline{Z}_1)$ is a supersolution of (1.9) for $M_\lambda \gg 1$.

Construction of a strict supersolution $(Z_2, \overline{Z}_2)$ for $\lambda < A_1$ when $(H_4)-(H_5)$ are satisfied. In particular, here we will construct this supersolution so that $\overline{Z}_2 = Z_2$.

Let $\lambda < A_1$. Take $(Z_2, Z_2) = (m_\lambda \theta_\lambda, m_\lambda \theta_\lambda)$ with $m_\lambda > 0$ and $l(s) = (\sigma_\lambda + \lambda)s - \lambda f(s)$. We note that $\sigma_\lambda > 0$ for $\lambda < A_1$. Then we have $l(0) = 0$ and $l'(0) = (\sigma_\lambda + \lambda) - \lambda f'(0) = \sigma_\lambda > 0$ since $f(0) = 0$ and $f'(0) = 1$. This implies that $-\Delta Z_2 = (\sigma_\lambda + \lambda) m_\lambda \theta_\lambda > \lambda f(m_\lambda \theta_\lambda) = \lambda f(Z_2)$ in Ω for $m_\lambda \approx 0$. Similarly, $-\Delta Z_2 > \lambda g(Z_2)$ for $\lambda < A_1$ and $m_\lambda \approx 0$. On the boundary, we have $\frac{\partial Z_2}{\partial \eta} + \sqrt{\lambda} Z_2 = 0$ since θ_λ satisfies this boundary condition. Thus (Z_2, Z_2) with $m_\lambda > 0$ and $m_\lambda \approx 0$ is a strict supersolution of (1.9).

Construction of a strict supersolution $(Z_3, \overline{Z}_3)$ for $\lambda \in \left(1, Q_1(a_1)\frac{1}{\|v\|_\infty}\right)$ when (H_4) and (H_6) are satisfied. In particular, here we will construct this supersolution so that $\overline{Z}_3 = Z_3$

Let $(Z_3, Z_3) = (\frac{a_1 v}{\|v\|_\infty}, \frac{a_1 v}{\|v\|_\infty})$ where v is as in (1.7). Then $-\Delta Z_3 = \frac{a_1}{\|v\|_\infty} > \lambda f(a_1) \geq \lambda f(Z_3)$ since $\lambda < Q_1(a_1)\frac{1}{\|v\|_\infty} \leq \frac{a_1}{f(a_1)}\frac{1}{\|v\|_\infty}$ and f is increasing. Similarly, $-\Delta Z_3 > \lambda g(Z_3)$. Further, Z_3 satisfies $\frac{\partial Z_3}{\partial \eta} + \sqrt{\lambda} Z_3 = \frac{a_1}{\|v\|_\infty}\frac{\partial v}{\partial \eta} + \sqrt{\lambda}\frac{a_1 v}{\|v\|_\infty} > \frac{a_1}{\|v\|_\infty}[\frac{\partial v}{\partial \eta} + v] = 0$ on $\partial\Omega$ since $\lambda > 1$. Thus (Z_3, Z_3) is a strict supersolution of (1.9) for $\lambda \in \left(1, Q_1(a_1)\frac{1}{\|v\|_\infty}\right)$.

Construction of a strict supersolution $(Z_4, \overline{Z}_4)$ for $\lambda \in \left(\frac{A_1}{2}, Q_1(a_2)\frac{1}{\|w\|_\infty}\right)$ when (H_4) and (H_7) are satisfied. In particular, here we will construct this supersolution so that $\overline{Z}_4 = Z_4$

Let $(Z_4, Z_4) = (\frac{a_2 w}{\|w\|_\infty}, \frac{a_2 w}{\|w\|_\infty})$ where w is as in (1.8). Then $-\Delta Z_4 = \frac{a_2}{\|w\|_\infty} > \lambda f(a_2) \geq \lambda f(Z_4)$ since $\lambda < Q_1(a_2)\frac{1}{\|w\|_\infty} \leq \frac{a_2}{f(a_2)}\frac{1}{\|w\|_\infty}$ and f is increasing. Similarly, $-\Delta Z_4 > \lambda g(Z_4)$. Further, Z_4 satisfies $\frac{\partial Z_4}{\partial \eta} + \sqrt{\lambda} Z_4 = \frac{a_2}{\|w\|_\infty}\frac{\partial w}{\partial \eta} + \sqrt{\lambda}\frac{a_2 w}{\|w\|_\infty} > \frac{a_2}{\|w\|_\infty}[\frac{\partial w}{\partial \eta} + \sqrt{\frac{A_1}{2}} w] = 0$ on $\partial\Omega$ since $\lambda > \frac{A_1}{2}$. Thus (Z_4, Z_4) is a strict supersolution of (1.9) for $\lambda \in \left(\frac{A_1}{2}, Q_1(a_2)\frac{1}{\|w\|_\infty}\right)$.

4.1 Proof of Theorem 1.5

a) Recall the construction of the supersolution $(Z_1, \overline{Z}_1)$ and the subsolution (ψ_2, ψ_2) (for $\lambda \geq A_1$). Choose $M_\lambda \gg 1$ and $n_\lambda \approx 0$ such that $(Z_1, \overline{Z}_1) \geq (\psi_2, \psi_2)$. By Lemma 2.1, (1.9) has a positive solution $(u_\lambda, v_\lambda) \in [(\psi_2, \psi_2), (Z_1, \overline{Z}_1)]$ for $\lambda \geq A_1$.

Now, recall the subsolution (ψ_3, ψ_3) of (1.9) and choose $M_\lambda \gg 1$ such that $(\psi_3, \psi_3) \leq (Z_1, \overline{Z}_1)$. Also, recall that $\|\psi_3\|_\infty \to \infty$ as $\lambda \to \infty$. Hence by Lemma 2.1, (1.9) has a positive solution $(u_\lambda, v_\lambda) \in [(\psi_3, \psi_3), (Z_1, \overline{Z}_1)]$ such that $\|u_\lambda\|_\infty, \|v_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$.

Next, let $\lambda \in [\overline{\lambda}, A_1)$ where $\overline{\lambda}$ is as in the construction of the strict subsolution (ψ_1, ψ_1). Note that $(\psi_0, \psi_0) = (0, 0)$ is a solution and hence a subsolution of (1.9). Recalling the strict supersolution (Z_2, Z_2) of (1.9), choose m_λ small enough such that $\|Z_2\|_\infty < \|\psi_1\|_\infty$. Next, choose $M_\lambda \gg 1$ such that $(\psi_1, \psi_1) \leq (Z_1, \overline{Z}_1)$ and $(Z_2, Z_2) \leq (Z_1, \overline{Z}_1)$. Hence by Lemma 2.2, (1.9) has at least two positive solutions

$(u_1, v_1) \in [(\psi_1, \psi_1), (Z_1, \overline{Z}_1)]$ and $(u_2, v_2) \in [(\psi_0, \psi_0), (Z_1, \overline{Z}_1)] \backslash ([(\psi_0, \psi_0), (Z_2, Z_2)] \cup [(\psi_1, \psi_1), (Z_1, \overline{Z}_1])$ for $\lambda \in [\overline{\lambda}, A_1)$. Since $(\psi_0, \psi_0) = (0, 0)$ is a solution, by using Lemma 2.2, we can guaranty here only two positive solutions.

b) Recall the strict subsolution $(\psi_4, \overline{\psi}_4)$ with $b = b_1$ and the strict supersolution (Z_3, Z_3) of (1.9). Choose n_λ small enough such that $(\psi_2, \psi_2) \leq (\psi_4, \overline{\psi}_4)$ and $(\psi_2, \psi_2) \leq (Z_3, Z_3)$. Next, choose $M_\lambda \gg 1$ such that $(\psi_4, \overline{\psi}_4) \leq (Z_1, \overline{Z}_1)$ and $(Z_3, Z_3) \leq (Z_1, \overline{Z}_1)$. We note that $\|\psi_4\|_\infty, \|\overline{\psi}_4\|_\infty \geq b_1 > a_1 = \|Z_3\|_\infty$. By Lemma 2.2, (1.9) has at least three positive solutions for $\lambda \in \left(\max\{A_1, Q_2(b_1)\frac{2NC_N}{R^2}, 1\}, \frac{Q_1(a_1)}{\|v\|_\infty}\right)$ which is the intersection of intervals of λ in the construction of (ψ_2, ψ_2), $(\psi_4, \overline{\psi}_4)$, $(Z_1, \overline{Z}_1)$, and (Z_3, Z_3). This completes the proof. $\square$

4.2 Proof of Theorem 1.6

Note that $(\psi_0, \psi_0) = (0, 0)$ is a solution and hence a subsolution of (1.9). Recall the strict supersolution (Z_4, Z_4) and the strict subsolution $(\psi_4, \overline{\psi}_4)$ with $b = b_2$. First, choose $\lambda^* > \max\{Q_2(b_2)\frac{2NC_N}{R^2}, \frac{A_1}{2}\}$, $\lambda^* < A_1$, and $\lambda^* \approx A_1$ (making $\delta_\lambda \approx 0$ in the construction of strict subsolution (ψ_1, ψ_1)) such that $(\psi_1, \psi_1) < (\psi_4, \overline{\psi}_4)$ and $(\psi_1, \psi_1) < (Z_4, Z_4)$ for $\lambda \in [\lambda^*, A_1)$. Recall (Z_2, Z_2) and choose m_λ small enough such that $\|Z_2\|_\infty < \|\psi_1\|_\infty$. Further, we can choose $M_\lambda \gg 1$ such that $(\psi_1, \psi_1) \leq (Z_1, \overline{Z}_1)$ and $(Z_2, Z_2) \leq (Z_1, \overline{Z}_1)$. Hence by Lemma 2.2, (1.9) has a positive solution $(u_1, v_1) \in [(\psi_0, \psi_0), (Z_1, \overline{Z}_1)] \backslash ([(\psi_0, \psi_0), (Z_2, Z_2)] \cup [(\psi_1, \psi_1), (Z_1, \overline{Z}_1)])$ for $\lambda \in [\lambda^*, A_1)$. We also have $(\psi_4, \overline{\psi}_4) \leq (Z_1, \overline{Z}_1)$, $(Z_4, Z_4) \leq (Z_1, \overline{Z}_1)$ for $M_\lambda \gg 1$ and $\|\psi_4\|_\infty \geq b_2 > a_2 = \|Z_4\|_\infty$, $\|\overline{\psi}_4\|_\infty \geq b_2 > a_2 = \|Z_4\|_\infty$. Again, by Lemma 2.2, (1.9) has at least three positive solutions $(u_2, v_2) \in [(\psi_1, \psi_1), (Z_4, Z_4)]$, $(u_3, v_3) \in [(\psi_4, \overline{\psi}_4), (Z_1, \overline{Z}_1)]$, and $(u_4, v_4) \in [(\psi_1, \psi_1), (Z_1, \overline{Z}_1)] \backslash ([(\psi_1, \psi_1), (Z_4, Z_4)] \cup [(\psi_4, \overline{\psi}_4),$

$(Z_1, \overline{Z}_1)])$ for $\lambda \in [\lambda^*, A_1)$. Noting that $(u_1, v_1) \notin [(\psi_1, \psi_1), (Z_1, \overline{Z}_1)]$ while $(u_i, v_i) \in [(\psi_1, \psi_1), (Z_1, \overline{Z}_1)]$; $i = 2, 3, 4$, (1.9) has at least four positive solutions for $\lambda \in [\lambda^*, A_1)$. This completes the proof $\square$

4.3 Proof of Corollary 1.7

We note that the proof of Corollary 1.7 is an immediate consequence of the proofs of Theorem 1.5 and Theorem 1.6. $\square$

CHAPTER V

PROOFS OF THEOREMS 1.8 - 1.11 STATED IN FOCUS 3

First, we recall (1.20) and consider either (1) $b = 0$ and define (i) $W_{1,\gamma_1} = W_{1,\gamma_1,0}$ and $E_1(1,\gamma_1) = E_1(1,0,\gamma_1)$ and (ii) $W_{r,\gamma_2} = W_{r,\gamma_2,0}$ and $E_1(r,\gamma_2) = E_1(r,0,\gamma_2)$, (2) $b = b_1, R = 1$, and $\gamma = \gamma_1$ and employ W_{1,γ_1,b_1} and $E_1(1,b_1,\gamma_1)$, or (3) $b = b_2, R = r$ and $\gamma = \gamma_2$ and employ W_{r,γ_2,b_2} and $E_1(r,b_2,\gamma_2)$.

Now, we consider the semitrivial steady states of (1.13) in which one population is present and the other is absent, namely:

$$\begin{cases} -\Delta W = \lambda W(1-W);\ \Omega \\ \frac{\partial W}{\partial \eta} + \sqrt{\lambda}\gamma_1 W = 0;\ \partial\Omega \end{cases} \tag{5.1}$$

and

$$\begin{cases} -\Delta W = \lambda r W(1-W);\ \Omega \\ \frac{\partial W}{\partial \eta} + \sqrt{\lambda}\gamma_2 W = 0;\ \partial\Omega. \end{cases} \tag{5.2}$$

Hence, (5.1) is (1.20) with $R = 1, b = 0$ and $\gamma = \gamma_1$, and it represents the governing steady state equation for species u in the absence of v. Then it has a unique positive solution $W \equiv W_{1,\gamma_1}$ whenever $\lambda > E_1(1,\gamma_1)$. Also (5.2) is (1.20) with $R = r, b = 0$ and $\gamma = \gamma_2$, and it represents the governing steady state equation for species v in the absence of u. Thus it has a unique positive solution $W \equiv W_{r,\gamma_2}$ whenever $\lambda > E_1(r,\gamma_2)$ (see (1.17)).

Let $\sigma_1 = \sigma_1(\lambda, \gamma_1)$ and $\sigma_2 = \sigma_2(\lambda, r, \gamma_2)$ be the principal eigenvalues of:

$$\begin{cases} -\Delta\phi_1 - \lambda\phi_1 = \sigma_1\phi_1; \ \Omega \\ \frac{\partial\phi_1}{\partial\eta} + \sqrt{\lambda}\gamma_1\phi_1 = 0; \ \partial\Omega \end{cases} \tag{5.3}$$

and

$$\begin{cases} -\Delta\phi_2 - \lambda r\phi_2 = \sigma_2\phi_2; \ \Omega \\ \frac{\partial\phi_2}{\partial\eta} + \sqrt{\lambda}\gamma_2\phi_2 = 0; \ \partial\Omega \end{cases} \tag{5.4}$$

with corresponding eigenfunctions ϕ_1, ϕ_2 which can be chosen such that $\phi_1, \phi_2 > 0; \overline{\Omega}$, respectively. The sign of these principal eigenvalues will determine whether or not a species can colonize the patch.

Finally, we consider two eigenvalue problems involving W_{1,γ_1} and W_{r,γ_2}:

$$\begin{cases} -\Delta\phi_3 - \lambda r(1 - b_2W_{1,\gamma_1})\phi_3 = \sigma_3\phi_3; \ \Omega \\ \frac{\partial\phi_3}{\partial\eta} + \sqrt{\lambda}\gamma_2\phi_3 = 0; \ \partial\Omega \end{cases} \tag{5.5}$$

and

$$\begin{cases} -\Delta\phi_4 - \lambda(1 - b_1W_{r,\gamma_2})\phi_4 = \sigma_4\phi_4; \ \Omega \\ \frac{\partial\phi_4}{\partial\eta} + \sqrt{\lambda}\gamma_1\phi_4 = 0; \ \partial\Omega. \end{cases} \tag{5.6}$$

Let $\sigma_3 = \sigma_3(\lambda, r, \gamma_2), \sigma_4 = \sigma_4(\lambda, \gamma_1)$ be the principal eigenvalues and $\phi_3, \phi_4 > 0; \overline{\Omega}$ be the corresponding eigenfunctions of (5.5) and (5.6), respectively. The sign of $\sigma_3(\sigma_4)$ will ultimately determine if $v(u)$ can invade the patch when rare if $u(v)$ is near its equilibrium.

In the absence of competition (i.e., $b_1 = 0 = b_2$) the principal eigenvalues, $E_1(1,\gamma_1)$ and $E_1(r,\gamma_2)$, can be employed to determine when one species has an advantage over the other in the sense that the species has a smaller minimum patch size allowing it to invade and colonize smaller patches than the other species. To see this, from the definition of λ we obtain the minimum patch size for u, $\ell_1^* = \sqrt{\frac{D_1 E_1(1,\gamma_1)}{r_1}}$, and for v, $\ell_2^* = \sqrt{\frac{D_1 E_1(r,\gamma_2)}{r_1}}$. Fixing r_1 and D_1, there are then three cases: 1) $E_1(1,\gamma_1) = E_1(r,\gamma_2)$ implying that $\ell_1^* = \ell_2^*$: neither species has an advantage as their minimum patch sizes are the same; 2) $E_1(1,\gamma_1) < E_1(r,\gamma_2)$ implying that $\ell_1^* < \ell_2^*$: u has an advantage being able to invade and colonize smaller patches than v; and 3) $E_1(1,\gamma_1) > E_1(r,\gamma_2)$ implying that $\ell_1^* > \ell_2^*$: v has an advantage being able to invade and colonize smaller patches than u. Crucial to this determination of advantage are the composite parameters, r, γ_1, γ_2, which encapsulate several biological mechanisms, i.e., r measures differences in the organisms in the patch and γ_1, γ_2 measure the combined effect of a hostile matrix on the respective organisms. To see this, we first assume that the matrix affects both species the same and there is no competition, i.e., $\gamma_1 = \gamma_2$ and $b_1 = 0 = b_2$. Note that r can be written as $r = \frac{\frac{r_2}{D_2}}{\frac{r_1}{D_1}}$ and interpreted as a means to compare the two species by their patch growth-to-diffusion (G-D) ratio which is defined as the ratio of patch intrinsic growth rate to patch diffusion rate. We explore three cases: 1) if $r = 1$, then both growth to diffusion ratios are the same, $E_1(1,\gamma_1) = E_1(r,\gamma_2)$ implying that $\ell_1^* = \ell_2^*$, and neither species has a G-D advantage; 2) if r $>$ 1 then v's growth to diffusion ratio is greater than u's, $E_1(1,\gamma_1) > E_1(r,\gamma_2)$ implying that $\ell_1^* > \ell_2^*$, and v has a G-D advantage in having a smaller minimum patch size; and 3) if $r < 1$ then u's ratio is greater than v's, $E_1(1, \gamma_1) < E_1(r, \gamma_2)$ implying that $\ell_1^* < \ell_2^*$, and u has a G-D advantage in having a smaller minimum patch size. Secondly, we assume there is no overall difference in G-D ratios of the organisms and no competition, i.e., $r = 1$ and $b_1 = 0 = b_2$. The combined effect of matrix hostility and behavior response to detecting a patch edge is measured in the respective γ_i-value. For example, a large γ_1-value could indicate a high matrix mortality rate (i.e. $S_1^* \gg 1$) and / or a propensity of organisms to recognize the patch edge, bias their movement, and leave the patch with a high probability (i.e., $\alpha_1 \approx 0$). We notice three cases: 1) if $\gamma_1 = \gamma_2$ then $E_1(1, \gamma_1) = E_1(1, \gamma_2)$, $\ell_1^* = \ell_2^*$, and the combined matrix effect benefits neither species over the other; 2) if $\gamma_1 > \gamma_2$ then $E_1(1, \gamma_1) > E_1(1, \gamma_2), \ell_1^* > \ell_2^*$, and the combined matrix effect causes more mortality in u through interactions with the hostile matrix, and thus, gives v a smaller minimum patch size and a matrix advantage; and 3) if $\gamma_1 < \gamma_2$, then $E_1(1, \gamma_1) < E_1(1, \gamma_2)$, $\ell_1^* < \ell_2^*$, and the combined matrix effect causes more mortality in v through interactions with the hostile matrix, and thus, gives u a smaller minimum patch size and a matrix advantage. Since larger patches have a correspondingly larger core area within the patch where organisms have little chance of encountering mortality at the patch/matrix interface, any differential matrix effect acting on the system will be more pronounced for small patch sizes and diminish as the patch size goes to infinity. As we will see in the sections that follow, advantage in growth-to-diffusion ratio and combined matrix effect will play vital roles in predicting the outcome of this competition system.

Now, we state some results that we will use in the proofs of our main results.

Theorem 5.1. *[Pao92], [Pao81] Let $r > 0, \gamma_1 = 0 = \gamma_2$, and $b_1, b_2 \geq 0$. Then for all $\lambda > 0$ the following hold:*

(A) If $b_1, b_2 < 1$ *(weak competition) then* (1.13) *has a globally asymptotically stable coexistence state given by:*

$$\left(\frac{1-b_1}{1-b_1b_2}, \frac{1-b_2}{1-b_1b_2}\right).$$

(B) If $b_1 < 1 \leq b_2$ *or* $b_2 < 1 \leq b_1$ *(semistrong competition), then no coexistence state of* (1.13) *exists.*

(C) If $b_1 = 1 = b_2$ *(neutral competition), then* (1.13) *has infinitely many asymptotically stable coexistence states of the form:*

$$(c, 1-c), c > 0.$$

Theorem 5.2. *[GMRS18] Let* $R > 0, b \in [0,1)$*, and* $\gamma \geq 0$*.*

(a) If $\sigma_0 \geq 0 \left(\lambda \leq \frac{E_1(R,b,\gamma)}{1-b}\right)$*, then* $W \equiv 0$ *is globally asymptotically stable and no positive solution exists for* (1.20)*.*

(b) If $\sigma_0 < 0 \left(\lambda > \frac{E_1(R,b,\gamma)}{1-b}\right)$*, then* $W \equiv 0$ *is unstable and there exists a unique globally asymptotically stable positive solution* $W_{R,\gamma,b}$ *for* (1.20)*. Moreover, the following properties of* $W_{R,\gamma,b}$ *hold:*

(i) $\frac{-\sigma(R,b,\gamma,\lambda)}{\lambda r}\phi_0 \leq W_{R,\gamma,b} \leq 1$*.*

(ii) For fixed x *and* λ*:*

(1) $W_{R,\gamma,b}$ *is increasing in* R *for fixed* b *and* γ*.*

(2) $W_{R,\gamma,b}$ *is decreasing in* b *for fixed* R *and* γ*.*

(3) $W_{R,\gamma,b}$ *is decreasing in* γ *for fixed* R *and* b*.*

(iii) $W_{R,\gamma,b} \to (1-b)$ *uniformly on every closed subset of* Ω *as* $\lambda \to \infty$ *(see Figure* 17*).*

Next, we state and prove some results that will be used in the proofs of our main theorems.

Lemma 5.3. *If* $\lambda > \max\{E_1(1,\gamma_1), E_1(r,\gamma_2)\}$ *and* $\sigma_3, \sigma_4 < 0$*, then* (1.18) *has a positive solution,* (u,v)*, which, for* $m \approx 0$*, satisfies:*

$$(m\phi_4, m\phi_3) < (u,v) < (W_{1,\gamma_1}, W_{r,\gamma_2}); \overline{\Omega}$$

where σ_3, σ_4 *are the principal eigenvalues with corresponding eigenfunctions* ϕ_3, ϕ_4 *of* (5.5), (5.6)*, respectively.*

Proof. Let $m > 0$ and define $\psi = (m\phi_4, m\phi_3)$ and $Z = (W_{1,\gamma_1}, W_{r,\gamma_2})$. By our choice of λ, we have $\sigma_1, \sigma_2 < 0$ ensuring that both W_{1,γ_1} and W_{r,γ_2} exist. We will now show that ψ and Z are a sub-supersolution pair for (1.18). First, we check (ψ_1, Z_2) :

$$\begin{aligned} -\Delta\psi_1 - \lambda\psi_1(1-\psi_1-b_1Z_2) &= m\sigma_4\phi_4 + m\lambda\phi_4 - m\lambda b_1 W_{r,\gamma_2}\phi_4 - \lambda m\phi_4 \\ &\quad + \lambda m^2\phi_4^2 + m\lambda b_1 W_{r,\gamma_2}\phi_4 \\ &= m\phi_4[\sigma_4 + \lambda m\phi_4] \\ &< 0 \qquad (5.7) \end{aligned}$$

for $m \approx 0$ since $\sigma_4 < 0$. Also, we have

$$\begin{aligned}-\Delta Z_2 - \lambda r Z_2(1 - Z_2 - b_2\psi_1) &= \lambda r W_{r,\gamma_2} - \lambda r W_{r,\gamma_2}^2 - \lambda r W_{r,\gamma_2} + \lambda r W_{r,\gamma_2}^2 \\ &\quad + \lambda r b_2 W_{r,\gamma_2} m\phi_4 \\ &= \lambda r b_2 W_{r,\gamma_2} m\phi_4 \\ &\geq 0 \qquad (5.8)\end{aligned}$$

since $W_{r,\gamma_2}, \phi_4 > 0; \Omega, \lambda, r > 0$, and $b_2 \geq 0$. It is easy to see that

$$\frac{\partial \psi_1}{\partial \eta} + \sqrt{\lambda}\gamma_1\psi_1 = 0 = \frac{\partial Z_2}{\partial \eta} + \sqrt{\lambda}\gamma_2 Z_2. \qquad (5.9)$$

Next, we check (Z_1, ψ_2):

$$\begin{aligned}-\Delta Z_1 - \lambda Z_1(1 - Z_1 - b_1\psi_2) &= \lambda W_{1,\gamma_1} - \lambda W_{1,\gamma_1}^2 - \lambda W_{1,\gamma_1} + \lambda W_{1,\gamma_1}^2 + \lambda b_1 W_{1,\gamma_1} m\phi_3 \\ &= \lambda b_1 W_{1,\gamma_1} m\phi_3 \\ &\geq 0 \qquad (5.10)\end{aligned}$$

since $W_{1,\gamma_1}, \phi_3 > 0; \Omega, \lambda, r > 0$, and $b_1 \geq 0$. Also, we have

$$\begin{aligned}-\Delta \psi_2 - \lambda r \psi_2(1 - \psi_2 - b_2 Z_1) &= m\sigma_3\phi_3 + m\lambda r\phi_3 - m\lambda r b_2 W_{1,\gamma_1}\phi_3 - \lambda r m\phi_3 \\ &\quad + \lambda r m^2\phi_3^2 + m\lambda r b_2 W_{1,\gamma_1}\phi_3 \\ &= m\phi_3[\sigma_3 + \lambda r m\phi_3] \\ &< 0 \qquad (5.11)\end{aligned}$$

for $m \approx 0$ since $\sigma_3 < 0$. It is easy to see that

$$\frac{\partial \psi_2}{\partial \eta} + \sqrt{\lambda}\gamma_2\psi_2 = 0 = \frac{\partial Z_1}{\partial \eta} + \sqrt{\lambda}\gamma_1 Z_1. \tag{5.12}$$

Also, we can choose $m \approx 0$ such that $\psi < Z; \overline{\Omega}$. Thus, ψ, Z are a strict sub-supersolution pair and (1.18) has at least one solution, (u, v), with

$$(\psi_1, \psi_2) < (u, v) < (Z_1, Z_2); \overline{\Omega}. \tag{5.13}$$

□

Lemma 5.4. *If $\lambda > \max\{E_1(1, \gamma_1), E_1(r, \gamma_2)\}$ then the following hold:*

(A) $\sigma_3 \int_\Omega W_{r,\gamma_2}\phi_3 dx = \lambda r \int_\Omega W_{r,\gamma_2}\phi_3[b_2 W_{1,\gamma_1} - W_{r,\gamma_2}]dx$

(B) $\sigma_4 \int_\Omega W_{1,\gamma_1}\phi_4 dx = \lambda \int_\Omega W_{1,\gamma_1}\phi_4[b_1 W_{r,\gamma_2} - W_{1,\gamma_1}]dx.$

Proof. We only present a proof of (A) as the proof of (B) is similar. Using Green's Identity, we have:

$$\int_\Omega (-\Delta W_{r,\gamma_2}\phi_3 + \Delta\phi_3 W_{r,\gamma_2})\, dx = \int_\Omega \left(-\frac{\partial W_{r,\gamma_2}}{\partial \eta}\phi_3 + \frac{\partial \phi_3}{\partial \eta} W_{r,\gamma_2}\right) ds. \tag{5.14}$$

It is easy to see that the right-hand side of (5.14) is zero. Thus

$$
\begin{aligned}
0 &= \int_\Omega \left(-\Delta W_{r,\gamma_2}\phi_3 + \Delta\phi_3 W_{r,\gamma_2}\right) dx \\
&= \int_\Omega \big(\lambda r W_{r,\gamma_2}\phi_3 - \lambda r W_{r,\gamma_2}^2\phi_3 - \sigma_3 W_{r,\gamma_2}\phi_3 - \lambda r W_{r,\gamma_2}\phi_3 \\
&\qquad\qquad + \lambda r b_2 W_{1,\gamma_1} W_{r,\gamma_2}\phi_3\big)\, dx \\
&= \int_\Omega (-\sigma_3 W_{r,\gamma_2}\phi_3 + \lambda r W_{r,\gamma_2}\phi_3[b_2 W_{1,\gamma_1} - W_{r,\gamma_2}])dx, \qquad (5.15)
\end{aligned}
$$

or, equivalently,

$$
\sigma_3 \int_\Omega W_{r,\gamma_2}\phi_3 dx = \int_\Omega \lambda r W_{r,\gamma_2}\phi_3[b_2 W_{1,\gamma_1} - W_{r,\gamma_2}]dx. \qquad (5.16)
$$

□

Lemma 5.5. *Considering σ_3, σ_4 as functions of $W_{1,\gamma_1}, W_{r,\gamma_2}$, respectively, the following hold:*

(A) σ_3, σ_4 is an increasing function of $W_{1,\gamma_1}, W_{r,\gamma_2}$, respectively

(B) if $\lambda > E_1(1,\gamma_1)$, then $\sigma_3(0) < \sigma_3(W_{1,\gamma_1}) < \sigma_3(1)$

(C) if $\lambda > E_1(r,\gamma_2)$, then $\sigma_4(0) < \sigma_4(W_{r,\gamma_2}) < \sigma_4(1)$.

The proof of Lemma 5.5 follows from Corollary 2.2 in [CC03].

Lemma 5.6. *If (u,v) is a positive solution of (1.18), then the following holds:*

$$
\lambda \int_\Omega uv[(1-r) + (rb_2 - 1)u + (r - b_1)v]dx = \sqrt{\lambda}(\gamma_1 - \gamma_2)\int_{\partial\Omega} uv ds. \qquad (5.17)
$$

Proof. By Green's Identity, we have that:

$$\int_{\Omega} (-\Delta uv + \Delta vu)\, dx = \int_{\partial\Omega} \left(-\frac{\partial u}{\partial \eta} v + \frac{\partial v}{\partial \eta} u \right) ds. \tag{5.18}$$

Thus, we have

$$\begin{aligned}
\int_{\Omega} (-\Delta uv + \Delta vu) dx &= \int_{\Omega} [\lambda u(1 - u - b_1 v)v - \lambda r v(1 - v - b_2 u)u] dx \\
&= \int_{\Omega} [\lambda uv - \lambda u^2 v - \lambda b_1 uv^2 - \lambda ruv + \lambda ruv^2 + \lambda r b_2 u^2 v] dx \\
&= \lambda \int_{\Omega} uv[(1 - r) + (rb_2 - 1)u + (r - b_1)v] dx
\end{aligned} \tag{5.19}$$

and

$$\int_{\Omega} \left(-\frac{\partial u}{\partial \eta} v + \frac{\partial v}{\partial \eta} u \right) ds = \sqrt{\lambda}(\gamma_1 - \gamma_2) \int_{\partial\Omega} uv ds$$

as desired. □

Lemma 5.7. *Suppose that* $D(x) := 1 - r + (rb_2 - 1)u(x) + (r - b_1)v(x)$. *If* $r > 0, b_1, b_2 \geq 0, \gamma_1, \gamma_2 \geq 0$ *and* (u, v) *is a positive solution of* (1.18), *then the following hold:*

(A) if $b_1 \leq 1 \leq b_2$ *and* $\frac{b_1}{b_2} \leq r \leq 1$, *then* $D(x) \geq 0$.

(B) if $b_2 \leq 1 \leq b_1$ *and* $1 \leq r \leq \frac{b_1}{b_2}$, *then* $D(x) \leq 0$.

Proof. To establish the result, we consider the following cases.

Case i: Assume that $r \leq \min \left\{ b_1, \frac{1}{b_2} \right\}$ which implies that $rb_2 - 1 \leq 0$ and $r - b_1 \leq 0$. Since $u, v > 0; \Omega$, if $r \geq 1$, then we have that:

$$D(x) = 1 - r + (rb_2 - 1)u(x) + (r - b_1)v(x) \leq 1 - r \leq 0; \Omega. \tag{5.20}$$

Also, since $u, v \leq 1; \Omega$, if $r \geq \frac{b_1}{b_2}$, then we have that:

$$\begin{aligned} D(x) = 1 - r + (rb_2 - 1)u(x) + (r - b_1)v(x) &\geq 1 - r + rb_2 - 1 + r - b_1 \\ &= rb_2 - b_1 \\ &\geq 0; \Omega. \end{aligned} \tag{5.21}$$

Notice that for (5.20) to hold, it is necessary that $b_2 \leq 1 \leq b_1$ and for (5.21) to hold, that $b_1 \leq 1 \leq b_2$. Also, $D(x) < 0; \Omega$ in (5.20) ($D(x) > 0; \Omega$ in (5.21)) if at least one of the inequalities is strict.

Case ii: Assume that $b_1 \leq r \leq \frac{1}{b_2}$, which implies that $rb_2 - 1 \leq 0$ and $r - b_1 \geq 0$. Since $u > 0$ and $v \leq 1; \Omega$, if $b_1 \geq 1$, then we have that:

$$\begin{aligned} D(x) = 1 - r + (rb_2 - 1)u(x) + (r - b_1)v(x) &\leq 1 - r + r - b_1 \\ &\leq 1 - b_1 \\ &\leq 0; \Omega. \end{aligned} \tag{5.22}$$

Also, since $u \leq 1$ and $v > 0; \Omega$, if $b_2 \geq 1$, then we have that:

$$\begin{aligned} D(x) = 1 - r + (rb_2 - 1)u(x) + (r - b_1)v(x) &\geq 1 - r + rb_2 - 1 \\ &= r(b_2 - 1) \\ &\geq 0; \Omega. \end{aligned} \tag{5.23}$$

Again, notice that for (5.22) to hold, it is necessary that $b_2 \leq 1 \leq b_1$ and for (5.23) to hold, that $b_1 \leq 1 \leq b_2$. Also, $D(x) < 0; \Omega$ in (5.22) ($D(x) > 0; \Omega$ in (5.23)) if at least one of the inequalities is strict.

Case iii: Assume that $\frac{1}{b_2} \leq r \leq b_1$, which implies that $rb_2 - 1 \geq 0$ and $r - b_1 \leq 0$. Since $u > 0$ and $v \leq 1; \Omega$, if $b_1 \leq 1$, then we have that:

$$\begin{aligned} D(x) = 1 - r + (rb_2 - 1)u(x) + (r - b_1)v(x) &\geq 1 - r + r - b_1 \\ &= 1 - b_1 \\ &\geq 0; \Omega. \end{aligned} \tag{5.24}$$

Also, since $u \leq 1$ and $v > 0; \Omega$, if $b_2 \leq 1$, then we have that:

$$\begin{aligned} D(x) = 1 - r + (rb_2 - 1)u(x) + (r - b_1)v(x) &\leq 1 - r + rb_2 - 1 \\ &= r(b_2 - 1) \\ &\leq 0; \Omega. \end{aligned} \tag{5.25}$$

Again, notice that for (5.24) to hold, it is necessary that $b_1 \leq 1 \leq b_2$, and for (5.25) to hold, that $b_2 \leq 1 \leq b_1$. Also, $D(x) > 0; \Omega$ in (5.24) ($D(x) < 0; \Omega$ in (5.25)) if at least one of the inequalities is strict.

Case iv: Assume that $\max\left\{\frac{1}{b_2}, b_1\right\} \leq r \leq 1$, which implies that $rb_2 - 1 \geq 0$ and $r - b_1 \geq 0$. Since $u, v > 0; \Omega$, we have that:

$$\begin{aligned} D(x) = 1 - r + (rb_2 - 1)u(x) + (r - b_1)v(x) &\geq 1 - r \\ &\geq 0; \Omega. \end{aligned} \tag{5.26}$$

Also, since $u, v \leq 1; \Omega$, if $r \leq \frac{b_1}{b_2}$, then we have that:

$$\begin{aligned} D(x) = 1 - r + (rb_2 - 1)u(x) + (r - b_1)v(x) &\leq 1 - r + rb_2 - 1 + r - b_1 \\ &= rb_2 - b_1 \\ &\leq 0; \Omega. \end{aligned} \tag{5.27}$$

Again, notice that for (5.26) to hold, it is necessary that $b_1 \leq 1 \leq b_2$, and for (5.27) to hold, that $b_2 \leq 1 \leq b_1$. Also, $D(x) > 0; \Omega$ in (5.26) ($D(x) < 0; \Omega$ in (5.27)) if at least one of the inequalities is strict.

The result now follows for (A) from (5.21). If $\frac{1}{b_2} \leq b_1$, then the result for (A) follows from (5.24), and if $\frac{1}{b_2} > b_1$, then the result follows from (5.23), and (5.26). Also, for (B), the result follows from (5.20). If $\frac{1}{b_2} \leq b_1$, then the result for (B) follows from (5.25), and if $\frac{1}{b_2} > b_1$, then the result follows from (5.22) and (5.27). □

Lemma 5.8. *If $b_1, b_2 < 1$ and (u, v) is a positive solution of* (1.18), *then the following hold:*

(A) if $z(x)$ is a smooth function that satisfies

$$\begin{cases} -\Delta z = \lambda z(1 - u - v); \ \Omega \\ \frac{\partial z}{\partial \eta} + \sqrt{\lambda}\gamma_1 z = 0; \ \partial\Omega, \end{cases} \tag{5.28}$$

then $z(x) \equiv 0$.

(B) if $z(x)$ *is a smooth function that satisfies*

$$\begin{cases} -\Delta z = \lambda r z(1-u-v);\ \Omega \\ \frac{\partial z}{\partial \eta} + \sqrt{\lambda}\gamma_2 z = 0;\ \partial\Omega, \end{cases} \tag{5.29}$$

then $z(x) \equiv 0$.

Proof. We only provide a proof for (A) as the proof for (B) is similar. Note that when $\mu = 0, w = u$ is a solution of

$$\begin{cases} -\Delta w - \lambda w(1-u-b_1 v) = \mu w;\ \Omega \\ \frac{\partial w}{\partial \eta} + \sqrt{\lambda}\gamma_1 w = 0;\ \partial\Omega. \end{cases} \tag{5.30}$$

Since $u > 0; \Omega$, the principal eigenvalue μ_1 of (5.30) is zero. But, for any $\phi \neq 0$ smooth, we must have:

$$\mu_1 = 0 \leq \frac{\int_\Omega \left(|\nabla\phi|^2 - \lambda(1-u-b_1 v)\phi^2\right) dx + \int_{\partial\Omega} \sqrt{\lambda}\gamma_1 \phi^2 ds}{\int_\Omega \phi^2 dx}, \tag{5.31}$$

as can be seen from page 97 of [CC03]. But, we also have

$$\int_\Omega -\Delta z z dx = \int_\Omega -\frac{\partial z}{\partial \eta} z ds + \int_\Omega |\nabla z|^2 dx,$$

where

$$\int_{\Omega} -\Delta z z dx = \int_{\Omega} \lambda(1-u-v)z^2 dx$$

and

$$\int_{\partial\Omega} -\frac{\partial z}{\partial \eta} z ds = \int_{\partial\Omega} \sqrt{\lambda}\gamma_1 z^2 ds$$

implying that

$$\int_{\Omega} |\nabla z|^2 dx - \int_{\Omega} \lambda(1-u-v)z^2 dx + \int_{\partial\Omega} \sqrt{\lambda}\gamma_1 z^2 ds = 0.$$

Now, using (5.31) we have

$$\begin{aligned} 0 &= \int_{\Omega} |\nabla z|^2 dx - \int_{\Omega} \lambda(1-u-b_1 v)z^2 dx + \int_{\partial\Omega} \sqrt{\lambda}\gamma_1 z^2 ds + \int_{\Omega} \lambda(1-b_1)vz^2 dx \\ &\geq \int_{\Omega} \lambda(1-b_1)vz^2 dx \end{aligned}$$

implying that

$$\int_{\Omega} \lambda(1-b_1)vz^2 dx \leq 0.$$

But, this is a contraction since $\lambda > 0, b_1 < 1$, and $v > 0$. Hence, $z \equiv 0$ as desired. □

Lemma 5.9. *The principal eigenvalue,* $E_1(r, \gamma)$*, which is defined in* (1.20) *has the following properties for all* $r > 0$ *and* $\gamma \geq 0$ *(note that* $b = 0$ *throughout this result):*

(A) For fixed $\gamma > 0$

(i) $E_1(r,\gamma)$ *is a decreasing function of* r

(ii) $E_1(r,\gamma) \to 0$ *as* $r \to \infty$

(iii) $E_1(r,\gamma) \to \infty$ *as* $r \to 0^+$

(B) *For fixed* $r > 0$

(i) $E_1(r,\gamma)$ *is an increasing function of* γ

(ii) $E_1(r,\gamma) \to \frac{E_1^D}{r}$ *as* $\gamma \to \infty$

(iii) $E_1(r,\gamma) \to 0$ *as* $\gamma \to 0^+$

(C) $E_1(r,\gamma) = \frac{E_1(1,\gamma)}{r}$

(D) *Fix* $\gamma_1 > 0$ *and* $\gamma_2 \geq 0$ *and let* $r^* = \frac{E_1(1,\gamma_2)}{E_1(1,\gamma_1)}$. *Then*

(i) *if* $r < r^*$ *then* $E_1(1,\gamma_1) < E_1(r,\gamma_2)$

(ii) *if* $r = r^*$ *then* $E_1(1,\gamma_1) = E_1(r,\gamma_2)$

(iii) *if* $r > r^*$ *then* $E_1(1,\gamma_1) > E_1(r,\gamma_2)$

(iv) *if* $\gamma_1 > \gamma_2$ *then* $r^* < 1$

(v) *if* $\gamma_1 = \gamma_2$ *then* $r^* = 1$

(vi) *if* $\gamma_1 < \gamma_2$ *then* $r^* > 1$.

The proof of (A) - (C) can be found in [CGMS20] and (D) follows immediately from (C).

Now we prove our main theorems.

5.1 Proof of Theorem 1.8

Assume that (u, v) is a positive solution of (1.18) for a fixed $\lambda > 0$.

(A) First, assume that $\lambda \leq E_1(1, \gamma_1)$ which implies that $\sigma_1 \geq 0$ (see Theorem 5.2). Using Green's Identity and the eigenfunction corresponding to σ_1, we have that:

$$\int_\Omega (-\Delta u \phi_1 + \Delta \phi_1 u)\, dx = \int_\Omega \left(-\frac{\partial u}{\partial \eta}\phi_1 + \frac{\partial \phi_1}{\partial \eta} u \right) ds. \tag{5.32}$$

But, the right-hand-side of (5.32) is clearly equal to zero, and we also have:

$$\begin{aligned} \int_\Omega (-\Delta u \phi_1 + \Delta \phi_1 u)\, dx &= \int_\Omega [\lambda u \phi_1 (1 - u - b_1 v) - u(\sigma_1 \phi_1 + \lambda \phi_1)]\, dx \\ &= \int_\Omega (\lambda u \phi_1 - \lambda u^2 \phi_1 - \lambda b_1 u v \phi_1 - u \sigma_1 \phi_1 - \lambda \phi_1 u) dx \\ &= \int_\Omega -u \phi_1 (u + b_1 v + \sigma_1) dx \\ &< 0 \end{aligned} \tag{5.33}$$

since $u, v, \phi_1 > 0; \Omega$ and $\sigma_1 \geq 0$. This contradiction ensures that no positive solution of (1.18) exists when $\lambda \leq E_1(1, \gamma_1)$. An almost identical argument follows when $\lambda \leq E_1(r, \gamma_2)$.

(B) - (D) Note that these parts follow immediately from Lemmas 5.6 and 5.7. For example, we provide a proof of (C): Note that (A) implies that $\lambda > \max\{E_1(1, \gamma_1), E_1(r, \gamma_2)\}$. Now, assuming $\gamma_1 > \gamma_2$ ensures that the right-hand-side of (5.17) is strictly positive, whereas the left-hand-side of (5.17) is nonpositive from Lemma 5.7 when $b_2 \leq 1 \leq b_1$ and $1 \leq r \leq \frac{b_1}{b_2}$ (since $u, v > 0; \Omega$ and $\lambda > 0$). This contradiction implies that no positive solution of (1.18) exists when $b_2 \le 1 \le b_1$ and $1 \le r \le \frac{b_1}{b_2}$.

(E) Assume that $b_1 > 1$ and $b_2 < \frac{b_1 - 1}{b_1}$. Since we wish to prove nonexistence for large λ-values, it suffices to show nonexistence for $\lambda > \frac{E_1(r,\gamma_2)}{1-b_2}$. Using Green's Identity, we have:

$$\int_\Omega (-\Delta u W_{1,\gamma_1} + \Delta W_{1,\gamma_1} u)\, dx = \int_{\partial\Omega} \left(-\frac{\partial u}{\partial \eta} W_{1,\gamma_1} + \frac{\partial W_{1,\gamma_1}}{\partial \eta} u \right) ds. \tag{5.34}$$

But, the right-hand-side of (5.34) is clearly equal to zero and the left-hand-side becomes:

$$\begin{aligned} \int_\Omega (-\Delta u W_{1,\gamma_1} + \Delta W_{1,\gamma_1} u)\, dx &= \int_\Omega [\lambda u(1 - u - b_1 v) W_{1,\gamma_1} - \lambda W_{1,\gamma_1}(1 - W_{1,\gamma_1}) u] dx \\ &= \int_\Omega \lambda u W_{1,\gamma_1}[W_{1,\gamma_1} - (u + b_1 v)] dx \\ &< \int_\Omega \lambda u W_{1,\gamma_1}[W_{1,\gamma_1} - b_1 W_{r,\gamma_2,b_2}] dx \end{aligned} \tag{5.35}$$

since $u > 0; \Omega$ and $v \ge W_{r,\gamma_2,b_2}; \Omega$ (see proof of (D) in Theorem 1.9 and note that for $\lambda > \frac{E_1(r,\gamma_2)}{1-b_2}$, Theorem 5.2 ensures that W_{r,γ_2,b_2} exists). Also, Theorem 5.2 ensures that:

$$W_{1,\gamma_1} - b_1 W_{r,\gamma_2,b_2} \to 1 - b_1(1 - b_2) \text{ on all closed subsets of } \Omega \text{ as } \lambda \to \infty.$$

Since $b_1 > 1$ and $b_2 < \frac{b_1-1}{b_1}$, we have that $1 - b_1(1 - b_2) < 0$ and can choose $\lambda \gg 1$ such that $\int_\Omega \lambda u W_{1,\gamma_1}[W_{1,\gamma_1} - b_1 W_{r,\gamma_2,b_2}] dx < 0$ which is a contradiction.

(F) We omit this proof as it is almost identical to the one for (E).

(G) Here, we show that there exists $\delta(b_2) > 0$ such that (1.18) has no positive solution for $\lambda < E_1(r, \gamma_2) + \delta(b_2)$. If $\lambda \le E_1(1, \gamma_1)$, then from (A) (1.18) has no positive solution. Thus, we assume (u, v) is a positive solution of (1.18) for some $\lambda \in (E_1(1, \gamma_1), E_1(r, \gamma_2))$ which implies that $\sigma_2 > 0$. By Green's Identity, we obtain:

$$\int_{\Omega} (-\Delta v \phi_2 + \Delta \phi_2 v)\, dx = \int_{\partial\Omega} \left(-\frac{\partial v}{\partial \eta}\phi_2 + \frac{\partial \phi_2}{\partial \eta} v \right) ds, \tag{5.36}$$

and it is easy to see that the right-hand-side of (5.36) is zero. Now, we also have that:

$$\begin{aligned}
\int_{\Omega} (-\Delta v \phi_2 + \Delta \phi_2 v)\, dx &= \int_{\Omega} (\lambda r v(1 - v - b_2 u)\phi_2 - (\lambda r + \sigma_2)\phi_2 v)\, dx \\
&= \int_{\Omega} (-\lambda r - \sigma_2 + \lambda r - \lambda r v - \lambda r b_2 u)\phi_2 v dx \\
&= \int_{\Omega} (-\sigma_2 - \lambda r v - \lambda r b_2 u)\phi_2 v dx \\
&= \lambda r \int_{\Omega} \left(\frac{-\sigma_2}{\lambda r} - v - b_2 u \right) \phi_2 v dx && (5.37) \\
&\le \lambda r \int_{\Omega} \left(\frac{-\sigma_2}{\lambda r} - v - b_2 \min_{\overline{\Omega}}\{u\} \right) \phi_2 v dx \\
&\le \lambda r \int_{\Omega} \left(\frac{-\sigma_2}{\lambda r} - b_2 \min_{\overline{\Omega}}\{u\} \right) \phi_2 v dx && (5.38)
\end{aligned}$$

which gives rise to a contradiction since $\sigma_2 > 0$. Further, from (5.37), we have $0 \le \min_{\overline{\Omega}}\{u\} \left[\frac{-\sigma_2}{\lambda r \min_{\overline{\Omega}}\{u\}} - b_2 \right]$, and we note that $\sigma_2 \to 0$ when $\lambda \to E_1(r, \gamma_2)$ and $\sigma_2 < 0$ when $\lambda > E_1(r, \gamma_2)$. Since $b_2 > 0$, there exists a $\delta(b_2) > 0$ such that (1.18) has no positive solution for $\lambda \in [E_1(r, \gamma_2), E_1(r, \gamma_2) + \delta(b_2))$, and hence a positive solution does not exist for $\lambda < E_1(r, \gamma_2) + \delta(b_2)$. Furthermore, it is clear that a necessary condition for existence of a positive solution is $H(\lambda, r) = \frac{-\sigma_2}{\lambda r \min_{\overline{\Omega}}\{u\}} \ge b_2$, as desired.

(H) We omit this proof as it is almost identical to the one for (G). $\square$

5.2 Proof of Theorem 1.9

(A) Assume that $b_1, b_2 < 1$ and $\lambda > \max\left\{\frac{E_1(r,\gamma_2)}{1-b_2}, \frac{E_1(1,\gamma_1)}{1-b_1}\right\}$. We first prove existence of a positive solution of (1.18). Note that this implies $\sigma_1, \sigma_2 < 0$ ensuring that $W_{1,\gamma_1}, W_{r,\gamma_2}$ (the unique positive solution of (1.20) with $R = 1$ and $R = r$, respectively) both exist. Now consider $\sigma_3(W_{1,\gamma_1})$ with $W_{1,\gamma_1} \equiv 1$ and $\sigma_4(W_{r,\gamma_2})$ with $W_{r,\gamma_2} \equiv 1$, namely,

$$\begin{cases} -\Delta\phi_3 - \lambda r(1-b_2)\phi_3 = \sigma_3\phi_3; \ \Omega \\ \frac{\partial\phi_3}{\partial\eta} + \sqrt{\lambda}\gamma_2\phi_3 = 0; \ \partial\Omega \end{cases} \tag{5.39}$$

and

$$\begin{cases} -\Delta\phi_4 - \lambda r(1-b_1)\phi_4 = \sigma_4\phi_4; \ \Omega \\ \frac{\partial\phi_4}{\partial\eta} + \sqrt{\lambda}\gamma_1\phi_4 = 0; \ \partial\Omega. \end{cases} \tag{5.40}$$

By Lemma 5.5, we have that $\sigma_3(W_{1,\gamma_1}) < \sigma_3(1)$ and $\sigma_4(W_{r,\gamma_2}) < \sigma_4(1)$. Thus by Lemma 5.3 it suffices to show that $\sigma_3(1), \sigma_4(1) < 0$ in order to prove existence. Comparing (5.39) with (1.22), uniqueness of the principal eigenvalue ensures that

$$\sigma_3(1) + \lambda r(1-b_2) = E_1(R,\gamma)R$$
$$\gamma = \gamma_2,$$

or equivalently,

$$\sigma_3(1) = E_1(R,\gamma)R - \lambda r(1-b_2). \tag{5.41}$$

Taking $\sigma_3(1) = 0$, we see that $R = r(1 - b_2)$ and $\lambda = E_1(r(1 - b_2), \gamma_2) = \frac{E_1(r,\gamma_2)}{1-b_2}$, by Lemma 5.9. Also, using (5.41) we have that $\sigma_3(1) < 0$ for $\lambda > \frac{E_1(r,\gamma_2)}{1-b_2}$.

Similarly, comparing (5.40) with (1.22), uniqueness of the principal eigenvalue ensures that

$$\sigma_4(1) + \lambda(1 - b_1) = E_1(R, \gamma)R$$

$$\gamma = \gamma_1$$

or, equivalently,

$$\sigma_4(1) = E_1(R, \gamma)R - \lambda(1 - b_1). \tag{5.42}$$

Again, taking $\sigma_4(1) = 0$, we see that $R = (1 - b_1)$ and $\lambda = E_1((1 - b_1), \gamma_1) = \frac{E_1(1,\gamma_1)}{1-b_1}$, by Lemma 5.9. Using (5.42), we have that $\sigma_4(1) < 0$ for $\lambda > \frac{E_1(1,\gamma_1)}{1-b_1}$. Thus, for $\lambda > \max\left\{\frac{E_1(r,\gamma_2)}{1-b_2}, \frac{E_1(1,\gamma_1)}{1-b_1}\right\}$, Lemma 5.3 ensures existence of a positive solution of (1.18) with $(m\phi_4, m\phi_3) \leq (u, v) \leq (W_{1,\gamma_1}, W_{r,\gamma_2}); \overline{\Omega}$ for $m \approx 0$.

(i) Now assume (u, v) is any positive solution of (1.18) with $\lambda > \max\{E_1(r, \gamma_2), E_1(1, \gamma_1)\}$. Then (u, v) also satisfies:

$$\begin{cases} -\Delta u - \lambda u(1 - u) = -\lambda b_1 uv; \Omega \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda}\gamma_1 u = 0; \partial\Omega, \end{cases} \tag{5.43}$$

implying that u is a strict subsolution of (5.1). Since $Z \equiv M > 1$ is a supersolution of (5.1) and $u \leq M; \overline{\Omega}$, uniqueness of W_{1,γ_1} gives that $u \leq W_{1,\gamma_1}; \overline{\Omega}$. A similar argument gives that $v \leq W_{r,\gamma_2}; \overline{\Omega}$.

(ii) We assume (u, v) is any positive solution of (1.18) with $\lambda > \max\left\{\frac{E_1(1,\gamma_1)}{1-b_1}, \frac{E_1(r,\gamma_2)}{1-b_2}\right\}$, which implies that $W_{1,\gamma_1,b_1}, W_{r,\gamma_2,b_2}$ both exist. Now, since $v \leq W_{r,\gamma_2} \leq 1; \overline{\Omega}$, we have that (u, v) satisfies:

$$\begin{cases} -\Delta u - \lambda u(1 - u - b_1) \geq -\Delta u - \lambda u(1 - u - b_1 v) = 0; \Omega \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda}\gamma_1 u = 0; \partial\Omega \end{cases} \tag{5.44}$$

implying that u is a supersolution of (1.20) with $R = 1, b = b_1$ and $\gamma = \gamma_1$. Using the principal eigenfunction, ϕ_0, corresponding to σ_1 (which is negative by our choice of λ) gives that $\psi = m\phi_0$ is a subsolution of (1.20) with $R = 1, b = b_1$, and $\gamma = \gamma_1$ and satisfies $m\phi_0 < u; \overline{\Omega}$ by choosing $m \approx 0$. Uniqueness of W_{1,γ_1,b_1} (the positive solution of (1.20) with $R = 1, b = b_1$ and $\gamma = \gamma_1$) gives that $W_{1,\gamma_1,b_1} \leq u; \overline{\Omega}$. A similar argument shows that $W_{r,\gamma_2,b_2} \leq v; \overline{\Omega}$.

(iii) Finally, assume that $r = 1$ and $\gamma_1 = \gamma_2$. We will show that $\left(\frac{1-b_1}{1-b_1 b_2} W_{1,\gamma_1}, \frac{1-b_2}{1-b_1 b_2} W_{r,\gamma_2}\right)$ will satisfy (1.18). To that end, we see that:

$-\Delta u - \lambda u(1 - u - b_1 v)$

$$\begin{aligned} &= \frac{1-b_1}{1-b_1 b_2}\lambda W_{1,\gamma_1}(1 - W_{1,\gamma_1}) \\ &\qquad - \lambda\left(\frac{1-b_1}{1-b_1 b_2}\right) W_{1,\gamma_1}\left(1 - \frac{1-b_1}{1-b_1 b_2} W_{1,\gamma_1} - \frac{b_1(1-b_2)}{1-b_1 b_2} W_{1,\gamma_1}\right) \\ &= \frac{1-b_1}{1-b_1 b_2}\lambda W_{1,\gamma_1}\left[1 - W_{1,\gamma_1} - 1 + \frac{1-b_1}{1-b_1 b_2} W_{1,\gamma_1} + \frac{b_1(1-b_2)}{1-b_1 b_2} W_{1,\gamma_1}\right] \\ &\qquad + \frac{1-b_1}{1-b_1 b_2}\lambda W_{1,\gamma_1}^2\left[\frac{b_1 b_2 - 1 + 1 - b_1 + b_1 - b_1 b_2}{1-b_1 b_2}\right] \\ &= 0 \end{aligned} \tag{5.45}$$

and

$$\frac{\partial u}{\partial \eta} + \sqrt{\lambda}\gamma_1 u = -\left(\frac{1-b_1}{1-b_1b_2}\right) W_{1,\gamma_1}\sqrt{\lambda}\gamma_1 + \left(\frac{1-b_1}{1-b_1b_2}\right) W_{1,\gamma_1}\sqrt{\lambda}\gamma_1 = 0; \partial\Omega. \quad (5.46)$$

A similar argument holds for v. Theorem 1.10 gives uniqueness of the solution in this case.

(B) Assume that $b_1 = b_2 = 1, \gamma_1 = \gamma_2$, $r = 1$ and $\lambda > E_1(1, \gamma_1)$. Notice that $\sigma_1 < 0$ in this case ensuring existence of W_{1,γ_1}. Fix $s \in (0, 1)$, and let $(u, v) = (sW_{1,\gamma_1}, (1-s)W_{1,\gamma_1})$. We will first show that (u, v) is a solution of (1.18). To that end, we see that:

$$\begin{aligned} -\Delta u - \lambda u(1-u-v) &= -\Delta sW_{1,\gamma_1} - \lambda sW_{1,\gamma_1}(1 - sW_{1,\gamma_1} - (1-s)W_{1,\gamma_1}) \\ &= s[-\Delta W_{1,\gamma_1} - \lambda W_{1,\gamma_1}(1 - W_{1,\gamma_1})] \\ &= 0 \qquad (5.47) \end{aligned}$$

and

$$\begin{aligned} -\Delta v - \lambda v(1-v-u) &= -\Delta(1-s)W_{1,\gamma_1} - \lambda(1-s)W_{1,\gamma_1}(1 - (1-s)W_{1,\gamma_1} - sW_{1,\gamma_1}) \\ &= (1-s)[-\Delta W_{1,\gamma_1} - \lambda W_{1,\gamma_1}(1 - W_{1,\gamma_1})] \\ &= 0 \qquad (5.48) \end{aligned}$$

with

$$
\begin{aligned}
\frac{\partial u}{\partial \eta} + \sqrt{\lambda}\gamma_1 u &= \frac{\partial s W_{1,\gamma_1}}{\partial \eta} + \sqrt{\lambda}\gamma_1 s W_{1,\gamma_1} \\
&= s\left[\frac{\partial W_{1,\gamma_1}}{\partial \eta} + \sqrt{\lambda}\gamma_1 W_{1,\gamma_1}\right] \\
&= 0 \qquad (5.49)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial v}{\partial \eta} + \sqrt{\lambda}\gamma_1 v &= \frac{\partial (1-s) W_{1,\gamma_1}}{\partial \eta} + \sqrt{\lambda}\gamma_1 (1-s) W_{1,\gamma_1} \\
&= (1-s)\left[\frac{\partial W_{1,\gamma_1}}{\partial \eta} + \sqrt{\lambda}\gamma_1 W_{1,\gamma_1}\right] \\
&= 0. \qquad (5.50)
\end{aligned}
$$

Now, we will show that all positive solutions of (1.18) must have the form $(sW_{1,\gamma_1}, (1-s)W_{1,\gamma_1})$. Assume that (u, v) is a positive solution of (1.18). Following the same argument as in the proof of Lemma 5.8, the principal eigenvalue of (5.30) with $b_1 = 1$, must be $\mu_1 = 0$. But, both u and v satisfy (5.30), and since μ_1 is simple, we must have that $u = cv$ where $c > 0$. Substituting (u, v) into (1.18) yields:

$$
\begin{aligned}
-\Delta u - \lambda u(1-u-v) &= -\Delta u - \lambda u\left(1 - u - \frac{1}{c}u\right) \\
&= -\Delta u - \lambda u\left(1 - \left(1 + \frac{1}{c}u\right)\right) \qquad (5.51)
\end{aligned}
$$

and

$$
\begin{aligned}
-\Delta v - \lambda v(1-v-u) &= -\Delta v - \lambda v\left(1 - v - cv\right) \\
&= -\Delta v - \lambda v\left(1 - (1+c)u\right). \qquad (5.52)
\end{aligned}
$$

It is now easy to see that $u = \frac{c}{c+1} W_{1,\gamma_1}$ and $v = \frac{1}{1+c} W_{r,\gamma_2}$. Let $s = \frac{c}{c+1} \in (0,1)$ which gives that $1 - s = \frac{1}{1+c}$, as desired.

(C) In this case, we assume that $b_1 < 1 \leq b_2, \gamma_1 > 0$, and $r > r^*$ (note that if $\gamma_2 = 0$ then there is no restriction on r), for which Lemma 5.9 implies that $E_1(r,\gamma_2) < E_1(1,\gamma_1)$. Fix $b_2 \geq 1$. By Lemma 5.3, it suffices to show that $\sigma_3(W_{1,\gamma_1}), \sigma_4(W_{r,\gamma_2}) < 0$. Since $E_1(r,\gamma_2) < E_1(1,\gamma_1)$, we have that $W_{1,\gamma_1}(x, E_1(1,\gamma_1)) \equiv 0$ and $W_{r,\gamma_2}(x, E_1(1,\gamma_1)) > 0; \Omega$. This implies that there exists a $\lambda_2(b_2) > (\approx) E_1(1,\gamma_1)$ such that $b_2 W_{1,\gamma_1}(x,\lambda) < W_{r,\gamma_2}(x,\lambda); \Omega$ for $\lambda \in (E_1(1,\gamma_1), \lambda_2(b_2))$. Now, fix $\lambda_0 \in (E_1(1,\gamma_1), \lambda_2(b_2))$, and choose b_1 such that

$$
b_1 < n_1(\lambda_0) := \min_{\Omega}\{W_{1,\gamma_1}(x,\lambda_0)\}. \qquad (5.53)
$$

Since $W_{r,\gamma_2}(x,\lambda) < 1; \Omega$, this choice ensures that $b_1 < \frac{W_{1,\gamma_1}(x,\lambda)}{W_{r,\gamma_2}(x,\lambda)}; \Omega$ for $\lambda \in (\lambda_1(b_1,b_2), \lambda_2(b_2))$, where $\lambda_1(b_1,b_2) := \lambda_0 - \delta_1$ for some $\delta_1(b_1,b_2) > (\approx) 0$. Thus, for $\lambda \in (\lambda_1(b_1,b_2), \lambda_2(b_2))$ and $b_1 < n_1(\lambda_0)$, we must have

$$
\begin{aligned}
&b_2 W_{1,\gamma_1}(x,\lambda) - W_{r,\gamma_2}(x,\lambda) < 0; \Omega, \\
&b_1 W_{r,\gamma_2}(x,\lambda) - W_{1,\gamma_1}(x,\lambda) < 0; \Omega.
\end{aligned}
$$

Lemma 5.4 now gives that $\sigma_3(W_{1,\gamma_1}), \sigma_4(W_{r,\gamma_2}) < 0$ for $\lambda \in (\lambda_1(b_1, b_2), \lambda_2(b_2))$. The furthermore statement follows from the proof of (A)(i)-(ii) for the bounds on u and from Lemma 5.3 for the bounds on v, as desired.

(D) In this case, we assume that $b_2 < 1 \leq b_1, \gamma_2 > 0$ and $r < r^*$ (note that if $\gamma_1 = 0$ then there is no restriction on r), for which Lemma 5.9 implies that $E_1(1, \gamma_1) < E_1(r, \gamma_2)$. Fix $b_1 \geq 1$. By Lemma 5.3, it suffices to show that $\sigma_3(W_1, \gamma_1), \sigma_4(W_{r,\gamma_2}) < 0$. Since $E_1(1, \gamma_1) < E_1(r, \gamma_2)$, we have that $W_{r,\gamma_2}(x, E_1(r, \gamma_2)) \equiv 0$ and $W_{1,\gamma_1}(x, E_1(r, \gamma_2)) > 0; \Omega$. This implies that there exists a $\lambda_2(b_1) > (\approx) E_1(r, \gamma_2)$ such that $b_1 W_{r,\gamma_2}(x, \lambda) < W_{1,\gamma_1}(x, \lambda); \Omega$ for $\lambda \in (E_1(r, \gamma_2), \lambda_2(b_1))$. Now, fix $\lambda_0 \in (E_1(r, \gamma_2), \lambda_2(b_2))$, and choose b_2 such that

$$b_2 < n_2(\lambda_0) := \min_{\Omega}\{W_{r,\gamma_2}(x, \lambda_0)\}. \tag{5.54}$$

Since $W_{1,\gamma_1}(x, \lambda) < 1; \Omega$, this choice ensures that $b_2 < \frac{W_{r,\gamma_2}(x,\lambda)}{W_{1,\gamma_1}(x,\lambda)}; \Omega$ for $\lambda \in (\lambda_1(b_1, b_2), \lambda_2(b_2))$, where $\lambda_1(b_1, b_2) := \lambda_0 - \delta_2$ for some $\delta_2(b_1, b_2) > (\approx) 0$. Thus, for $\lambda \in (\lambda_1(b_1, b_2), \lambda_2(b_2))$ and $b_2 < n_2(\lambda_0)$, we must have

$$b_2 W_{1,\gamma_1}(x, \lambda) - W_{r,\gamma_2}(x, \lambda) < 0; \Omega,$$
$$b_1 W_{r,\gamma_2}(x, \lambda) - W_{1,\gamma_1}(x, \lambda) < 0; \Omega.$$

Lemma 5.4 now gives that $\sigma_3(W_{1,\gamma_1}), \sigma_4(W_{r,\gamma_2}) < 0$ for $\lambda \in (\lambda_1(b_1, b_2), \lambda_2(b_2))$. The furthermore statement follows from the proof of (A)(i) for the bounds on v and from

Lemma 5.3 for the bounds on u, as desired.

(E) In the case of $b_1, b_2 > 1$, the argument in (A)(ii) gives existence of at least one positive solution of the specified form. However, uniqueness is still open. □

5.3 Proof of Theorem 1.10

(A) We assume that $b_1, b_2 < 1, r = 1$ and $\gamma_1 = \gamma_2$. Now, suppose that (u, v) is any positive solution of (1.18), which we rewrite as:

$$\begin{cases} -\Delta u - \lambda u(1-u-v) - \lambda(1-b_1)uv = 0;\ \Omega \\ -\Delta v - \lambda v(1-v-u) - \lambda(1-b_2)uv = 0;\ \Omega \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda}\gamma_1 u = 0;\ \partial\Omega \\ \frac{\partial v}{\partial \eta} + \sqrt{\lambda}\gamma_1 v = 0;\ \partial\Omega. \end{cases} \tag{5.55}$$

Now, multiply the first and third equations in (5.55) by $(1-b_2)$ and the second and fourth equations by $(1-b_1)$ and subtract the second from the first and then the fourth from the third giving:

$$\begin{cases} -\Delta \psi - \lambda \psi(1-u-v) = 0;\ \Omega \\ \frac{\partial \psi}{\partial \eta} + \sqrt{\lambda}\gamma_1 \psi = 0;\ \partial\Omega, \end{cases} \tag{5.56}$$

where $\psi = (1-b_2)u-(1-b_1)v$. By Lemma 5.8, $\psi \equiv 0$ giving that $(1-b_2)u \equiv (1-b_1)v$. In other words, we have that $v = Ru$ and $R = \frac{1-b_2}{1-b_1}$. But, this gives

$$1 + Rb_1 = 1 + \frac{b_1(1-b_2)}{1-b_1} = \frac{1-b_1b_2}{1-b_1} \tag{5.57}$$

and, hence,

$$
\begin{aligned}
0 &= -\Delta u - \lambda u(1 - u - b_1 v) \\
&= -\Delta u - \lambda u(1 - (1 + Rb_1)u) \\
&= -\Delta u - \lambda u \left(1 - \frac{1 - b_1 b_2}{1 - b_1} u\right); \Omega.
\end{aligned} \tag{5.58}
$$

Thus, u satisfies

$$
\begin{cases}
-\Delta u - \lambda u \left(1 - \left(\frac{1-b_1 b_2}{1-b_1}\right) u\right) = 0; \ \Omega \\
\frac{\partial u}{\partial \eta} + \sqrt{\lambda}\gamma_1 u = 0; \ \partial\Omega.
\end{cases} \tag{5.59}
$$

From Theorem 5.2, it is now easy to see that $u = \frac{1-b_1}{1-b_1 b_2} W_{1,\gamma_1}$ and, since $v = Ru$, $v = \frac{1-b_2}{1-b_1 b_2} W_{1,\gamma_1}$. This fact combined with Theorem 1.9 (A) (ii) and (iii) gives the result.

(B) Here, we assume that $r > 0, \gamma_1, \gamma_2 > 0$ and $b_1, b_2 < 1$ with (u_1, v_1) and (u_2, v_2) both positive solutions of (1.18). Let $p = u_1 - u_2$ and $q = v_1 - v_2$. Then we must have

$-\Delta p$

$$
\begin{aligned}
&= \lambda u_1(1 - u_1 - b_1 v_1) - \lambda u_2(1 - u_2 - b_1 v_2) \\
&= \lambda u_1 - \lambda u_1^2 - \lambda b_1 u_1 v_1 - \lambda u_2 + \lambda u_2^2 + \lambda b_1 u_2 v_2 + \lambda u_1 u_2 + \lambda b_1 u_2 v_1 - \lambda u_1 u_2 - \lambda b_1 u_2 v_1 \\
&= \lambda(u_1 - u_2)(1 - u_1 - b_1 v_1) - \lambda u_2(u_1 - u_2) - \lambda b_1 u_2(v_1 - v_2) \\
&= \lambda p(1 - u_1 - b_1 v_1) - \lambda u_2 p - \lambda b_1 u_2 q; \Omega,
\end{aligned}
$$

and, similarly,

$$-\Delta q = \lambda r q(1 - v_2 - b_2 u_2) - \lambda r b_2 v p - \lambda r v_1 q; \Omega. \tag{5.60}$$

Also,

$$\frac{\partial p}{\partial \eta} + \sqrt{\lambda}\gamma_1 p = \frac{\partial u_1}{\partial \eta} - \frac{\partial u_2}{\partial \eta} + \sqrt{\lambda}\gamma_1(u_1 - u_2) = 0; \partial\Omega \tag{5.61}$$

and, similarly,

$$\frac{\partial q}{\partial \eta} + \sqrt{\lambda}\gamma_2 q = 0; \partial\Omega. \tag{5.62}$$

Thus, (p, q) satisfies

$$\begin{cases} -\Delta p - \lambda p(1 - u_1 - b_1 v_1) + \lambda u_2 p + \lambda b_1 u_2 q = 0;\ \Omega \\ -\Delta q - \lambda r q(1 - v_2 - b_2 u_2) + \lambda r b_2 v_1 p + \lambda r v_1 q = 0 \\ \frac{\partial p}{\partial \eta} + \sqrt{\lambda}\gamma_1 p = 0;\ \partial\Omega \\ \frac{\partial q}{\partial \eta} + \sqrt{\lambda}\gamma_1 q = 0;\ \partial\Omega. \end{cases} \tag{5.63}$$

From the proof of Lemma 5.8, if z is a smooth function that satisfies

$$\begin{cases} -\Delta z = \lambda z(1 - u - b_1 v);\ \Omega \\ \frac{\partial z}{\partial \eta} + \sqrt{\lambda}\gamma_1 z = 0;\ \partial\Omega \end{cases} \tag{5.64}$$

then z also satisfies

$$\int_{\Omega} |\nabla z|^2 dx - \int_{\Omega} \lambda(1 - u - b_1 v) z^2 dx + \int_{\Omega} \sqrt{\lambda}\gamma_1 z^2 ds \geq 0.$$

Similarly, if w satisfies

$$\begin{cases} -\Delta w = \lambda r w(1 - v - b_2 u); \ \Omega \\ \frac{\partial w}{\partial \eta} + \sqrt{\lambda}\gamma_2 w = 0; \ \partial\Omega, \end{cases} \tag{5.65}$$

then w also satisfies

$$\int_{\Omega} |\nabla w|^2 dx - \int_{\Omega} \lambda(1 - v - b_2 u) w^2 dx + \int_{\Omega} \sqrt{\lambda}\gamma_2 w^2 ds \geq 0.$$

Hence, the following hold:

$$\int_{\Omega} z[-\Delta z - \lambda z(1 - u_1 - b_1 v_1)] dx \geq 0 \tag{5.66}$$

$$\int_{\Omega} w[-\Delta w - \lambda r w(1 - v_2 - b_2 u_2)] dx \geq 0. \tag{5.67}$$

Now, we multiplying the first equation in (5.63) by p and the second by q and integrating both of them over Ω yields

$$\int_{\Omega} \left(p[-\Delta p - \lambda p(1 - u_1 - b_1 v_1)] + \lambda u_2 p^2 + \lambda b_1 u_2 pq\right) dx = 0 \tag{5.68}$$

$$\int_\Omega \left(q[-\Delta q - \lambda rq(1 - v_2 - b_2u_2)] + \lambda rb_2v_1pq + \lambda rv_1q^2\right) dx = 0. \tag{5.69}$$

Adding (5.68) to (5.69) gives

$$\int_\Omega \{p[-\Delta p - \lambda p(1 - u_1 - b_1v_1)] + q[-\Delta q - \lambda rq(1 - v_2 - b_2u_2)] \\ + \lambda u_2p^2 + \lambda b_1u_2pq + \lambda rb_2v_1pq + \lambda rv_1q^2\Big\} dx = 0. \tag{5.70}$$

Employing (5.66) and (5.67) we further obtain

$$\lambda \int_\Omega (u_2p^2 + (b_1u_2 + rb_2v_1)pq + rv_1q^2)dx \leq 0.$$

Let $Q_x(s,t) := u_2(x)s^2 + [b_1u_2(x) + rb_2v_1(x)]st + rv_1(x)t^2$. If $Q_x(s,t)$ is positive definite for all $x \in \Omega$, then $p, q \equiv 0$ proving uniqueness. To that end, if the following holds, then we are ensured the result:

$$(b_1u_2 + rb_2v_1)^2 - 4u_2rv_1 < 0, \tag{5.71}$$

or, equivalently,

$$4 > \frac{b_1^2u_2}{rv_1} + 2b_1b_2 + rb_2^2\frac{v_1}{u_1}; \Omega.$$

It is now clear that if (1.23) holds, then so does (5.71), giving the result. The final statement of the theorem follows immediately from the fact that both W_{1,γ_1} and W_{r,γ_2} are bounded above and below (and in this case, away from zero). Thus, taking

$b_1, b_2 \approx 0$ and $\lambda > \max\left\{\frac{E_1(1,\gamma_1)}{1-b_1}, \frac{E_1(r,\gamma_2)}{1-b_2}\right\}$, Theorem 1.8 and the previous argument together ensure existence of a unique positive solution for (1.18). $\square$

5.4 Proof of Theorem 1.11

Here, we assume that $r > 0, b_1, b_2 \geq 0, \gamma_1, \gamma_2 \geq 0$ and $\lambda > 0$ are such that $\sigma_1, \sigma_2 < 0$. We note that (A) and (B) are standard, omit their proofs, and direct the interested reader to, e.g., [Smi08]. In particular, the author in [Smi08] proves in Theorem 7.6.2 that if a positive solution, (u, v), of (1.18) is stable, then it is also asymptotically stable. (Even though Theorem 7.6.2 specifically addresses a quasi-monotone nondecreasing system, a change of variables as suggested in [Smi08] allows the theorem to apply to our quasimonotone nonincreasing system, see also [Pao92]). Also, note that (i)-(iii) of (C) follows immediately from our construction of sub- and supersolutions of (1.18) in Lemma 5.3 and Theorem 5.2, Theorem 5.5 in Chapter 10 of [Pao92].

To prove (iv) of (C), fix $\lambda > 0$ such that $\sigma_3, \sigma_4 < 0$ and assume that there exists a sequence of asymptotically stable positive solutions of (1.18), $\{(u_n, v_n)\}_{n=1}^{\infty}$, converging to $(0, W_{r,\gamma_2})$ as $n \to \infty$. Choose $M > 1$ such that for all $n > M$ we have

$$\frac{-\sigma_4}{\lambda} > |u_n - b_1(W_{r,\gamma_2} - v_n)|; \overline{\Omega}.$$

Thus, there exists an $\epsilon > 0$ such that

$$\frac{-\sigma_4}{\lambda} > \epsilon > |u_n - b_1(W_{r,\gamma_2} - v_n)|; \overline{\Omega}.$$

Now, we have that:

$$\begin{aligned} u_t &= \frac{1}{\lambda}\Delta u + u(1-u-b_1 v) \\ &= \frac{1}{\lambda}\Delta u + u(1-b_1W_{r,\gamma_2} - [u - b_1(W_{r,\gamma_2}+v)]) \\ &\geq \frac{1}{\lambda}\Delta u + u(1-b_1W_{r,\gamma_2} - \epsilon); t>0, x\in\Omega \end{aligned} \quad (5.72)$$

as long as $\epsilon > |u - b_1(W_{r,\gamma_2}+v)|$. Fix an $n > M$ and $u(0,x), v(0,x) > 0; \overline{\Omega}$ with $u(0,x) \approx 0$ and $v(0,x) \approx W_{r,\gamma_2}$ on $\overline{\Omega}$. There must exist a $K > 0$ such that $u(0,x) > K\phi_4(x); \overline{\Omega}$, where ϕ_4 is the eigenfunction corresponding to σ_4 chosen such that $\phi_4(x) > 0; \overline{\Omega}$ and $\|\phi_4\|_\infty = 1$. Also, we can choose $t_0 > 0$ such that

$$\frac{-\sigma_4}{\lambda} > \epsilon > |u(t,x) - b_1(W_{r,\gamma_2} - v(t,x)|; x \in \overline{\Omega}$$

for all $t > t_0$.

Define $\psi(t,x) = Ke^{(\frac{-\sigma_4}{\lambda}-\epsilon)t}\phi_4(x)$ and $h(x) = 1 - b_1W_{r,\gamma_2}$. For all $t > 0$, we have that:

$$\begin{aligned} \psi_t - \frac{1}{\lambda}\Delta\psi - (h(x)-\epsilon)\psi &= K\left(\frac{-\sigma_4}{\lambda} - \epsilon\right)e^{\left(\frac{-\sigma_4}{\lambda}-\epsilon\right)t}\phi_4(x) + \frac{K}{\lambda}e^{\left(\frac{-\sigma_4}{\lambda}-\epsilon\right)t}[\sigma_4 \\ &\quad + \lambda h(x)]\phi_4(x) - Ke^{\left(\frac{-\sigma_4}{\lambda}-\epsilon\right)t}[h(x)-\epsilon]\phi_4(x) \\ &= 0 \end{aligned} \quad (5.73)$$

and, clearly,

$$\frac{\partial \psi}{\partial \eta} + \sqrt{\lambda}\gamma_1 \psi = 0; \partial\Omega.$$

Thus, $u(t,x)$ is a supersolution and $\psi(t,x)$ is a solution of:

$$\begin{cases} W_t = \frac{1}{\lambda}\Delta W + (h(x) - \epsilon)W; t > 0, x \in \Omega \\ \qquad W(0,x) = K\phi_4(x); x \in \Omega \\ \frac{\partial W}{\partial \eta} + \sqrt{\lambda}\gamma_1 W = 0;\ t > 0, x \in \partial\Omega. \end{cases} \tag{5.74}$$

A standard argument now implies that $u(t,x) \geq \psi(t,x) = Ke^{\left(\frac{-\sigma_4}{\lambda} - \epsilon\right)}\phi_4(x); x \in \overline{\Omega}$ for $t > t_0$. But, our choice of ϵ implies that $\frac{-\sigma_4}{\lambda} - \epsilon > 0$ giving that $u(t,x)$ is unbounded as $t \to \infty$. This is a contradiction, and, hence, no such sequence can exist. An almost identical argument holds for the case that (u_n, v_n) converges to $(W_{1,\gamma_1}, 0)$ as $t \to \infty$ and is omitted. □

CHAPTER VI

COMPUTATIONALLY GENERATED BIFURCATION CURVES AND SOLUTIONS IN DIMENSION $N = 2$ FOR EXAMPLES IN FOCUS 4

6.1 Part 1

We have the system of equations to solve

$$\begin{cases} -\Delta u = \lambda f(u); \quad \Omega = (0,1) \times (0,1) \\ \frac{\partial u}{\partial n} + \sqrt{\lambda} u = 0; \quad \partial\Omega, \end{cases} \tag{6.1}$$

where

$$f(u) = \begin{cases} e^{\frac{cu}{c+u}} - 1; & u \leq k \\ \left[e^{\frac{\alpha u}{\alpha+u}} - e^{\frac{\alpha k}{\alpha+k}}\right] + \left[e^{\frac{ck}{c+k}} - 1\right]; & u > k. \end{cases} \tag{6.2}$$

Here $c = 2.5$ is a fixed number, $\alpha > 0$ and $k > 0$ are parameters.

We build a regular mesh of triangular finite elements on Ω, as pictured in Figure 25

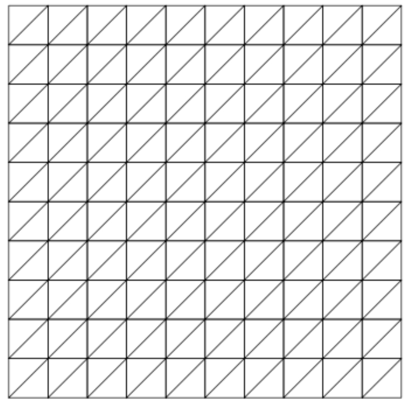


Figure 25. Regular mesh of triangular finite elements on a unit square.

Applying standard finite element procedures as described in Section 2.4, we get a system of nonlinear equations that can be written in the matrix form

$$AU + \sqrt{\lambda} C U - \lambda R(U) = 0, \tag{6.3}$$

where

$$A_{i,j} = \int_{\Omega} \nabla \varphi_i \nabla \varphi_j dx$$

is the stiffness matrix, $\{\varphi_i\}$ is the basis finite element functions,

$$C_{i,j} = \int_{\Omega} \varphi_j \varphi_i ds$$

is a matrix related to boundary condition, and

$$R_i(U) = \int_\Omega f\left(\sum_{k=1}^{n} u_k \varphi_k\right) \varphi_i \, dx$$

is the nonlinear functional.

$$U = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

is a vector of nodal values, n is the number of mesh nodes, The system (6.3) could be written as

$$\mathbf{F}(U) = \vec{0}, \tag{6.4}$$

where $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear vector-valued function with dimension n. The vector equation is solved by Newton iterations starting from some initial approximation U^0:

$$U^{n+1} = U^n - \left(\frac{\partial \mathbf{F}(U^n)}{\partial U}\right)^{-1} \mathbf{F}(U^n),$$

where the Jacobian matrix is calculated by differentiating the left side of (6.3) by u_j:

$$\frac{\partial \mathbf{F}_i}{\partial u_j} = A_{i,j} + \sqrt{\lambda} C_{i,j} - \lambda \int_\Omega \frac{\partial f\left(\sum_{k=1}^{n} u_k \varphi_k\right)}{\partial u_j} \varphi_i \, dx$$

with

$$\frac{\partial f\left(\sum_{k=1}^{n} u_k\varphi_k\right)}{\partial u_j} = \begin{cases} e^{\frac{c\ \mathcal{U}}{c+\mathcal{U}}}\ \frac{c^2\ \varphi_j}{(c+\mathcal{U})^2}; & \mathcal{U} \le k \\ \\ e^{\frac{\alpha\ \mathcal{U}}{\alpha+\mathcal{U}}}\ \frac{\alpha^2\ \varphi_j}{(\alpha+\mathcal{U})^2}; & \mathcal{U} > k, \end{cases} \tag{6.5}$$

where we have introduced the notation $\mathcal{U} = \sum_{k=1}^{n} u_k\varphi_k$.

6.1.1 Numerical results

We chose constant values for U^0 as the initial approximations:

$$U^0 = \begin{pmatrix} u_0 \\ u_0 \\ \vdots \\ u_0 \end{pmatrix}.$$

Our goal is to detect all branches of solutions U depending on the parameter λ. We perform calculations for a grid of values for u_0, λ. First, we use grid

$$\lambda = 1,\ 1.4,\ 1.8,\ ...,\ 9; \qquad \delta\lambda = 0.4,$$

$$u_0 = 0.5,\ 0.8,\ 1.1,\ ...,\ 20; \quad \delta u_0 = 0.3.$$

The computations are performed with 20×20 mesh subdivisions. For parameters $k = 3,\ \alpha = 3.1$ we obtain the following points:

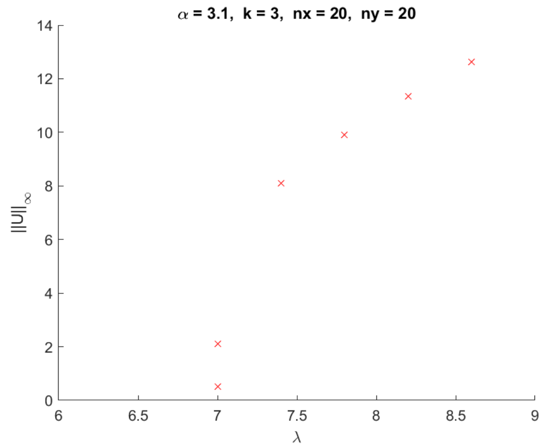


Figure 26. Starting points for continuation process.

Each point corresponds to "good" solution for which $U \geq 0$. Those points serve as starting points for continuation processes to get the branches of the bifurcation curve. During the simulation we observed that the Newton algorithm still works well despite the discontinuity of (6.5). To resolve the turning points for the branches during the continuation process, we used an adaptive refinement of the λ step when the derivative of the branch curve became big or changed quickly. Continuation yielded the branches presented in the Figure 27.

Observe that some part is seemingly absent. The reason is that our initial search corresponding to the red crosses did not yield results in the corresponding sub-region. Looking on the Figure 27, we tried to use a more dense grid for λ :

$$\lambda = 6.8, 6.9, 7.0, ..., 9; \qquad \delta\lambda = 0.1$$

which yielded the more complete diagram (see Figure 28).

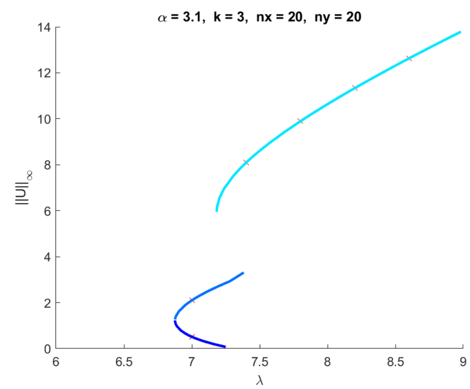


Figure 27. Discontinuous bifurcation curve when $\alpha = 3.1, k = 3$, and mesh 20×20.

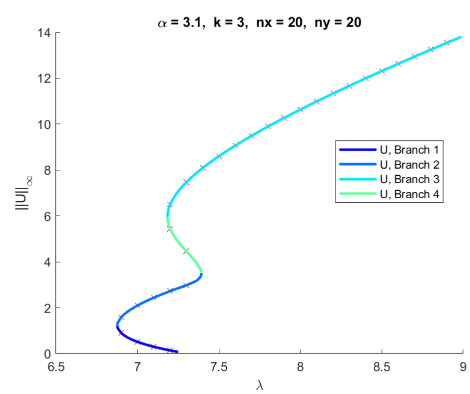


Figure 28. Approximate bifurcation curve when $\alpha = 3.1, k = 3$, and mesh 20×20.

Since the computations on a finer mesh for λ took much more time, we chose to use a non-uniform grid for λ :

$$\lambda = 6.0,\ 6.5,\ 7.0,\ 7.2,\ 7.3,\ 7.4,\ 7.5,\ 8.0,\ 8.5,\ 9$$

The λ values were inspired by Figure 27 and aim to resolve the segment $7.2 < \lambda < 7.4$. The process is a manual adjustment, and it is not suitable for automatic processing of arbitrary values of k, α.

We now provide more detailed figures for the example. The solution shape for branch 1 is pictured in Figure 29.

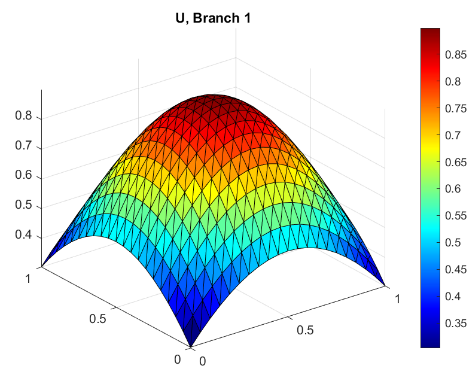


Figure 29. Solution shape for branch 1 when $\lambda = 7, k = 3, \alpha = 3.1$.

A bifurcation diagram for parameters $k = 5,\ \alpha = 5.5$ is given below (see Figure 30).

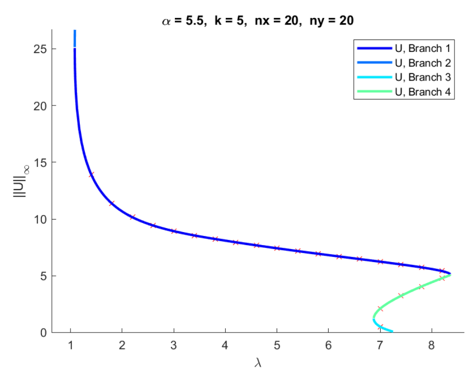


Figure 30. Approximate bifurcation diagram when $k = 5, \alpha = 5.5$, and mesh 20×20.

6.1.2 *Approximation consistency*

We know the finite element method should converge by [NW76]. Note that the asymptotic result should not change with change significantly due to changes in the mesh dimensions. Our tests show good stability results for different meshes (including non square e.g. 20×40). The result for parameters $k = 3$, $\alpha = 3.1$ and a 40×40 mesh are given in Figure 31

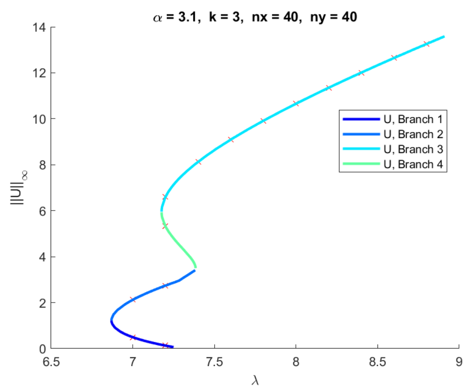


Figure 31. Approximate bifurcation diagram when $\alpha = 3.1, k = 3$, and mesh 40×40.

We see close agreement with the 20×20 results in Figure 28.

6.2 Part 2

We approximate the system of equations

$$\begin{cases} -\Delta u = \lambda u(1 - u - b_1 v); \quad \Omega \\ -\Delta v = \lambda r v(1 - v - b_2 u); \quad \Omega \\ \frac{\partial u}{\partial n} + \sqrt{\lambda}\gamma_1 u = 0; \quad \partial\Omega \\ \frac{\partial v}{\partial n} + \sqrt{\lambda}\gamma_2 v = 0; \quad \partial\Omega. \end{cases} \tag{6.6}$$

We divide $\Omega = (0,1) \times (0,1)$ into triangular finite elements and seek approximations of the form

$$\begin{aligned} u &= \sum_i^n u_i \varphi_i(x) \\ v &= \sum_i^n v_i \varphi_i(x) \end{aligned} \tag{6.7}$$

where n is the number of mesh nodes.

We again use a Galerkin formulation to project (6.6) onto a finite dimensional formulation for (u, v). The weak formulation of the first equation of the system (6.6) is:

$$\int_\Omega (-\Delta u)\, w dx = \int_\Omega \lambda u(1 - u - b_1 v)\, w dx.$$

Integrating the left side by parts yields:

$$\int_\Omega \nabla u\, \nabla w dx - \int_{\partial\Omega} \frac{\partial u}{\partial n}\, w ds = \int_\Omega \lambda u(1 - u - b_1 v)\, w dx.$$

Substituting the boundary condition for $\partial u / \partial n$ in (6.6) yields

$$\int_\Omega \nabla u\, \nabla w(x) dx + \sqrt{\lambda}\gamma_1 \int_{\partial\Omega} u\, w ds = \int_\Omega \lambda u(1 - u - b_1 v)\, w dx.$$

Substituting φ_i for weight function w and (6.7) for u we get n equations:

$$\begin{aligned}\sum_{j=1}^{n} u_j \int_\Omega \nabla\varphi_j \nabla\varphi_i dx + \sqrt{\lambda}\gamma_1 \sum_{j=1}^{n} u_j \int_{\partial\Omega} \varphi_j\varphi_i ds = \lambda \sum_{j=1}^{n} u_j \int_\Omega \varphi_j\varphi_i dx - \\ - \ \lambda \int_\Omega \Big[\big(\sum_{j=1}^{n} u_j\varphi_j\big)^2 + b_1 \big(\sum_{j=1}^{n} u_j\varphi_j\big)\big(\sum_{j=1}^{n} v_k\varphi_k\big)\Big] \varphi_i dx\end{aligned} \tag{6.8}$$

for all $i = 1, 2, ..., n$. The equations can be written in matrix form as

$$\begin{aligned}&A\,U + \sqrt{\lambda}\gamma_1 C\,U - \lambda B\,U \\ &\qquad + \lambda \begin{pmatrix} \vdots \\ \int_\Omega \Big[\big(\sum_{j=1}^{n} u_j\varphi_j\big)^2 + b_1 \big(\sum_{j=1}^{n} u_j\varphi_j\big)\big(\sum_{j=1}^{n} v_k\varphi_k\big)\Big] \varphi_i dx \\ \vdots \end{pmatrix} = \vec{0},\end{aligned} \tag{6.9}$$

where

$$U = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

is the vector of nodal values;

$$A = \int_\Omega \nabla\varphi_i \nabla\varphi_j dx$$

is the stiffness matrix,

$$B = \int_\Omega \varphi_i \varphi_j dx$$

is the mass matrix, and

$$C = \int_{\partial\Omega} \varphi_j \varphi_i ds$$

is the matrix related to the boundary conditions.

The last term in (6.9) contains the non-linear terms $u^2,\ u\,v$.

The equation for v could be stated similarly:

$$\begin{aligned} &A\,V + \sqrt{\lambda}\gamma_2 C\,V - \lambda r B\,V \\ &\qquad + \lambda r \begin{pmatrix} \vdots \\ \int_\Omega \left[\left(\sum_{j=1}^n v_j \varphi_j \right)^2 + b_2 \left(\sum_{j=1}^n u_j \varphi_j \right) \left(\sum_{j=1}^n v_k \varphi_k \right) \right] \varphi_i dx \\ \vdots \end{pmatrix} = \vec{0}. \end{aligned} \tag{6.10}$$

To formulate a united problem, we introduce the vector W that combines the vectors U and V:

$$W = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \\ v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

The combined system is

$$K\,W + \lambda \begin{pmatrix} \vdots \\ \int_\Omega \left[\left(\sum_{j=1}^n u_j \varphi_j\right)^2 + b_1 \left(\sum_{j=1}^n u_j \varphi_j\right)\left(\sum_{j=1}^n v_k \varphi_k\right) \right] \varphi_i dx \\ \vdots \\ r \int_\Omega \left[\left(\sum_{j=1}^n v_j \varphi_j\right)^2 + b_2 \left(\sum_{j=1}^n u_j \varphi_j\right)\left(\sum_{j=1}^n v_k \varphi_k\right) \right] \varphi_i dx \\ \vdots \end{pmatrix} = 0 \qquad (6.11)$$

where K is a $2n \times 2n$ matrix that can be written via $n \times n$ blocks:

$$K = \begin{pmatrix} A + \sqrt{\lambda}\gamma_1 C - \lambda B & 0 \\ & \\ 0 & A + \sqrt{\lambda}\gamma_2 C - \lambda r B \end{pmatrix}.$$

The system (6.11) can be written as

$$\mathbf{F}(W) = \vec{0},$$

where $\mathbf{F}(W)$ is a non linear vector valued function with dimension $2n$. The vector equation is solved by Newton iterations starting from some initial approximation W^0:

$$W^{n+1} = W^n - \left(\frac{\partial \mathbf{F}(W^n)}{\partial W}\right)^{-1} \mathbf{F}(W^n).$$

The Jacobian matrix is calculated by differentiating the left side of (6.11) by w_j:

$$\frac{\partial \mathbf{F}(W^n)}{\partial W} = \frac{\partial \mathbf{F}_i}{\partial w_j} = K + \lambda \begin{pmatrix} \int_\Omega \left(\sum_{k=1}^{n} {}_k(2u_k + b_1 v_k)\varphi_k \right) \varphi_i \varphi_j dx & b_1 \int_\Omega \left(\sum_{k=1}^{n} {}_k u_k \varphi_k \right) \varphi_i \varphi_j dx \\ rb_2 \int_\Omega \left(\sum_{k=1}^{n} {}_k v_k \varphi_k \right) \varphi_i \varphi_j dx & r \int_\Omega \left(\sum_{k=1}^{n} {}_k(2v_k + b_2 u_k)\varphi_k \right) \varphi_i \varphi_j dx \end{pmatrix}. \tag{6.12}$$

We choose constant values for $U^0 = U_0$ and $V^0 = V_0$ as initial approximations so that

$$W^0 = \begin{pmatrix} U_0 \\ U_0 \\ \vdots \\ U_0 \\ V_0 \\ V_0 \\ \vdots \\ V_0 \end{pmatrix}$$

To detect as many branches of solutions as possible we performed calculations for some grid of $U_0,\ V_0$ values ($U_0, V_0 = 0.2 - 4.0$ with step 0.2) for several values of λ $(20, 35, 50)$. Then, for each detected branch starting point U_0, V_0, λ is chosen and the branch is calculated via continuation process with step $d\lambda = 0.5$ in both directions (decreasing and increasing λ) from the starting point. Approximation consistency is verified as in Subsection 6.1.2 [NW76]. Results for some set of parameters γ_1, γ_2, r, b_1, b_2 are presented in Figures 32-37.

Blue and red curves represent the bifurcation curves corresponding the independent u and v solutions respectively. The bifurcation curves of the coupled solutions are represented by green and purple curves where green corresponds to the u component and purple corresponds to the v component.

The diagrams show the exact results when the dimension $N = 2$ and domain $\Omega = (0, 1) \times (0, 1)$ supporting the results obtained in Focus 3 analytically.

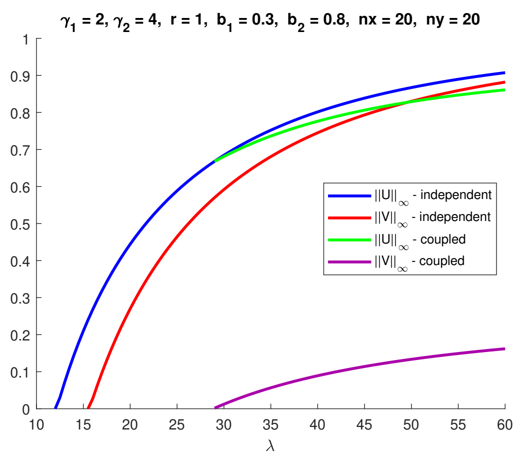


Figure 32. Approximate bifurcation curves for the positive solutions of (6.6) when $\gamma_1 = 2, \gamma_2 = 4, r = 1, b_1 = 0.3$ & $b_2 = 0.8$.

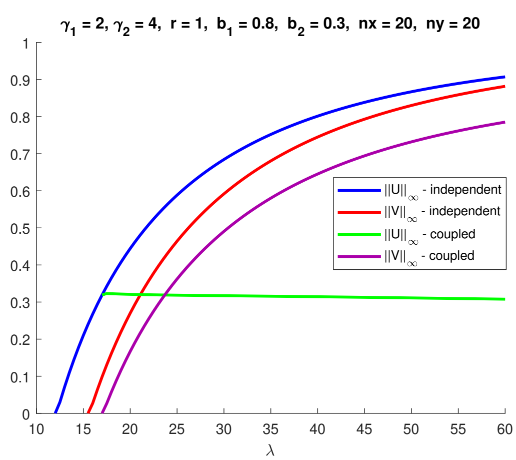


Figure 33. Approximate bifurcation curves for the positive solutions of (6.6) when $\gamma_1 = 2, \gamma_2 = 4, r = 1, b_1 = 0.8$ & $b_2 = 0.3$.

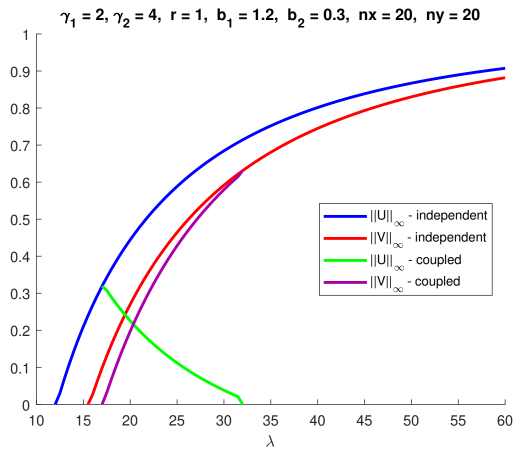


Figure 34. Approximate bifurcation curves for the positive solutions of (6.6) when $\gamma_1 = 2, \gamma_2 = 4, r = 1, b_1 = 1.2$ & $b_2 = 0.3$.

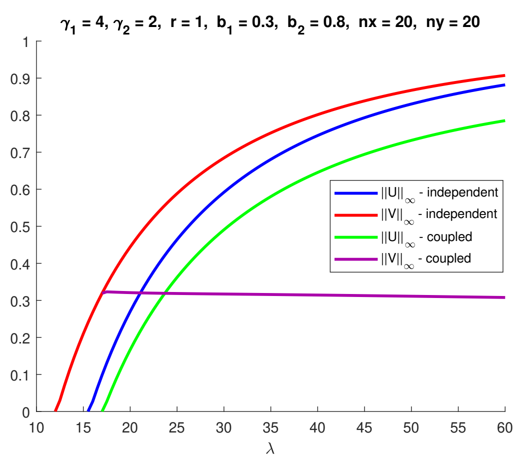


Figure 35. Approximate bifurcation curves for the positive solutions of (6.6) when $\gamma_1 = 4, \gamma_2 = 2, r = 1, b_1 = 0.3$ & $b_2 = 0.8$.

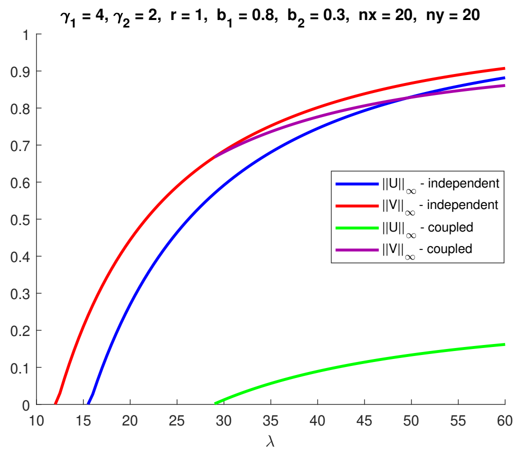


Figure 36. Approximate bifurcation curves for the positive solutions of (6.6) when $\gamma_1 = 4, \gamma_2 = 2, r = 1, b_1 = 0.8$ & $b_2 = 0.3$.

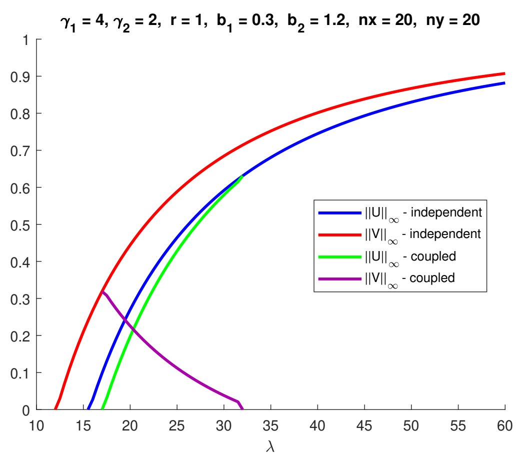


Figure 37. Approximate bifurcation curves for the positive solutions of (6.6) when $\gamma_1 = 4, \gamma_2 = 2, r = 1, b_1 = 0.3$ & $b_2 = 1.2$.

CHAPTER VII

CONCLUSIONS AND FUTURE DIRECTIONS

7.1 Conclusions

In this dissertation, we analyze positive solutions for classes of steady state nonlinear reaction diffusion equations and systems. First, we establish the occurrence of a Σ-shaped bifurcation curve for certain classes of reaction terms. Then we extended the study to a coupled system. Next, we analyze a diffusive Lotka-Volterra competition model with two species in fragmented patches. We analyze the minimum patch size as well as the maximum patch size for the existence of non trivial coupled solutions as the competition rates vary. Finally, we use the finite element method to obtain the bifurcation diagrams when $N = 2$ for an example in Focus 1 and for the model in Focus 3.

7.2 Future Directions

(1) Explore the uniqueness of the positive solution of the problem in Focus 1 for $\lambda \gg 1$.

(2) Explore $\Sigma-$shaped bifurcation curves for the positive solutions for problems with nonlinear boundary condition, namely, for the systems of the form:

$$\begin{cases} -\Delta u = \lambda f_1(v);\ \Omega \\ -\Delta v = \lambda f_2(u);\ \Omega \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda} g(u,v)u = 0;\ \partial\Omega \\ \frac{\partial v}{\partial \eta} + \sqrt{\lambda} h(u,v)v = 0;\ \partial\Omega \end{cases}$$

where $\lambda > 0$, f_1, f_2 are continuous increasing functions such that $f_1(0) = 0 = f_2(0)$, and $\lim\limits_{s\to\infty} \frac{f_1(Mf_2(s))}{s} = 0$ for all $M > 0$ (combined sublinearity), and $g, h \in C^1([0,\infty) \times [0,\infty), (0,\infty))$.

(3) Extend the study in Focus 3 when the species interact at the boundary as well, namely, study the systems of the form:

$$\begin{cases} -\Delta u = \lambda u(1 - u - b_1 v);\ \Omega \\ -\Delta v = \lambda r v(1 - v - b_2 u);\ \Omega \\ \frac{\partial u}{\partial \eta} + \sqrt{\lambda} g(u,v) u = 0;\ \partial\Omega \\ \frac{\partial v}{\partial \eta} + \sqrt{\lambda} h(u,v) v = 0;\ \partial\Omega \end{cases}$$

$\lambda > 0$,$\gamma_1, \gamma_2 > 0$, $r > 0$, $b_1, b_2 \geq 0$, and $g, h \in C^1([0,\infty), (0,\infty))$.

(4) Explore the study in Focus 4 considering non-convex domains such as L shaped domains in $\mathbb{R}^2$.

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