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New *p*-adic hypergeometric functions and syntomic regulators

par Masanori ASAKURA

Résumé. Nous introduisons un nouveau type de fonctions hypergéométriques p-adiques, que nous appelons fonctions hypergéométriques p-adiques de type logarithmique. Le premier résultat principal de cet article est la preuve des relations de congruence similaires à celles de Dwork. Le deuxième résultat principal est que les valeurs spéciales de nos nouvelles fonctions apparaissent dans le calcul des régulateurs syntomiques pour les courbes hypergéométriques, courbes de Fermat et certaines courbes elliptiques. D'après la conjecture de Beilinson p-adique de Perrin-Riou, on s'attend à ce qu'elles soient liées aux valeurs spéciales des fonctions L p-adiques. Nous en donnons un exemple.

Abstract. We introduce a new type of p-adic hypergeometric functions, which we call the p-adic hypergeometric functions of logarithmic type. The first main result is to prove the congruence relations that are similar to Dwork's. The second main result is that the special values of our new functions appear in the syntomic regulators for hypergeometric curves, Fermat curves and some elliptic curves. According to the p-adic Beilinson conjecture by Perrin-Riou, they are expected to be related with the special values of p-adic L-functions. We provide one example for this.

1. Introduction

Let $s \geq 1$ be an integer. For a s-tuple $\underline{a} = (a_1, \ldots, a_s) \in \mathbb{Z}_p^s$ of p-adic integers, let

$$F_{\underline{a}}(t) = {}_sF_{s-1}\left(\begin{matrix} a_1, \ldots, a_s \\ 1, \ldots, 1 \end{matrix} : t\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n}{n!} \cdots \frac{(a_s)_n}{n!} t^n$$

be the hypergeometric power series where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ denotes the Pochhammer symbol. In his seminal paper [12], B. Dwork discovered that certain ratios of the hypergeometric power series are the uniform limit of rational functions. We call his functions *Dwork's p-adic*

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hypergeometric functions. Let α' denote the Dwork prime, which is defined to be $(\alpha+l)/p$ where $l \in \{0,1,\dots,p-1\}$ is the unique integer such that $\alpha+l \equiv 0 \bmod p$. The i-th Dwork prime $a^{(i)}$ is defined by $a^{(i+1)} = (a^{(i)})'$ and $a^{(0)} = a$. Write $\underline{a}' = (a_1', \dots, a_s')$ and $\underline{a}^{(i)} = (a_1^{(i)}, \dots, a_s^{(i)})$. Then Dwork's p-adic hypergeometric function is defined to be

$$\mathscr{F}_{\underline{a}}^{\mathrm{Dw}}(t) = F_{\underline{a}}(t)/F_{\underline{a}'}(t^p). \tag{1.1}$$

This is a convergent function in the sense of Krasner. More precisely Dwork proved the congruence relations ([12, p. 41, Thm. 3])

$$\mathscr{F}_{\underline{a}}^{\mathrm{Dw}}(t) \equiv \frac{[F_{\underline{a}}(t)]_{<p^n}}{[F_{\underline{a}'}(t^p)]_{<p^n}} \mod p^n\mathbb{Z}_p[\![t]\!] \tag{1.2}$$

where for a power series $f(t) = \sum c_n t^n$, we write $[f(t)]_{<m} := \sum_{n<m} c_n t^n$ the truncated polynomial. This implies that $\mathscr{F}_{\underline{a}}^{\mathrm{Dw}}(t)$ is a convergent function. More precisely, for $f(t) \in \mathbb{Z}_p[t]$, let $\overline{f(t)} \in \mathbb{F}_p[t]$ denote the reduction modulo p. Let $I \subset \mathbb{Z}_{\geq 0}$ be a finite subset such that $\{\overline{[F_{\underline{a}^{(i)}}(t)]_{<p}}\}_{i\in\mathbb{Z}_{\geq 0}} = \{\overline{[F_{\underline{a}^{(i)}}(t)]_{<p}}\}_{i\in I}$ as sets. Put $h(t) = \prod_{i\in I}[F_{\underline{a}^{(i)}}(t)]_{<p}$. Then (1.2) implies

$$\mathscr{F}_{\underline{a}}^{\mathrm{Dw}}(t) \in \mathbb{Z}_p\langle t, h(t)^{-1}\rangle := \varprojlim_{n\geq 1}(\mathbb{Z}_p/p^n\mathbb{Z}_p[t, h(t)^{-1}]),$$

and hence that $\mathscr{F}_{\underline{a}}^{\mathrm{Dw}}(t)$ is a convergent function on a domain $\{z \in \mathbb{C}_p \mid |h(z)|_p = 1\}$.

Dwork showed a geometric aspect of his p-adic hypergeometric functions. Let E be the elliptic curve over $\mathbb{F}_p$ defined by a Weierstrass equation $y^2 = x(1-x)(1-ax)$ with $a \in \mathbb{F}_p \setminus \{0,1\}$. Suppose that E is ordinary, which means that $p \nmid a_p$ where $T^2 - a_pT + p$ is the characteristic polynomial of the Frobenius on E. Let α_E be the root of $T^2 - a_pT + p$ in $\mathbb{Z}_p$ such that $|\alpha_E|_p = 1$, which is often referred to as the unit root. Then Dwork proved a formula

$$\alpha_E = (-1)^{\frac{p-1}{2}}\mathscr{F}_{\frac{1}{2},\frac{1}{2}}^{\mathrm{Dw}}(\widehat{a})$$

where $\widehat{a} \in \mathbb{Z}_p^\times$ is the Teichmüller lift of $a \in \mathbb{F}_p^\times$. This is now called Dwork's unit root formula (cf. [24, §7])

In this paper, we introduce new p-adic hypergeometric functions, which we call the *p-adic hypergeometric functions of logarithmic type*. We shall first introduce the p-adic polygamma functions $\psi_p^{(r)}(z)$ in Section 2, which are slight modifications of Diamond's p-adic polygamma functions [11]. Let $W = W(\overline{\mathbb{F}}_p)$ be the Witt ring of $\overline{\mathbb{F}}_p$, and $K = \operatorname{Frac} W$ the fractional field. Let σ be a p-th Frobenius on $W[\![t]\!]$ given by $\sigma(t) = ct^p$ with $c \in 1 + pW$. Let $\log : \mathbb{C}_p^\times \to \mathbb{C}_p$ be the Iwasawa logarithmic function. Let

$$G_{\underline{a}}(t) := \psi_p(a_1) + \cdots + \psi_p(a_s) + s\gamma_p - p^{-1}\log(c) + \int_0^t (F_{\underline{a}}(t) - F_{\underline{a}'}(t^\sigma))\frac{\mathrm{d}t}{t}$$

be a power series where $\int_0^t(\,\cdot\,)\frac{dt}{t}$ means the operator which sends $\sum_{n\geq 1} a_n t^n$ to $\sum_{n\geq 1} \frac{a_n}{n} t^n$. It is not hard to show $G_{\underline{a}}(t) \in W[\![t]\!]$ (Lemma 3.2). Then our new function is defined to be a ratio

$$\mathscr{F}_{\underline{a}}^{(\sigma)}(t) := G_{\underline{a}}(t)/F_{\underline{a}}(t).$$

Notice that $\mathscr{F}_{\underline{a}}^{(\sigma)}(t)$ is also p-adically continuous with respect to $\underline{a}$. In case $a_1 = \cdots = a_s = c = 1$, one has $\mathscr{F}_{\underline{a}}^{(\sigma)}(t) = (1-t)\ln_1^{(p)}(t)$ the p-adic logarithm. In this sense, we can regard $\mathscr{F}_{\underline{a}}^{(\sigma)}(t)$ as a deformation of the p-adic logarithm.

The first main result of this paper is the congruence relations for $\mathscr{F}_{\underline{a}}^{(\sigma)}(t)$ that are similar to Dwork's.

Theorem 1.1 (Theorem 3.3). *Suppose that $a_i \notin \mathbb{Z}_{\leq 0}$ for all i. Then*

$$\mathscr{F}_{\underline{a}}^{(\sigma)}(t) \equiv \frac{[G_{\underline{a}}(t)]_{<p^n}}{[F_{\underline{a}}(t)]_{<p^n}} \mod p^n W[\![t]\!]$$

if $c \in 1 + 2pW$. If $p = 2$ and $c \in 1 + 2W$, then the congruence holds modulo p^{n-1}.

Thanks to this, $\mathscr{F}_{\underline{a}}^{(\sigma)}(t)$ is a convergent function,

$$\mathscr{F}_{\underline{a}}^{(\sigma)}(t) \in \mathbb{Z}_p\langle t, h(t)^{-1}\rangle$$

and then the special value at $t=\alpha$ is defined for $\alpha \in \mathbb{C}_p$ such that $|h(\alpha)|_p = 1$.

The second main result is to give a geometric aspect of our $\mathscr{F}_{\underline{a}}^{(\sigma)}(t)$, which concerns with the *syntomic regulator map*. Let X be a smooth variety over W. Let $H^\bullet_{\mathrm{syn}}(X, \mathbb{Q}_p(j))$ be the rigid syntomic cohomology groups by Besser [6] (see also [21, 1B]), which agree with the syntomic cohomology groups by Fontaine–Messing [14] (see also [16]) when X is projective. Let $K_i(X)$ be Quillen's algebraic K-groups. Then there is the syntomic regulator map

$$\mathrm{reg}^{i,j}_{\mathrm{syn}} : K_i(X) \longrightarrow H^{2j-i}_{\mathrm{syn}}(X, \mathbb{Q}_p(j))$$

for each $i, j \geq 0$ ([6, Thm. 7.5], [21, Thm. A]). We shall concern ourselves with only $\mathrm{reg}^{2,2}_{\mathrm{syn}}$, which we abbreviate to $\mathrm{reg}_{\mathrm{syn}}$ in this paper. Note that there is the natural isomorphism $H^2_{\mathrm{syn}}(X, \mathbb{Q}_p(2)) \cong H^1_{\mathrm{dR}}(X_K/K)$ where $K = \mathrm{Frac}\, W$ is the fractional field and $X_K := X \times_W K$. Our second main result is to relate $\mathscr{F}_{a,b}^{(\sigma)}(t)$ with the syntomic regulator of a certain element of K_2 of a *hypergeometric curve*, which is defined in the following way. Let $N, M \geq 2$ be integers, and p a prime such that $p \nmid NM$. Let $\alpha \in W$ satisfy that $\alpha \not\equiv 0, 1 \mod p$. Then we define a hypergeometric curve X_α to be a projective smooth scheme over W defined by a bihomogeneous equation

$$(X_0^N - X_1^N)(Y_0^M - Y_1^M) = \alpha X_0^N Y_0^M$$

in $\mathbb{P}^1_W(X_0, X_1) \times \mathbb{P}^1_W(Y_0, Y_1)$ (Section 4.1). We put $x = X_1/X_0$ and $y = Y_1/Y_0$.

Theorem 1.2 (Corollary 4.10)**.** *Suppose $p > \max(N, M)$. Let*

$$\xi = \left\{ \frac{x-1}{x-\nu_1}, \frac{y-1}{y-\nu_2} \right\} \in K_2(X_\alpha) \otimes \mathbb{Q}$$

for $(\nu_1, \nu_2) \in \mu_N(K) \times \mu_M(K)$ where $\mu_m(K)$ denotes the group of m-th roots of unity in K (cf. (4.24)*). Let*

$$Q : H^1_{\mathrm{dR}}(X_{\alpha,K}/K) \otimes H^1_{\mathrm{dR}}(X_{\alpha,K}/K) \longrightarrow H^2_{\mathrm{dR}}(X_{\alpha,K}/K) \cong K$$

be the cup-product pairing. Suppose that $h(\alpha) \not\equiv 0 \mod p$ where $h(t)$ is as above. For a pair of integers (i, j) such that $0 < i < N$ and $0 < j < M$, we put $\omega_{i,j} := N x^{i-1} y^{j-M}/(1 - x^N)\,\mathrm{d}x$ a regular 1-form (cf. (4.4)*), and $e^{\mathrm{unit}}_{i,j}$ the unit root vectors which are explicitly given in Theorem 4.7. Then we have*

$$Q(\mathrm{reg}_{\mathrm{syn}}(\xi), e^{\mathrm{unit}}_{N-i,M-j}) = -\frac{(1-\nu_1^{-i})(1-\nu_2^{-j})}{NM} \mathscr{F}^{(\sigma_\alpha)}_{a_i,b_j}(\alpha) Q(\omega_{i,j}, e^{\mathrm{unit}}_{N-i,M-j}).$$

For some elliptic curves, we also have similar results to the above (see Sections 4.6–4.7). For the proof of Theorem 1.2, we employ the main result in [4] as a fundamental tool to compute the syntomic regulators. We share a part of the technique in the proof of [4, Thm. 4.8] (see also Remark 4.23 below). A new ingredient is our function $\mathscr{F}^{(\sigma)}_{a,b}(t)$ (indeed Theorem 1.1 plays a key role in the proof of Theorem 1.2). This paper focuses on the cup-product of the regulator with a generator of the unit root subspace, that appears in the p-adic Beilinson conjecture by Perrin-Riou.

A more striking application of our new function $\mathscr{F}^{(\sigma)}_{\underline{a}}(t)$ is that one can describe the syntomic regulators of the Ross symbols in K_2 of the Fermat curves.

Theorem 1.3 (Theorem 4.11)**.** *Suppose that $N|(p-1)$ and $M|(p-1)$. Let F be the Fermat curve over K defined by an affine equation $z^N + w^M = 1$. Let $\{1-z, 1-w\} \in K_2(F)$ be the Ross symbol* [26]*. Let*

$$\mathrm{reg}_{\mathrm{syn}}(\{1-z, 1-w\}) = \sum_{(i,j)\in I} A_{i,j} M^{-1} z^{i-1} w^{j-M} \mathrm{d}z \in H^1_{\mathrm{dR}}(F/K)$$

where the notation be as in the beginning of Section 4.5. Then

$$A_{i,j} = \mathscr{F}^{(\sigma)}_{a_i,b_j}(1)$$

for (i, j) such that $i/N + j/M < 1$.

As long as the author knows, this is the first explicit description of the Ross symbol in p-adic cohomology.

In the last section, we discuss the p-adic Beilinson conjecture by Perrin-Riou [23, 4.2.2] (see also [10, Conj. 2.7]) for K_2 of elliptic curves. Let E be an elliptic curve over $\mathbb{Q}$. Let p be a prime at which E has a good ordinary reduction. Let $\alpha_{E,p}$ be the unit root of the reduction $\overline{E}$ at p, and $e_{\mathrm{unit}} \in H^1_{\mathrm{dR}}(E/\mathbb{Q}_p)$ the eigenvector with eigenvalue $\alpha_{E,p}$. Let $\omega_E \in \Gamma(E, \Omega^1_{E/\mathbb{Q}})$ be a regular 1-form. Let $L_p(E,\chi,s)$ be the p-adic L-function defined by Mazur and Swinnerton–Dyer [20]. Then as a consequence of the p-adic Beilinson conjecture for elliptic curves, one can expect that there is an element $\xi \in K_2(E)$ which is integral in the sense of [27] such that

$$(1-p\alpha_{E,p}^{-1})\frac{Q(\mathrm{reg}_{\mathrm{syn}}(\xi), e_{\mathrm{unit}})}{Q(\omega_E, e_{\mathrm{unit}})} \sim_{\mathbb{Q}^\times} L_p(E,\omega^{-1},0)$$

where $x \sim_{\mathbb{Q}^\times} y$ means $xy \neq 0$ and $x/y \in \mathbb{Q}^\times$. We also refer to [3, Conj. 3.3] for more precise statement. According to our main results, we can replace the left hand side with the special values of the p-adic hypergeometric functions of logarithmic type. For example, let E_a be the elliptic curve defined by $y^2 = x(1-x)(1-(1-a)x)$ with $a \in \mathbb{Q} \setminus \{0,1\}$ and $p > 3$ a prime where E_a has a good ordinary reduction. Then one predicts

$$(1-p\alpha_{E_a,p}^{-1})\mathscr{F}^{(\sigma_a)}_{\frac{1}{2},\frac{1}{2}}(a) \sim_{\mathbb{Q}^\times} L_p(E_a,\omega^{-1},0)$$

if $a = -1, \pm 2, \pm 4, \pm 8, \pm 16, \pm\frac{1}{2}, \pm\frac{1}{8}, \pm\frac{1}{4}, \pm\frac{1}{16}$ (Conjecture 5.2). See Section 5.2 for other cases. The author has no idea how to attack the question in general, while we have one example (the proof relies on Brunault's paper [7] and his appendix in [3]).

Theorem 1.4 (Theorem 5.3)**.** $(1-p\alpha_{E_4,p}^{-1})\mathscr{F}^{(\sigma_4)}_{\frac{1}{2},\frac{1}{2}}(4) = -L_p(E_4,\omega^{-1},0)$.

We note that this is a p-adic counterpart of a formula of Rogers and Zudilin ([25, Thm. 2, p. 399 and (6), p. 386])

$$2L'(E_4,0) = \mathrm{Re}\left[\log 4 - {}_4F_3\left({\frac{3}{2},\frac{3}{2},1,1 \atop 2,2,2};4\right)\right] \left(= {}_3F_2\left({\frac{1}{2},\frac{1}{2},\frac{1}{2} \atop \frac{3}{2},1};\frac{1}{4}\right)\right).$$

The conjectures in Section 5.2 give the first formulation of the p-adic counterparts of Rogers–Zudilin type formulas. We hope that our results will provide a new direction of the study of the p-adic Beilinson conjecture.

This paper is organized as follows. Section 2 is the preliminary section on Diamond's p-adic polygamma functions. More precisely we shall give a slight modification of Diamond's polygamma functions (though it might be known to the experts). We give a self-contained exposition, because the author does not find a suitable reference, especially concerning with our modified functions. In Section 3, we introduce the p-adic hypergeometric functions of logarithmic type, and prove the congruence relations. In Section 4, we show that our new p-adic hypergeometric functions appear in the syntomic regulators of the hypergeometric curves. Finally we discuss the p-adic Beilinson conjecture for K_2 of elliptic curves in Section 5.

Notation. For a field K, let $\mu_n(K) \subset K^\times$ denote the group of n-th roots of unity. We write $\mu_\infty(K) = \bigcup_{n\geq 1} \mu_n(K)$. If there is no fear of confusion, we drop "K" and simply write μ_n. For a power series $f(t) = \sum_{i=0}^\infty a_i t^i \in R[\![t]\!]$ with coefficients in a commutative ring R, we write the truncated polynomial $\sum_{i=0}^{n-1} a_i t^i$ by $[f(t)]_{<n}$.

2. p-adic polygamma functions

The complex analytic polygamma functions are the r-th derivative

$$\psi^{(r)}(z) := \frac{\mathrm{d}^r}{\mathrm{d}z^r}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right), \quad r \in \mathbb{Z}_{\geq 0}.$$

In his paper [11], Jack Diamond gave a p-adic counterpart of the polygamma functions $\psi^{(r)}_{D,p}(z)$ which are given in the following way.

$$\psi^{(0)}_{D,p}(z) = \lim_{s\to\infty} \frac{1}{p^s} \sum_{n=0}^{p^s-1} \log(z+n), \tag{2.1}$$

$$\psi^{(r)}_{D,p}(z) = (-1)^{r+1} r! \lim_{s\to\infty} \frac{1}{p^s} \sum_{n=0}^{p^s-1} \frac{1}{(z+n)^r}, \quad r \geq 1, \tag{2.2}$$

where $\log : \mathbb{C}_p^\times \to \mathbb{C}_p$ is the Iwasawa logarithmic function which is characterized as a continuous homomorphism satisfying $\log(p) = 0$ and

$$\log(z) = -\sum_{n=1}^\infty \frac{(1-z)^n}{n}, \quad |z-1|_p < 1.$$

It should be noticed that the series (2.1) and (2.2) converge only when $z \notin \mathbb{Z}_p$, and hence $\psi^{(r)}_{D,p}(z)$ turn out to be locally analytic functions on $\mathbb{C}_p \setminus \mathbb{Z}_p$. This causes inconvenience in our discussion. In this section we give a continuous function $\psi^{(r)}_p(z)$ on $\mathbb{Z}_p$ which is a slight *modification* of $\psi^{(r)}_{D,p}(z)$. See Section 2.2 for the definition and also Section 2.4 for an alternative definition in terms of p-adic measure.

2.1. p-adic polylogarithmic functions. Let x be an indeterminate. For an integer $r \in \mathbb{Z}$, the r-th p-adic polylogarithmic function $\ln^{(p)}_r(x)$ is defined as a formal power series

$$\ln^{(p)}_r(x) := \sum_{k\geq 1, p\nmid k} \frac{x^k}{k^r} = \lim_{s\to\infty}\left(\frac{1}{1-x^{p^s}} \sum_{1\leq k<p^s,\, p\nmid k} \frac{x^k}{k^r}\right) \in \mathbb{Z}_p[\![x]\!] \tag{2.3}$$

which belongs to the ring

$$\mathbb{Z}_p\langle x,(1-x)^{-1}\rangle := \varprojlim_s (\mathbb{Z}/p^s\mathbb{Z}[x,(1-x)^{-1}])$$

of convergent power series. If $r \leq 0$, this is a rational function, more precisely

$$\ln_0^{(p)}(x) = \frac{1}{1-x} - \frac{1}{1-x^p}, \quad \ln_{-r}^{(p)}(x) = \left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^r \ln_0^{(p)}(x).$$

If $r > 0$, this is known to be an *overconvergent function*, more precisely it has a (unique) analytic continuation to the domain $|x-1| > |1-\zeta_p|$ where $\zeta_p \in \overline{\mathbb{Q}}_p$ is a primitive p-th root of unity.

Let $W(\overline{\mathbb{F}}_p)$ be the Witt ring of $\overline{\mathbb{F}}_p$ and F the p-th Frobenius endomorphism. Define the *p-adic logarithmic function*

$$\log^{(p)}(z) := \frac{1}{p}\log\left(\frac{z^p}{F(z)}\right) = -\sum_{n=1}^{\infty}\frac{p^{-1}}{n}\left(1-\frac{z^p}{F(z)}\right)^n \tag{2.4}$$

for $z \in W(\overline{\mathbb{F}}_p)^\times$, where $\log(z)$ is the Iwasawa logarithmic function.

Lemma 2.1. *The function* $\ln_1^{(p)}(x)$ *agrees with*

$$-\frac{1}{p}\log\left(\frac{(1-x)^p}{1-x^p}\right) = \sum_{n=1}^{\infty}\frac{p^{n-1}w(x)^n}{n} \in \mathbb{Z}_p\langle x,(1-x)^{-1}\rangle$$

where $w(x) := 1-(1-x)^p/(1-x^p)$. *In particular, evaluating at* $x = z$ *for* $z \in W(\overline{\mathbb{F}}_p)^\times$ *such that* $F(z) = z^p$ *and* $z \not\equiv 1$ *mod* p, *one has* $\ln_1^{(p)}(z) = -\log^{(p)}(1-z)$.

Proof. We have the power series expression

$$-\frac{1}{p}\log\left(\frac{(1-x)^p}{1-x^p}\right) = \sum_{k\geq 1,\, p\nmid k}\frac{x^k}{k}$$

in $\mathbb{Z}_p[\![x]\!]$, and this agrees with the expression (2.3) of $\ln_1^{(p)}(x)$. Then the assertion is immediate as $\mathbb{Z}_p\langle x,(1-x)^{-1}\rangle \to \mathbb{Z}_p[\![x]\!]$ is injective. □

Proposition 2.2 (cf. [9, IV Prop. 6.1, 6.2])**.** *Let* $r \in \mathbb{Z}$ *be an integer. Then*

$$\ln_r^{(p)}(x) = x\frac{\mathrm{d}}{\mathrm{d}x}\ln_{r+1}^{(p)}(x), \tag{2.5}$$

$$\ln_r^{(p)}(x) = (-1)^{r+1}\ln_r^{(p)}(x^{-1}), \tag{2.6}$$

$$\sum_{\zeta\in\mu_N}\ln_r^{(p)}(\zeta x) = \frac{1}{N^{r-1}}\ln_r^{(p)}(x^N) \quad \textit{(distribution formula).} \tag{2.7}$$

Proof. One can derive (2.5) and (2.7) immediately from the series expansion $\ln_r^{(p)}(x) = \sum_{k\geq 1, p\nmid k} x^k/k^r$. On the other hand, (2.6) follows from the fact

$$\frac{1}{1-x^{-p^s}} \sum_{1\leq k<p^s,\, p\nmid k} \frac{x^{-k}}{k^r} = \frac{-1}{1-x^{p^s}} \sum_{1\leq k<p^s,\, p\nmid k} \frac{x^{p^s-k}}{k^r}$$
$$\equiv \frac{(-1)^{r+1}}{1-x^{p^s}} \sum_{1\leq k<p^s,\, p\nmid k} \frac{x^{p^s-k}}{(p^s-k)^r}$$

modulo $p^s\mathbb{Z}[x,(1-x)^{-1}]$. □

Lemma 2.3. *Let $m, N \geq 2$ be integers prime to p. Let $\varepsilon \in \mu_m \setminus \{1\}$. Then for any $n \in \{0, 1, \ldots, N-1\}$, we have*

$$N^r \sum_{\nu^N=\varepsilon} \nu^{-n} \ln_{r+1}^{(p)}(\nu) = \lim_{s\to\infty} \frac{1}{1-\varepsilon^{p^s}} \sum_{\substack{0\leq k<p^s \\ k+n/N\not\equiv 0 \bmod p}} \frac{\varepsilon^k}{(k+n/N)^{r+1}}.$$

Proof. Note $\sum_{\nu^N=\varepsilon} \nu^i = N\varepsilon^{i/N}$ if $N|i$ and $=0$ otherwise. We have

$$\begin{aligned}
N^r \sum_{\nu^N=\varepsilon} \nu^{-n} \ln_{r+1}^{(p)}(\nu x) &= N^r \sum_{k\geq 1, p\nmid k} \sum_{\nu^N=\varepsilon} \frac{\nu^{k-n} x^k}{k^{r+1}} \\
&= N^{r+1} \sum_{N|(k-n), p\nmid k} \frac{\varepsilon^{(k-n)/N} x^k}{k^{r+1}} \\
&= \sum_{n+\ell N\not\equiv 0 \bmod p,\, \ell\geq 0} \frac{(\varepsilon x^N)^\ell x^n}{(\ell+n/N)^{r+1}} \quad (\ell = (k-n)/N) \\
&\equiv \frac{1}{1-(\varepsilon x^N)^{p^s}} \sum_{\substack{0\leq \ell<p^s \\ n+\ell N\not\equiv 0 \bmod p}} \frac{(\varepsilon x^N)^\ell x^n}{(\ell+n/N)^{r+1}}
\end{aligned}$$

modulo $p^s\mathbb{Z}[x,(1-\varepsilon x^N)^{-1}]$. Since $\varepsilon \not\equiv 1 \bmod p$, the evaluation at $x=1$ makes sense, and then we have the desired equation. □

The following theorem is well-known to experts as Coleman's formula.

Theorem 2.4 (Coleman)**.** *Let $r \neq 1$ be an integer. Then for any integer $N \geq 2$ prime to p,*

$$\sum_{\varepsilon\in\mu_N\setminus\{1\}} \ln_r^{(p)}(\varepsilon) = -(1-N^{1-r}) L_p(r, \omega^{1-r}) \tag{2.8}$$

where $L_p(s,\chi)$ is the p-adic L-function and ω is the Teichmüller character.

Proof. We give a self-contained and straightforward proof for convenience of the reader, because the author does not find a suitable literature (note that (2.8) is not covered by [9, I, (3)]).

We first show (2.8) in case $r=-m$ with $m\in\mathbb{Z}_{\geq 1}$. Note that $\ln^{(p)}_{-m}(x)$ is a rational function. More precisely, let

$$\ln_0(x):=\frac{x}{1-x},\quad \ln_{-m}(x):=\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^m\ln_0(x),$$

then $\ln^{(p)}_{-m}(x)=\ln_{-m}(x)-p^m\ln_{-m}(x^p)$. Therefore

$$\sum_{\varepsilon\in\mu_N\setminus\{1\}}\ln^{(p)}_{-m}(\varepsilon)=(1-p^m)\sum_{\varepsilon\in\mu_N\setminus\{1\}}\ln_{-m}(\varepsilon).$$

Since $L_p(-m,\omega^{1+m})=-(1-p^m)B_{m+1}/(m+1)$ where B_n are the Bernoulli numbers, the equation (2.8) for $r=-m$ is equivalent to

$$(1-N^{m+1})\frac{B_{m+1}}{m+1}=\sum_{\varepsilon\in\mu_N\setminus\{1\}}\ln_{-m}(\varepsilon). \tag{2.9}$$

Put $\ell_r(x):=\ln_r(x)-N^{1-r}\ln_r(x^N)$. By the distribution property

$$\sum_{\varepsilon\in\mu_N}\ln_r(\varepsilon x)=N^{1-r}\ln_r(x^N)$$

which can be easily shown by a computation of power series expansions, the right hand side of (2.9) equals to the evaluation $-\ell_{-m}(x)|_{x=1}$ at $x=1$, and hence

$$\sum_{\varepsilon\in\mu_N\setminus\{1\}}\ln_{-m}(\varepsilon)=-\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^m\ell_0(x)\bigg|_{x=1}. \tag{2.10}$$

On the other hand, letting $x=e^z$, one has

$$\begin{aligned}\ell_0(e^z)=\frac{e^z}{1-e^z}-\frac{Ne^{Nz}}{1-e^{Nz}}&=-\sum_{n=1}^{\infty}\left(B_n\frac{z^{n-1}}{n!}-B_n\frac{N^nz^{n-1}}{n!}\right)\\&=-\sum_{n=1}^{\infty}(1-N^n)B_n\frac{z^{n-1}}{n!},\end{aligned}$$

and hence

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^m\ell_0(x)\bigg|_{x=1}=\frac{\mathrm{d}^m}{\mathrm{d}z^m}\ell_0(e^z)\bigg|_{z=0}=-(1-N^{m+1})\frac{B_{m+1}}{m+1}. \tag{2.11}$$

Now (2.9) follows from (2.10) and (2.11).

We have shown (2.8) for negative r. Let $r\neq 1$ be an arbitrary integer. Since $\ln^{(p)}_r(x)=\sum_{p\nmid k}x^k/k^r$, one has that for any integers r,r' such that $r\equiv r'\bmod (p-1)p^{s-1}$,

$$\ln^{(p)}_r(x)\equiv\ln^{(p)}_{r'}(x)\mod p^s\mathbb{Z}_p[\![x]\!]$$

and hence modulo $p^s\mathbb{Z}_p\langle x,(1-x)^{-1}\rangle$. This implies

$$\ln_r^{(p)}(\varepsilon)\equiv\ln_{r'}^{(p)}(\varepsilon)\mod p^sW(\overline{\mathbb{F}}_p).$$

Take $r'=r-p^{s+a-1}(p-1)$ with $a\gg 0$. It follows that

$$(1-N^{1-r'})L_p(r',\omega^{1-r'})=(1-N^{1-r'})L_p(r',\omega^{1-r})\to(1-N^{1-r})L_p(r,\omega^{1-r})$$

as $a\to\infty$ by the continuity of the p-adic L-functions. Since $r'<0$, one can apply (2.8) and then

$$-(1-N^{1-r})L_p(r,\omega^{1-r})\equiv\sum_{\varepsilon\in\mu_N\setminus\{1\}}\ln_r(\varepsilon)\mod p^sW(\overline{\mathbb{F}}_p)$$

for any $s>0$. This completes the proof. □

2.2. p-adic polygamma functions.

Lemma 2.5.

$$\sum_{1\le k<p^s,p\nmid k}k^m\equiv\begin{cases}-p^{s-1} & p\ge 3 \text{ and } (p-1)|m\\ 2^{s-1} & p=2 \text{ and } 2|m\\ 1 & p=2 \text{ and } s=1\\ 0 & \text{otherwise}\end{cases}\mod p^s.$$

Proof. Let $p>2$. Let $\zeta\in\mathbb{Z}_p$ be a primitive $(p-1)$-th root of unity. Then the set $\{\zeta^i(1+p)^i\mid 0\le i<p^{s-1}(p-1)\}$ is a representative of $(\mathbb{Z}/p^s\mathbb{Z})^\times$. Hence

$$\begin{aligned}\sum_{1\le k<p^s,p\nmid k}k^m&\equiv\sum_{0\le i<p^{s-1}(p-1)}(\zeta(1+p))^{mi}\mod p^s\\&=\frac{1-(\zeta(1+p))^{mp^{s-1}(p-1)}}{1-(\zeta(1+p))^m}=\frac{1-(1+p)^{mp^{s-1}(p-1)}}{1-(\zeta(1+p))^m}.\end{aligned}$$

Note that $(1+p)^{m_0p^j}\equiv 1+m_0p^{j+1}\mod p^{j+2}$ for $p\nmid m_0$ and $j\ge 0$. Therefore, when $(p-1)\nmid m$, the last term vanishes. If $(p-1)|m$, then the last term is equivalent to $mp^s(p-1)/mp=p^{s-1}(p-1)\equiv -p^{s-1}$ modulo p^s. This completes the proof in case $p>2$. Let $p=2$. When $s=1$, the proof is obvious. Suppose $s\ge 2$. Then the set $\{\pm 5^i\mid 0\le i<2^{s-2}\}$ is a representative of $(\mathbb{Z}/2^s\mathbb{Z})^\times$. Therefore

$$\sum_{1\le k<2^s,\,2\nmid k}k^m\equiv\sum_{0\le i<2^{s-2}}5^{mi}+(-1)^m5^{mi}\mod 2^s.$$

This vanishes when m is odd. If m is even, then the right hand side is

$$2\sum_{0\le i<2^{s-2}}5^{mi}=2\frac{1-5^{2^{s-2}m}}{1-5^m}\equiv 2\frac{2^sm}{4m}=2^{s-1}\mod 2^s$$

as $(1+4)^{m_0 2^j} \equiv 1+2^{j+2}m_0 \bmod 2^{j+3}$ for odd m_0 and $j \geq 0$. This completes the proof in case $p = 2$. □

Let $r \in \mathbb{Z}$ be an integer. For $z \in \mathbb{Z}_p$, define

$$\widetilde{\psi}_p^{(r)}(z) := \lim_{n \in \mathbb{Z}_{>0}, n \to z} \sum_{1 \leq k < n, p \nmid k} \frac{1}{k^{r+1}}. \tag{2.12}$$

The limit exists by Lemma 2.5, and moreover it satisfies

$$\widetilde{\psi}_p^{(r)}(z) - \widetilde{\psi}_p^{(r)}(z') \equiv \begin{cases} 0 \bmod p^s & p \geq 3 \text{ and } (p-1) \nmid (r+1) \\ 0 \bmod p^s & p = 2,\ s \geq 2 \text{ and } 2 \nmid (r+1) \\ 0 \bmod p^{s-1} & \text{othewise.} \end{cases} \tag{2.13}$$

for $z \equiv z' \bmod p^s$. In particular, $\widetilde{\psi}_p^{(r)}(z)$ is a p-adic continuous function on $\mathbb{Z}_p$. Define the *p-adic Euler constant* [1] by

$$\gamma_p := -\lim_{s \to \infty} \frac{1}{p^s} \sum_{0 \leq j < p^s, p \nmid j} \log(j), \quad (\log = \text{Iwasawa log}).$$

where the convergence follows by

$$\begin{aligned} \sum_{0 \leq j < p^{s+1}, p \nmid j} \log(j) - p \sum_{0 \leq j < p^s, p \nmid j} \log(j) &= \sum_{k=0}^{p-1} \sum_{0 \leq j < p^s, p \nmid j} \log\left(1 + \frac{kp^s}{j}\right) \\ &\equiv \sum_{k=0}^{p-1} \sum_{0 \leq j < p^s, p \nmid j} \frac{kp^s}{j} \mod p^{2s-1} \\ &\equiv 0 \mod p^{2s-1} \quad \text{(Lemma 2.5).} \end{aligned} \tag{2.14}$$

We define the r-th *p-adic polygamma function* to be

$$\psi_p^{(r)}(z) := \begin{cases} -\gamma_p + \widetilde{\psi}_p^{(0)}(z) & r = 0 \\ -L_p(1+r, \omega^{-r}) + \widetilde{\psi}_p^{(r)}(z) & r \neq 0. \end{cases} \tag{2.15}$$

If $r = 0$, we also write $\psi_p(z) = \psi_p^{(0)}(z)$ and call it the *p-adic digamma function.*

2.3. Formulas on p-adic polygamma functions.

Theorem 2.6.

(1) $\widetilde{\psi}_p^{(r)}(0) = \widetilde{\psi}_p^{(r)}(1) = 0$ *or equivalently* $\psi_p^{(r)}(0) = \psi_p^{(r)}(1) = -\gamma_p$ *or* $= -L_p(r+1, \omega^{-r})$.

(2) $\widetilde{\psi}_p^{(r)}(z) = (-1)^r \widetilde{\psi}_p^{(r)}(1-z)$ *or equivalently* $\psi_p^{(r)}(z) = (-1)^r \psi_p^{(r)}(1-z)$ *(note* $L_p(1+r, \omega^{-r}) = 0$ *for odd* r*).*

---

[1] This is different from Diamond's p-adic Euler constant. His constant is $p/(p-1)\gamma_p$, [11, §7].

$$(3)\ \widetilde{\psi}_p^{(r)}(z+1)-\widetilde{\psi}_p^{(r)}(z)=\psi_p^{(r)}(z+1)-\psi_p^{(r)}(z)=\begin{cases}z^{-r-1} & z\in\mathbb{Z}_p^\times\\ 0 & z\in p\mathbb{Z}_p.\end{cases}$$

Compare the above with the formulas on the complex analytic polygamma functions, [22, 5.15.2, 5.15.5, 5.15.6].

Proof. (1) follows from Lemma 2.5, and (3) are immediate from definition. We show (2). Since $\mathbb{Z}_{>0}$ is a dense subset in $\mathbb{Z}_p$, it is enough to show in case $z=n>0$ an integer. Let $s>0$ be arbitrary such that $p^s>n$. Then

$$\begin{aligned}\widetilde{\psi}_p^{(r)}(n)&\equiv\sum_{1\le k<n,p\nmid k}\frac{1}{k^{r+1}}\equiv(-1)^{r+1}\sum_{-n<k\le -1,p\nmid k}\frac{1}{k^{r+1}}\\&\equiv(-1)^{r+1}\sum_{p^s-n+1\le k<p^s,p\nmid k}\frac{1}{k^{r+1}}\\&\equiv(-1)^{r+1}\sum_{0\le k<p^s,p\nmid k}\frac{1}{k^{r+1}}-(-1)^{r+1}\sum_{0\le k<p^s-n+1,p\nmid k}\frac{1}{k^{r+1}}\\&\equiv(-1)^{r}\sum_{0\le k<p^s-n+1,p\nmid k}\frac{1}{k^{r+1}}\\&\equiv(-1)^r\widetilde{\psi}_p^{(r)}(1-n)\end{aligned}$$

modulo p^s or p^{s-1}. Since s is an arbitrary large integer, this means $\widetilde{\psi}_p^{(r)}(n)=(-1)^r\widetilde{\psi}_p^{(r)}(1-n)$ as required. □

Theorem 2.7. *Let $0\le n<N$ be integers and suppose $p\nmid N$. Then*

$$\widetilde{\psi}_p^{(r)}\left(\frac{n}{N}\right)=N^r\sum_{\varepsilon\in\mu_N\setminus\{1\}}(1-\varepsilon^{-n})\ln_{r+1}^{(p)}(\varepsilon).\tag{2.16}$$

For example

$$\psi_p^{(r)}\left(\frac{1}{2}\right)=-L_p(1+r,\omega^{-r})+2^{r+1}\ln_{r+1}^{(p)}(-1)=(1-2^{r+1})L_p(1+r,\omega^{-r}).$$

Compare this with [22, 5.15.3] the formula on the complex analytic polygamma functions.

Proof. We may assume $n>0$. Let $s>0$ be an integer such that $p^s\equiv 1 \mod N$. Write $p^s-1=lN$. We have

$$\begin{aligned}S:=\sum_{\varepsilon\in\mu_N\setminus\{1\}}(1-\varepsilon^{-n})\ln_{r+1}^{(p)}(\varepsilon)&\overset{(2.3)}{\equiv}\sum_{1\le k<p^s,\,p\nmid k}\left(\sum_{\varepsilon\in\mu_N\setminus\{1\}}\frac{1-\varepsilon^{-n}}{1-\varepsilon^{p^s}}\frac{\varepsilon^k}{k^{r+1}}\right)\\&=\sum_{1\le k<p^s,\,p\nmid k}\left(\sum_{\varepsilon\in\mu_N\setminus\{1\}}\frac{\varepsilon^k+\cdots+\varepsilon^{k+N-n-1}}{k^{r+1}}\right)\end{aligned}$$

modulo p^s. Note $\sum_{\varepsilon\in\mu_N\setminus\{1\}}\varepsilon^i = N-1$ if $N|i$ and $=-1$ otherwise. Let I be the set of integers k satisfying that $0\le k<p^s$, $p\nmid k$ and that there is an integer $0\le i<N-n$ such that $k+i\equiv 0 \bmod N$. Then we have

$$S\equiv\sum_{k\in I}\frac{N}{k^{r+1}} \mod p^{s-1}$$

by Lemma 2.5. Hence

$$\begin{aligned}
N^rS &\equiv \sum_{k\in I}\frac{1}{(k/N)^{r+1}} = \sum_{k\equiv 0 \bmod N} + \sum_{k\equiv -1 \bmod N} + \cdots + \sum_{k\equiv n-N+1 \bmod N}\\
&= \sum_{\substack{1\le j<p^s/N\\ j\not\equiv 0 \bmod p}}\frac{1}{j^{r+1}} + \sum_{\substack{1\le j<(p^s+1)/N\\ j-1/N\not\equiv 0 \bmod p}}\frac{1}{(j-1/N)^{r+1}} +\\
&\qquad\qquad \cdots + \sum_{\substack{1\le j<(p^s+N-n-1)/N\\ j-(N-n-1)/N\not\equiv 0 \bmod p}}\frac{1}{(j-(N-n-1)/N)^{r+1}}\\
&\equiv \sum_{\substack{1\le j\le l\\ j\not\equiv 0 \bmod p}}\frac{1}{j^{r+1}} + \sum_{\substack{1\le j\le l\\ j+l\not\equiv 0 \bmod p}}\frac{1}{(j+l)^{r+1}} +\\
&\qquad\qquad \cdots + \sum_{\substack{1\le j\le l\\ j+l(N-n-1)\not\equiv 0 \bmod p}}\frac{1}{(j+l(N-n-1))^{r+1}}\\
&= \sum_{\substack{1\le j\le l(N-n)\\ j\not\equiv 0 \bmod p}}\frac{1}{j^{r+1}} = \sum_{\substack{0\le j<l(N-n)+1\\ j\not\equiv 0 \bmod p}}\frac{1}{j^{r+1}}.
\end{aligned}$$

Since $l(N-n)+1\equiv n/N \mod p^s$, the last summation is equivalent to $\widetilde{\psi}^{(r)}(n/N)$ modulo p^{s-1} by definition. □

Remark 2.8. The complex analytic analogy of Theorem 2.7 is the following. Let $\ln_r(z)=\ln_r^{an}(z)=\sum_{n=1}^{\infty} z^n/n^r$ be the analytic polylog. Then

$$\begin{aligned}
&N^r\sum_{k=1}^{N-1}(1-e^{-2\pi ikn/N})\ln_{r+1}(e^{2\pi ik/N})\\
&=\sum_{m=1}^{\infty}\sum_{k=1}^{N-1}\frac{N^r}{m^{r+1}}(e^{2\pi ikm/N}-e^{2\pi ik(m-n)/N})\\
&=\sum_{k=1}^{\infty}\frac{N^{r+1}}{(kN)^{r+1}}-\frac{N^{r+1}}{(kN-N+n)^{r+1}}=\sum_{k=1}^{\infty}\frac{1}{k^{r+1}}-\frac{1}{(k-1+n/N)^{r+1}}.
\end{aligned}$$

If $r=0$, then this is equal to $\psi(z)-\psi(1)$ ([22, 5.7.6]). If $r\ge 1$, then this is equal to $\zeta(r+1)+(-1)^r/r!\psi^{(r)}(n/N)$ ([22, 5.15.1]).

Theorem 2.9. *Let $m \geq 1$ be an positive integer prime to p.*

(1) *Let $\psi_p(z) = \psi_p^{(0)}(z)$ be the p-adic digamma function. Then*

$$\psi_p(mz) - \log^{(p)}(m) = \frac{1}{m}\sum_{i=0}^{m-1}\psi_p\Big(z + \frac{i}{m}\Big),$$

(see (2.4) for the definition of $\log^{(p)}(z)$).

(2) *If $r \neq 0$, we have*

$$\psi_p^{(r)}(mz) = \frac{1}{m^{r+1}}\sum_{i=0}^{m-1}\psi_p^{(r)}\Big(z + \frac{i}{m}\Big).$$

See [22, 5.15.7] for the corresponding formula on the complex analytic polygamma functions.

Proof. By Theorem 2.4 (and Lemma 2.1 in case $r = 0$), the assertions are equivalent to

$$\frac{1}{m^{r+1}}\sum_{i=0}^{m-1}\widetilde{\psi}_p^{(r)}\Big(z + \frac{i}{m}\Big) = \widetilde{\psi}_p^{(r)}(mz) + \sum_{\varepsilon\in\mu_m\setminus\{1\}}\ln_{r+1}^{(p)}(\varepsilon) \tag{2.17}$$

for all $r \in \mathbb{Z}$. Since $\mathbb{Z}_{(p)} \cap [0,1)$ is a dense subset in $\mathbb{Z}_p$, it is enough to show the above in case $z = n/N$ with $0 \leq n < N$, $p \nmid N$. By Theorem 2.7,

$$\begin{aligned}\frac{1}{m^{r+1}}\sum_{i=0}^{m-1}\widetilde{\psi}_p^{(r)}\Big(z + \frac{i}{m}\Big) &= \frac{1}{m^{r+1}}\sum_{i=0}^{m-1}\widetilde{\psi}_p^{(r)}\Big(\frac{nm+iN}{mN}\Big)\\ &= \frac{N^r}{m}\sum_{i=0}^{m-1}\sum_{\nu\in\mu_{mN}\setminus\{1\}}(1-\nu^{-nm-iN})\ln_{r+1}^{(p)}(\nu).\end{aligned}$$

The last summation is divided into the following two terms

$$\sum_{i=0}^{m-1}\sum_{\nu\in\mu_N\setminus\{1\}}(1-\nu^{-nm})\ln_{r+1}^{(p)}(\nu) = m\sum_{\nu\in\mu_N\setminus\{1\}}(1-\nu^{-nm})\ln_{r+1}^{(p)}(\nu),$$

$$\begin{aligned}\sum_{i=0}^{m-1}\sum_{\varepsilon\in\mu_m\setminus\{1\}}\sum_{\nu^N=\varepsilon}(1-\nu^{-nm}\varepsilon^{-i})\ln_{r+1}^{(p)}(\nu) &= m\sum_{\varepsilon\in\mu_m\setminus\{1\}}\sum_{\nu^N=\varepsilon}\ln_{r+1}^{(p)}(\nu)\\ &= \frac{m}{N^r}\sum_{\varepsilon\in\mu_m\setminus\{1\}}\ln_{r+1}^{(p)}(\varepsilon)\end{aligned}$$

where the last equality follows from the distribution formula (2.7). Since the former is equal to $\widetilde{\psi}_p^{(r)}(nm/N)$ by Theorem 2.7, the equality (2.17) follows. □

The relation with Morita's p-adic Gamma function $\Gamma_p(z)$ (e.g. [8, 11.6]) is as follows.

Theorem 2.10. *Let $B_p(x,y)=\Gamma_p(x)\Gamma_p(y)/\Gamma_p(x+y)$. Then*

$$\log B_p(z,q)=\sum_{i=1}^{\infty}\widetilde{\psi}_p^{(i-1)}(z)\frac{(-1)^i q^i}{i}$$

for $z\in\mathbb{Z}_p$ and $q\in p\mathbb{Z}_p$ where $\log:1+p\mathbb{Z}_p\to p\mathbb{Z}_p$ is the Iwasawa logarithm.

Proof. Fix $q\in p\mathbb{Z}_p$. The functions

$$\mathbb{Z}_p\longrightarrow p\mathbb{Z}_p,\quad z\longmapsto \log B_p(z,q)$$

and

$$\mathbb{Z}_p\longrightarrow p\mathbb{Z}_p,\quad z\longmapsto \sum_{i=1}^{\infty}\widetilde{\psi}_p^{(i-1)}(z)\frac{(-q)^i}{i}$$

are continuous. Therefore it is enough to show

$$\log B_p(n,q)=\sum_{i=1}^{\infty}\widetilde{\psi}_p^{(i-1)}(n)\frac{(-q)^i}{i} \tag{2.18}$$

for all $n\in\mathbb{Z}_{\geq 0}$ as the set $\mathbb{Z}_{\geq 0}$ is dense in $\mathbb{Z}_p$. We show it by induction on n. The case $n=0$ is trivial. Suppose that (2.18) is true for n. If $p|n$, then $B_p(n+1,q)=B_p(n,q)$ ([8, 11.6.8.(3)]) and $\widetilde{\psi}_p^{(j)}(n+1)=\widetilde{\psi}_p^{(j)}(n)$ (Theorem 2.6 (3)), so that (2.18) is true for $n+1$. If $p\nmid n$, then

$$\begin{aligned}
\log B_p(n+1,q)&=\log\frac{\Gamma_p(n+1)\Gamma_p(q)}{\Gamma_p(n+1+q)}\\
&=\log\left(\frac{-n}{-(n+q)}\frac{\Gamma_p(n)\Gamma_p(q)}{\Gamma_p(n+q)}\right) && \text{([8, 11.6.8.(3)])}\\
&=-\log\left(1+\frac{q}{n}\right)+\log B_p(n,q)\\
&=\sum_{i=1}^{\infty}\frac{1}{n^i}\frac{(-q)^i}{i}+\sum_{i=1}^{\infty}\widetilde{\psi}_p^{(i-1)}(n)\frac{(-q)^i}{i}\\
&=\sum_{i=1}^{\infty}\widetilde{\psi}_p^{(i-1)}(n+1)\frac{(-q)^i}{i}, && \text{(Theorem 2.6 (3))}
\end{aligned}$$

so that (2.18) is true for $n+1$. This completes the proof. □

2.4. *p*-adic measure. For a function $g:\mathbb{Z}_p\to\mathbb{C}_p$, the Volkenborn integral is defined by

$$\int_{\mathbb{Z}_p} g(t)\,\mathrm{d}t=\lim_{s\to\infty}\frac{1}{p^s}\sum_{0\leq j<p^s}g(j)$$

if the limit exists. We refer [8, 11.1.2] for a general theory on Volkenborn integrals

Theorem 2.11. *Let* $\log : \mathbb{C}_p^\times \to \mathbb{C}_p$ *be the Iwasawa logarithmic function. Let*

$$\mathbb{1}_{\mathbb{Z}_p^\times}(z) := \begin{cases} 1 & z \in \mathbb{Z}_p^\times \\ 0 & z \in p\mathbb{Z}_p \end{cases}$$

be the characteristic function. Then

$$\psi_p(z) = \int_{\mathbb{Z}_p} \log(z+t)\mathbb{1}_{\mathbb{Z}_p^\times}(z+t)\,\mathrm{d}t.$$

Proof. Using the computation in (2.14), one can easily show that the Volkenborn integral $Q(z) := \int_{\mathbb{Z}_p} \mathbb{1}_{\mathbb{Z}_p^\times}(z+t)\log(z+t)\,\mathrm{d}t$ is defined, and it is continuous with respect to $z \in \mathbb{Z}_p$. Moreover we have

$$Q(z+1) - Q(z) \equiv \begin{cases} p^{-s}(\log(z) - \log(z+p^s)) & z \in \mathbb{Z}_p^\times \\ 0 & z \in p\mathbb{Z}_p \end{cases} \mod p^{s'}$$

where $s' = s-1$ if $p = 2$ and $s' = s$ if $p \geq 3$. For $z \in \mathbb{Z}_p^\times$, since

$$p^{-s}(\log(z) - \log(z+p^s)) = -p^{-s}\log(1+z^{-1}p^s) \equiv z^{-1} \mod p^{s'},$$

it follows from Theorem 2.6 (3) that $Q(z)$ differs from $\psi_p(z)$ by a constant. Since

$$Q(0) = \lim_{s\to\infty} \frac{1}{p^s} \sum_{0 \leq j < p^s, p\nmid j} \log(j) = -\gamma_p = \psi_p(0),$$

we obtain $Q(z) = \psi_p(z)$. □

Theorem 2.12. *If* $r \neq 0$*, then*

$$\psi_p^{(r)}(z) = -\frac{1}{r}\int_{\mathbb{Z}_p} (z+t)^{-r}\mathbb{1}_{\mathbb{Z}_p^\times}(z+t)\,\mathrm{d}t$$

where $\mathbb{1}_{\mathbb{Z}_p^\times}(z)$ *denotes the characteristic function as in Theorem 2.11.*

Proof. By Lemma 2.5, the Volkenborn integral

$$Q(z) = -\frac{1}{r}\int_{\mathbb{Z}_p} (z+t)^{-r}\mathbb{1}_{\mathbb{Z}_p^\times}(z+t)\,\mathrm{d}t$$

is defined. Moreover, if $z \in \mathbb{Z}_p^\times$, then

$$Q(z+1) - Q(z) \equiv \frac{-1}{rp^s}\left(\frac{1}{(z+p^s)^r} - \frac{1}{z^r}\right) \equiv z^{-1-r} \mod p^{s-\mathrm{ord}_p(r)},$$

and if $z \in p\mathbb{Z}_p$, then $Q(z+1) \equiv Q(z)$. This shows that $Q(z) - \psi_p^{(r)}(z)$ is a constant by Theorem 2.6 (3). We show $Q(0) = \psi_p^{(r)}(0)$. By definition

$$Q(0) = \lim_{n\to\infty} \frac{-1}{rp^n} \sum_{0 \leq k < p^n, p\nmid k} \frac{1}{k^r}.$$

Recall the original definition of the p-adic L-function by Kubota–Leopoldt in [19]

$$L_p(s,\chi)=\frac{1}{s-1}\lim_{n\to\infty}\frac{1}{fp^n}\sum_{0\le k<fp^n,p\nmid k}\chi(k)\langle k\rangle^{1-s},\quad \langle k\rangle:=k/\omega(k)$$

for a primitive Dirichlet character χ with conductor $f\ge 1$. This immediately implies $Q(0)=-L_p(1+r,\omega^{-r})=\psi_p^{(r)}(0)$. □

3. p-adic hypergeometric functions of logarithmic type

We write the Pochhammer symbol by $(a)_n$,

$$(a)_0:=1,\quad (a)_n:=a(a+1)\cdots(a+n-1),\ n\ge 1.$$

For $a\in\mathbb{Z}_p$, the *Dwork prime* a' is defined to be $(a+l)/p$ where $l\in\{0,1,\dots,p-1\}$ is the unique integer such that $a+l\equiv 0 \bmod p$. The i-th Dwork prime is denoted by $a^{(i)}$ which is defined to be $(a^{(i-1)})'$ with $a^{(0)}=a$.

3.1. Definition. Let $s\ge 1$ be a positive integer. Let $a_i,b_j\in\mathbb{Q}_p$ with $b_j\notin\mathbb{Z}_{\le 0}$. Let

$${}_sF_{s-1}\left(\begin{matrix}a_1,\dots,a_s\\ b_1,\dots b_{s-1}\end{matrix}:t\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_s)_n}{(b_1)_n\cdots(b_{s-1})_n}\frac{t^n}{n!}.$$

be the *hypergeometric power series* with coefficients. In what follows we only consider the cases $a_i\in\mathbb{Z}_p$ and $b_j=1$, and then we write

$$F_{\underline{a}}(t):={}_sF_{s-1}\left(\begin{matrix}a_1,\dots,a_s\\ 1,\dots 1\end{matrix}:t\right)\in\mathbb{Z}_p[\![t]\!]$$

for $\underline{a}=(a_1,\dots,a_s)\in\mathbb{Z}_p^s$.

Definition 3.1 (p-adic hypergeometric functions of logarithmic type)**.** Write $\underline{a}^{(i)}=(a_1^{(i)},\dots,a_s^{(i)})$ where $(\,\cdot\,)^{(i)}$ denotes the i-th Dwork prime. Let $W=W(\overline{\mathbb{F}}_p)$ be the Witt ring of $\overline{\mathbb{F}}_p$. Let $\sigma: W[\![t]\!]\to W[\![t]\!]$ be the p-th Frobenius endomorphism given by $\sigma(t)=ct^p$ with $c\in 1+pW$, compatible with the Frobenius on W. Put a power series

$$G_{\underline{a}}(t):=\psi_p(a_1)+\cdots+\psi_p(a_s)+s\gamma_p-p^{-1}\log(c)+\int_0^t(F_{\underline{a}}(t)-F_{\underline{a}^{(1)}}(t^\sigma))\frac{dt}{t}$$

where $\psi_p(z)$ is the p-adic digamma function defined in Section 2.2, and $\log(z)$ is the Iwasawa logarithmic function. Then we define

$$\mathscr{F}^{(\sigma)}_{\underline{a}}(t)=G_{\underline{a}}(t)/F_{\underline{a}}(t),$$

and call the *p-adic hypergeometric functions of logarithmic type.*

Lemma 3.2. *$G_{\underline{a}}(t)\in W[\![t]\!]$. Hence it follows $\mathscr{F}^{(\sigma)}_{\underline{a}}(t)\in W[\![t]\!]$.*

Proof. Let $G_{\underline{a}}(t)=\sum B_it^i$. Let $F_{\underline{a}}(t)=\sum A_it^i$ and $F_{\underline{a}^{(1)}}(t)=\sum \widetilde{A}_it^i$. If $p\nmid i$, then $B_i=A_i/i$ is obviously a p-adic integer. For $i=mp^k$ with $k\geq 1$ and $p\nmid m$, one has

$$B_i=B_{mp^k}=\frac{A_{mp^k}-c^{mp^{k-1}}\widetilde{A}_{mp^{k-1}}}{mp^k}.$$

Since $c^{mp^{k-1}}\equiv 1 \bmod p^k$, it is enough to see $A_{mp^k}\equiv \widetilde{A}_{mp^{k-1}} \bmod p^k$. However this follows from [12, p. 36, Cor. 1]. □

3.2. Congruence relations. For a power series $f(t)=\sum_{n=0}^{\infty}A_nt^n$, we write $[f(t)]_{<m}:=\sum_{n<m}A_nt^n$ the truncated polynomial.

Theorem 3.3. *Suppose that $a_i\notin\mathbb{Z}_{\leq 0}$ for all i. Let us write $\mathscr{F}^{(\sigma)}_{\underline{a}}(t)=G_{\underline{a}}(t)/F_{\underline{a}}(t)$. If $c\in 1+2pW$, then for all $n\geq 1$*

$$\mathscr{F}^{(\sigma)}_{\underline{a}}(t)\equiv\frac{[G_{\underline{a}}(t)]_{<p^n}}{[F_{\underline{a}}(t)]_{<p^n}} \mod p^nW[\![t]\!]. \tag{3.1}$$

If $p=2$ and $c\in 1+2W$ (not necessarily $c\in 1+4W$), then the above holds modulo p^{n-1}.

Corollary 3.4. *Suppose that there exists an integer $r\geq 0$ such that $a_i^{(r+1)}=a_i$ for all i where $(\,\cdot\,)^{(r)}$ denotes the r-th Dwork prime. Then $\mathscr{F}^{(\sigma)}_{\underline{a}}(t)$ belongs to the ring*

$$W\langle t,[F_{\underline{a}}(t)]^{-1}_{<p},\dots,[F_{\underline{a}^{(r)}}(t)]^{-1}_{<p}\rangle:=\varprojlim_n(W/p^n[t,[F_{\underline{a}}(t)]^{-1}_{<p},\dots,[F_{\underline{a}^{(r)}}(t)]^{-1}_{<p}]).$$

Hence, for $\alpha\in W$ such that $[F_{\underline{a}^{(i)}}(\alpha)]_{<p}\not\equiv 0 \bmod p$ for all i, the special value of $\mathscr{F}^{(\sigma)}_{\underline{a}}(t)$ at $t=\alpha$ is defined, and it is explicitly given by

$$\mathscr{F}^{(\sigma)}_{\underline{a}}(\alpha)=\lim_{n\to\infty}\frac{[G_{\underline{a}}(\alpha)]_{<p^n}}{[F_{\underline{a}}(\alpha)]_{<p^n}}.$$

3.3. Proof of Congruence relations: Reduction to the case $c=1$. Throughout the Sections 3.3, 3.4 and 3.5, we use the following notation. Fix $s\geq 1$ and $\underline{a}=(a_1,\dots,a_s)$ with $a_i\notin\mathbb{Z}_{\leq 0}$. Let $\sigma(t)=ct^p$ be the Frobenius. Put

$$A_n:=\frac{(a_1)_n}{n!}\cdots\frac{(a_s)_n}{n!},\quad \widetilde{A}_n:=\frac{(a_1^{(1)})_n}{n!}\cdots\frac{(a_s^{(1)})_n}{n!} \tag{3.2}$$

for $n\geq 0$. Let B_n be defined by $G_{\underline{a}}(t)=\sum_{n=0}^{\infty}B_nt^n$, or explicitly

$$B_0=\psi_p(a_1)+\cdots+\psi_p(a_s)+s\gamma_p, \tag{3.3}$$

$$B_n=\frac{A_n}{n},\ (p\nmid n),\quad B_{mp^k}=\frac{A_{mp^k}-c^{mp^{k-1}}\widetilde{A}_{mp^{k-1}}}{mp^k},\ (m,k\geq 1). \tag{3.4}$$

Lemma 3.5. *The proof of Theorem 3.3 is reduced to the case $\sigma(t) = t^p$ (i.e. $c = 1$).*

Proof. Write $[f(t)]_{\geq m} := f(t) - [f(t)]_{<m}$. Put $n^* := n$ if $c \in 1 + 2pW$ and $n^* = n - 1$ if $p = 2$ and $c \notin 1 + 4W$. Theorem 3.3 is equivalent to saying

$$F_{\underline{a}}(t) \cdot [G_{\underline{a}}(t)]_{\geq p^n} \equiv [F_{\underline{a}}(t)]_{\geq p^n} \cdot G_{\underline{a}}(t) \mod p^{n^*} W[\![t]\!],$$

namely

$$\sum_{i+j=m,\, i,j\geq 0} A_{i+p^n} B_j - A_i B_{j+p^n} \equiv 0 \mod p^{n^*}$$

for all $m \geq 0$. Suppose that this is true when $c = 1$, namely

$$\sum_{i+j=m} A_{i+p^n} B_j^\circ - A_i B_{j+p^n}^\circ \equiv 0 \mod p^{n^*} \tag{3.5}$$

where B_i° are the coefficients (3.3) or (3.4) when $c = 1$. Suppose that $c \in 1 + pW$ is an arbitrary element, and let B_i be as in (3.3) or (3.4). We then want to show

$$\sum_{i+j=m} A_{i+p^n}(B_j^\circ - B_j) - A_i(B_{j+p^n}^\circ - B_{j+p^n}) \equiv 0 \mod p^{n^*}. \tag{3.6}$$

Let $c = 1 + pe$ with $e \neq 0$ (if $e = 0$, there is nothing to prove). Then

$$\begin{aligned}
&\sum_{i+j=m} A_{i+p^n}(B_j^\circ - B_j) \\
&= A_{m+p^n} p^{-1} \log(c) + \sum_{1\leq j\leq m} p^{-1} \frac{(c^{j/p} - 1) A_{m+p^n-j} \widetilde{A}_{j/p}}{j/p} \\
&= A_{m+p^n} \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} p^{i-1} e^i + \sum_{1\leq j\leq m} (j/p)^{-1} \sum_{i=1}^{\infty} \binom{j/p}{i} p^{i-1} e^i A_{m+p^n-j} \widetilde{A}_{j/p} \\
&= \sum_{i=1}^{\infty} \left(A_{m+p^n} \frac{(-1)^{i+1}}{i} + \sum_{1\leq j\leq m} (j/p)^{-1} \binom{j/p}{i} A_{m+p^n-j} \widetilde{A}_{j/p} \right) p^{i-1} e^i \\
&= \sum_{i=1}^{\infty} \left(A_{m+p^n} \frac{(-1)^{i+1}}{i} + \sum_{1\leq j\leq m} i^{-1} \binom{j/p-1}{i-1} A_{m+p^n-j} \widetilde{A}_{j/p} \right) p^{i-1} e^i \\
&= \sum_{i=1}^{\infty} \left(\sum_{0\leq j\leq m} i^{-1} \binom{j/p-1}{i-1} A_{m+p^n-j} \widetilde{A}_{j/p} \right) p^{i-1} e^i
\end{aligned}$$

where we always mean $A_{j/p}=\widetilde{A}_{j/p}=0$ unless $p|j$. Similarly

$$\sum_{i+j=m} A_i(B^\circ_{j+p^n}-B_{j+p^n}) \\ =\sum_{i=1}^{\infty}\left(\sum_{0\le j\le m} i^{-1}\binom{(m+p^n-j)/p-1}{i-1}A_j\widetilde{A}_{(m+p^n-j)/p}\right)p^{i-1}e^i.$$

Therefore it is enough to show that

$$\frac{p^{i-1}e^i}{i}\sum_{0\le j\le m}\binom{j/p-1}{i-1}A_{m+p^n-j}\widetilde{A}_{j/p} \\ \equiv \frac{p^{i-1}e^i}{i}\sum_{0\le j\le m}\binom{(m+p^n-j)/p-1}{i-1}A_j\widetilde{A}_{(m+p^n-j)/p} \mod p^{n^*},$$

or equivalently

$$(3.7)\quad \sum_{0\le j\le m}(1-j/p)_{i-1}A_{m+p^n-j}\widetilde{A}_{j/p} \\ \equiv \sum_{0\le j\le m}(1-(m+p^n-j)/p)_{i-1}A_j\widetilde{A}_{(m+p^n-j)/p} \mod p^{n^*-i+1}i!e^{-i}$$

for all $i\ge 1$ and $m\ge 0$. Recall the Dwork congruence

$$(3.8)\qquad \frac{F_{\underline{a}^{(1)}}(t^p)}{F_{\underline{a}}(t)}\equiv\frac{[F_{\underline{a}^{(1)}}(t^p)]_{<p^m}}{[F_{\underline{a}}(t)]_{<p^m}} \mod p^l\mathbb{Z}_p[\![t]\!],\quad m\ge l$$

from [12, p. 37, Thm. 2, p. 45]. This immediately imples (3.7) in case $i=1$. Suppose $i\ge 2$. To show (3.7), it is enough to show

$$(3.9)\quad \sum_{0\le j\le m}(j/p)^kA_{m+p^n-j}\widetilde{A}_{j/p} \\ \equiv \sum_{0\le j\le m}((m+p^n-j)/p)^kA_j\widetilde{A}_{(m+p^n-j)/p} \mod p^{n^*-i+1}i!e^{-i}$$

for each $k\ge 0$. Let $D=t\frac{\mathrm{d}}{\mathrm{d}t}$, and put $F^*(t):=D^kF_{\underline{a}^{(1)}}(t)=\sum_{j=0}^{\infty}j^k\widetilde{A}_jt^j$. Then (3.9) is equivalent to saying

$$(3.10)\qquad [F_{\underline{a}}(t)]_{<p^n}F^*(t^p)\equiv F_{\underline{a}}(t)[F^*(t^p)]_{<p^n} \mod p^{n^*-i+1}i!e^{-i}\mathbb{Z}_p[\![t]\!].$$

We show (3.10), which finishes the proof of Lemma 3.5. Using the Dwork congruence (3.8), one can show

$$\frac{\frac{\mathrm{d}^k}{\mathrm{d}t^k}F_{\underline{a}^{(1)}}(t)}{F_{\underline{a}^{(1)}}(t)}\equiv\frac{\frac{\mathrm{d}^k}{\mathrm{d}t^k}([F_{\underline{a}^{(1)}}(t)]_{<p^m})}{[F_{\underline{a}^{(1)}}(t)]_{<p^m}} \mod p^m\mathbb{Z}_p[\![t]\!]$$

in the same way as the proof of [12, p. 45, (3.14)]. Hence

$$\frac{F^*(t)}{F_{\underline{a}^{(1)}}(t)} \equiv \frac{[F^*(t)]_{<p^m}}{[F_{\underline{a}^{(1)}}(t)]_{<p^m}} \mod p^l\mathbb{Z}_p[\![t]\!], \quad m \geq l$$

and this implies

$$\frac{F^*(t^p)}{F_{\underline{a}^{(1)}}(t^p)} \equiv \frac{[F^*(t^p)]_{<p^n}}{[F_{\underline{a}^{(1)}}(t^p)]_{<p^n}} \mod p^{n-1}\mathbb{Z}_p[\![t]\!].$$

Therefore we have

$$\begin{aligned}\frac{F^*(t^p)}{F_{\underline{a}}(t)} &= \frac{F_{\underline{a}^{(1)}}(t^p)}{F_{\underline{a}}(t)}\frac{F^*(t^p)}{F_{\underline{a}^{(1)}}(t^p)} \\ &\equiv \frac{[F_{\underline{a}^{(1)}}(t^p)]_{<p^n}}{[F_{\underline{a}}(t)]_{<p^n}}\frac{[F^*(t^p)]_{<p^n}}{[F_{\underline{a}^{(1)}}(t^p)]_{<p^n}} = \frac{[F^*(t^p)]_{<p^n}}{[F_{\underline{a}}(t)]_{<p^n}} \mod p^{n-1}\mathbb{Z}_p[\![t]\!].\end{aligned}$$

If $p \geq 3$, then $\mathrm{ord}_p(p^{n^*-i+1}i!) = \mathrm{ord}_p(p^{n-i+1}i!) \leq n-1$ for any $i \geq 2$, and hence (3.10) follows. If $p = 2$, then $\mathrm{ord}_p(p^{n-i+1}i!) \leq n$ but not necessarily $\mathrm{ord}_p(p^{n-i+1}i!) \leq n-1$. If $e \in 2W \setminus \{0\}$, then $\mathrm{ord}_p(p^{n^*-i+1}i!e^{-i}) = \mathrm{ord}_p(p^{n-i+1}i!e^{-i}) \leq n-i < n-1$, and hence (3.10) follows. If e is a unit, then $\mathrm{ord}_p(p^{n^*-i+1}i!e^{-i}) = \mathrm{ord}_p(p^{n-i}i!) \leq n-1$ for any $i \geq 2$, as required again. This completes the proof. □

3.4. Proof of Congruence relations : Preliminary lemmas. Until the end of Section 3.5, let σ be the Frobenius given by $\sigma(t) = t^p$ (i.e. $c = 1$). Therefore

$$(3.11) \qquad B_0 = \psi_p(a_1) + \cdots + \psi_p(a_s) + s\gamma_p, \quad B_i = \frac{A_i - \widetilde{A}_{i/p}}{i}, \quad i \in \mathbb{Z}_{\geq 1}$$

where the notation is as in (3.2) and we always mean $A_{i/p} = \widetilde{A}_{i/p} = 0$ unless $p|i$.

Lemma 3.6. *For an p-adic integer $\alpha \in \mathbb{Z}_p$ and $n \in \mathbb{Z}_{\geq 1}$, we define*

$$\{\alpha\}_n := \prod_{\substack{1\leq i\leq n \\ p\nmid(a+i-1)}} (\alpha + i - 1),$$

and $\{\alpha\}_0 := 1$. Let $a \in \mathbb{Z}_p \setminus \mathbb{Z}_{\leq 0}$, and $l \in \{0, 1, \ldots, p-1\}$ the integer such that $a \equiv -l$ mod p. Then for any $m \in \mathbb{Z}_{\geq 0}$, we have

$$m \equiv 0, 1, \ldots, l \bmod p \Rightarrow \frac{(a)_m}{m!}\left(\frac{(a')_{\lfloor m/p\rfloor}}{\lfloor m/p\rfloor!}\right)^{-1} = \frac{\{a\}_m}{\{1\}_m} \in \mathbb{Z}_p^\times,$$

$$m \equiv l+1, \ldots, p-1 \bmod p \Rightarrow \frac{(a)_m}{m!}\left(\frac{(a')_{\lfloor m/p\rfloor}}{\lfloor m/p\rfloor!}\right)^{-1} = \left(a + l + p\left\lfloor\frac{m}{p}\right\rfloor\right)\frac{\{a\}_m}{\{1\}_m}$$

where $a' = a^{(1)}$ is the Dwork prime.

Proof. Straightforward. □

Lemma 3.7 (Dwork). *For any $m \in \mathbb{Z}_{\geq 0}$, $A_m/\widetilde{A}_{\lfloor m/p \rfloor}$ are p-adic integers, and*

$$m_1 \equiv m_2 \mod p^n \quad \Longrightarrow \quad A_{m_1}/\widetilde{A}_{\lfloor m_1/p \rfloor} \equiv A_{m_2}/\widetilde{A}_{\lfloor m_2/p \rfloor} \mod p^n.$$

Proof. This is [12, p. 36, Cor. 1], or one can easily show this by using Lemma 3.6 on noticing the fact that

$$\{\alpha\}_{p^n} \equiv \prod_{i \in (\mathbb{Z}/p^n\mathbb{Z})^\times} i \equiv \begin{cases} 1 & p=2,\, n \neq 2 \\ -1 & \text{otherwise} \end{cases} \mod p^n. \qquad \square$$

Lemma 3.8. *Let $a \in \mathbb{Z}_p \setminus \mathbb{Z}_{\leq 0}$ and $m, n \in \mathbb{Z}_{\geq 1}$. Then*

$$1 - \frac{(a')_{mp^{n-1}}}{(mp^{n-1})!}\left(\frac{(a)_{mp^n}}{(mp^n)!}\right)^{-1} \equiv mp^n(\psi_p(a) + \gamma_p) \mod p^{2n}. \tag{3.12}$$

Moreover $\widetilde{A}_{mp^{n-1}}/A_{mp^n}$ and B_k/A_k are p-adic integers for all $k, m \geq 0$, $n \geq 1$, and

$$\frac{\widetilde{A}_{mp^{n-1}}}{A_{mp^n}} \equiv 1 - mp^n(\psi_p(a_1) + \cdots + \psi_p(a_s) + s\gamma_p) \mod p^{2n}, \tag{3.13}$$

$$p \nmid m \quad \Longrightarrow \quad \frac{B_{mp^n}}{A_{mp^n}} = \frac{1 - \widetilde{A}_{mp^{n-1}}/A_{mp^n}}{mp^n} \equiv B_0 \mod p^n. \tag{3.14}$$

Proof. We already see that $\widetilde{A}_{mp^{n-1}}/A_{mp^n} \in \mathbb{Z}_p$ in Lemma 3.6. It is enough to show (3.12). Indeed (3.13) is immediate from (3.12), and (3.14) is immediate from (3.13). Moreover (3.13) also implies that $B_k/A_k \in \mathbb{Z}_p$ for any $k \in \mathbb{Z}_{\geq 0}$.

Let us show (3.12). Let $a = -l + p^n b$ with $l \in \{0, 1, \ldots, p^n - 1\}$. Then

$$\begin{aligned} \frac{(a')_{mp^{n-1}}}{(mp^{n-1})!}\left(\frac{(a)_{mp^n}}{(mp^n)!}\right)^{-1} &= \frac{\{1\}_{mp^n}}{\{a\}_{mp^n}} \\ &= \prod_{\substack{l<k<mp^n \\ k-l \not\equiv 0 \bmod p}} \frac{k-l}{k-l+p^n b} \times \prod_{\substack{0 \leq k < l \\ k-l \not\equiv 0 \bmod p}} \frac{k-l+mp^n}{k-l+p^n b} \end{aligned}$$

by Lemma 3.6. If $(p,n)\neq(2,1)$, we have

$$\begin{aligned}\frac{\{1\}_{mp^n}}{\{a\}_{mp^n}} &\equiv \prod_{\substack{l<k<mp^n\\ p\nmid k-l}}\left(1-\frac{p^nb}{k-l}\right)\prod_{\substack{0\le k<l\\ p\nmid k-l}}\left(1-\frac{p^n(b-m)}{k-l}\right)\\ &\equiv 1-p^n\left(\sum_{\substack{l<k<mp^n\\ p\nmid k-l}}\frac{b}{k-l}+\sum_{\substack{0\le k<l\\ p\nmid k-l}}\frac{b-m}{k-l}\right)\\ &\overset{(*)}{\equiv} 1-mp^n\sum_{\substack{l<k<mp^n\\ p\nmid k-l}}\frac{1}{k-l}=1-mp^n\sum_{1\le k<mp^n-l,p\nmid k}\frac{1}{k}\\ &\overset{(2.13)}{\equiv} 1-mp^n(\psi_p(a)+\gamma_p)\end{aligned}$$

modulo p^{2n}, where $(*)$ follows from Lemma 2.5. This completes the proof of (3.12) in case $(p,n)\neq(2,1)$. In case $(p,n)=(2,1)$, we need another observation (since the equivalence $(*)$ breaks down). In this case we have

$$\begin{aligned}\frac{\{1\}_{2m}}{\{a\}_{2m}} &\equiv 1-2\left(\sum_{\substack{l<k<2m\\ 2\nmid k-l}}\frac{b}{k-l}+\sum_{\substack{0\le k<l\\ 2\nmid k-l}}\frac{b-m}{k-l}\right) \mod 4\\ &= 1-2\left(\sum_{\substack{l<k<2m\\ 2\nmid k-l}}\frac{m}{k-l}+\sum_{\substack{0\le k<2m\\ 2\nmid k-l}}\frac{b-m}{k-l}\right)\\ &\equiv 1-2m\left(\sum_{0<k<2m-l,\, 2\nmid k}\frac{1}{k}+b-m\right) \mod 4\end{aligned}$$

and

$$\psi_2(a)+\gamma_2\equiv\sum_{0<k<L,\, 2\nmid k}\frac{1}{k} \mod 2$$

where $L\in\{0,1,2,3\}$ such that $a=-l+2b\equiv L \mod 4$. Therefore (3.12) is equivalent to

$$m\left(\sum_{0<k<2m-l,\, 2\nmid k}\frac{1}{k}-\sum_{0<k<L,\, 2\nmid k}\frac{1}{k}+b-m\right)\equiv 0 \mod 2.$$

We may assume that $m>0$ is odd and $b=0,1$ (hence $a=0,\pm1,2$). Then one can check this on a case-by-case analysis. □

Lemma 3.9. *For any* $m_1, m_2 \in \mathbb{Z}_{\geq 0}$ *and* $n \in \mathbb{Z}_{\geq 1}$*, we have*

$$m_1 \equiv m_2 \mod p^n \quad \Longrightarrow \quad \frac{B_{m_1}}{A_{m_1}} \equiv \frac{B_{m_2}}{A_{m_2}} \mod p^n.$$

Proof. If $p \nmid m_i$, then $B_{m_i}/A_{m_i} = 1/m_i$ and hence the assertion is obvious. Let $m_1 = kp^i$ with $i \geq 1$ and $p \nmid k$. It is enough to show the assertion in case $m_2 = m_1 + p^n$. If $n \leq i$, then

$$\frac{B_{m_1}}{A_{m_1}} \equiv \frac{B_{m_2}}{A_{m_2}} \equiv B_0 \mod p^n$$

by (3.14). Suppose $n \geq i$. Notice that for $m \in p\mathbb{Z}_{\geq 0}$

$$1 - m\frac{B_m}{A_m} = \frac{\widetilde{A}_{m/p}}{A_m} = \prod_{r=1}^{s} \frac{\{1\}_m}{\{a_r\}_m}$$

by (3.11) and Lemma 3.6. We have

$$\begin{aligned}
1 - m_2\frac{B_{m_2}}{A_{m_2}} &\\
&= \prod_r \frac{\{1\}_{kp^i+p^n}}{\{a_r\}_{kp^i+p^n}} = \prod_r \frac{\{1\}_{kp^i}}{\{a_r\}_{kp^i}} \frac{\{1+kp^i\}_{p^n}}{\{a_r+kp^i\}_{p^n}} \\
&= \left(1 - m_1\frac{B_{m_1}}{A_{m_1}}\right) \prod_r \frac{\{1+kp^i\}_{p^n}}{\{a_r+kp^i\}_{p^n}} \\
&= \left(1 - m_1\frac{B_{m_1}}{A_{m_1}}\right) \prod_r \frac{\{1\}_{p^n}}{\{a_r+kp^i\}_{p^n}} \frac{\{1+kp^i\}_{p^n}}{\{1\}_{p^n}} \\
&\overset{(*)}{\equiv} \left(1 - m_1\frac{B_{m_1}}{A_{m_1}}\right) \prod_r (1 - p^n(\psi_p(a_r+kp^i) - \psi_p(1+kp^i))) \mod p^{2n} \\
&\overset{(**)}{\equiv} \left(1 - m_1\frac{B_{m_1}}{A_{m_1}}\right)(1 - p^nB_0) \mod p^{n+i}.
\end{aligned}$$

Here $(*)$ follows from Lemmas 3.6 and 3.8. The equivalence $(**)$ follows from Theorem 2.6 (1) and (2.13) in case $(p,i) \neq (2,1)$, and in case $(p,i) = (2,1)$, it does from the fact that

$$\psi_2(z+2) - \psi_2(z) \equiv 1 \mod 2, \quad z \in \mathbb{Z}_2.$$

Therefore we have

$$kp^i\left(\frac{B_{m_2}}{A_{m_2}} - \frac{B_{m_1}}{A_{m_1}}\right) \equiv p^n\left(B_0 - \frac{B_{m_2}}{A_{m_2}}\right) \mod p^{i+n}.$$

By (3.14), the right hand side vanishes. This is the desired assertion. □

Lemma 3.10. *Put* $S_m := \sum_{i+j=m} A_{i+p^n}B_j - A_iB_{j+p^n}$ *for* $m \in \mathbb{Z}_{\geq 0}$*. Then*

$$S_m \equiv \sum_{i+j=m} (A_{i+p^n}A_j - A_iA_{j+p^n})\frac{B_j}{A_j} \mod p^n.$$

Proof.

$$\begin{aligned}S_m &= \sum_{i+j=m} A_{i+p^n}B_j - A_iA_{j+p^n}\frac{B_{j+p^n}}{A_{j+p^n}} \\ &\equiv \sum_{i+j=m} A_{i+p^n}B_j - A_iA_{j+p^n}\frac{B_j}{A_j} \mod p^n \quad \text{(Lemma 3.9)} \\ &= \sum_{i+j=m} (A_{i+p^n}A_j - A_iA_{j+p^n})\frac{B_j}{A_j}\end{aligned}$$

as required. □

Lemma 3.11.

$$S_m \equiv \sum_{i+j=m} (\widetilde{A}_{\lfloor j/p \rfloor}\widetilde{A}_{\lfloor i/p \rfloor + p^{n-1}} - \widetilde{A}_{\lfloor i/p \rfloor}\widetilde{A}_{\lfloor j/p \rfloor + p^{n-1}})\frac{A_i}{\widetilde{A}_{\lfloor i/p \rfloor}}\frac{A_j}{\widetilde{A}_{\lfloor j/p \rfloor}}\frac{B_j}{A_j} \mod p^n.$$

Proof. This follows from Lemma 3.10 and Lemma 3.7. □

Lemma 3.12. *For all $m, k, s \in \mathbb{Z}_{\geq 0}$ and $0 \leq l \leq n$, we have*

$$\sum_{\substack{i+j=m \\ i \equiv k \bmod p^{n-l}}} A_iA_{j+p^{n-1}} - A_jA_{i+p^{n-1}} \equiv 0 \mod p^l. \tag{3.15}$$

Proof. There is nothing to prove in case $l = 0$. If $l = n$, then (3.15) is obvious as

$$\text{LHS} = \sum_{i+j=m} A_iA_{j+p^{n-1}} - A_jA_{i+p^{n-1}} = 0.$$

Suppose $1 \leq l \leq n-1$. We simply write

$$F^{(r)}(t) = F_{\underline{a}^{(r)}}(t) = \sum_{i=0}^{\infty} \frac{(a_1^{(r)})_i}{i!} \cdots \frac{(a_s^{(r)})_i}{i!} t^i, \quad F(t) = F^{(0)}(t).$$

For $k \in \mathbb{Z}_{\geq 0}$, we put

$$F_k^{(r)}(t) := \sum_{i \equiv k \bmod p^{n-l}} \frac{(a_1^{(r)})_i}{i!} \cdots \frac{(a_s^{(r)})_i}{i!} t^i = \frac{1}{p^{n-l}} \sum_{s=0}^{p^{n-l}-1} \zeta^{-sk} F(\zeta^s t) \tag{3.16}$$

where ζ is a primitive p^{n-l}-th root of unity. Then (3.15) is equivalent to

$$F_k(t) \cdot [F_{m-k}(t)]_{<p^{n-1}} \equiv [F_k(t)]_{<p^{n-1}} \cdot F_{m-k}(t) \mod p^l. \tag{3.17}$$

It follows from the Dwork congruence [12, p. 37, Thm. 2] that one has

$$\frac{F^{(i)}(t)}{F^{(i+1)}(t^p)} \equiv \frac{[F^{(i)}(t)]_{<p^m}}{[F^{(i+1)}(t^p)]_{<p^m}} \mod p^n$$

for any $m \geq n \geq 1$. This implies

$$\frac{F^{(i)}(t^p)}{F^{(i+1)}(t^{p^2})} \equiv \frac{[F^{(i)}(t^p)]_{<p^{n+1}}}{[F^{(i+1)}(t^{p^2})]_{<p^{n+1}}} \mod p^n,$$
$$\frac{F^{(i)}(t^{p^2})}{F^{(i+1)}(t^{p^3})} \equiv \frac{[F^{(i)}(t^{p^2})]_{<p^{n+2}}}{[F^{(i+1)}(t^{p^3})]_{<p^{n+2}}} \mod p^n,$$

and so on. Hence we have

$$\begin{aligned}\frac{F(t)}{F^{(n-l)}(t^{p^{n-l}})} &= \frac{F(t)}{F^{(1)}(t^p)}\frac{F^{(1)}(t^p)}{F^{(2)}(t^{p^2})}\cdots\frac{F^{(n-l-1)}(t^{p^{n-l-1}})}{F^{(n-l)}(t^{p^{n-l}})}\\ &\equiv \frac{[F(t)]_{<p^d}}{[F^{(1)}(t^p)]_{<p^d}}\frac{[F(t^p)]_{<p^d}}{[F^{(1)}(t^{p^2})]_{<p^d}}\cdots\frac{[F^{(n-l-1)}(t^{p^{n-l-1}})]_{<p^d}}{[F^{(n-l)}(t^{p^{n-l}})]_{<p^d}}\\ &= \frac{[F(t)]_{<p^d}}{[F^{(n-l)}(t^{p^{n-l}})]_{<p^d}}\end{aligned}$$

modulo $p^{d-n+l+1}\mathbb{Z}_p[\![t]\!]$, namely there are $a_i \in \mathbb{Z}_p$ such that

$$\frac{F(t)}{F^{(n-l)}(t^{p^{n-l}})} = \frac{[F(t)]_{<p^d}}{[F^{(n-l)}(t^{p^{n-l}})]_{<p^d}} + p^{d-n+l+1}\sum_i a_i t^i.$$

Substitute t for $\zeta^s t$ in the above and multiply it by

$$\left(\frac{F(t)}{F^{(n-l)}(t^{p^{n-l}})}\right)^{-1} = \left(\frac{[F(t)]_{<p^d}}{[F^{(n-l)}(t^{p^{n-l}})]_{<p^d}} + p^{d-n+l+1}\sum_i a_i t^i\right)^{-1}.$$

Then we have

$$F(\zeta^s t)\cdot[F(t)]_{<p^d} - [F(\zeta^s t)]_{<p^d}\cdot F(t) = p^{d-n+l+1}\sum_{i=0}^{\infty} b_i(\zeta^s)t^i$$

where $b_i(x) \in \mathbb{Z}_p[x]$ are polynomials which do not depend on s. Applying $\sum_{s=0}^{p^{n-l}-1}\zeta^{-sk}(\,\cdot\,)$ on both side, one has

$$p^{n-l}F_k(t)[F(t)]_{<p^d} - p^{n-l}[F_k(t)]_{<p^d}F(t) = p^{d-n+l+1}\sum_{i=0}^{\infty}\sum_{s=0}^{p^{n-l}-1}\zeta^{-sk}b_i(\zeta^s)t^i$$

by (3.16). Since $\sum_{s=0}^{p^{n-l}-1}\zeta^{sj} = 0$ or p^{n-l}, the right hand side is zero modulo p^{d+1}. Therefore

$$\frac{F_k(t)}{F(t)} \equiv \frac{[F_k(t)]_{<p^d}}{[F(t)]_{<p^d}} \mod p^{d-n+l+1}\mathbb{Z}_p[\![t]\!].$$

This implies

$$\frac{F_k(t)[F_j(t)]_{<p^d} - [F_k(t)]_{<p^d} F_j(t)}{F(t)} \equiv \frac{[F_k(t)]_{<p^d}[F_j(t)]_{<p^d} - [F_k(t)]_{<p^d}[F_j(t)]_{<p^d}}{[F(t)]_{<p^d}}$$

modulo $p^{d-n+l+1}$, and the right hand side vanishes. Now (3.17) is the case $(d, j) = (n-1, s-k)$. ☐

3.5. Proof of Congruence relations : End of proof. We finish the proof of Theorem 3.3. Let S_m be as in Lemma 3.10. The goal is to show

$$S_m \equiv 0 \mod p^n, \quad \forall\ m \geq 0.$$

Let us put

$$q_i := A_i/\widetilde{A}_{\lfloor i/p \rfloor},\ A(i,j) := \widetilde{A}_i \widetilde{A}_j,\ A^*(i,j) := A(j, i+p^{n-1}) - A(i, j+p^{n-1})$$
$$B(i,j) := A^*(\lfloor i/p \rfloor, \lfloor j/p \rfloor).$$

Then

$$S_m \equiv \sum_{i+j=m} B(i,j) q_i q_j \frac{B_j}{A_j} \mod p^n$$

by Lemma 3.11. It follows from Lemma 3.9 and Lemma 3.7 that we have

$$k \equiv k' \mod p^i \implies \frac{B_k}{A_k} \equiv \frac{B_{k'}}{A_{k'}},\ q_k \equiv q_{k'} \mod p^i. \tag{3.18}$$

By Lemma 3.12, we have

$$\sum_{\substack{i+j=s \\ i \equiv k \bmod p^{n-l}}} A^*(i,j) \equiv 0 \mod p^l, \quad 0 \leq l \leq n \tag{3.19}$$

for all $s \geq 0$. Let $m = l + sp$ with $l \in \{0, 1, \dots, p-1\}$. Note

$$B(i, m-i) = \begin{cases} A^*(k, s-k) & kp \leq i \leq kp + l \\ A^*(k, s-k-1) & kp + l < i \leq (k+1)p - 1. \end{cases}$$

Therefore

$$\begin{aligned}
S_m &\equiv \sum_{i+j=m} B(i,j) q_i q_j \frac{B_j}{A_j} \mod p^n \\
&= \sum_{i=0}^{p-1} \sum_{k=0}^{\lfloor (m-i)/p \rfloor} B(i+kp, m-(i+kp)) q_{i+kp} q_{m-(i+kp)} \frac{B_{m-(i+kp)}}{A_{m-(i+kp)}} \\
&= \sum_{k=0}^{s} B(i+kp, m-(i+kp)) \sum_{i=0}^{l} q_{i+kp} q_{m-(i+kp)} \frac{B_{m-(i+kp)}}{A_{m-(i+kp)}} \\
&\quad + \sum_{k=0}^{s-1} B(i+kp, m-(i+kp)) \sum_{i=l+1}^{p-1} q_{i+kp} q_{m-(i+kp)} \frac{B_{m-(i+kp)}}{A_{m-(i+kp)}} \\
&= \sum_{k=0}^{s} A^*(k, s-k) \overbrace{\left(\sum_{i=0}^{l} q_{i+kp} q_{m-(i+kp)} \frac{B_{m-(i+kp)}}{A_{m-(i+kp)}} \right)}^{P_k} \\
&\quad + \sum_{k=0}^{s-1} A^*(k, s-k-1) \underbrace{\left(\sum_{i=l+1}^{p-1} q_{i+kp} q_{m-(i+kp)} \frac{B_{m-(i+kp)}}{A_{m-(i+kp)}} \right)}_{Q_k}.
\end{aligned}$$

We show that the first term vanishes modulo p^n. It follows from (3.18) that we have

$$k \equiv k' \mod p^i \implies P_k \equiv P_{k'} \mod p^{i+1}. \tag{3.20}$$

Therefore one can write

$$\sum_{k=0}^{s} A^*(k, s-k) P_k \equiv \sum_{i=0}^{p^{n-1}-1} P_i \overbrace{\left(\sum_{k \equiv i \bmod p^{n-1}} A^*(k, s-k) \right)}^{(*)} \mod p^n.$$

It follows from (3.19) that $(*)$ is zero modulo p. Therefore, again by (3.20), one can rewrite

$$\sum_{k=0}^{s} A^*(k, s-k) P_k \equiv \sum_{i=0}^{p^{n-2}-1} P_i \overbrace{\left(\sum_{k \equiv i \bmod p^{n-2}} A^*(k, s-k) \right)}^{(**)} \mod p^n.$$

It follows from (3.19) that $(**)$ is zero modulo p^2, so that one has

$$\sum_{k=0}^{s} A^*(k, s-k) P_k \equiv \sum_{i=0}^{p^{n-3}-1} P_i \left(\sum_{k \equiv i \bmod p^{n-3}} A^*(k, s-k) \right) \mod p^n$$

by (3.20). Continuing the same discussion, one finally obtains

$$\sum_{k=0}^{s} A^*(k, s-k)P_k \equiv \sum_{k=0}^{s} A^*(k, s-k) = 0 \mod p^n$$

the vanishing of the first term. In the same way one can show the vanishing of the second term,

$$\sum_{k=0}^{s} A^*(k, s-1-k)Q_k \equiv 0 \mod p^n.$$

We thus have $S_m \equiv 0 \bmod p^n$. This completes the proof of Theorem 3.3.

4. Geometric aspect of *p*-adic hypergeometric functions of logarithmic type

We mean by a *fibration* over a ring R a projective flat morphism of quasi-projective smooth R-schemes. Let X be a smooth R-scheme. We mean by a relative *normal crossing divisor* (abbreviated to NCD) in X over R a divisor in X which is locally defined by an equation $x_1^{r_1} \cdots x_s^{r_s}$ where $r_i > 0$ are integers and $(x_1, \ldots, x_n)$ is a local coordinate of X/R. We say a divisor D simple if D is a union of R-smooth divisors.

4.1. Hypergeometric Curves. Let A be a commutative ring. Let $\mathbb{P}^1_A(Z_0, Z_1)$ denote the projective line over A with homogeneous coordinate $(Z_0 : Z_1)$. Let $N, M \geq 2$ be integers which are invertible in A. Let $t \in A$ such that $t(1-t) \in A^\times$. Define X to be a projective scheme over A defined by a bihomogeneous equation

$$(X_0^N - X_1^N)(Y_0^M - Y_1^M) = tX_0^N Y_0^M \tag{4.1}$$

in $\mathbb{P}^1_A(X_0, X_1) \times \mathbb{P}^1_A(Y_0, Y_1)$. We call it a *hypergeometric curve* over A. The morphism $X \to \operatorname{Spec} A$ is smooth projective with connected fibers of relative dimension one, and the genus of a geometric fiber is $(N-1)(M-1)$ (e.g. Hurwitz formula). We put $x := X_1/X_0$ and $y := Y_1/Y_0$, and often refer to an affine equation

$$(1 - x^N)(1 - y^M) = t.$$

In what follows, we only consider the case $A = W[t, (t-t^2)^{-1}]$ where W is a commutative ring in which NM is invertible, and t is an indeterminate.

Lemma 4.1. *Then the morphism $X \to \operatorname{Spec} W[t, (t-t^2)^{-1}]$ extends to a projective flat morphism*

$$f : Y \longrightarrow \mathbb{P}^1_W = \mathbb{P}^1_W(T_0, T_1)$$

of smooth projective W-schemes satisfying the following conditions.

(1) *f has a semistable reduction at $t=0$. The fiber $D := f^{-1}(t=0)$ is a relative simple NCD over W, and the multiplicity of each component is one.*
(2) *The fiber $f^{-1}(t=1)$ is a relative simple NCD over W, and the multiplicity of each component is either of $1, iN, jM$ with $i \in \{1,\dots,M\}$, $\le NM$.*
(3) *The fiber $f^{-1}(t=\infty)$ is a relative simple NCD over W, and the multiplicity of each component is $\le \max(N,M)$.*

Proof. See [2, §3.1] where the explicit construction of resolution of singularities is done in [2, App. B]. □

4.2. Gauss–Manin connection. In this section we assume that W is an integral domain of characteristic zero. Let $K = \operatorname{Frac} W$ be the fractional field. Let $A = W[t,(t-t^2)^{-1}]$ and $S = \operatorname{Spec} A$. For a W-scheme T and a W-algebra R, we write $T_R = T\times_W R$. The group $\mu_N\times\mu_M = \mu_N(\overline{K})\times\mu_M(\overline{K})$ acts on $X_{\overline{K}}$ in the following way

$$[\zeta,\nu]\cdot(x,y,t) = (\zeta x, \nu y, t), \quad (\zeta,\nu)\in\mu_N\times\mu_M. \tag{4.2}$$

For a $\overline{K}$-module V with an action of $\mu_N\times\mu_M$, let $V(i,j)$ denote the submodule on which (ζ,ν) acts by multiplication by $\zeta^i\nu^j$ for all $(\zeta,\nu)\in\mu_N\times\mu_M$. Then one has the eigen decomposition

$$H^1_{\mathrm{dR}}(X_{\overline{K}}/S_{\overline{K}}) = \bigoplus_{i=1}^{N-1}\bigoplus_{j=1}^{M-1} H^1_{\mathrm{dR}}(X_{\overline{K}}/S_{\overline{K}})(i,j),$$

and each eigenspace $H^1_{\mathrm{dR}}(X_{\overline{K}}/S_{\overline{K}})(i,j)$ is free of rank 2 over $\mathscr{O}(S_{\overline{K}})$ ([1, Lem. 2.2]). Put

$$a_i := 1-\frac{i}{N}, \quad b_j := 1-\frac{j}{M}. \tag{4.3}$$

Let

$$\omega_{i,j} := N\,\frac{x^{i-1}y^{j-M}}{1-x^N}\mathrm{d}x = -M\,\frac{x^{i-N}y^{j-1}}{1-y^M}\mathrm{d}y, \tag{4.4}$$

$$\eta_{i,j} := \frac{1}{x^N-1+t}\omega_{i,j} = Mt^{-1}x^{i-N}y^{j-M-1}\mathrm{d}y \tag{4.5}$$

for integers i,j such that $1\le i\le N-1$, $1\le j\le M-1$. Then $\omega_{i,j}$ is the 1st kind, and $\eta_{i,j}$ is the 2nd kind. They form a $\mathscr{O}(S_{\overline{K}})$-free basis of $H^1_{\mathrm{dR}}(X_{\overline{K}}/S_{\overline{K}})(i,j)$. According to this, we put

$$H^1_{\mathrm{dR}}(X_K/S_K)(i,j) := \mathscr{O}(S_K)\omega_{i,j} + \mathscr{O}(S_K)\eta_{i,j} \subset H^1_{\mathrm{dR}}(X_K/S_K).$$

Let

$$F_{a,b}(t) := {}_2F_1\left({a,b \atop 1};t\right) = \sum_{i=0}^{\infty}\frac{(a)_i}{i!}\frac{(b)_i}{i!}t^i \in K[\![t]\!]$$

be the hypergeometric series. Put

$$\begin{aligned}\widetilde{\omega}_{i,j} &:= \frac{1}{F_{a_i,b_j}(t)}\omega_{i,j},\\ \widetilde{\eta}_{i,j} &:= -t(1-t)^{a_i+b_j}(F'_{a_i,b_j}(t)\omega_{i,j}+b_jF_{a_i,b_j}(t)\eta_{i,j}).\end{aligned} \tag{4.6}$$

For the later use, we give notation $V(i,j), \omega_{i,j}, \eta_{i,j}, \widetilde{\omega}_{i,j}, \widetilde{\eta}_{i,j}$ for (i,j) not necessarily a pair of integers. Let $(i,j) = (q/r, q'/r') \in \mathbb{Q}^2$ such that $\gcd(r,N) = \gcd(r',M) = 1$ and $N \nmid q$ and $M \nmid q'$. Let i_0, j_0 be the unique integers such that $i_0 \equiv i \bmod N$, $j_0 \equiv j \bmod M$ and $1 \le i_0 < N$, $1 \le j_0 < M$. Then we define

$$V(i,j) = V(i_0,j_0), \quad \omega_{i,j} = \omega_{i_0,j_0}, \quad \dots \quad \widetilde{\eta}_{i,j} = \widetilde{\eta}_{i_0,j_0}. \tag{4.7}$$

Proposition 4.2. *Let* $\nabla : H^1_{\mathrm{dR}}(X_K/S_K) \to \mathscr{O}(S_K)\mathrm{d}t \otimes H^1_{\mathrm{dR}}(X_K/S_K)$ *be the Gauss–Manin connection. It naturally extends on* $K((t)) \otimes_{\mathscr{O}(S)} H^1_{\mathrm{dR}}(X_K/S_K)$ *which we also write by* ∇. *Then*

$$\begin{pmatrix}\nabla(\omega_{i,j}) & \nabla(\eta_{i,j})\end{pmatrix} = \mathrm{d}t \otimes \begin{pmatrix}\omega_{i,j} & \eta_{i,j}\end{pmatrix}\begin{pmatrix}0 & -a_i(t-t^2)^{-1}\\ -b_j & (-1+(1+a_i+b_j)t)(t-t^2)^{-1}\end{pmatrix},$$

$$\begin{pmatrix}\nabla(\widetilde{\omega}_{i,j}) & \nabla(\widetilde{\eta}_{i,j})\end{pmatrix} = \mathrm{d}t \otimes \begin{pmatrix}\widetilde{\omega}_{i,j} & \widetilde{\eta}_{i,j}\end{pmatrix}\begin{pmatrix}0 & 0\\ t^{-1}(1-t)^{-a_i-b_j}F_{a_i,b_j}(t)^{-2} & 0\end{pmatrix}.$$

Proof. We may replace the base field with $\mathbb{C}$. Since ∇ is commutative with the action of $\mu_N(\mathbb{C}) \times \mu_M(\mathbb{C})$, ∇ preserves the eigen components $H^1_{\mathrm{dR}}(X/S)(i,j)$. We think X and S of being complex manifolds. For $\alpha \in \mathbb{C} \setminus \{0,1\}$ we write $X_\alpha = f^{-1}(t = \alpha)$. Then there is a homology cycle $\delta_\alpha \in H_1(X_\alpha, \mathbb{Q})$ such that

$$\int_{\delta_\alpha} \omega_{i,j} = 2\pi\sqrt{-1}\,{}_2F_1\left({a_i, b_j \atop 1}; \alpha\right)$$

([1, Lem. 2.3]). Let $\partial_t = \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}}$ be the differential operator on $\mathscr{O}(S)^{an} \otimes_{\mathscr{O}(S)} H^1_{\mathrm{dR}}(X/S)$. Put $D = t\partial_t$ and $P_{\mathrm{HG}} = D^2 - t(D+a_i)(D+b_j)$ the hypergeometric differential operator. Since the series ${}_2F_1\left({a_i,b_j \atop 1};t\right)$ is annihilated by P_{HG}, we have

$$\int_{\delta_\alpha} P_{\mathrm{HG}}(\omega_{i,j}) = 0.$$

Since $H_1(X_\alpha, \mathbb{C})(i,j)$ is a 2-dimensional irreducible $\pi_1(S,\alpha)$-module, we have $\int_\gamma P_{\mathrm{HG}}(\omega_{i,j}) = 0$ for all $\gamma \in \pi_1(S,\alpha)$, which means

$$P_{\mathrm{HG}}(\omega_{i,j}) = (D^2 - t(D+a_i)(D+b_j))(\omega_{i,j}) = 0. \tag{4.8}$$

Next we show

$$\partial_t(\omega_{i,j}) = -b_j\eta_{i,j}. \tag{4.9}$$

Put $\phi := N\frac{x^{i-1}}{1-x^N}\mathrm{d}x \in \Gamma(U,\Omega^1_{X/\mathbb{C}})$ with $U = \{x \neq \infty, y \neq \infty\} \subset X$, and $\Omega_{i,j} := y^{j-M}\phi$ a lifting of $\omega_{i,j}$. Since ϕ is a linear combination of $\mathrm{d}x/(x-\nu)$'s, one has $d(\phi) = 0$. Therefore

$$d(\Omega_{i,j}) = d(y^{j-M}) \wedge \phi = (j-M)y^{j-M-1}\mathrm{d}y \wedge \phi \in \Gamma(U,\Omega^2_{X/\mathbb{C}}).$$

Taking $\wedge\phi$ on both sides of

$$\frac{\mathrm{d}t}{t} = \frac{Nx^{N-1}\mathrm{d}x}{x^N-1} + \frac{My^{M-1}\mathrm{d}y}{y^M-1},$$

we have

$$\frac{\mathrm{d}t}{t} \wedge \phi = \frac{My^{M-1}\mathrm{d}y}{y^M-1} \wedge \phi \iff \frac{y^M-1}{My^{M-1}}\frac{\mathrm{d}t}{t} \wedge \phi = \mathrm{d}y \wedge \phi,$$

and hence

$$\begin{aligned} d(\Omega_{i,j}) &= -(1-j/M)y^{j-M}\frac{y^M-1}{y^M}\frac{\mathrm{d}t}{t} \wedge \phi \\ &= -(1-j/M)\frac{y^M-1}{ty^M}\mathrm{d}t \wedge \Omega_{i,j} = -b_j\mathrm{d}t \wedge \frac{1}{x^N-1+t}\Omega_{i,j}. \end{aligned}$$

Since $(x^N-1+t)^{-1}\Omega_{i,j}$ is a lifting of $\eta_{i,j}$, this shows $\nabla(\omega_{i,j}) = -b_j\mathrm{d}t \otimes \eta_{i,j}$. This completes the proof of (4.9). Now all the formulas on ∇ follow from (4.8) and (4.9). □

The following is straightforward from Proposition 4.2.

Corollary 4.3. *Let $\nabla_{i,j}$ be the connection on the eigen component*

$$H_{i,j} := K((t)) \otimes_{\mathscr{O}(S)} H^1_{\mathrm{dR}}(X_K/S_K)(i,j).$$

Then $\operatorname{Ker}\nabla_{i,j} = K\widetilde{\eta}_{i,j}$. *Moreover let $\overline{\nabla}_{i,j}$ be the connection on $M_{i,j} := H_{i,j}/K((t))\widetilde{\eta}_{i,j}$ induced from $\nabla_{i,j}$. Then* $\operatorname{Ker}\overline{\nabla}_{i,j} = K\widetilde{\omega}_{i,j}$.

We mean by a semistable family $g : \mathscr{X} \to \operatorname{Spec} R[\![t]\!]$ over a commutative ring R that g is a proper flat morphism, smooth over $\operatorname{Spec} R((t))$ and it is locally described by

$$g : \operatorname{Spec} R[\![x_1,\ldots,x_n]\!] \longrightarrow \operatorname{Spec} R[\![t]\!], \quad g^*(t) = x_1\cdots x_r$$

in each formal neighborhood. Let D be the fiber at $\operatorname{Spec} R[\![t]\!]/(t)$, which is a relative NCD in $\mathscr{X}$ over R with no multiplicities. Let $\mathscr{U} := \mathscr{X} \setminus D$. We define the log de Rham complex $\omega^\bullet_{\mathscr{X}/R[\![t]\!]}$ to be the subcomplex of $\Omega^\bullet_{\mathscr{U}/R((t))}$ generated by $\mathrm{d}x_i/x_i$ ($1 \le i \le r$) and dx_j ($j > r$) over $\mathscr{O}_{\mathscr{X}}$. Equivalently,

$$\omega^\bullet_{\mathscr{X}/R[\![t]\!]} := \operatorname{Coker}\left[\frac{\mathrm{d}t}{t} \otimes \Omega^{\bullet-1}_{\mathscr{X}/R}(\log D) \to \Omega^\bullet_{\mathscr{X}/R}(\log D)\right] \tag{4.10}$$

where $\Omega^\bullet_{\mathscr{X}/R}(\log D)$ denotes the t-adic completion of the complex of the algebraic Kähler differentials,

$$\Omega^\bullet_{\mathscr{X}/R}(\log D) := \varprojlim_{n\geq 1} (\Omega^{\bullet,\mathrm{alg}}_{\mathscr{X}/R}(\log D)/t^n\Omega^{\bullet,\mathrm{alg}}_{\mathscr{X}/R}(\log D))$$

Corollary 4.4. *Let $f : Y \to \mathbb{P}^1_W$ be the morphism of projective smooth W-schemes in Lemma 4.1. Let $Y_K := Y \times_W K$. Let $\Delta_K := \operatorname{Spec} K[\![t]\!] \hookrightarrow \mathbb{P}^1_K$ be the formal neighborhood, and put $\mathscr{Y}_K := f^{-1}(\Delta_K)$. Let $D_K \subset \mathscr{Y}_K$ be the fiber at $t = 0$. Let*

$$\begin{array}{ccccc} D_K & \longrightarrow & \mathscr{Y}_K & \longrightarrow & Y_K \\ \downarrow & & \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \Delta_K & \longrightarrow & \mathbb{P}^1_K \end{array}$$

Put $H_K := H^1_{\mathrm{zar}}(\mathscr{Y}_K, \omega^\bullet_{\mathscr{Y}_K/K[\![t]\!]})$. It follows from [28, (2.18)–(2.20)] *or* [30, (17)] *that $H_K \to K((t)) \otimes_{\mathscr{O}(S)} H^1_{\mathrm{dR}}(X/S)$ is injective. We identify H_K with its image. Then the eigen component $H_K(i,j)$ is a free $K[\![t]\!]$-module with basis $\{\widetilde{\omega}_{i,j}, \widetilde{\eta}_{i,j}\}$.*

Proof. H_K is called *Deligne's canonical extension*, and is characterized by the following conditions ([30, (17)]).

- (D1) H_K is a free $K[\![\lambda]\!]$-module such that $K((t)) \otimes H_K = K((t)) \otimes_{\mathscr{O}(S)} H^1_{\mathrm{dR}}(X/S)$,
- (D2) the connection extends to have log pole, $\nabla : H_K \to \frac{dt}{t} \otimes H_K$,
- (D3) each eigenvalue α of $\mathrm{Res}(\nabla)$ satisfies $0 \leq \mathrm{Re}(\alpha) < 1$, where $\mathrm{Res}(\nabla)$ is the K-linear endomorphism defined by a commutative diagram

$$\begin{array}{ccc} H_K & \xrightarrow{\nabla} & \frac{\mathrm{d}t}{t} \otimes H_K \\ \downarrow & & \downarrow {\scriptstyle \mathrm{Res}\otimes 1} \\ H_K/tH_K & \xrightarrow{\mathrm{Res}(\nabla)} & H_K/tH_K. \end{array}$$

Put $H^0_K := \bigoplus_{i,j} K[\![t]\!]\widetilde{\omega}_{i,j} + K[\![t]\!]\widetilde{\eta}_{i,j}$. We can directly check that H^0_K satisfies (D1)–(D3) by Proposition 4.2. We then conclude $H_K = H^0_K$ thanks to the uniqueness of Deligne's canonical extension. ☐

4.3. Rigid cohomology and a category Fil-*F*-MIC(*S*). In what follows, let the base ring W be the Witt ring $W(\overline{\mathbb{F}}_p)$ of the algebraic closure $\overline{\mathbb{F}}_p$ with $p \nmid NM$. Let F be the p-th Frobenius on W, and $K := \operatorname{Frac} W$ the fractional field.

Let $f : X \to S$ be the hypergeometric curve as before. Write $X_{\overline{\mathbb{F}}_p} := X \times_W \overline{\mathbb{F}}_p$ and $S_{\overline{\mathbb{F}}_p} := S \times_W \overline{\mathbb{F}}_p$. Let σ be a F-linear p-th Frobenius on $W[t, (t - t^2)^{-1}]^\dagger$ the ring of overconvergent power series, which naturally extends on $K[t,(t-t^2)^{-1}]^\dagger := K\otimes W[t,(t-t^2)^{-1}]^\dagger$. Then the i-th *rigid cohomology* group

$$H^i_{\mathrm{rig}}(X_{\overline{\mathbb{F}}_p}/S_{\overline{\mathbb{F}}_p}) := \Gamma(S_K^{\mathrm{an}}, R^i f_{\mathrm{rig}} j_X^\dagger \mathscr{O}_{X_K^{\mathrm{an}}}),$$

is defined where $R^i f_{\mathrm{rig}} j_X^\dagger \mathscr{O}_{X_K^{\mathrm{an}}}$ is the i-th relative rigid cohomology sheaf (cf. [4, Def. 2.12] for the notation and remark on the definition). The required properties in below is the following (loc.cit).

- $H^\bullet_{\mathrm{rig}}(X_{\overline{\mathbb{F}}_p}/S_{\overline{\mathbb{F}}_p})$ is a finitely generated $\mathscr{O}(S)^\dagger = K[t,(t-t^2)^{-1}]^\dagger$-module.
- (Frobenius) The p-th Frobenius $\Phi_{X/S}$ on $H^\bullet_{\mathrm{rig}}(X_{\overline{\mathbb{F}}_p}/S_{\overline{\mathbb{F}}_p})$ (depending on σ) is defined in a natural way. This is a σ-linear endomorphism:

$$\Phi_{X/S}(h(t)x) = \sigma(h(t))\Phi_{X/S}(x), \quad \text{for } x\in H^\bullet_{\mathrm{rig}}(X_{\overline{\mathbb{F}}_p}/S_{\overline{\mathbb{F}}_p}),\ h(t)\in \mathscr{O}(S)^\dagger.$$

- (Comparison) There is the comparison isomorphism with the algebraic de Rham cohomology,

$$c : H^\bullet_{\mathrm{rig}}(X_{\overline{\mathbb{F}}_p}/S_{\overline{\mathbb{F}}_p}) \cong H^\bullet_{\mathrm{dR}}(X/S)\otimes_{\mathscr{O}(S)} \mathscr{O}(S)^\dagger.$$

In [4, §2.1] we introduce a category Fil-F-MIC(S) = Fil-F-MIC(S,σ). It consists of collections of datum $(H_{\mathrm{dR}}, H_{\mathrm{rig}}, c, \Phi, \nabla, \mathrm{Fil}^\bullet)$ such that

- H_{dR} is a finitely generated $\mathscr{O}(S)$-module,
- H_{rig} is a finitely generated $\mathscr{O}(S)^\dagger$-module,
- $c\colon H_{\mathrm{rig}} \cong H_{\mathrm{dR}}\otimes_{\mathscr{O}(S)} \mathscr{O}(S)^\dagger$, the comparison
- $\Phi\colon \sigma^* H_{\mathrm{rig}} \xrightarrow{\cong} H_{\mathrm{rig}}$ is an isomorphism of $\mathscr{O}(S)^\dagger$-module,
- $\nabla\colon H_{\mathrm{dR}} \to \Omega^1_{S/\mathbb{Q}_p}\otimes H_{\mathrm{dR}}$ is an integrable connection that satisfies $\Phi\nabla = \nabla\Phi$.
- $\mathrm{Fil}^\bullet$ is a finite descending filtration on H_{dR} of locally free $\mathscr{O}(S)$-module (i.e. each graded piece is locally free), that satisfies $\nabla(\mathrm{Fil}^i)\subset \Omega^1\otimes \mathrm{Fil}^{i-1}$.

Let $\mathrm{Fil}^\bullet$ denote the Hodge filtration on the de Rham cohomology, and ∇ the Gauss–Manin connection. Let

$$H^i(X/S) := (H^i_{\mathrm{dR}}(X/S), H^i_{\mathrm{rig}}(X_{\overline{\mathbb{F}}_p}/S_{\overline{\mathbb{F}}_p}), c, \Phi_{X/S}, \nabla, \mathrm{Fil}^\bullet)$$

be an object of Fil-F-MIC(S).

For an integer r, the Tate object $\mathscr{O}_S(r)\in$ Fil-F-MIC(S) is defined in a customary way (loc. cit.). We simply write

$$M(r) = M\otimes \mathscr{O}_S(r)$$

for an object $M\in$ Fil-F-MIC(S).

Lemma 4.5. *Suppose that σ is given by $\sigma(t) = ct^p$ with some $c\in 1+pW$. Then, with the notation in Corollary 4.4, the Frobenius $\Phi_{X/S}$ induces the action on H_K in a natural way.*

Proof. Let $W((t))^\wedge$ be the p-adic completion and write $K((t))^\wedge := K \otimes_W W((t))^\wedge$ on which σ extends as $\sigma(t) = ct^p$. The Frobenius $\Phi_{X/S}$ on $H^1_{\mathrm{dR}}(X/S) \otimes_{\mathscr{O}(S)} \mathscr{O}(S)^\dagger$ naturally extends on $H^1_{\mathrm{dR}}(X/S) \otimes_{\mathscr{O}(S)} K((t))^\wedge$ via the homomorphism $\mathscr{O}(S)^\dagger \to K((t))^\wedge$. We show that the action of $\Phi_{X/S}$ preserves the subspace H_K

Let $f : Y \to \mathbb{P}^1_W$ be the morphism of projective smooth W-schemes in Lemma 4.1. Let

$$\begin{array}{ccccc} D_W & \longrightarrow & \mathscr{Y} & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Delta_W & \longrightarrow & \mathbb{P}^1_W \end{array}$$

where $\Delta_W := \operatorname{Spec} W[\![t]\!] \hookrightarrow \mathbb{P}^1_W$ is the formal neighborhood, and $0 = \operatorname{Spec} W[\![t]\!]/(t)$. Note that D_W is reduced, namely $\mathscr{Y}/\Delta_W$ has a semistable reduction. Write $\mathscr{Y}_{\overline{\mathbb{F}}_p} := \mathscr{Y} \times_W \overline{\mathbb{F}}_p$ etc. We employ the log-crystalline cohomology

$$H^\bullet_{\text{log-crys}}((\mathscr{Y}_{\overline{\mathbb{F}}_p}, D_{\overline{\mathbb{F}}_p})/(\Delta_W, 0)) \tag{4.11}$$

where $(\mathscr{X}, \mathscr{D})$ denotes the log scheme with log structure induced by the divisor $\mathscr{D}$. There is the comparison theorem by Kato [17, Thm. 6.4],

$$H^\bullet_{\text{log-crys}}((\mathscr{Y}_{\overline{\mathbb{F}}_p}, D_{\overline{\mathbb{F}}_p})/(\Delta_W, 0)) \cong H^\bullet(\mathscr{Y}, \omega^\bullet_{\mathscr{Y}/W[\![t]\!]}) \tag{4.12}$$

(see (4.10) for the complex $\omega^\bullet_{\mathscr{Y}/W[\![t]\!]}$). The log-crystalline cohomology is endowed with the p-th Frobenius $\Phi_{(\mathscr{Y}, D_W)}$ which is compatible with $\Phi_{X/S}$ under the map

$$\begin{aligned} H^1(\mathscr{Y}, \omega^\bullet_{\mathscr{Y}/W[\![t]\!]}) &\to H^1(\mathscr{Y}_K, \omega^\bullet_{\mathscr{Y}_K/K[\![t]\!]})) \\ &\hookrightarrow H^1_{\mathrm{dR}}(X/S) \otimes_{\mathscr{O}(S)} K((t)) \\ &\hookrightarrow H^1_{\mathrm{dR}}(X/S) \otimes_{\mathscr{O}(S)} K((t))^\wedge \end{aligned}$$

where $\mathscr{Y}_K := \mathscr{Y} \times_{W[\![t]\!]} K[\![t]\!]$ and $D_K := D_W \times_W K$. Thus the assertion follows. □

Proposition 4.6. *Let $\widetilde{\omega}_{i,j}, \widetilde{\eta}_{i,j}$ be as in* (4.4) *and* (4.5)*. Suppose that σ is given by $\sigma(t) = ct^p$ with some $c \in 1 + pW$. Then*

$$\Phi_{X/S}(\widetilde{\eta}_{p^{-1}i,p^{-1}j}) \in K\widetilde{\eta}_{i,j}, \quad \Phi_{X/S}(\widetilde{\omega}_{p^{-1}i,p^{-1}j}) \equiv p\widetilde{\omega}_{i,j} \mod K((t))\widetilde{\eta}_{i,j}$$

where we use the notation (4.7)*.*

Proof. Let ∇ be the Gauss–Manin connection on $H^1_{\mathrm{dR}}(X/S) \otimes_{\mathscr{O}(S)} K((t))$. Since $\Phi_{X/S}\nabla = \nabla\Phi_{X/S}$, we have $\Phi_{X/S} \operatorname{Ker}(\nabla) \subset \operatorname{Ker}(\nabla)$. Moreover, $\Phi_{X/S}$ sends the eigencomponents $H_{i,j} := H^1_{\mathrm{dR}}(X/S)(i,j) \otimes_{\mathscr{O}(S)} K((t))$ onto the component $H_{pi,pj}$ as $\Phi_{X/S}[\zeta, \nu] = [\zeta, \nu]\Phi_{X/S}$. Therefore we have

$$\Phi_{X/S}(\widetilde{\eta}_{p^{-1}i,p^{-1}j}) \in K\widetilde{\eta}_{i,j}$$

by Corollary 4.3. We show the latter. By Lemma 4.5 together with Corollary 4.4, there are $f_{i,j}(t), g_{i,j}(t) \in K[\![t]\!]$ such that $\Phi_{X/S}(\widetilde{\omega}_{p^{-1}i,p^{-1}j}) = f_{i,j}(t)\widetilde{\omega}_{i,j} + g_{i,j}(t)\widetilde{\eta}_{i,j}$. Put $M_{i,j} := H_{i,j}/K((t))\widetilde{\eta}_{i,j}$. Then $\Phi_{X/S}(M_{p^{-1}i,p^{-1}j}) \subset M_{i,j}$ and $\Phi_{X/S}$ is commutative with the connection $\overline{\nabla}$ on $M_{i,j}$. Therefore $f_{i,j}(t) = C_{i,j}$ is a constant as $\mathrm{Ker}(\overline{\nabla}) = K\widetilde{\omega}_{i,j}$ by Corollary 4.3,

$$\Phi_{X/S}(\widetilde{\omega}_{p^{-1}i,p^{-1}j}) = C_{i,j}\widetilde{\omega}_{i,j} + g_{i,j}(t)\widetilde{\eta}_{i,j}. \tag{4.13}$$

We want to show $C_{i,j} = p$. To do this, we recall the log-crystalline cohomology (4.11)

$$H^{\bullet}_{\text{log-crys}}((\mathscr{Y}_{\overline{\mathbb{F}}_p}, D_{\overline{\mathbb{F}}_p})/(\Delta_W, 0)) \cong H^{\bullet}(\mathscr{Y}, \omega^{\bullet}_{\mathscr{Y}/W[\![t]\!]}).$$

where we keep the notation in the proof of Lemma 4.5. Let Z_W be the intersection locus of D_W. This is a disjoint union of NM-copies of $\mathrm{Spec}\, W$. More precisely, let $P_{\zeta,\nu}$ be the point of Z_W defined by $x = \zeta$ and $y = \nu$. Then $Z_W = \{P_{\zeta,\nu} \mid (\zeta,\nu) \in \mu_N \times \mu_M\}$. We consider the composition of morphisms

$$\omega^{\bullet}_{\mathscr{Y}/W[\![t]\!]} \xrightarrow{\wedge \frac{\mathrm{d}t}{t}} \Omega^{\bullet+1}_{\mathscr{Y}/W}(\log D) \xrightarrow{\mathrm{Res}} \mathscr{O}_{Z_W}[-1]$$

of complexes where Res is the Poincare residue. This gives rise to the map

$$R : H^1(\mathscr{Y}_K, \Omega^{\bullet}_{\mathscr{Y}_K/K[\![t]\!]}(\log D_K)) \longrightarrow H^0(Z_K, \mathscr{O}_{Z_K}) = \bigoplus_{\zeta,\nu} K \cdot P_{\zeta,\nu}$$

which is compatible with respect to the Frobenius $\Phi_{X/S}$ on the left and the Frobenius Φ_Z on the right in the sense that

$$R \circ \Phi_{X/S} = p\Phi_Z \circ R. \tag{4.14}$$

Notice that Φ_Z is a F-linear map such that $\Phi_Z(P_{\zeta,\nu}) = P_{\zeta,\nu}$ where F is the Frobenius on W. We claim

$$R(\widetilde{\eta}_{i,j}) = 0, \tag{4.15}$$

and

$$R(\widetilde{\omega}_{i,j}) = \sum_{\zeta,\nu} \zeta^i \nu^j P_{\zeta,\nu}. \tag{4.16}$$

To show (4.15), we recall the definition (4.6). Since $R(t\widetilde{\omega}_{i,j}) = 0$ obviously, it is enough to show $R(t\eta_{i,j}) = 0$. However since $t\eta_{i,j} = Mx^{i-N}y^{j-M}\mathrm{d}y$ by (4.5), we have

$$R(t\eta_{i,j}) = \mathrm{Res}\left(Mx^{i-N}y^{j-M}\mathrm{d}y\frac{\mathrm{d}t}{t}\right) = 0$$

as required. One can show (4.16) as follows.

$$\begin{aligned}R(\widetilde{\omega}_{i,j})=R(\omega_{i,j})&=\mathrm{Res}\left(M\,\frac{x^{i-N}y^{j-1}}{y^M-1}\mathrm{d}y\wedge\frac{\mathrm{d}t}{t}\right)\\&=\mathrm{Res}\left(\frac{Mx^{i-N}y^{j-1}}{y^M-1}\mathrm{d}y\wedge\frac{Nx^{N-1}}{x^N-1}\mathrm{d}x\right)\\&=\sum_{\zeta,\nu}\zeta^i\nu^jP_{\zeta,\nu}.\end{aligned}$$

We turn to the proof of $C_{i,j}=p$ in (4.13). Apply R on the both side of (4.13). By (4.15), the right hand side is $\alpha_{i,j}R(\widetilde{\omega}_{i,j})$, and the left hand side is $p\Phi_Z\circ R(\widetilde{\omega}_{p^{-1}i,p^{-1}j})$ by (4.14),

$$C_{i,j}R(\widetilde{\omega}_{i,j})=p\Phi_Z\circ R(\widetilde{\omega}_{p^{-1}i,p^{-1}j}).$$

Apply (4.15) to the above. We have

$$C_{i,j}\left(\sum_{\zeta,\nu}\zeta^i\nu^jP_{\zeta,\nu}\right)=p\Phi_Z\left(\sum_{\zeta,\nu}\zeta^{p^{-1}i}\nu^{p^{-1}j}\cdot P_{\zeta,\nu}\right)=p\left(\sum_{\zeta,\nu}\zeta^i\nu^jP_{\zeta,\nu}\right)$$

and hence $C_{i,j}=p$ as required. □

Theorem 4.7 (Unit root formula)**.** *Suppose $\sigma(t)=t^p$. Let (i,j) be a pair of integers (i,j) with $0<i<N$ and $0<j<M$. Put*

$$e^{\mathrm{unit}}_{i,j}:=(1-t)^{-a_i-b_j}F_{a_i,b_j}(t)^{-1}\widetilde{\eta}_{i,j}.$$

Let $s\geq 0$ be the minimal integer such that $a_i^{(s+1)}=a_i$ and $b_j^{(s+1)}=b_j$ [2]*. Put $h(t):=\prod_{m=0}^s[F_{a_i^{(m)},b_j^{(m)}}(t)]_{<p}$. Then*

$$e^{\mathrm{unit}}_{i,j}\in H^1_{\mathrm{dR}}(X/S)\otimes_{\mathscr{O}(S)}K\langle t,(t-t^2)^{-1},h(t)^{-1}\rangle \tag{4.17}$$

and

$$\Phi_{X/S}(e^{\mathrm{unit}}_{p^{-1}i,p^{-1}j})=\frac{(1-t)^{a_i+b_j}}{(1-t^p)^{a_i^{(1)}+b_j^{(1)}}}\mathscr{F}^{\mathrm{Dw}}_{a_i,b_j}(t)e^{\mathrm{unit}}_{i,j} \tag{4.18}$$

where $\mathscr{F}^{\mathrm{Dw}}_{a_i,b_j}(t)$ is the Dwork p-adic hypergeometric function (1.1)*, and we apply the convention* (4.7) *to the notation $e^{\mathrm{unit}}_{p^{-1}i,p^{-1}j}$. In particular $e^{\mathrm{unit}}_{p^{-1}i,p^{-1}j}$*

[2]For any $a\in\mathbb{Z}_{(p)}$ with $0<a<1$, there exists $i>0$ such that $a^{(i)}=a$. Let $n>0$ be an integer prime to p. For $a=l/n$ with $0<l<n$, the Dwork prime $a^{(1)}=k/n$ is characterized by $pk\equiv l \bmod n$ and $0<k<n$. Therefore, the map $a\mapsto a^{(i)}$ induces a bijection on the set $\{1/n,2/n,\dots,(n-1)/n\}$, and it is identity if and only if $p^i\equiv 1 \bmod n$.

is the eigen vector of $\Phi^{s+1}_{X/S}=\overbrace{\Phi_{X/S}\circ\cdots\circ\Phi_{X/S}}^{s+1}$, *and*

$$\Phi^{s+1}_{X/S}(e^{\mathrm{unit}}_{i,j})=\left(\prod_{m=0}^{s}\frac{(1-t^{p^m})^{a_i^{(m)}+b_j^{(m)}}}{(1-t^{p^{m+1}})^{a_i^{(m+1)}+b_j^{(m+1)}}}\mathscr{F}^{\mathrm{Dw}}_{a_i^{(m)},b_j^{(m)}}(t^{p^m})\right)e^{\mathrm{unit}}_{i,j}. \tag{4.19}$$

Notice that $(1-t)^{a_i+b_j}\notin\mathbb{Z}_p\langle t,(1-t)^{-1}\rangle$ but $(1-t)^{a_i+b_j}/(1-t^p)^{a_i^{(1)}+b_j^{(1)}}\in\mathbb{Z}_p\langle t,(1-t)^{-1}\rangle$.

Proof. Since

$$\frac{F'_{a_i,b_j}(t)}{F_{a_i,b_j}(t)}\in\mathbb{Z}_p\langle t,(t-t^2)^{-1},h(t)^{-1}\rangle \tag{4.20}$$

by [12, p. 45, Lem. 3.4], (4.17) follows. We show (4.18), which is equivalent to

$$\Phi_{X/S}(\widetilde{\eta}_{p^{-1}i,p^{-1}j})=\widetilde{\eta}_{i,j}\in H^1_{\mathrm{dR}}(X/S)\otimes_{\mathscr{O}(S)}K((t)). \tag{4.21}$$

Let

$$Q:H^1_{\mathrm{dR}}(X_K/S_K)\otimes H^1_{\mathrm{dR}}(X_K/S_K)\longrightarrow H^2_{\mathrm{dR}}(X_K/S_K)\cong\mathscr{O}(S_K)$$

be the cup-product pairing which is anti-symmetric and non-degenerate. This extends on $H^1_{\mathrm{dR}}(X/S)\otimes_{\mathscr{O}(S)}K((t))$ which we also write by Q. Then the following is satisfied.

- (Q1) $Q(\Phi_{X/S}(x),\Phi_{X/S}(y))=pQ(x,y)^\sigma$ for $x,y\in H^1_{\mathrm{dR}}(X/S)\otimes_{\mathscr{O}(S)}K((t))$,
- (Q2) $Q(gx,gy)=Q(x,y)$ for $g=(\zeta,\nu)\in\mu_N\times\mu_M$,
- (Q3) $Q(F^1,F^1)=0$ where $F^1=\Gamma(X_K,\Omega^1_{X_K/S_K})$ is the Hodge filtration,
- (Q4) $Q(\nabla(x),y)+Q(x,\nabla(y))=dQ(x,y)$.

Put $H_{i,j}:=H^1_{\mathrm{dR}}(X/S)(i,j)\otimes_{\mathscr{O}(S)}K((t))$ eigen components. By (Q2), Q induces a perfect pairing $H_{i,j}\otimes H_{N-i,M-j}\to\mathscr{O}(S)$. Therefore $Q(\widetilde{\omega}_{i,j},\widetilde{\eta}_{N-i,M-j})\neq0$ by (Q3). We claim

$$Q(\widetilde{\eta}_{i,j},\widetilde{\eta}_{N-i,M-j})=0, \tag{4.22}$$

$$Q(\widetilde{\omega}_{i,j},\widetilde{\eta}_{N-i,M-j})\in\mathbb{Q}^\times. \tag{4.23}$$

To show (4.22), we recall H_K in Corollary 4.4. Since Q is the cup-product pairing, this induces a pairing $H_K\otimes H_K\to K[\![t]\!]$, and hence

$$\overline{Q}:H_K/tH_K\otimes_K H_K/tH_K\longrightarrow K.$$

Since $\nabla(\widetilde{\eta}_{i,j})=0$, $Q(\widetilde{\eta}_{i,j},\widetilde{\eta}_{N-i,M-j})$ is a constant by (Q4). Therefore if one can show $\overline{Q}(\widetilde{\eta}_{i,j},\widetilde{\eta}_{N-i,M-j})=0$, then (4.22) follows. It follows from (Q4) that

$$\overline{Q}(\mathrm{Res}(\nabla)(x),y)+\overline{Q}(x,\mathrm{Res}(\nabla)(y))=0,\quad\forall\ x,y\in H_K/tH_K$$

where $\mathrm{Res}(\nabla)$ is as in the proof of Corollary 4.4. Since $\mathrm{Res}(\nabla)(\widetilde{\omega}_{i,j}) = \widetilde{\eta}_{i,j}$ and $\mathrm{Res}(\nabla)\widetilde{\eta}_{i,j} = 0$ by Proposition 4.2, one has

$$\begin{aligned}\overline{Q}(\widetilde{\eta}_{i,j}, \widetilde{\eta}_{N-i,M-j}) &= \overline{Q}(\mathrm{Res}(\nabla)\widetilde{\omega}_{i,j}, \widetilde{\eta}_{N-i,M-j}) \\ &= -\overline{Q}(\widetilde{\omega}_{i,j}, \mathrm{Res}(\nabla)\widetilde{\eta}_{N-i,M-j}) = 0\end{aligned}$$

as required. We show (4.23). Since $\nabla(\widetilde{\omega}_{i,j}) \in K((t))\widetilde{\eta}_{i,j}$, we have

$$dQ(\widetilde{\omega}_{i,j}, \widetilde{\eta}_{N-i,M-j}) = Q(\nabla(\widetilde{\omega}_{i,j}), \widetilde{\eta}_{N-i,M-j}) = 0$$

by (4.22) which means that $Q(\widetilde{\omega}_{i,j}, \widetilde{\eta}_{N-i,M-j})$ is a constant. Since X/S, Q and $\widetilde{\omega}_{i,j}, \widetilde{\eta}_{i,j}$ are defined over $\mathbb{Q}((t))$, the constant should belong to $\mathbb{Q}^\times$. This completes the proof of (4.23).

We turn to the proof of (4.21). By Proposition 4.6, there is a constant $\alpha \in K$ such that $\Phi_{X/S}(\widetilde{\eta}_{p^{-1}i,p^{-1}j}) = \alpha\widetilde{\eta}_{i,j}$. Put $c := Q(\widetilde{\omega}_{i,j}, \widetilde{\eta}_{N-i,M-j})$ which belongs to $\mathbb{Q}^\times$ by (4.23). By (Q1), we have

$$Q(\Phi_{X/S}(\widetilde{\omega}_{i,j}), \Phi_{X/S}(\widetilde{\eta}_{N-i,M-j})) = pQ(\widetilde{\omega}_{i,j}, \widetilde{\eta}_{N-i,M-j})^\sigma = pc,$$

and hence

$$\alpha Q(\Phi_{X/S}(\widetilde{\omega}_{i,j}), \widetilde{\eta}_{N-i,M-j}) = pc.$$

It follows from (4.22) and Proposition 4.6 that the left hand side is

$$p\alpha Q(\widetilde{\omega}_{i,j}, \widetilde{\eta}_{N-i,M-j}) = p\alpha c.$$

Therefore $\alpha = 1$. This completes the proof. ☐

4.4. Syntomic Regulators of hypergeometric curves. Let $f_R : Y_R \to \mathbb{P}^1_R$ be the fibration of hypergeometric curves in Lemma 4.1 that is defined over a ring $R := \mathbb{Z}[\zeta_N, \zeta_M, (NM)^{-1}]$ where ζ_n is a primitve n-th root of unity in $\overline{\mathbb{Q}}$. Let $S_R := \mathrm{Spec}\, R[t, (t-t^2)^{-1}]$ and $X_R := f_R^{-1}(S_R)$ as before. Let $\overline{X}_R := f_R^{-1}(\mathrm{Spec}\, R[t, t^{-1}])$ and $\overline{U}_R := \mathrm{Spec}\, R[x, y, t, t^{-1}]/((1-x^N)(1-y^M)-t)$. Put $Z_R := \overline{X}_R \backslash \overline{U}_R$. By the construction in Section 4.1, Z_R consists of disjoint $(N+M)$-components, and every components are isomorphic to $\mathbb{G}_{m,R} = \mathrm{Spec}\, R[t, t^{-1}]$. For $(\nu_1, \nu_2) \in \mu_N(R) \times \mu_M(R)$, let

$$\xi = \xi(\nu_1, \nu_2) = \left\{ \frac{x-1}{x-\nu_1}, \frac{y-1}{y-\nu_2} \right\} \in K_2^M(\mathscr{O}(\overline{U}_R)) \tag{4.24}$$

be a Milnor symbol in K_2. Let $K_2(\overline{U}_R)^{(2)} \xrightarrow{\partial} K_1(Z_R)^{(1)} = (R[t, t^{-1}]^\times)^{\oplus} \otimes \mathbb{Q}$ be the boundary map where $K_i(\,\cdot\,)^{(j)}$ denotes the Adams weight piece, which is explicitly described by

$$\{f, g\} \longmapsto (-1)^{\mathrm{ord}_P(f)\,\mathrm{ord}_P(g)} \left.\frac{g^{\mathrm{ord}_P(f)}}{f^{\mathrm{ord}_P(g)}}\right|_P, \quad P \in Z_R.$$

It is a simple exercise to show that $\partial(\xi) = 0$ and hence ξ lies in the image of $K_2(\overline{X}_R)^{(2)}$. Since $K_2(\overline{X}_R)^{(2)} \to K_2(\overline{U}_R)^{(2)}$ is injective as $K_2(\mathbb{G}_{m,R})^{(1)} = 0$, we have an element in $K_2(\overline{X}_R)^{(2)}$ which we also write by ξ. Let dlog :

$K_2(\overline{X}_R)^{(2)} \to \Gamma(\overline{X}, \Omega^2_{\overline{X}_R/R})\otimes\mathbb{Q}$ be the dlog map which is given by $\{f,g\} \mapsto \frac{\mathrm{d}f}{f} \wedge \frac{\mathrm{d}g}{g}$. One immediately has

$$\mathrm{dlog}(\xi) = N^{-1}M^{-1}\sum_{i=1}^{N-1}\sum_{j=1}^{M-1}(1-\nu_1^{-i})(1-\nu_2^{-j})\frac{\mathrm{d}t}{t}\wedge\omega_{i,j}. \tag{4.25}$$

Let $p > \max(N,M)$ be a prime. Let $W = W(\overline{\mathbb{F}}_p)$ be the Witt ring and $K := \operatorname{Frac} W$ the fractional field. Fix an embedding $R \hookrightarrow W$. Write $Y := Y \times_R W$, $\overline{X} := \overline{X}_R \times_R W$ etc. Let $D_i = f^{-1}(t=i)$ be the fiber at $t=i$ for $i \in \{0,1,\infty\}$. By Lemma 4.1, the morphism

$$f : (Y, D_0 \cup (D_1)_{\mathrm{red}} \cup (D_\infty)_{\mathrm{red}}) \longrightarrow (\mathbb{P}^1, \{0,1,\infty\})$$

of log schemes is smooth where $(\,\cdot\,)_{\mathrm{red}}$ denotes the reduced part, and a pair (V,D) denotes the log scheme whose log structure is defined by the divisor D. The boundary $Z \subset \overline{X}$ consists of sections $\{X_0 - \zeta X_1 = Y_0 = 0\}$ and $\{Y_0 - \zeta' Y_1 = X_0 = 0\}$ in the equation (4.1). One easily sees that the closure $\overline{Z} \subset Y$ in Y also consists of sections which are disjoint, and each section intersects with regular reduced locus of D_i transversally for every $i \in \{0,1,\infty\}$. For $c \in 1 + pW$, there is a p-th Frobenius σ on the weak completion $\mathscr{O}(S)^\dagger$ given by $\sigma(t) = ct^p$ compatible with the Frobenius on W. This setting is under the setting in [4, §4.1], so that the comparison map

$$\mathscr{O}(S)^\dagger_K \otimes_{\mathscr{O}(S_K)} H^i_{\mathrm{dR}}(U_K/S_K) \longrightarrow H^i_{\mathrm{rig}}(U_{\overline{\mathbb{F}}_p}/S_{\overline{\mathbb{F}}_p})$$

is bijective, and the symbol map

$$[\,\cdot\,]_{U/S} : K_2^M(\mathscr{O}(U)) \longrightarrow \mathrm{Ext}^1_{\text{Fil-}F\text{-MIC}(S,\sigma)}(\mathscr{O}_S, H^1(U/S)(2))$$

is defined. Since the Milnor symbol $\xi \in K_2^M(\mathscr{O}(U))$ has no boundary at $X \setminus U$, the symbol map $[-]_{U/S}$ also defines a 1-extension

$$0 \longrightarrow H^1(X/S)(2) \longrightarrow M_\xi(X/S) \longrightarrow \mathscr{O}_S \longrightarrow 0 \tag{4.26}$$

in the exact category Fil-F-MIC(S,σ) ([4, Prop. 4.3]). Let $e_\xi \in \operatorname{Fil}^0 M_\xi(X/S)_{\mathrm{dR}}$ be the unique lifting of $1 \in \mathscr{O}_S(S)$. Let $\varepsilon_k^{(i,j)}(t)$ and $E_k^{(i,j)}(t)$ be defined by

$$\begin{aligned} & e_\xi - \Phi(e_\xi) \\ &= N^{-1}M^{-1}\sum_{i=1}^{N-1}\sum_{j=1}^{M-1}(1-\nu_1^{-i})(1-\nu_2^{-j})[\varepsilon_1^{(i,j)}(t)\omega_{i,j}+\varepsilon_2^{(i,j)}(t)\eta_{i,j}] &(4.27)\\ &= N^{-1}M^{-1}\sum_{i=1}^{N-1}\sum_{j=1}^{M-1}(1-\nu_1^{-i})(1-\nu_2^{-j})[E_1^{(i,j)}(t)\widetilde{\omega}_{i,j}+E_2^{(i,j)}(t)\widetilde{\eta}_{i,j}]. &(4.28)\end{aligned}$$

Notice that $\varepsilon_k^{(i,j)}(t)$ and $E_k^{(i,j)}(t)$ depend on the choice of the Frobenius σ. The relation between $\varepsilon_k^{(i,j)}(t)$ and $E_k^{(i,j)}(t)$ is explicitly given by

$$\varepsilon_1^{(i,j)}(t) = E_1^{(i,j)}(t)F_{a_i,b_j}(t)^{-1} - t(1-t)^{a_i+b_j}F'_{a_i,b_j}(t)E_2^{(i,j)}(t) \tag{4.29}$$

$$\varepsilon_2^{(i,j)}(t) = -b_j t(1-t)^{a_i+b_j}F_{a_i,b_j}(t)E_2^{(i,j)}(t). \tag{4.30}$$

By the definition $\varepsilon_k^{(i,j)}(t)$ are automatically overconvergent functions,

$$\varepsilon_k^{(i,j)}(t) \in K[t,(t-t^2)^{-1}]^\dagger.$$

On the other hand, since $F'_{a_i,b_j}(t)/F_{a_i,b_j}(t)$ is a convergent function (cf. (4.20)), so is $E_1^{(i,j)}(t)/F_{a_i,b_j}(t)$,

$$\frac{E_1^{(i,j)}(t)}{F_{a_i,b_j}(t)} \in K\langle t,(t-t^2)^{-1},h(t)^{-1}\rangle, \quad h(t) := \prod_{m=0}^{s}[F_{a_i^{(m)},b_j^{(m)}}(t)]_{<p} \tag{4.31}$$

where $s \geq 0$ is the minimal integer such that $a_i^{(s+1)} = a_i$ and $b_j^{(s+1)} = b_j$.

The following is the main theorem in this paper, which provides a geometric aspect of $\mathscr{F}_{a,b}^{(\sigma)}(t)$ the p-adic hypergeometric function of logarithmic type defined in Section 3.1.

Theorem 4.8. *Suppose $p > \max(N,M)$. We have*

$$\frac{E_1^{(i,j)}(t)}{F_{a_i,b_j}(t)} = -\mathscr{F}_{a_i,b_j}^{(\sigma)}(t). \tag{4.32}$$

Hence

$$e_\xi - \Phi(e_\xi) \equiv \sum_{i=1}^{N-1}\sum_{j=1}^{M-1}\frac{(1-\nu_1^{-i})(1-\nu_2^{-j})}{NM}\mathscr{F}_{a_i,b_j}^{(\sigma)}(t)\omega_{i,j}$$

modulo $\sum_{i,j} K\langle t,(t-t^2),h(t)^{-1}\rangle e_{i,j}^{\mathrm{unit}}$.

Proof. The Frobenius σ extends on $K((t))$, and Φ also extends on

$$K((t))\otimes H^1_{\mathrm{dR}}(X/S) \cong H^1_{\text{log-crys}}((\mathscr{Y}_{\overline{\mathbb{F}}_p}, D_{\overline{\mathbb{F}}_p})/(\Delta_W,0))\otimes_{W[\![t]\!]}K((t))$$

in the natural way where the isomorphism follows from (4.12). Apply the Gauss–Manin connection ∇ on (4.28). Since $\nabla\Phi = \Phi\nabla$ and $\nabla(e_\xi) = -\operatorname{dlog}\xi$ ([4, (2.30)]), we have

$$(1-\Phi)\left(-N^{-1}M^{-1}\sum_{i=1}^{N-1}\sum_{j=1}^{M-1}(1-\nu_1^{-i})(1-\nu_2^{-j})\frac{\mathrm{d}t}{t}\omega_{i,j}\right) \tag{4.33}$$

$$= N^{-1}M^{-1}\sum_{i=1}^{N-1}\sum_{j=1}^{M-1}(1-\nu_1^{-i})(1-\nu_2^{-j})\nabla(E_1^{(i,j)}(t)\widetilde{\omega}_{i,j}+E_2^{(i,j)}(t)\widetilde{\eta}_{i,j}) \tag{4.34}$$

by (4.25). Let $\Phi_{X/S}$ denote the p-th Frobenius on $H^1_{\mathrm{rig}}(X_{\overline{\mathbb{F}}_p}/S_{\overline{\mathbb{F}}_p})$. Then the Φ on $H^1_{\mathrm{rig}}(X/S)(2)$ agrees with $p^{-2}\Phi_{X/S}$ by definition of Tate twists. It follows from Proposition 4.6 that we have

$$\Phi_{X/S}(\widetilde{\omega}_{p^{-1}i,p^{-1}j}) \equiv p\widetilde{\omega}_{i,j} \mod K((t))\widetilde{\eta}_{i,j}$$

where $m \in \{1,\dots,N-1\}$ with $pm \equiv n \bmod N$. Therefore

$$(4.33) \equiv -N^{-1}M^{-1}\sum_{i=1}^{N-1}\sum_{j=1}^{M-1}(1-\nu_1^{-i})(1-\nu_2^{-j})(F_{a_i,b_j}(t)-F_{a_i^{(1)},b_j^{(1)}}(t^\sigma))\frac{\mathrm{d}t}{t}\widetilde{\omega}_{i,j}$$

modulo $\sum_{i,j} K((t))\widetilde{\eta}_{i,j}$. On the other hand,

$$(4.34) \equiv N^{-1}M^{-1}\sum_{i=1}^{N-1}\sum_{j=1}^{M-1}(1-\nu_1^{-i})(1-\nu_2^{-j})t\frac{\mathrm{d}}{\mathrm{d}t}(E_1^{(i,j)}(t))\cdot\frac{\mathrm{d}t}{t}\widetilde{\omega}_{i,j} \mod \sum_{i,j}K((t))\widetilde{\eta}_{i,j}$$

by Proposition 4.2. We thus have

$$t\frac{\mathrm{d}}{\mathrm{d}t}E_1^{(i,j)}(t) = -F_{a_i,b_j}(t) + F_{a_i^{(1)},b_j^{(1)}}(t^\sigma), \tag{4.35}$$

and hence

$$E_1^{(i,j)}(t) = -\left(C + \int_0^t F_{a_i,b_j}(t) - F_{a_i^{(1)},b_j^{(1)}}(t^\sigma)\frac{\mathrm{d}t}{t}\right)$$

for some constant $C \in K$. We determine the constant C in the following way. Firstly $E_1^{(i,j)}(t)/F_{a_i,b_j}(t)$ is a convergent function by (4.31). If $C = \psi_p(a_i)+\psi_p(b_j)+2\gamma_p-p^{-1}\log(c)$, then $E_1^{(i,j)}(t)/F_{a_i,b_j}(t) = \mathscr{F}^{(\sigma)}_{a_i,b_j}(t)$ belongs to $K\langle t,(t-t^2)^{-1},h(t)^{-1}\rangle$ by Corollary 3.4. If there is another C' such that $E_1^{(i,j)}(t)/F_{a_i,b_j}(t) \in K\langle t,(t-t^2)^{-1},h(t)^{-1}\rangle$, then it follows

$$\frac{C-C'}{F_{a_i,b_j}(t)} \in K\langle t,(t-t^2)^{-1},h(t)^{-1}\rangle.$$

We show that this is impossible. We recall from Theorem 4.7 the formula (4.19)

$$\begin{aligned}\Phi^{s+1}_{X/S}(e^{\mathrm{unit}}_{i,j}) &= \left(\prod_{m=0}^{s}\frac{(1-t^{p^m})^{a_i^{(m)}+b_j^{(m)}}}{(1-t^{p^{m+1}})^{a_i^{(m+1)}+b_j^{(m+1)}}}\mathscr{F}^{\mathrm{Dw}}_{a_i^{(m)},b_j^{(m)}}(t^{p^m})\right)e^{\mathrm{unit}}_{i,j}\\ &= \left(\frac{(1-t)^{a_i+b_j}}{(1-t^{p^{s+1}})^{a_i+b_j}}\right)\frac{F_{a_i,b_j}(t)}{F_{a_i,b_j}(t^{p^{s+1}})}e^{\mathrm{unit}}_{i,j}.\end{aligned}$$

Iterating $\Phi^{s+1}_{X/S}$ to the above, we have

$$(\Phi^{s+1}_{X/S})^n(e^{\mathrm{unit}}_{i,j}) = \overbrace{\left(\frac{(1-t)^{a_i+b_j}}{(1-t^{p^{n(s+1)}})^{a_i+b_j}}\right)}^{\mu(t)} \frac{F_{a_i,b_j}(t)}{F_{a_i,b_j}(t^{p^{n(s+1)}})} e^{\mathrm{unit}}_{i,j}.$$

Put $q := p^{n(s+1)}$. Let $\alpha \in W$ satisfy $\alpha^q = \alpha$ and $(\alpha-\alpha^2)h(\alpha) \not\equiv 0 \bmod p$. Then the evaluation $\mu(\alpha)$ is a root of unity. Suppose $g(t) := F_{a_i,b_j}(t)^{-1} \in K\langle t,(t-t^2)^{-1},h(t)^{-1}\rangle$. Let $g(t) = (t-\alpha)^k g_0(t)$ with $g_0(t) \in K\langle t,(t-t^2)^{-1}, h(t)^{-1}\rangle$ such that $g_0(\alpha) \neq 0$. Then we have

$$\left.\frac{F_{a_i,b_j}(t)}{F_{a_i,b_j}(t^q)}\right|_{t=\alpha} = \left.\frac{(t^q-\alpha)^k g_0(t^q)}{(t-\alpha)^k g_0(t)}\right|_{t=\alpha} = (q\alpha^{q-1})^k = q^k.$$

Since the first evaluation is a unit in W, we have $k=0$. Thus an eigen value of $(\Phi^{s+1}_{X/S})^n|_{t=\alpha}$ is a root of unity. This contradicts with the Weil–Riemann hypothesis. □

Theorem 4.9 (Syntomic Regulator Formula)**.** *Let $\alpha \in W$ such that $\alpha \not\equiv 0,1 \bmod p$. Let σ_α be the Frobenius given by $t^\sigma = F(\alpha)\alpha^{-p}t^p$ where F is the Frobenius on W. Let X_α be the fiber at $t=\alpha$ ($\Leftrightarrow \lambda = 1-\alpha$), which is a smooth projective variety over W of relative dimension one. Let*

$$\mathrm{reg}_{\mathrm{syn}} : K_2(X_\alpha) \longrightarrow H^2_{\mathrm{syn}}(X_\alpha, \mathbb{Q}_p(2)) \cong H^1_{\mathrm{dR}}(X_\alpha/K)$$

be the syntomic regulator map. Then

$$\mathrm{reg}_{\mathrm{syn}}(\xi|_{X_\alpha}) = -\sum_{i=1}^{N-1}\sum_{j=1}^{M-1} \frac{(1-\nu_1^{-i})(1-\nu_2^{-j})}{NM}[\varepsilon_1^{(i,j)}(\alpha)\omega_{i,j} + \varepsilon_2^{(i,j)}(\alpha)\eta_{i,j}].$$

Proof. This follows from [4, Thm. 4.4]. □

Corollary 4.10. *Let the notation and assumption be as in Theorem 4.9. Suppose further that $h(\alpha) \not\equiv 0 \bmod p$ where $h(t)$ is as in (4.31). Let $e^{\mathrm{unit}}_{N-i,M-j}$ be as in Theorem 4.7, and $Q : H^1_{\mathrm{dR}}(X_\alpha/K) \otimes H^1_{\mathrm{dR}}(X_\alpha/K) \to H^2_{\mathrm{dR}}(X_\alpha/K) \cong K$ the cup-product pairing. Then we have*

$$\begin{aligned} &Q(\mathrm{reg}_{\mathrm{syn}}(\xi|_{X_\alpha}), e^{\mathrm{unit}}_{N-i,M-j}) \\ &\qquad = -\frac{(1-\nu_1^{-i})(1-\nu_2^{-j})}{NM} \mathscr{F}^{(\sigma_\alpha)}_{a_i,b_j}(\alpha) Q(\omega_{i,j}, e^{\mathrm{unit}}_{N-i,M-j}). \end{aligned}$$

Proof. Noticing $Q(e^{\mathrm{unit}}_{i,j}, e^{\mathrm{unit}}_{N-i,M-j}) = 0$ by (4.22), this is immediate from Theorem 4.9. □

4.5. Syntomic regulator of the Ross symbols of Fermat curves. We apply Theorem 4.8 to the study of the syntomic regulator of the *Ross symbol* [26]

$$\{1-z, 1-w\} \in K_2(F) \otimes \mathbb{Q}$$

of the (projective smooth) Fermat curve F defined by an affine equation $z^N + w^M = 1$ over a field K of characteristic zero. The group $\mu_N \times \mu_M$ acts on F by $(\varepsilon_1, \varepsilon_2) \cdot (z, w) = (\varepsilon_1 z, \varepsilon_2 w)$. Let $H^1_{\mathrm{dR}}(F/K)(i,j)$ denote the subspace on which $(\varepsilon_1, \varepsilon_2)$ acts by multiplication by $\varepsilon_1^i \varepsilon_2^j$. Let

$$I = \left\{ (i,j) \in \mathbb{Z}^2 \,\middle|\, 1 \le i \le N-1, 1 \le j \le M-1, \ \frac{i}{N} + \frac{j}{M} \ne 1 \right\},$$

then

$$H^1_{\mathrm{dR}}(F/K) = \bigoplus_{(i,j)\in I} H^1_{\mathrm{dR}}(F/K)(i,j).$$

Each eigen space $H^1_{\mathrm{dR}}(F/K)(i,j)$ is one-dimensional with basis $z^{i-1}w^{j-M}\mathrm{d}z = -N^{-1}Mz^{i-N}w^{j-1}\mathrm{d}w$, and

$$H^1_{\mathrm{dR}}(F/K)(i,j) \subset \Gamma(F, \Omega^1_{F/K}) \quad \Longleftrightarrow \quad \frac{i}{N} + \frac{j}{M} < 1$$

(e.g. [15, §2]). In particular, the genus of F is $1 + \frac{1}{2}(NM - N - M - \gcd(N,M))$.

Theorem 4.11. *Let $p > \max(N, M)$ be a prime and $W = W(\overline{\mathbb{F}}_p)$ the Witt ring and $K = \mathrm{Frac}(W)$. Let F be the Fermat curve defined by an affine equation $z^N + w^M = 1$ that is smooth and projective over W. Let*

$$\mathrm{reg}_{\mathrm{syn}} : K_2(F) \otimes \mathbb{Q} \longrightarrow H^2_{\mathrm{syn}}(F, \mathbb{Q}_p(2)) \cong H^1_{\mathrm{dR}}(F/K)$$

be the syntomic regulator map and let $A^{(i,j)} \in K$ be defined by

$$\mathrm{reg}_{\mathrm{syn}}(\{1-z, 1-w\}) = \sum_{(i,j)\in I} A^{(i,j)} M^{-1} z^{i-1} w^{j-M} \mathrm{d}z.$$

Suppose that $(i,j) \in I$ satisfies

$$\text{(4.36)} \qquad \text{(i) } \frac{i}{N} + \frac{j}{M} < 1, \quad \text{(ii) } [F_{\frac{i}{N},\frac{j}{M}}(t)]_{<p^n}|_{t=1} \not\equiv 0 \bmod p, \quad \forall\, n \ge 1,$$

where $f(t)|_{t=a}$ denotes the evaluation $f(a)$ at $t = a$. Then we have

$$\text{(4.37)} \qquad A^{(i,j)} = \mathscr{F}^{(\sigma)}_{\frac{i}{N},\frac{j}{M}}(1)$$

where $\sigma = \sigma_1$ (i.e. $\sigma(t) = t^p$).

The following lemma gives a sufficient condition for that the conditions (4.36) are satisfied.

Lemma 4.12.

(1) *Let* $a, b \in \mathbb{Z}_p$. *Then* $[F_{a,b}(t)]_{<p^n}|_{t=1} \not\equiv 0 \bmod p$ *for all* $n \geq 1$ *if and only if* $[F_{a^{(k)},b^{(k)}}(t)]_{<p}|_{t=1} \not\equiv 0 \bmod p$ *for all* $k \geq 0$ *where* $a^{(k)}$ *denotes the Dwork k-th prime.*

(2) *Let* $a_0, b_0 \in \{0, 1, , \ldots, p-1\}$ *satisfy* $a \equiv -a_0$ *and* $b \equiv -b_0 \bmod p$. *Then*

$$[F_{a,b}(t)]_{<p}|_{t=1} \equiv \frac{\Gamma(1+a_0+b_0)}{\Gamma(1+a_0)\Gamma(1+b_0)} = \frac{(a_0+b_0)!}{a_0!b_0!} \mod p.$$

In particular

$$[F_{a,b}(t)]_{<p}|_{t=1} \not\equiv 0 \iff a_0 + b_0 \leq p-1.$$

(3) *Suppose that* $N|(p-1)$ *and* $M|(p-1)$. *Then for any* (i, j) *such that* $0 < i < N$ *and* $0 < j < M$ *and* $i/N + j/M < 1$, *the conditions* (4.36) *hold.*

Proof. (1) is a consequence of the Dwork congruence (1.2). We show (2). Obviously $[F_{a,b}(t)]_{<p} \equiv [F_{-a_0,-b_0}(t)]_{<p} \bmod p\mathbb{Z}_p[t]$, and $[F_{-a_0,-b_0}(t)]_{<p} = F_{-a_0,-b_0}(t)$ as a_0 and b_0 are non-positive integers greater than $-p$. Then apply Gauss' formula (e.g. [22] 15.4.20)

$${}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b) > 0.$$

To see (3), letting $a = i/N$ and $b = j/M$, we note that $a^{(k)} = a$, $b^{(k)} = b$ and $a_0 = i(p-1)/N$, $b_0 = j(p-1)/M$. Then the condition (4.36) (ii) follows by (1) and (2). □

Proof of Theorem 4.11. We show that the theorem is reduced to the case $M = N$. Let L be the least common multiple of M, N, and F_1 the Fermat curve defined by an affine equation $F_1 : z_1^L + w_1^L = 1$. There is a finite surjective map $\rho : F_1 \to F$ given by $\rho^*(z) = z_1^A$ $\rho^*(w) = w_1^B$ where $AN = BM = L$. There is a commutative diagram

$$\begin{array}{ccc} K_2(F_1) \otimes \mathbb{Q} & \longrightarrow & H^2_{\mathrm{syn}}(F_1, \mathbb{Q}_p(2)) \\ \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \rho_*} \\ K_2(F) \otimes \mathbb{Q} & \longrightarrow & H^2_{\mathrm{syn}}(F, \mathbb{Q}_p(2)) \end{array}$$

with surjective vertical arrows. It is a simple exercise to show that $\rho_*\{1-z_1, 1-w_1\} = \{1-z.1-w\}$ and $\rho_*(L^{-1}z_1^{i-1}w_1^{j-L}\mathrm{d}z_1) = M^{-1}z^{i'-1}w^{j'-M}\mathrm{d}z$ if $(i, j) = (i'A, j'B)$ and $= 0$ otherwise. Thus the theorem for F can be deduced from the theorem for F_1. □

We assume $N = M$ until the end of the proof. Let $f : Y_s \to \mathbb{P}^1_W$ be the curve (4.1), which has bad fibers at $t = 0, 1, \infty$. Let $\lambda := 1 - t$ be a new parameter, and let $\lambda_0^N=\lambda$. Let $\overline{S}_0:=\operatorname{Spec} W[\lambda_0,(1-\lambda_0^N)^{-1}]\to\mathbb{P}^1_W$ and $S_0:=\operatorname{Spec} W[\lambda_0,\lambda_0^{-1},(1-\lambda_0^N)^{-1}]\subset\overline{S}_0$. Let $\overline{X}_s:=Y_s\times_{\mathbb{P}^1_W}\overline{S}$. Then $\overline{X}_s$ has a unique singular point $(x,y,\lambda_0)=(0,0,0)$ in an affine open set

$$U_s=\operatorname{Spec} W[x,y,\lambda_0,(1-\lambda_0^N)^{-1}]/(x^Ny^N-x^N-y^N-\lambda_0^N)\subset\overline{X}_s.$$

Let $\overline{X}_0\to\overline{X}_s$ be the blow-up at $(x,y,\lambda_0)=(0,0,0)$. Then $\overline{X}_0\to\operatorname{Spec} W$ is smooth, and the morphism

$$f_0:\overline{X}_0\longrightarrow\overline{S}_0 \tag{4.38}$$

is projective flat such that $X_0:=f_0^{-1}(S_0)\to S_0$ smooth and f_0 has a semistable reduction at $\lambda_0=0$. The fiber $Z:=f^{-1}(\lambda_0=0)$ is a reduced divisor with two irreducible components F and E where F is the proper transform of the curve $x^Ny^N-x^N-y^N=0\Leftrightarrow z^N+w^N=1$ ($z:=x^{-1},w:=y^{-1}$), and E is the exceptional curve. Both curves are isomorphic to the Fermat curve $u^N+v^N=1$. Moreover E and F intersects transversally at N-points.

We recall the K_2-symbols $\xi(\nu_1,\nu_2)$ in (4.24). We think them to be elements of $K_2(\overline{X}_0)\otimes\mathbb{Q}$, and put

$$\Xi:=\sum_{(\nu_1,\nu_2)\in\mu_N\times\mu_N}\xi(\nu_1,\nu_2)=\left\{\frac{(x-1)^N}{x^N-1},\frac{(y-1)^N}{y^N-1}\right\}\in K_2(\overline{X}_0)\otimes\mathbb{Q}.$$

Then the restriction of Ξ on F is

$$\begin{aligned}&\{(1-z)^N,(1-w)^N\}-\{(1-z)^N,1-w^N\}\\&\qquad-\{1-z^N,(1-w)^N\}+\{1-z^N,1-w^N\}\\&=\{(1-z)^N,(1-w)^N\}-\{(1-z)^N,z^N\}\\&\qquad-\{w^N,(1-w)^N\}+\{w^N,1-w^N\}\\&=N^2\{1-z,1-w\}.\end{aligned}$$

This is the Ross symbol. We thus have

$$N^2\operatorname{reg}_{\mathrm{syn}}(\{1-z,1-w\})=\sum_{(\nu_1,\nu_2)\in\mu_N\times\mu_N}\operatorname{reg}_{\mathrm{syn}}(\xi(\nu_1,\nu_2)|_F). \tag{4.39}$$

Write $\xi=\xi(\nu_1,\nu_2)$ simply. Let σ be the p-th Frobenius on $W[\lambda_0,\lambda_0^{-1}]$ such that $\sigma(t)=t^p\Leftrightarrow\sigma(\lambda_0)=(1-(1-\lambda_0^N)^p)^{\frac{1}{N}}$. Recall (4.27) and (4.28),

$$\begin{aligned}e_\xi-\Phi_\sigma(e_\xi)&=N^{-2}\sum_{1\le i,j\le N-1}(1-\nu_1^{-i})(1-\nu_2^{-j})[\varepsilon^{(i,j)}_{1,\sigma}(t)\omega_{i,j}+\varepsilon^{(i,j)}_{2,\sigma}(t)\eta_{i,j}]\\&=N^{-2}\sum_{1\le i,j\le N-1}(1-\nu_1^{-i})(1-\nu_2^{-j})[E^{(i,j)}_{1,\sigma}(t)\widetilde{\omega}_{i,j}+E^{(i,j)}_{2,\sigma}(t)\widetilde{\eta}_{i,j}]\end{aligned}$$

where we write "$(\,\cdot\,)_\sigma$" to emphasize that they depend on σ. Let τ be the p-th Frobenius on $W[\![\lambda_0]\!]$ such that $\tau(\lambda_0)=\lambda_0^p$. Let

$$
\begin{aligned}
(4.40)\quad & e_\xi-\Phi_\tau(e_\xi)\\
&=N^{-2}\sum_{1\le i,j\le N-1}(1-\nu_1^{-i})(1-\nu_2^{-j})[\varepsilon_{1,\tau}^{(i,j)}(\lambda)\omega_{i,j}+\varepsilon_{2,\tau}^{(i,j)}(\lambda)\eta_{i,j}]
\end{aligned}
$$

be defined in the same way. This is related to (4.39) in the following way. Let $\Delta:=\operatorname{Spec}W[\![\lambda_0]\!]\to\overline{S}_0$, and $\mathscr{X}:=f^{-1}(\Delta)$. We have the syntomic regulator

$$\mathrm{reg}_{\mathrm{syn}}(\xi)\in H^2_{\mathrm{syn}}(\mathscr{X},\mathbb{Z}_p(2))$$

in the syntomic cohomology group. We endow the log structure on Δ (resp. $\mathscr{X}$) defined by the divisor $O=\operatorname{Spec}W[\![\lambda_0]\!]/(\lambda_0)$ (resp. $E+F$) which is denoted by the same notation O (resp. $E+F$). Let $\omega_{\mathscr{X}/\Delta}$ be the log de Rham complex for $(\mathscr{X},E+F)/(\Delta,O)$. Recall the log syntomic cohomology groups (e.g. [29, §2])

$$H^i_{\mathrm{syn}}((X,M),\mathbb{Z}_p(j))$$

of a log scheme (X,M) satisfying several conditions (all log schemes appearing in this proof satisfy them). Moreover one can further define the syntomic cohomology groups $H^i_{\mathrm{syn}}((\mathscr{X},E+F)/(\Delta,O,\tau),\mathbb{Z}_p(j))$ following the construction in [4, §3.1], where we note that τ induces the p-th Frobenius on (Δ,O) (while so does not σ). Let

$$
\begin{aligned}
\rho_\tau:\ & H^2_{\mathrm{syn}}(\mathscr{X},\mathbb{Z}_p(2))\\
&\longrightarrow H^2_{\mathrm{syn}}((\mathscr{X},E+F)/(\Delta,O,\tau),\mathbb{Z}_p(2))\overset{\cong}{\longleftarrow}H^1_{\mathrm{zar}}(\mathscr{X},\omega^\bullet_{\mathscr{X}/\Delta})
\end{aligned}
$$

be the composition of natural maps. We endow the log structure on F defined by the divisor $T:=E\cap F$ which is denoted by T. Put $U:=F\backslash T$. Let $\omega^\bullet_F:=\Omega^\bullet_{F/W}(\log T)$ the log de Rham complex for $(F,T)/W$. Let $\iota:F\hookrightarrow\mathscr{X}$ be the closed immersion. Then there is a commutative diagram

$$
\begin{array}{ccccc}
H^2_{\mathrm{syn}}(\mathscr{X},\mathbb{Z}_p(2)) & \longrightarrow & H^2_{\mathrm{syn}}((F,T),\mathbb{Z}_p(2)) & & \\
\downarrow{\scriptstyle \rho_\tau} & & \uparrow{\scriptstyle \cong} & & \\
H^1_{\mathrm{zar}}(\mathscr{X},\omega^\bullet_{\mathscr{X}/\Delta}) & \xrightarrow{\ \iota^*\ } & H^1_{\mathrm{zar}}(F,\omega^\bullet_F) & \xrightarrow{\ \subset\ } & H^1_{\mathrm{dR}}(U/K)\\
\downarrow{\scriptstyle \pi} & & & & \\
W((\lambda_0))\otimes H^1_{\mathrm{dR}}(X_0/S_0) & & & &
\end{array}
$$

and we have

$$(4.41)\qquad (\iota^*\circ\rho_\tau)(\mathrm{reg}_{\mathrm{syn}}(\xi))=\mathrm{reg}_{\mathrm{syn}}(\xi|_F)\in H^1_{\mathrm{dR}}(U/K).$$

Moreover it follows from [4, Thm. 4.5] that

$$(\pi \circ \rho_\tau)(\mathrm{reg}_{\mathrm{syn}}(\xi)) = \Phi(e_\xi) - e_\xi \in W((\lambda_0)) \otimes H^1_{\mathrm{dR}}(X_0/S_0). \tag{4.42}$$

Note that $H^1_{\mathrm{dR}}(F/K) \to H^1_{\mathrm{dR}}(U/K)$ is injective, and the above element belongs to the image of $H^1_{\mathrm{dR}}(F/K)$, so that we may replace $H^1_{\mathrm{dR}}(U/K)$ with $H^1_{\mathrm{dR}}(F/K)$ in (4.41).

Write $X_{0,K} := X_0 \times_W K$ etc. Let $\Delta_K := \operatorname{Spec} K[\![\lambda_0]\!]$ and $\mathscr{X}_K := \overline{X}_0 \times_{\overline{S}_0} \Delta_K$. Put

$$H_K := H^1(\mathscr{X}_K, \omega^\bullet_{\mathscr{X}_K/\Delta_K}) \hookrightarrow K((\lambda_0)) \otimes H^1_{\mathrm{dR}}(X_0/S_0).$$

Lemma 4.13. *Put $s := (a_i+b_j)N$ which is a positive integer. If $a_i+b_j < 1$, then the eigencomponent $H_K(i,j)$ is a free $K[\![\lambda_0]\!]$-module of rank two with a basis $\{\omega_{i,j}, \lambda_0^s \eta_{i,j}\}$.*

Proof. This is proven in the same way as the proof of Corollary 4.4. □

Lemma 4.14. *Let $1 \le i,j \le N-1$ be integers, and put $a_i := 1-i/N$ and $b_j := 1-j/N$. Put*

$$f_n(t) = f_{n,i,j}(t) := -\frac{(1-\nu_1^{-i})(1-\nu_2^{-j})}{N^2} \frac{1}{F_{a_i,b_j}(t)} \left(\frac{d^{n-1}}{dt^{n-1}} \left(\frac{F_{a_i^{(1)},b_j^{(1)}}(t)}{t} \right) \right)^\sigma$$

for $n \in \mathbb{Z}_{\ge 1}$. Then

$$\varepsilon^{(i,j)}_{1,\tau}(\lambda) - \mathscr{F}^{(\sigma)}_{a_i,b_j}(t) = \sum_{n=1}^\infty \frac{(t^\tau - t^\sigma)^n}{n!} p^{-1} f_n(t) + b_j^{-1} \frac{F'_{a_i,b_j}(t)}{F_{a_i,b_j}(t)} \varepsilon^{(i,j)}_{2,\tau}(\lambda).$$

Notice that $f_n(t)$ is a convergent function on the rigion $\{[F_{a_i,b_j}(t)]_{<p^n} \not\equiv 0\}$ by [12, p. 37, Thm. 2, p. 45 Lem. 3.4]

Proof. The relation between $\varepsilon^{(i,j)}_{k,\sigma}(t)$ and $\varepsilon^{(i,j)}_{k,\tau}(t)$ is the following (e.g. [13, 6.1], [18, 17.3.1])

$$\Phi_\tau(e_\xi) - \Phi_\sigma(e_\xi) = \sum_{n=1}^\infty \frac{(t^\tau - t^\sigma)^n}{n!} \Phi_\sigma \partial_t^n e_\xi \tag{4.43}$$

where $\partial_t = \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}}$ is the differential operator on $M_\xi(X/S)_{\mathrm{dR}}$. By (4.25),

$$\begin{aligned}\partial_t(e_\xi) &= -\sum_{1\le i,j\le N-1} \frac{(1-\nu_1^{-i})(1-\nu_2^{-j})}{N^2} \frac{1}{t} \omega_{i,j} \\ &= -\sum_{1\le i,j\le N-1} \frac{(1-\nu_1^{-i})(1-\nu_2^{-j})}{N^2} \frac{F_{a_i,b_j}(t)}{t} \widetilde{\omega}_{i,j}.\end{aligned}$$

Let $\eta^*_{i,j} := (1-t)^{-a_i-b_j}F_{a_i,b_j}(t)^{-1}\widetilde{\eta}_{i,j} \in H^1_{\mathrm{dR}}(X/S)\otimes K\langle t,(t-t^2)^{-1},h(t)^{-1}\rangle$ where $h(t)=\prod_{m=0}^N F_{a_i^{(m)},b_j^{(m)}}(t)$ with $N \gg 0$. By Proposition 4.2,

$$\partial_t^n(e_\xi) = -\sum_{1\le i,j\le N-1}\frac{(1-\nu_1^{-i})(1-\nu_2^{-j})}{N^2}\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}\left(\frac{F_{a_i,b_j}(t)}{t}\right)\widetilde{\omega}_{i,j}+(\cdots)\widetilde{\eta}_{i,j}$$

and hence

$$\Phi_\sigma\partial_t^n(e_\xi) \equiv \sum_{1\le i,j\le N-1} p^{-1}f_{n,i,j}(t) \mod K\langle t,(t-t^2)^{-1},h(t)^{-1}\rangle\eta^*_{i,j}$$

by Proposition 4.6. Take the reduction of the both side of (4.43) modulo $K\langle t,(t-t^2)^{-1},h(t)^{-1}\rangle\eta^*_{i,j}$. We then have

$$\varepsilon^{(i,j)}_{1,\tau}(\lambda)-\varepsilon^{(i,j)}_{1,\sigma}(t)-b_j^{-1}\frac{F'_{a_i,b_j}(t)}{F_{a_i,b_j}(t)}(\varepsilon^{(i,j)}_{2,\tau}(\lambda)-\varepsilon^{(i,j)}_{2,\sigma}(t)) = \sum_{n=1}^{\infty}\frac{(t^\tau-t^\sigma)^n}{n!}p^{-1}f_n(t).$$

On the other hand,

$$\varepsilon^{(i,j)}_{1,\sigma}(t) = \mathscr{F}^{(\sigma)}_{a_i,b_j}(t)+b_j^{-1}\frac{F'_{a_i,b_j}(t)}{F_{a_i,b_j}(t)}\varepsilon^{(i,j)}_{2,\sigma}(t)$$

by (4.29), (4.30) and Theorem 4.8. Hence

$$\varepsilon^{(i,j)}_{1,\tau}(\lambda)-\mathscr{F}^{(\sigma)}_{a_i,b_j}(t) = \sum_{n=1}^{\infty}\frac{(t^\tau-t^\sigma)^n}{n!}p^{-1}f_n(t)+b_j^{-1}\frac{F'_{a_i,b_j}(t)}{F_{a_i,b_j}(t)}\varepsilon^{(i,j)}_{2,\tau}(\lambda)$$

as required. □

Lemma 4.15. *If $a_i+b_j<1$, then*

$$\mathrm{ord}_{\lambda=0}(\varepsilon^{(i,j)}_{1,\tau}(\lambda)) \ge 0 \quad \textit{and} \quad \mathrm{ord}_{\lambda=0}(\varepsilon^{(i,j)}_{2,\tau}(\lambda)) \ge 1.$$

Proof. Since $e_\xi - \Phi_\tau(e_\xi) \in H_K$, we have

$$\varepsilon^{(i,j)}_{1,\tau}(\lambda)\omega_{i,j}+\varepsilon^{(i,j)}_{2,\tau}(\lambda)\eta_{i,j} \in H_K(i,j).$$

If $a_i+b_j<1$, then this means

$$\varepsilon^{(i,j)}_{1,\tau}(\lambda_0^N),\, \lambda_0^{-s}\varepsilon^{(i,j)}_{2,\tau}(\lambda_0^N) \in K[\![\lambda_0]\!].$$

by Lemma 4.13. Since $s=(a_i+b_j)N<N$, the assertion follows. □

Lemma 4.16. *If $a_i+b_j<1$ and $F_{a_i,b_j}(1)_{<p^n} \not\equiv 0$ mod p for all $n\ge 1$, then*

$$\varepsilon^{(i,j)}_{1,\tau}(0) = \mathscr{F}^{(\sigma)}_{a_i,b_j}(1)$$

where the left hand side denotes the evaluation at $\lambda=0$ ($\Leftrightarrow t=1$) and the right hand side denotes the evaluation at $t=1$. Note that the left value is defined by Lemma 4.15.

Proof. This is straightforward from Lemma 4.14 on noticing that $F'_{a_i,b_j}(t)/F_{a_i,b_j}(t)$ and $f_n(t)$ are convergent at $t=1$ by [12, p. 45, Lem. 3.4] under the condition that $[F_{a_i,b_j}(t)]_{<p^n}|_{t=1}\not\equiv 0 \bmod p$ for all $n\geq 1$. □

We finish the proof of Theorem 4.11. Let (i,j) satisfy $a_i+b_j<1$. Let

$$\rho_\tau(\mathrm{reg}_{\mathrm{syn}}(\xi))(i,j)\in H_K(i,j)$$

be the eigencomponent of $\psi_\tau(\mathrm{reg}_{\mathrm{syn}}(\xi))$, which agrees with

$$-N^{-2}(1-\nu_1^{-i})(1-\nu_2^{-j})[\varepsilon_{1,\tau}^{(i,j)}(t)\omega_{i,j}+\varepsilon_{2,\tau}^{(i,j)}(t)\eta_{i,j}]$$

by (4.40) and (4.42). It is straightforward to see $\iota^*(\omega_{i,j})=Nz^{N-i-1}w^{-j}\mathrm{d}z$. Then

$$\begin{aligned}&\iota^*[\varepsilon_{1,\tau}^{(i,j)}(\lambda)\omega_{i,j}+\varepsilon_{2,\tau}^{(i,j)}(\lambda)\eta_{i,j}]\\ &=\varepsilon_{1,\tau}^{(i,j)}(\lambda_0^L)|_{\lambda_0=0}\cdot\iota^*(\omega_{i,j})+(\lambda_0^{-s}\varepsilon_{2,\tau}^{(i,j)}(\lambda_0^L))|_{\lambda_0=0}\cdot\iota^*(\lambda_0^s\eta_{i,j})] \quad \text{(Lemma 4.13)}\\ &=\varepsilon_{1,\tau}^{(i,j)}(0)\cdot Nz^{N-i-1}w^{-j}\mathrm{d}z \quad \text{(Lemma 4.15 and } s<L)\\ &=\mathscr{F}_{a_i,b_j}^{(\sigma)}(1)\cdot Nz^{N-i-1}w^{-j}\mathrm{d}z \quad \text{(Lemma 4.16).}\end{aligned}$$

Therefore

$$\mathrm{reg}_{\mathrm{syn}}(\xi|_F)=-N^{-2}\sum_{a_i+b_j<1}(1-\nu_1^{-i})(1-\nu_2^{-j})\mathscr{F}_{a_i,b_j}^{(\sigma)}(1)Nz^{N-i-1}w^{-j}\mathrm{d}z + \sum_{a_i+a_j>1}(\cdot)$$

by (4.41). Taking the summation over $(\nu_1,\nu_2)\in\mu_N\times\mu_M$, we have

$$N^2\,\mathrm{reg}_{\mathrm{syn}}(\{1-z,1-w\})=-\sum_{a_i+a_j<1}\mathscr{F}_{a_i,b_j}^{(\sigma)}(1)Nz^{N-i-1}w^{-j}dz+\sum_{a_i+a_j>1}(\cdot)$$

by (4.39). This finishes the proof of Theorem 4.11.

In [26], Ross showed the non-vanishing of the Beilinson regulator

$$\mathrm{reg}_B\{1-z,1-w\}\in H^2_{\mathscr{D}}(F,\mathbb{R}(2))\cong H^1_B(F,\mathbb{R})^{F_\infty=-1}$$

of his element in the Deligne–Beilinson cohomology group. We expect the non-vanishing also in the p-adic situation.

Conjecture 4.17. *Under the condition* (4.36), $\mathscr{F}^{(\sigma)}_{\frac{i}{N},\frac{j}{M}}(1)\neq 0$.

By the congruence relation for $\mathscr{F}^{(\sigma)}_{\underline{a}}(t)$ (Theorem 3.3), we have

$$\mathscr{F}^{(\sigma)}_{\frac{i}{N},\frac{j}{M}}(1)\neq 0 \iff [G^{(\sigma)}_{\frac{i}{N},\frac{j}{M}}(t)]_{<p^n}|_{t=1}\not\equiv 0 \mod p^n \text{ for some } n>0.$$

A number of computations by computer indicate that this holds (possibly $n\neq 1$). Moreover if the Fermat curve has a quotient to an elliptic curve over $\mathbb{Q}$, one can expect that the syntomic regulator agrees with the special value of the p-adic L-function according to the p-adic Beilinson conjecture by Perrin-Riou [23, 4.2.2]. See Conjecture 5.7 below for detail.

4.6. Syntomic Regulators of Hypergeometric curves of Gauss type. Let $W = W(\overline{\mathbb{F}}_p)$ and $K = \operatorname{Frac} W$. Let $N \geq 2$, $A, B > 0$ be integers such that $0 < A, B < N$ and $\gcd(N, A) = \gcd(N, B) = 1$. Let $X_{\mathrm{Gauss},K} \to \operatorname{Spec} K[\lambda, (\lambda - \lambda^2)^{-1}]$ a smooth projective morphism of relative dimension one whose generic fiber is defined from an affine equation

$$v^N = u^A(1-u)^B(1-\lambda u)^{N-B}.$$

We call $X_{\mathrm{Gauss},K}$ the *hypergeometric curve of Gauss type* ([1, §2.3]). The genus of a smooth fiber is $N-1$. Let X be the hypergeometric curve in Section 4.1 defined by an affine equation $(1-x^N)(1-y^N) = t$. Then there is a finite cyclic covering

$$\rho : X \times_W K \longrightarrow X_{\mathrm{Gauss},K}, \qquad \begin{cases} \rho^*(u) = x^{-N} \\ \rho^*(v) = x^{-A}(1-x^{-N})y^{N-B} \\ \rho^*(\lambda) = 1-t \end{cases} \tag{4.44}$$

of degree N whose Galois group is generated by an automorphism $g_{A,B} := [\zeta_N^B, \zeta_N^{-A}] \in \mu_N(K) \times \mu_N(K)$ (see (4.2) for the notation) where ζ_N is a fixed primitive N-th root of unity (ρ is a generalization of the Fermat quotient, e.g. [15, p. 211]).

Suppose that N is prime to p. We construct an integral model X_{Gauss} over W in the following way. The cyclic group $\langle g_{A,B} \rangle$ generated by $g_{A,B}$ acts on X, and it is a free action. We define X_{Gauss} to be the quotient

$$f : X_{\mathrm{Gauss}} \overset{\mathrm{def}}{=} X/\langle g_{A,B} \rangle \longrightarrow S = \operatorname{Spec} W[t, (t-t^2)^{-1}].$$

of X by the cyclic group $\langle g_{A,B} \rangle$. Since it is a free action, X_{Gauss} is smooth over W. The cyclic group $\mu_N(K) \times \mu_N(K)/\langle g_{A,B} \rangle$ acts on $X_{\mathrm{Gauss},K}$, and it is generated by an automorphism h given by $(u, v) \mapsto (u, \zeta_N^{-1} v)$. For an integer n, let $V(n)$ be the eigenspace on which h acts by multiplication by ζ_N^n for all $\zeta_N \in \mu_N(K)$. Then the pull-back ρ^* satisfies

$$\rho^*(H^1_{\mathrm{dR}}(X_{\mathrm{Gauss},K}/S_K)(n)) = H^1_{\mathrm{dR}}(X_K/S_K)(nA, nB), \quad 0 < n < N \tag{4.45}$$

and the push-forward ρ_* satisfies

$$\rho_*(H^1_{\mathrm{dR}}(X_K/S_K)(i,j)) = \begin{cases} H^1_{\mathrm{dR}}(X_{\mathrm{Gauss},K}/S_K)(n) & (i,j) \equiv (nA, nB) \bmod N \\ 0 & \text{otherwise} \end{cases} \tag{4.46}$$

for $0 < i, j < N$. Put

$$\omega_n := \rho_*(\omega_{nA,nB}), \quad \eta_n := \rho_*(\eta_{nA,nB}) \tag{4.47}$$

a basis of $H^1_{\mathrm{dR}}(X_{\mathrm{Gauss},K}/S_K)(n)$ (see (4.7) for the notation $(\cdot)_{nA,nB}$). Recall $e^{\mathrm{unit}}_{i,j}$ from Theorem 4.7. Put

$$e^{\mathrm{unit}}_n := \rho_* e^{\mathrm{unit}}_{nA,nB} \in H^1_{\mathrm{dR}}(X_{\mathrm{Gauss},K}/S_K)(n) \otimes K\langle t,(t-t^2)^{-1},h(t)^{-1}\rangle$$

for $0<n<N$. Notice that $\rho^*(\omega_n) = N\omega_{nA,nB}$, $\rho^*(\eta_n) = N\eta_{nA,nB}$ and $\rho^* e^{\mathrm{unit}}_n = N e^{\mathrm{unit}}_{nA,nB}$ by (4.45) and (4.46) together with the fact that $\rho_*\rho^* = N$. We put

$$(4.48)\quad \xi_{\mathrm{Gauss}} = \xi_{\mathrm{Gauss}}(\nu_1,\nu_2) := \rho_*\xi(\nu_1,\nu_2) \in K_2(X_{\mathrm{Gauss}})^{(2)} \subset K_2(X_{\mathrm{Gauss}})\otimes\mathbb{Q}$$

where $\xi(\nu_1,\nu_2)$ is as in the beginning of Section 4.4. Let σ_α be the Frobenius given by $t^\sigma = ct^p$ with $c \in 1+pW$. Taking the fixed part of (4.26) by $\langle g_{A,B}\rangle$, we have a 1-extension

$$0 \longrightarrow H^1(X_{\mathrm{Gauss}}/S)(2) \longrightarrow M_{\xi_{\mathrm{Gauss}}}(X_{\mathrm{Gauss}}/S) \longrightarrow \mathscr{O}_S \longrightarrow 0$$

in the exact category Fil-F-MIC(S). Let $e_{\xi_{\mathrm{Gauss}}} \in \mathrm{Fil}^0 M_{\xi_{\mathrm{Gauss}}}(X_{\mathrm{Gauss}}/S)_{\mathrm{dR}}$ be the unique lifting of $1 \in \mathscr{O}(S)$.

Theorem 4.18. *Put $a_n := -nA/N - \lfloor -nA/N\rfloor$ and $b_n := -nB/N - \lfloor -nB/N\rfloor$. Let $h(t) = \prod_{m=0}^{s}[F_{a_n^{(m)},b_n^{(m)}}(t)]_{<p}$ where s is the minimal integer such that $(a_n^{(s+1)}, b_n^{(s+1)}) = (a_n,b_n)$ for all $n \in \{1,2,\ldots,N-1\}$. Then*

$$e_{\xi_{\mathrm{Gauss}}} - \Phi(e_{\xi_{\mathrm{Gauss}}}) \equiv -\sum_{n=1}^{N-1}\frac{(1-\nu_1^{-nA})(1-\nu_2^{-nB})}{N^2}\mathscr{F}^{(\sigma)}_{a_n,b_n}(t)\omega_n$$

modulo $\sum_{n=1}^{N-1} K\langle t,(t-t^2),h(t)^{-1}\rangle e^{\mathrm{unit}}_n$.

Proof. This is immediate by applying ρ_* on the formula in Theorem 4.8. □

Corollary 4.19. *Suppose $p>N$. Let $a \in W$ such that $a \not\equiv 0,1 \bmod p$. Let σ_a be the Frobenius given by $t^\sigma = F(a)a^{-p}t^p$ where F is the Frobenius on W. Let $X_{\mathrm{Gauss},a}$ be the fiber at $t=a$ ($\Leftrightarrow \lambda = 1-a$), which is a smooth projective variety over W of relative dimension one. Let*

$$\mathrm{reg}_{\mathrm{syn}} : K_2(X_{\mathrm{Gauss},a}) \longrightarrow H^2_{\mathrm{syn}}(X_{\mathrm{Gauss},a},\mathbb{Q}_p(2)) \cong H^1_{\mathrm{dR}}(X_{\mathrm{Gauss},a}/K)$$

be the syntomic regulator map. Let Q be the cup-product pairing on $H^1_{\mathrm{dR}}(X_{\mathrm{Gauss},a}/K)$. Then

$$Q(\mathrm{reg}_{\mathrm{syn}}(\xi_{\mathrm{Gauss}}|_{X_a}), e^{\mathrm{unit}}_n) = N^{-2}(1-\nu_1^{-nA})(1-\nu_2^{-nB})\mathscr{F}^{(\sigma_a)}_{a_n,b_n}(a)Q(\omega_n, e^{\mathrm{unit}}_n).$$

Proof. This is immediate from Theorem 4.18 on noticing $Q(e^{\mathrm{unit}}_n, e^{\mathrm{unit}}_m) = 0$ for any n,m, cf. Corollary 4.10. □

4.7. Syntomic Regulators of elliptic curves. The methods in Section 4.4 work not only for the hypergeometric curves but also for the elliptic fibrations listed in [1, §5]. We here give the results together with a sketch of the proof because the discussion is similar to before.

Theorem 4.20. *Let $p > 5$ be a prime number. Let $W = W(\overline{\mathbb{F}}_p)$ be the Witt ring and $F := \mathrm{Frac}(W)$ the fractional field. Let $f : X \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be the elliptic fibration over W defined by a Weierstrass equation $3y^2 = 2x^3 - 3x^2 + 1 - t$ over W. Put $\omega = dx/y$. Let*

$$\xi := \left\{ \frac{y - x + 1}{y + x - 1}, \frac{t}{2(x-1)^3} \right\} \in K_2(X).$$

Let $a \in W$ satisfy that $a \not\equiv 0, 1 \mod p$ and X_a has a good ordinary reduction where X_a is the fiber at $\mathrm{Spec}\, W[t]/(t - a)$. Let $e_{\mathrm{unit}} \in H^1_{\mathrm{dR}}(X_a/K)$ be the eigen vector of the unit root with respect to the p-th Frobenius Φ. Let σ_a denote the p-th Frobenius given by $\sigma_a(t) = F(a)a^{-p}t^p$. Then, we have

$$Q(\mathrm{reg}_{\mathrm{syn}}(\xi|_{X_a}), e_{\mathrm{unit}}) = \mathscr{F}^{(\sigma_a)}_{\frac{1}{6},\frac{5}{6}}(a) Q(\omega, e_{\mathrm{unit}})$$

Sketch of the proof. Let $U \subset X$ be the complement of divisors $\{y = \pm(x - 1)\}$, $\{x = 1\}$ and $\{x = \infty\}$ so that the symbol ξ lies in the image of $K_2^M(\mathscr{O}(U))$. It is not hard to construct an elliptic fibration $f : Y \to \mathbb{P}^1_W$ such that the union the closure of $X \setminus U$ and singular fibers of f is a relative simple NCD over W and that the multiplicity of any component of the singular fibers is at most 6. Therefore this setting is under the setting in [4, §4.1]. Let $\mathscr{E}$ be the fiber over the formal neighborhood $\mathrm{Spec}\, W[\![t]\!] \hookrightarrow \mathbb{P}^1_W$. Let $\rho : \mathbb{G}_m \to \mathscr{E}$ be the uniformization, and u the uniformizer of $\mathbb{G}_m$. Then we have

$$\rho^*\omega = F(t) \frac{\mathrm{d}u}{u}$$

and a formal power series $F(t) \in W[\![t]\!]$ can be computed by the usual method of the Picard–Fuchs equation. One sees that $F(t)$ is a solution of the differential equation

$$(t - t^2)\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + (1 - 2t)\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{5}{36} y = 0,$$

and therefore it agrees with the hypergeometric power series

$$F_{\frac{1}{6},\frac{5}{6}}(t) = {}_2F_1\left({\frac{1}{6}, \frac{5}{6} \atop 1} ; t \right)$$

up to scalar. Looking at the residue of ω at the point $(x, y, t) = (1, 0, 0)$, one finds that the constant term of $F(t)$ is 1, and hence we have

$$\rho^*\omega = F_{\frac{1}{6},\frac{5}{6}}(t) \frac{\mathrm{d}u}{u}.$$

It is straightforward to show

$$\mathrm{dlog}(\xi) = \frac{\mathrm{d}x}{y}\frac{\mathrm{d}t}{t} = \omega \wedge \frac{\mathrm{d}t}{t}.$$

Then the rest of the proof goes in the same way as the proof of Theorem 4.8. ☐

The following theorems are proven by the same argument as in the proof of Theorem 4.20.

Theorem 4.21. *Let $p > 3$ be a prime and $W = W(\overline{\mathbb{F}}_p)$ the Witt ring. Let $f : X \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be the elliptic fibration over W defined by a Weierstrass equation $y^2 = x^3 + (3x + 4t)^2$, and*

$$\xi := \left\{ \frac{y - 3x - 4t}{-8t}, \frac{y + 3x + 4t}{8t} \right\}.$$

Then, under the same notation in Theorem 4.20, we have

$$Q(\mathrm{reg}_{\mathrm{syn}}(\xi|_{X_a}), e_{\mathrm{unit}}) = 3\mathscr{F}^{(\sigma_a)}_{\frac{1}{3},\frac{2}{3}}(a)Q(\omega, e_{\mathrm{unit}}).$$

Theorem 4.22. *Let $p > 3$ be a prime and $W = W(\overline{\mathbb{F}}_p)$ the Witt ring. Let $f : X \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be the elliptic fibration defined by a Weierstrass equation $y^2 = x^3 - 2x^2 + (1 - t)x$, and*

$$\xi := \left\{ \frac{y - (x - 1)}{y + (x - 1)}, \frac{-tx}{(x - 1)^3} \right\}.$$

Then, under the same notation in Theorem 4.20, we have

$$Q(\mathrm{reg}_{\mathrm{syn}}(\xi|_{X_a}), e_{\mathrm{unit}}) = \mathscr{F}^{(\sigma_a)}_{\frac{1}{4},\frac{3}{4}}(a)Q(\omega, e_{\mathrm{unit}}).$$

Remark 4.23. The syntomic regulator in Theorem 4.21 is also considered in [4, Thm. 4.8], where the authors give a full description of the element $\mathrm{reg}_{\mathrm{syn}}(\xi|_{X_a})$ in $H^1_{\mathrm{dR}}(X_a/K)$ (not only the cup-product with a unit root vector).

5. *p*-adic Beilinson conjecture for elliptic curves over $\mathbb{Q}$

5.1. Statement. The Beilinson regulator is a generalization of Dirichlet's regulators of number fields. in higher K-groups of varieties. He conjectured the formulas on the regulators and special values of motivic L-functions which generalize the analytic class number formula. For an elliptic curve E over $\mathbb{Q}$, Beilinson proved that there is an integral symbol $\xi \in K_2(E)$ such that the real regulator $\mathrm{reg}_{\mathbb{R}}(\xi) \in H^2_{\mathscr{D}}(E, \mathbb{R}(2)) \cong \mathbb{R}$ agrees with the special value of the L-function $L(E, s)$ of E,

$$\mathrm{reg}_{\mathbb{R}}(\xi) \sim_{\mathbb{Q}^\times} L'(E, 0) \quad (\Longleftrightarrow \mathrm{reg}_{\mathbb{R}}(\xi)/L'(E, 0) \in \mathbb{Q}^\times)$$

where $L'(E, s) = \frac{\mathrm{d}}{\mathrm{d}s}L(E, s)$ ([5, Thm. 1.3]).

The p-adic counterpart of the Beilinson conjecture was formulated by Perrin-Riou [23, 4.2.2], which we call the *p-adic Beilinson conjecture*. See also [10] for a general survey. Her conjecture is formulated in terms of the p-adic etale regulators (which are compatible with the syntomic regulators thanks to Besser's theorem) and the conjectural p-adic measures which provide the p-adic L-functions of motives. However there are only a few of literatures due to the extremal difficulty of the statement. In a joint paper [3] with Chida, we give a concise statement of the p-adic Beilinson conjecture by restricting ourselves to the case of elliptic curves over $\mathbb{Q}$.

Conjecture 5.1 (Weak p-adic Beilinson conjecture)**.** *Let E be an elliptuic curve over $\mathbb{Q}$. Let $p > 2$ be a prime at which E has a good ordinary reduction. Let $E_{\mathbb{Q}_p} := E \times_{\mathbb{Q}} \mathbb{Q}_p$ and let $e_{\mathrm{unit}} \in H^1_{\mathrm{dR}}(X_\alpha/K)$ be the eigen vector for the unit root $\alpha_{E,p}$ of the p-th Frobenius Φ. Let $L_p(E, \chi, s)$ be the p-adic L-function by Mazur and Swinnerton–Dyer* [20]*. Let $Q : H^1_{\mathrm{dR}}(E_{\mathbb{Q}_p}/\mathbb{Q}_p)^{\otimes 2} \to H^2_{\mathrm{dR}}(E_{\mathbb{Q}_p}/\mathbb{Q}_p) \cong \mathbb{Q}_p$ be the cup-product pairing. Let*

$$\mathrm{reg}_{\mathrm{syn}} : K_2(E) \to H^2_{\mathrm{syn}}(E, \mathbb{Q}_p(2)) \cong H^1_{\mathrm{dR}}(E_{\mathbb{Q}_p}/\mathbb{Q}_p)$$

be the syntomic regulator map. Fix a regular 1-form $\omega_E \in \Gamma(E, \Omega^1_{E/\mathbb{Q}})$. Let $\omega : \mathbb{Z}_p^\times \to \mu_{p-1}$ be the Teichmüller character. Then there is a constant $C \in \mathbb{Q}^\times$ which does not depend on p such that

$$(1 - p\alpha_{E,p}^{-1}) \frac{Q(\mathrm{reg}_{\mathrm{syn}}(\xi), e_{\mathrm{unit}})}{Q(\omega_E, e_{\mathrm{unit}})} = C L_p(E, \omega^{-1}, 0).$$

In [3, Conj. 3.3], we give a precise description of the constant C in terms of the real regulator.

5.2. Conjecture on Rogers–Zudilin type formulas. In their paper [25], Rogers and Zudilin give descriptions of special values of L-functions of elliptic curves in terms of the hypergeometric functions ${}_3F_2$ or ${}_4F_3$. Apply theorems in Section 4.7 to Conjecture 5.1, we have a statement of the p-adic counterpart of the Rogers–Zudilin type formulas.

The following is obtained from Conjecture 5.1 and Corollary 4.19 (note that the symbol (5.1) below agrees with $\xi_{\mathrm{Gauss}}|_{X_a}$ in Section 4.6 up to a constant).

Conjecture 5.2. *Let $X \to \mathbb{P}^1$ be the elliptic fibration defined by a Weierstrass equation $y^2 = x(1-x)(1-(1-t)x)$ (i.e. the hypergeometric curve of Gauss type, cf. Section 4.6). Let X_a be the fiber at $t = a \in \mathbb{Q} \setminus \{0, 1\}$. Suppose that the symbol*

$$\xi_a = \left\{ \frac{y - 1 + x}{y + 1 - x}, \frac{ax^2}{(1-x)^2} \right\} \in K_2(X_a) \tag{5.1}$$

is integral in the sense of Scholl [27]*. Let $p > 2$ be a prime such that X_a has a good ordinary reduction at p. Let $\sigma_a : \mathbb{Z}_p[\![t]\!] \to \mathbb{Z}_p[\![t]\!]$ be the p-th Frobenius given by $\sigma_a(t) = a^{1-p}t^p$. Let $\alpha_{X_a,p}$ be the unit root. Then there is a rational number $C_a \in \mathbb{Q}^\times$ not depending on p such that*

$$(1 - p\alpha_{X_a,p}^{-1})\mathscr{F}^{(\sigma_a)}_{\frac{1}{2},\frac{1}{2}}(a) = C_a L_p(X_a, \omega^{-1}, 0)$$

where ω is the Teichmüller character.

Here are examples of a such that the symbol (5.1) is integral (cf. [1, 5.4])

$$a = -1, \pm 2, \pm 4, \pm 8, \pm 16, \pm\frac{1}{2}, \pm\frac{1}{8}, \pm\frac{1}{4}, \pm\frac{1}{16}.$$

F. Brunault compared the symbol (5.1) with the Beilinson–Kato element in case $a = 4$ ([3, App. B]). Then, thanks to the main result of his paper [7], it follows that $\mathrm{reg}_{\mathrm{syn}}(\xi_a)$ gives the p-adic L-value of X_4 (see [3, Thm. 5.2] for the precise statement). We thus have

Theorem 5.3. *When $a = 4$, Conjecture 5.2 is true and $C_4 = -1$.*

We obtain the following statements from Theorems 4.20, 4.21 and 4.22.

Conjecture 5.4. *Let $a \in \mathbb{Q} \setminus \{0, 1\}$ and let X_a be the ellptic curve over $\mathbb{Q}$ defined by an affine equation $3y^2 = 2x^3 - 3x^2 + 1 - a$. Suppose that the symbol*

$$\left\{\frac{y - x + 1}{y + x - 1}, \frac{a}{2(x-1)^3}\right\} \in K_2(X_a) \tag{5.2}$$

is integral. Let $p > 5$ be a prime such that X_a has a good ordinary reduction at p. Then there is a rational number $C_a \in \mathbb{Q}^\times$ not depending on p such that

$$(1 - p\alpha_{X_a,p}^{-1})\mathscr{F}^{(\sigma_a)}_{\frac{1}{6},\frac{5}{6}}(a) = C_a L_p(X_a, \omega^{-1}, 0).$$

There are infinitely many a such that the symbol (5.2) is integral. For example, if $a = 1/n$ with $n \in \mathbb{Z}_{\geq 2}$ and $n \equiv 0, 2 \bmod 6$, then the symbol (5.2) is integral (cf. [1, 5.4]).

Conjecture 5.5. *Let $a \in \mathbb{Q} \setminus \{0, 1\}$ and let X_a be the ellptic curve over $\mathbb{Q}$ defined by an affine equation $y^2 = x^3 + (3x + 4a)^2$. Suppose that the symbol*

$$\left\{\frac{y - 3x - 4a}{-8a}, \frac{y + 3x + 4a}{8a}\right\} \in K_2(X_a) \tag{5.3}$$

is integral. Let $p > 3$ be a prime such that X_a has a good ordinary reduction at p. Then there is a rational number $C_a \in \mathbb{Q}^\times$ not depending on p such that

$$(1 - p\alpha_{X_a,p}^{-1})\mathscr{F}^{(\sigma_a)}_{\frac{1}{3},\frac{2}{3}}(a) = C_a L_p(X_a, \omega^{-1}, 0).$$

If $a = \frac{1}{6n}$ with $n \in \mathbb{Z}_{\geq 1}$ arbitrary, then the symbol (5.3) is integral (cf. [1, 5.4]).

Conjecture 5.6. *Let $a \in \mathbb{Q} \setminus \{0,1\}$ and let X_a be the elliptic curve over $\mathbb{Q}$ defined by an affine equation $y^2 = x^3 - 2x^2 + (1-a)x$. Suppose that the symbol*

$$\left\{ \frac{y-(x-1)}{y+(x-1)}, \frac{-ax}{(x-1)^3} \right\} \in K_2(X_a) \tag{5.4}$$

is integral. Let $p > 2$ be a prime such that X_a has a good ordinary reduction at p. Then there is a rational number $C_a \in \mathbb{Q}^\times$ not depending on p such that

$$(1 - p\alpha_{X_a,p}^{-1})\mathscr{F}^{(\sigma_a)}_{\frac{1}{4},\frac{3}{4}}(a) = C_a L_p(X_a, \omega^{-1}, 0).$$

If the denominator of $j(X_a) = 64(1+3a)^3/(a(1-a)^2)$ is prime to a (e.g. $a = 1/n$, $n \in \mathbb{Z}_{\geq 2}$), then the symbol (5.4) is integral.

From Corollary 4.10 and Theorem 4.11, we have the following conjectures.

Conjecture 5.7. *Let $a \in \mathbb{Q} \setminus \{0,1\}$ and let X_a be the ellptic curve over $\mathbb{Q}$ defined by an affine equation $(x^2-1)(y^2-1) = a$. Suppose that the symbol*

$$\left\{ \frac{x-1}{x+1}, \frac{y-1}{y+1} \right\} \in K_2(X_a) \tag{5.5}$$

is integral. Let $p > 2$ be a prime such that X_a has a good ordinary reduction at p. Then there is a rational number $C_a \in \mathbb{Q}^\times$ not depending on p such that

$$(1 - p\alpha_{X_a,p}^{-1})\mathscr{F}^{(\sigma_a)}_{\frac{1}{2},\frac{1}{2}}(1) = C_a L_p(X_a, \omega^{-1}, 0).$$

If the denominator of $j(X_a) = 16(a^2 - 16a + 16)^3/((1-a)a^4)$ is prime to a (e.g. $a = \pm 2^n$, $n \in \{\pm 1, \pm 2, \pm 3\}$), then the symbol (5.5) is integral.

Conjecture 5.8. *Let $F_{N,M}$ be the Fermat curve defined by an affine equation $z^N + w^M = 1$, and $F^*_{2,4}$ the curve $z^2 = w^4 + 1$. Let $\sigma = \sigma_1$ (i.e. $\sigma(t) = t^p$). Then there are rational numbers $C, C', C'' \in \mathbb{Q}^\times$ not depending on p such that*

$$(1 - p\alpha_{F_{3,3},p}^{-1})\mathscr{F}^{(\sigma)}_{\frac{1}{3},\frac{1}{3}}(1) = C L_p(F_{3,3}, \omega^{-1}, 0),$$
$$(1 - p\alpha_{F_{2,4},p}^{-1})\mathscr{F}^{(\sigma)}_{\frac{1}{2},\frac{1}{4}}(1) = C' L_p(F_{2,4}, \omega^{-1}, 0),$$
$$(1 - p\alpha_{F^*_{2,4},p}^{-1})\mathscr{F}^{(\sigma)}_{\frac{1}{4},\frac{1}{4}}(1) = C'' L_p(F^*_{2,4}, \omega^{-1}, 0).$$

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