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# Twisted Alexander polynomials, chirality, and local deformations of hyperbolic 3-cone-manifolds

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**Abstract**

In this paper, we discuss a relationship between the chirality of knots and higher-dimensional twisted Alexander polynomials associated with holonomy representations of hyperbolic 3-cone-manifolds. In particular, we provide a new necessary condition for a knot, that appears in a hyperbolic 3-cone-manifold of finite volume as a singular set, to be amphicheiral. Moreover, we can detect the chirality of hyperbolic twist knots, according to our criterion, using low-dimensional irreducible representations.

## 1. Introduction

The Alexander polynomial is one of the fundamental invariants of a knot in the 3-sphere $S^3$. It is determined by the maximal metabelian quotient of the fundamental group of the complement of a knot (namely, the knot group), and hence is far from a complete invariant. In particular, it often fails to detect geometric or topological properties of a knot. For example, the Alexander polynomial has mutation invariance, and cannot detect the chirality of knots, because a given knot and its mirror image have isomorphic knot groups.

The twisted Alexander polynomial was originally introduced by Lin [18] for knots in $S^3$ and by Wada for finitely presentable groups [33]. It is defined for a group and its representation and provides a natural generalization of the Alexander polynomial. Wada shows in [33] that Kinoshita–Terasaka and Conway's 11 crossing knots are distinguished by the collection of twisted Alexander polynomials associated with representations over a finite field. That is, we might be able to detect mutant knots using twisted Alexander polynomials, even though such knots share many polynomial invariants (see [24] for example). As for the chirality of a knot, Dunfield, Friedl and Jackson presented a criterion for determining whether a given hyperbolic knot, that is, a knot whose complement admits a complete hyperbolic metric of finite volume, is amphicheiral by means of the normalized twisted Alexander polynomial, the hyperbolic torsion polynomial, associated with a lift of the holonomy representation into $\mathrm{SL}(2,\mathbb{C})$. See the survey papers [7, 22]

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and the references therein for recent developments on twisted Alexander polynomials and their applications.

The purpose of this paper is to give a new necessary condition for a given knot that appears in a hyperbolic 3-cone-manifold of finite volume as a singular set to be amphicheiral using higher-dimensional twisted Alexander polynomials associated to a lift of the holonomy representation of a 3-cone-manifold (Corollary 3.2). This result generalizes the result of Dunfield, Friedl and Jackson mentioned above in two directions. One is for higher-dimensional irreducible representations of knot groups, and the other is for deformation of hyperbolic 3-cone-manifolds. To the best of our knowledge, the latter is a new application of the twisted Alexander polynomials to local deformations of hyperbolic 3-cone-manifolds. Furthermore, for hyperbolic twist knots, an infinite family of hyperbolic two-bridge knots, we can detect their chirality in hyperbolic 3-cone-manifolds using our criterion with low-dimensional irreducible representations of knot groups. Roughly speaking, the following three conditions are equivalent (see Theorem 4.2 for a more precise statement): (i) a hyperbolic twist knot is amphicheiral, (ii) the hyperbolic torsion polynomial is a real polynomial, (iii) either the adjoint torsion polynomial (see Subsection 2.3 for the definition) is a real polynomial or its coefficients are pure imaginary.

The remainder of this paper is organized as follows. In the next section, we review some basic notions, twisted Alexander polynomials and higher-dimensional irreducible representations of the special linear group $\mathrm{SL}(2, \mathbb{C})$. In Section 3, we state our criterion for a given knot to be amphicheiral. In the final section, we discuss a characterization of the chirality of a twist knot in a hyperbolic 3-cone-manifold.

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## 2. **Preliminaries**

### 2.1. **Basic notions**

We review basic notions on knots in accordance with [29]. A *knot* $K$ is a smoothly embedded circle in the 3-sphere $S^3$. Two knots $K$ and $K'$ are said to be *equivalent*, $K \cong K'$, if there is a self-homeomorphism $f$ of $S^3$ such that $f(K) = K'$, i.e., the pair $(S^3, K)$ is homeomorphic to the pair $(S^3, K')$. If the homeomorphism $f$ preserves the orientation of $S^3$ and hence is isotopic to the identity homeomorphism, then $K$ and $K'$ are said to be *isotopic*. Every knot is represented by a *knot diagram*, a 4-valent planar graph whose vertices are endowed with over/under information. A vertex of a knot diagram with over/under information is called a *crossing*.

For a knot $K$, the knot $K^*$, *the mirror image* of $K$, is the image of $K$ under an orientation-reversing homeomorphism of $S^3$. $K^*$ is represented by the knot diagram obtained from that of $K$ by reversing the over/under information at every crossing. A knot $K$ is *amphicheiral* (or *achiral*) if $K^*$ is isotopic to $K$; otherwise, it is *chiral*. It is well-known that the trefoil knot is chiral, and that the figure-eight knot is amphicheiral.

An *oriented knot* is a knot $K$ for which the circle $K$ is also endowed with an orientation. (We assume that $S^3$ has the standard orientation.) Two oriented knots $K$ and $K'$ are said to be *isotopic*, if there is an orientation-preserving self-homeomorphism $f$ of $S^3$ with $f(K) = K'$ such that $f|_K : K \to K'$ is also orientation-preserving. This is equivalent to the condition that there is an isotopy of $S^3$ that carries the oriented circle $K$ to the oriented circle $K'$. For a specific oriented knot $K$, we obtain the following three (possibly isotopic) oriented knots by reversing one or both of the orientations of $S^3$ and the circle $K$:

$$-K := (S^3, -K),\ K^* := (-S^3, K) \cong (S^3, K^*),\ -K^* := (-S^3, -K) \cong (S^3, -K^*).$$

Let $M$ be a complete hyperbolic 3-manifold of finite volume. There is a faithful representation $\rho_0 : \pi_1(M) \to \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C})$, where $\mathbb{H}^3$ denotes the upper half-space model of the hyperbolic 3-space with discrete image such that $\mathbb{H}^3/\mathrm{Im}\,\rho_0 \cong M$. The representation $\rho_0$ is called a *holonomy representation* and is unique up to conjugation. Since conjugate representations correspond to the same geometric structure and from the Mostow–Prasad rigidity theorem the hyperbolic metric is unique as long as it is complete. Thus, the unique complete hyperbolic structure of a complete hyperbolic 3-manifold corresponds to the discrete faithful representation. It is also known that a peripheral torus subgroup of $\rho_0(\pi_1(M)) \subset \mathrm{PSL}(2, \mathbb{C})$ is conjugate to a group of cosets of matrices of the from $\left(\begin{smallmatrix} 1 & \nu \\ 0 & 1 \end{smallmatrix}\right)$, where $\nu \in \mathbb{C}$. In particular, the traces (defined up to sign) of the elements of such a group are $\pm 2$.

Let $C$ be an orientable hyperbolic 3-cone-manifold of finite volume with 1-dimensional compact singularity $\Sigma$. $C$ carries a nonsingular but incomplete hyperbolic structure on the complement of the singularity $N = C - \Sigma$. $C$ itself inherits a metric induced from a Riemannian metric on $N$. We assume that $C$ is complete with respect to this metric. In particular, the metric completion of $N$ is identical to $C$. As in the case of a complete hyperbolic 3-manifold of finite volume, we have a developing map of $N$ from its universal covering space $\widetilde{N}$, say, $D_C : \widetilde{N} \to \mathbb{H}^3$, and a holonomy representation $\rho_C : \pi_1(N) \to \mathrm{PSL}(2, \mathbb{C})$. These are called a *developing map* and a *holonomy representation* of the cone-manifold $C$. In this case, a developing map is a local isometry, but is never injective. A holonomy representation of $C$ is hardly discrete nor faithful.

In this paper, we assume that $C = S^3$ and that the singularity $\Sigma$ is connected, i.e., it forms a knot $K$ in $S^3$ with cone angle $\alpha \in (0, \infty)$. Let $m$ be an oriented meridian loop for the singular set $K$. The image $\rho_C(m)$ under the holonomy representation is an elliptic element rotating $\mathbb{H}^3$ by $\alpha$ about the axis, though the rotation angle of $\rho_C(m)$ makes sense only modulo $2\pi$.

Hereafter, we regard a cusp as an empty singular set, following which we can consider a finite volume hyperbolic 3-manifold $M$ as a 3-cone-manifold of cone angle zero. In this sense, we denote a holonomy representation of a cone-manifold $C$ with cone angle $\alpha$ by $\rho_\alpha$. In [30], Thurston shows that $\rho_0$ can be deformed into a one-parameter family $\{\rho_\alpha\}_\alpha$ of representations to yield a corresponding one-parameter family $\{C_\alpha\}_\alpha$ of singular complete hyperbolic 3-manifolds. These $C_\alpha$'s are called the *hyperbolic* 3*-cone-manifolds with cone angle $\alpha$ along $K$*.

The holonomy representation $\rho_0$ of a complete hyperbolic 3-manifold is known to lift into $\mathrm{SL}(2,\mathbb{C})$ in [30]. Kojima proved that the holonomy representation of a compact orientable hyperbolic 3-cone-manifold can be lifted to an $\mathrm{SL}(2,\mathbb{C})$-representation, if the cone angle is at most $\pi$ ([17, Corollary 2]). An element of $\mathrm{SL}(2,\mathbb{C})$ is a nontrivial rotation of $\mathbb{H}^3$ if and only if its trace is real and contained in $(-2, 2)$, and that

$$\mathrm{tr}(\rho_\alpha(m)) = \pm 2\cos(\alpha/2). \tag{2.1}$$

The sign of this formula depends only on the choice of the lift to $\mathrm{SL}(2,\mathbb{C})$ of the representation into $\mathrm{PSL}(2,\mathbb{C})$. By abuse of notation, we denote by $\rho_\alpha$ a lift of the holonomy representation of the hyperbolic 3-cone-manifold with cone angle $\alpha$.

## 2.2. Twisted Alexander polynomial

Let $K$ be an oriented knot in the 3-sphere $S^3$, and $E_K = S^3 - \mathrm{int}(N(K))$ the exterior of $K$ in $S^3$. Here, $N(K)$ is a closed tubular neighborhood of $K$. We denote $\pi_1(E_K)$ by $G(K)$ and call it the *knot group*. We choose and fix a Wirtinger presentation of $G(K)$:

$$G(K) = \langle x_1, \ldots, x_\ell \mid r_1, \ldots, r_{\ell-1}\rangle$$

where every generator corresponds to an arc in a knot diagram $D(K)$ of $K$ and every relator comes from a crossing in $D(K)$. The abelianization homomorphism

$$\mathfrak{a} \;:\; G(K) \to H_1(E_K;\mathbb{Z}) \cong \mathbb{Z} = \langle t\rangle$$

is provided by assigning each generator $x_i$ to the meridian element $t \in H_1(E_K;\mathbb{Z})$.

In this paper, we consider a representation of $G(K)$ into the $k$-dimensional special linear group $\mathrm{SL}(k,\mathbb{C})$, say, $\rho : G(K) \to \mathrm{SL}(k,\mathbb{C})$. The maps $\rho$ and $\mathfrak{a}$ naturally induce two ring homomorphisms $\widetilde{\rho} : \mathbb{Z}[G(K)] \to M(k,\mathbb{C})$ and $\widetilde{\mathfrak{a}} : \mathbb{Z}[G(K)] \to \mathbb{Z}[t^{\pm 1}]$, where $\mathbb{Z}[G(K)]$ is the group ring of $G(K)$ and $M(k,\mathbb{C})$ is the matrix algebra of degree $k$ over

$\mathbb{C}$. Then $\widetilde{\mathfrak{a}} \otimes \widetilde{\rho}$ defines a ring homomorphism $\mathbb{Z}[G(K)] \to M(k, \mathbb{C}[t^{\pm 1}])$. Let $F_\ell$ denote the free group on generators $x_1, \ldots, x_\ell$ and

$$\Phi \ : \ \mathbb{Z}[F_\ell] \to M(k, \mathbb{C}[t^{\pm 1}])$$

the composition of the surjection $\widetilde{\phi} \ : \ \mathbb{Z}[F_\ell] \to \mathbb{Z}[G(K)]$ induced by the presentation of $G(K)$ and the map $\widetilde{\mathfrak{a}} \otimes \widetilde{\rho} : \mathbb{Z}[G(K)] \to M(k, \mathbb{C}[t^{\pm 1}])$.

Let us consider the $(\ell - 1) \times \ell$ matrix $A$ whose $(i, j)$-entry is the $k \times k$ matrix

$$\Phi\left(\frac{\partial r_i}{\partial x_j}\right) \in M(k, \mathbb{C}[t^{\pm 1}]),$$

where $\frac{\partial}{\partial x} : \mathbb{Z}[F_\ell] \to \mathbb{Z}[F_\ell]$ is the free differential. We call $A$ the *Alexander matrix* of the knot group $G(K)$ associated with $\rho$. For $1 \leq j \leq \ell$, let us denote by $A_j$ the $(\ell - 1) \times (\ell - 1)$ matrix obtained from $A$ by removing the $j$-th column. We regard $A_j$ as a $k(\ell - 1) \times k(\ell - 1)$ matrix with coefficients in $\mathbb{C}[t^{\pm 1}]$. The *twisted Alexander polynomial* of a knot $K$ associated with a representation $\rho : G(K) \to \mathrm{SL}(k, \mathbb{C})$ is the rational function

$$\Delta_{K,\rho}(t) = \frac{\det A_j}{\det \Phi(x_j - 1)}$$

and is well-defined up to multiplication by $t^j$ $(j \in \mathbb{Z})$ if $k$ is even, and by $\pm t^j$ if $k$ is odd (see [33] for details). In particular, it does not depend on the choice of a presentation of $G(K)$.

*Remark 2.1.* Let $G(K_i)$ $(i = 1, 2)$ be the knot groups with abelianizations $\mathfrak{a}_i : G(K_i) \to \mathbb{Z}$. If there is an isomorphism $\psi : G(K_1) \to G(K_2)$ such that $\mathfrak{a}_1 = \mathfrak{a}_2 \circ \psi$, then for any representation $\rho : G(K_1) \to \mathrm{SL}(k, \mathbb{C})$ of $G(K_1)$, we have $\Delta_{K_1,\rho}(t) \doteq \Delta_{K_2,\rho\circ\psi^{-1}}(t)$ (see [33, Section 3]).

## 2.3. Irreducible representation of $\mathrm{SL}(2, \mathbb{C})$

A representation $\rho : G \to \mathrm{SL}(k, \mathbb{C})$ of a group $G$ is called irreducible if there is no proper invariant subspace of $\mathbb{C}^k$ under the action of $\rho(G)$. The group $\mathrm{SL}(2, \mathbb{C})$ acts naturally on the 2-dimensional vector space $\mathbb{C}^2$. The symmetric product $\mathrm{Sym}^{k-1}(\mathbb{C}^2)$ and the induced action by $\mathrm{SL}(2, \mathbb{C})$ provide a $k$-dimensional irreducible representation of $\mathrm{SL}(2, \mathbb{C})$. In fact, $\mathrm{Sym}^{k-1}(\mathbb{C}^2)$ can be identified with the vector space $V_k$ of homogeneous polynomials on $\mathbb{C}^2$ with degree $k - 1$, i.e.,

$$V_k = \mathrm{span}_{\mathbb{C}}\langle z_1^{k-1}, z_1^{k-2} z_2, \ldots, z_1 z_2^{k-2}, z_2^{k-1}\rangle.$$

The action of $P \in \mathrm{SL}(2, \mathbb{C})$ on $V_k$ is

$$P \cdot p(\mathbf{z}) := p\left(P^{-1}\mathbf{z}\right), \ \mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

which induces a representation $\sigma_k : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(V_k)$. For $\bar{P} \in \mathrm{SL}(2,\mathbb{C})$, the complex conjugate of $P$, we have

$$\bar{P} \cdot p(\mathbf{z}) = p\left(\bar{P}^{-1}\mathbf{z}\right) = p\left(\overline{P^{-1}\mathbf{z}}\right) = \overline{p\left(P^{-1}\mathbf{z}\right)} = \overline{P \cdot p(\mathbf{z})}.$$

This equality shows that $\sigma_k(\bar{P}) = \overline{\sigma_k(P)}$. It is well-known that the image of $\sigma_k$ is actually contained in $\mathrm{SL}(k,\mathbb{C})$. For a representation $\rho : G(K) \to \mathrm{SL}(2,\mathbb{C})$, we denote by $\rho^{(k)}$ the composition $\sigma_k \circ \rho : G(K) \to \mathrm{SL}(k,\mathbb{C})$, whereby $\rho^{(2)} = \rho$.

Let Ad be the adjoint action of $\mathrm{SL}(2,\mathbb{C})$ on the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, $\mathrm{Ad} : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{Aut}(\mathfrak{sl}(2,\mathbb{C}))$; $A \mapsto (\mathrm{Ad}_A : x \mapsto AxA^{-1})$. It is known that Ad is faithful and irreducible, and that $\mathrm{Ad}\circ\rho : G(K) \to \mathrm{Aut}(\mathfrak{sl}(2,\mathbb{C})) \leq \mathrm{SL}(3,\mathbb{C})$ is equivalent to $\rho^{(3)}$. We call $\Delta_{K,Ad\circ\rho}(t)$ the adjoint twisted Alexander polynomial of $K$, and in particular, we call $\Delta_{K,Ad\circ\rho_0}(t)$ the adjoint torsion polynomial of the hyperbolic knot $K$.

*Remark 2.2.* It is known that $\Delta_{K,\rho^{(2)}}(t)$ is a polynomial if $\rho : G(K) \to \mathrm{SL}(2,\mathbb{C})$ is a non-abelian representation (see [16]). More generally, if $\rho^{(k)}|_{[G(K),G(K)]}$ is nontrivial for an irreducible representation $\rho : G(K) \to \mathrm{SL}(2,\mathbb{C})$, then $\Delta_{K,\rho^{(k)}}(t)$ is a polynomial (see [6]).

*Remark 2.3.* Suppose $\rho$ is a non-trivial representation such that the twisted homology group $H_*(E_K,\rho)$ vanishes. Substituting $t = 1$ for the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ yields the Reidemeister torsion $\tau(K,\rho) = \Delta_{K,\rho}(1)$ of the exterior $E_K$ (see [15] and Proposition A.2 in Appendix). If $\rho$ is a lift of the holonomy representation of the complete hyperbolic structure, then $H_*(E_K,\rho^{(k)}) = 0$ and $\tau(K,\rho^{(k)}) = \Delta_{K,\rho^{(k)}}(1)$ when $k$ is even, while $H_*(E_K,\rho^{(k)}) \neq 0$ when $k$ is odd. In this case the Reidemeister torsion depends on the choice of some bases $\mathbf{h} = \{h^1, h^2\}$ for the twisted homology groups. If we choose the longitude of $K$ as the base, the Reidemeister torsion $\tau(K,\rho^{(k)},\mathbf{h})$ is equal to $\lim_{t\to 1} \Delta_{K,\rho^{(k)}}(t)/(t-1)$. See Appendix for details.

## 3. **Chirality and twisted Alexander polynomial**

In this section, we describe a relationship between a higher-dimensional twisted Alexander polynomial of a hyperbolic knot $K$ and its mirror image $K^*$, following which we provide a new criterion for a given hyperbolic knot to be amphicheiral.

### 3.1. **Twisted Alexander polynomial of mirror image**

Let us denote by $\bar{R}(t)$ the rational function whose coefficients are the complex conjugates of those of a rational function $R(t) \in \mathbb{C}(t)$.

**Theorem 3.1.** *Let $\rho_\alpha : G(K) \to \mathrm{SL}(2,\mathbb{C})$ be a lift of the holonomy representation of a hyperbolic 3-cone-manifold with cone angle $\alpha \in [0,\pi)$ along the singularity $K$. For the mirror image $K^*$ of $K$ and its lift of the holonomy representation $\rho^*_\alpha : G(K^*) \to \mathrm{SL}(2,\mathbb{C})$, $\Delta_{K^*,\rho^{*}_\alpha{}^{(k)}}(t) = \overline{\Delta_{K,\rho_\alpha^{(k)}}}(t)$ holds.*

*Proof.* Let $\psi : G(K) \to G(K^*)$ be the isomorphism induced from an orientation-reversing self-homeomorphism of $S^3$ taking $K$ to itself. A lift of the holonomy representation of the mirror image $K^*$ is $\rho^*_\alpha = \bar{\rho}_\alpha \circ \psi^{-1} : G(K^*) \to \mathrm{SL}(2,\mathbb{C})$, where each $\bar{\rho}_\alpha(x)$ is the matrix that is the complex conjugate of $\rho_\alpha(x)$. Thus, by Remark 2.1 and $\sigma_k(\bar{P}) = \overline{\sigma_k(P)}$,

$$
\begin{aligned}
\Delta_{K^*,\rho^{*}_\alpha{}^{(k)}}(t) &= \Delta_{K^*,\sigma_k\circ\rho^*_\alpha}(t) \\
&= \Delta_{K,\sigma_k\circ(\rho^*_\alpha\circ\psi)}(t) = \Delta_{K,\sigma_k\circ\bar{\rho}_\alpha}(t) = \Delta_{K,\overline{\sigma_k\circ\rho_\alpha}}(t) = \overline{\Delta_{K,\sigma_k\circ\rho_\alpha}}(t) \\
&= \overline{\Delta_{K,\rho_\alpha^{(k)}}}(t).
\end{aligned}
$$

This completes the proof. □

As an immediate corollary of Theorem 3.1, we have the following.

**Corollary 3.2.** *Under the same assumption as in Theorem 3.1, if $K$ is amphicheiral, then every coefficient of $\Delta_{K,\rho_\alpha^{(k)}}(t)$ is real if $k$ is even, and is real or pure imaginary if $k$ is odd.*

*Remark 3.3.* When $k = 2$, Theorem 3.1 and Corollary 3.2 were proven originally by Dunfield, Friedl and Jackson ([5, Theorem 1.2]), though they consider only the holonomy representation corresponding to the complete hyperbolic structure of the complement of a hyperbolic knot in $S^3$. Porti shows that similar properties hold for Reidemeister torsions associated with higher-dimensional representations of closed hyperbolic 3-manifolds ([27, Section 4]). Moreover, Dubois [3] refined the Reidemeister torsion, which is a real number with a well-defined sign, has the property to change its sign when the knot changes in its mirror image.

## 3.2. Example

Let us consider the figure-eight knot $4_1$, which is one of the hyperbolic two-bridge knots and amphicheiral. By [1, 2, 25, 28], for a hyperbolic two-bridge knot $K$, there exists an angle $\alpha_K \in [2\pi/3, \pi)$ such that $C_\alpha$ has the following types of cone-manifold structures: (i) hyperbolic for $\alpha \in (0, \alpha_K)$, (ii) Euclidean for $\alpha = \alpha_K$, and (iii) spherical for $\alpha \in (\alpha_K, \pi)$. In particular, $\alpha_{4_1}$ is known to be $2\pi/3$ (see [10, 19, 26]).

Now, let us compute the twisted Alexander polynomial of $4_1$ associated with 2- and 3-dimensional representations (see [4, Example 4.1] for adjoint representation). The knot group $G(4_1)$ has the following presentation:

$$G(4_1) = \langle a,\ b \mid aba^{-1}b^{-1}ab^{-1}a^{-1}bab^{-1} \rangle.$$

See Figure 1 for the definition of $a$ and $b$. Suppose $\rho : G(4_1) \to \mathrm{SL}(2,\mathbb{C})$ is an irreducible representation. Up to conjugation, we can assume that

$$\rho(a) = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix} \quad \text{and} \quad \rho(b) = \begin{pmatrix} s & 0 \\ 2-y & s^{-1} \end{pmatrix}$$

where $s \neq 0 \in \mathbb{C}$ and $y \neq 2 \in \mathbb{C}$ satisfy the equation

$$y^2 - (s^{-2} + s^2 + 1)y + (s^{-2} + s^2 + 1) = 0. \tag{3.1}$$

We set $x = \operatorname{tr}\rho(a) = \operatorname{tr}\rho(b) = s + s^{-1}$ and note that $y = \operatorname{tr}\rho(ab^{-1})$. Using the letter $x$, this equation becomes

$$y^2 - (x^2 - 1)y + x^2 - 1 = 0. \tag{3.2}$$

Denoting the relator of $G(4_1)$ by $r = aba^{-1}b^{-1}ab^{-1}a^{-1}bab^{-1}$, we have

$$\begin{aligned}\frac{\partial r}{\partial a} &= 1 - aba^{-1} + aba^{-1}b^{-1} - aba^{-1}b^{-1}ab^{-1}a^{-1} + aba^{-1}b^{-1}ab^{-1}a^{-1}b \\ &= 1 - aba^{-1} + aba^{-1}b^{-1} - ba^{-1}b^{-1} + ba^{-1},\end{aligned}$$

and

$$\begin{aligned}\det A_2 &= \left|\Phi\left(\frac{\partial r}{\partial a}\right)\right| = \left|I - t\rho\left(aba^{-1}\right) + \rho\left(aba^{-1}b^{-1}\right) - t^{-1}\rho\left(ba^{-1}b^{-1}\right) + \rho\left(ba^{-1}\right)\right| \\ &= \frac{1}{t^2} - \frac{3}{t}(s^{-1} + s) + 2(s^{-2} + s^2 + 3) - 3t(s^{-1} + s) + t^2,\end{aligned}$$

where we have used the relation (3.1). On the other hand, the denominator of the twisted Alexander polynomial is $\det\Phi(b-1) = 1 - (s^{-1} + s)t + t^2$. Consequently,

$$\Delta_{4_1,\rho}(t) = \frac{\det A_2}{\det\Phi(b-1)} = \frac{1}{t^2}\left(1 - 2(s^{-1} + s)t + t^2\right) \doteq 1 - 2xt + t^2.$$

Substituting $t = 1$ for $\Delta_{4_1,\rho}(t)$ yields the Reidemeister torsion $\tau(4_1,\rho) = 2 - 2x$ (see [27, Example 3.18]).

Next, setting $z = y - 2$ yields the adjoint representation

$$\rho^{(3)}(a) = \sigma_3 \circ \rho(a) = \begin{pmatrix} s^{-2} & 0 & 0 \\ -2s^{-1} & 1 & 0 \\ 1 & -s & s^2 \end{pmatrix} \quad \text{and} \quad \rho^{(3)}(b) = \begin{pmatrix} s^{-2} & zs^{-1} & z^2 \\ 0 & 1 & 2zs \\ 0 & 0 & s^2 \end{pmatrix}.$$

Thereby a similar computation shows that

$$\begin{aligned}\Delta_{4_1,\rho^{(3)}}(t) &= \frac{-1+t}{t^3}\left(1-(2s^{-2}+1+2s^2)t+t^2\right)\\ &\doteq -1+2(-1+x^2)t-2(-1+x^2)t^2+t^3\end{aligned}$$

and

$$\tau(4_1,\rho^{(3)},\mathbf{h}) = \lim_{t\to 1}\frac{\Delta_{4_1,\rho^{(3)}}(t)}{t-1} = 5-2x^2.$$

If $x=\pm 2\cos(\alpha/2)$ (refer to (2.1)), the coefficients of the twisted Alexander polynomials, and hence the Reidemeister torsions, are all real. According to (3.2), $y\in\mathbb{C}\setminus\mathbb{R}$, if $1<x^2<5$. The case $x=1$ corresponds to $\alpha=2\pi/3=\alpha_{4_1}$. These results were obtained using Mathematica.

## 4. On the converse of Corollary 3.2

In this section, we show that the converse of Corollary 3.2 holds for 2- and 3-dimensional representations of hyperbolic twist knots, one of the infinite families of hyperbolic genus one two-bridge knots. Cf. [5, Open problem 5 in §1.7].

### 4.1. Genus one two-bridge knot

In accordance with [13], let $K=J(k,l)$ be a two-bridge knot, as shown in Figure 1. A positive number corresponds to the right-handed twist, and a negative number corresponds to the left-handed twist. Note that $J(k,l)$ is a knot if and only if $kl$ is even and is the trivial knot if $kl=0$. Furthermore, $J(k,l)$ is the mirror image of $J(l,k)=J(-k,-l)$.

It is known that any *genus one two-bridge knot* is equivalent to $J(2m,2n)$ for some $m,n$. In particular, $J(2,2n)$ is called the *twist knot*, denoted $K_{2n}$ for simplicity. The typical examples are the trefoil knot $J(2,2)=K_2$ and the figure-eight knot $J(2,-2)=K_{-2}$. It is also known that $J(2m,2n)$ is hyperbolic if $(m,n)\neq(1,1)$. Hereafter, we assume that $K=J(2m,2n)$ is a hyperbolic knot.

The knot group of $K=J(2m,2n)$ has a presentation

$$G(K)=\langle a,b \mid w^n a = b w^n\rangle$$

where $a,b$ are meridians and $w=(ba^{-1})^m(b^{-1}a)^m$. Suppose that $\rho: G(K)\to \mathrm{SL}(2,\mathbb{C})$ is an irreducible representation. Up to conjugation, we can assume that

$$\rho(a)=\begin{pmatrix} s & 1\\ 0 & s^{-1}\end{pmatrix} \quad\text{and}\quad \rho(b)=\begin{pmatrix} s & 0\\ 2-y & s^{-1}\end{pmatrix}$$

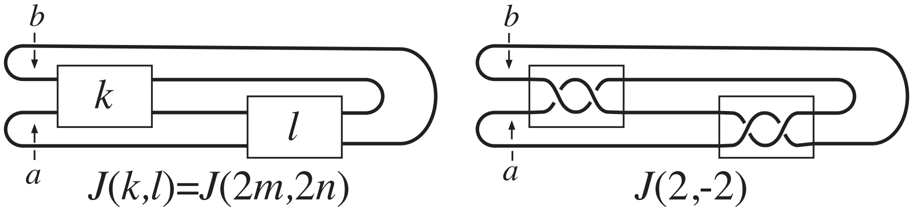


FIGURE 4.1. Genus one two-bridge knot $J(2m, 2n)$

where complex numbers $s \neq 0$ and $y \neq 2$ satisfy the Riley equation $\phi_K(s, y) = 0$. We call $\phi_K(s, y) \in \mathbb{Z}[s^{\pm 1}, y]$ the *Riley polynomial* of $K$. Note that $y = \operatorname{tr} \rho(ab^{-1})$. Hence, the set

$$X_K = \{(s, y) \in \mathbb{C}^2 \mid \phi_K(s, y) = 0, s \neq 0, y \neq 2\}$$

describes every conjugacy class of the irreducible representation $\rho : G(K) \to \mathrm{SL}(2, \mathbb{C})$.

In [31], Tran shows that the Riley polynomial can be expressed explicitly as

$$\phi_K(s, y) = S_{n-2}(\lambda) - \Big(1 - (y + 2 - x^2)S_{m-1}(y)\big(S_{m-1}(y) - S_{m-2}(y)\big)\Big)S_{n-1}(\lambda)$$

where $x = \operatorname{tr} \rho(a) = \operatorname{tr} \rho(b) = s + s^{-1}$ and

$$\lambda := \operatorname{tr} \rho(w) = 2 + (y - 2)\left(y + 2 - x^2\right) S_{m-1}^2(y).$$

This is equivalent to

$$\lambda = 2S_m^2(y) - 2yS_m(y)S_{m-1}(y) + \left(-x^2y + 2x^2 + y^2 - 2\right) S_{m-1}^2(y).$$

Note that an equation corresponding to this in [31, p. 2 line 15] contains an error. Here, the $S_l(v)$'s are the Chebyshev polynomials of the second kind defined by

$$S_0(v) = 1,\ S_1(v) = v \quad \text{and} \quad S_l(v) = vS_{l-1}(v) - S_{l-2}(v)$$

for all integers $l$. We can easily check that

$$S_l^2(v) - vS_l(v)S_{l-1}(v) + S_{l-1}^2(v) = 1. \tag{4.1}$$

The formulas of the twisted Alexander polynomials for 2- and 3-dimensional representations are provided by Tran in [31, 32].

**Lemma 4.1** ([32, Theorem 1], [31, Theorem 1.1] )**.** *Suppose $\rho : G(K) \to \mathrm{SL}(2,\mathbb{C})$ is an irreducible representation of a genus one two-bridge knot $K = J(2m, 2n)$.*

(1) *The twisted Alexander polynomial of $K$ associated with $\rho$ is*

$$\Delta_{K,\rho}(t) = (t + t^{-1} - x)\left(\frac{S_m(y) - S_{m-2}(y) - 2}{y-2}\right)\left(\frac{S_n(\lambda) - S_{n-2}(\lambda) - 2}{\lambda - 2}\right) + xS_{m-1}(y)S_{n-1}(\lambda).$$

(2) *The adjoint twisted Alexander polynomial of $K$ associated with $\rho^{(3)} = \mathrm{Ad}\circ\rho$ is*

$$\Delta_{K,\mathrm{Ad}\circ\rho}(t) = \frac{t-1}{(y+2-x^2)\big(4 - x^2 + (y-2)(y+2-x^2)S_{m-1}^2(y)\big)} \times \left(mnt^2 - \frac{A(y)x^4 + B(y)x^2 + C(y)}{4 + (y-2)(y+2-x^2)S_{m-1}^2(y)}t + mn\right)$$

*where $A(y)$, $B(y)$, and $C(y)$ are rational functions in $y$.*

Using these formulas, we can reproduce those in Subsection 3.2. An application of Lemma 4.1 (2) to the fibering and genus detecting problems for hyperbolic knots is discussed in [23].

## 4.2. Chirality of twist knots

In Corollary 3.2, we presented a necessary condition for a hyperbolic knot to be amphicheiral. For hyperbolic twist knot $K_{2n}$, we can show that the converse is also true for 2- and 3-dimensional representations.

**Theorem 4.2.** *Let $\rho_\alpha : G(K) \to \mathrm{SL}(2,\mathbb{C})$ be a lift of the holonomy representation of the hyperbolic 3-cone-manifold with cone angle $\alpha$ along the hyperbolic twist knot $K = K_{2n}$. There exists $\varepsilon > 0$ such that the following three conditions are equivalent for $\alpha \in [0, \varepsilon)$.*

(1) *$K$ is amphicheiral.*

(2) $\Delta_{K,\rho_\alpha}(t) \in \mathbb{R}[t^{\pm 1}]$.

(3) $\Delta_{K,\mathrm{Ad}\circ\rho_\alpha}(t) \in \mathbb{R}[t^{\pm 1}]$ *or* $\Delta_{K,\mathrm{Ad}\circ\rho_\alpha}(t) \in i\mathbb{R}[t^{\pm 1}]$.

*Remark 4.3.* Note that $\Delta_{K,\rho_\alpha}(t)$ and $\Delta_{K,Ad\circ\rho_\alpha}(t)$ are both polynomials (see Remark 2.2). In fact, $\rho_\alpha : G(K) \to \mathrm{SL}(2,\mathbb{C})$ is irreducible, and $\rho^{(3)}|_{[G(K),G(K)]}$ is nontrivial, because Ad is faithful and irreducible. These facts also follow from the formulas of Tran in Lemma 4.1.

To prove Theorem 4.2, we must describe a solution of the Riley equation corresponding to the complete hyperbolic structure of the complement of twist knots according to [12]. We choose a lift of the holonomy representation $\rho_0 : G(K_{2n}) \to \mathrm{SL}(2, \mathbb{C})$ such that $x = s + s^{-1} = 2$. That is to say, we consider one of the solutions of the restricted Riley equation $\phi_K(1, y) = 0$, which describes *parabolic* representations of the twist knot $K_{2n}$. Here, we note that the variable $z$ in [12] corresponds to $y - 2$ in our notation. In particular, we denote by $y_0$ the solution of $\phi_K(1, y) = 0$ which corresponds to $z_0$ in [12, Theorem 1].

As in [12], $\arg(z)$ denotes the principal argument of $z$ lying in $(-\pi, \pi]$ and $\log z = \ln|z| + i \arg(z)$.

**Lemma 4.4.** *For the hyperbolic twist knot $K = K_{2n}$ $(n \neq 0, 1)$, let $z_0$ be the complex number appearing in* [12], *Theorem 1. Then, $z_0$ has the argument*

$$\begin{cases} \dfrac{2n-3}{4n}\pi < \arg(z_0) < \dfrac{\pi}{2}, & \textit{if } n > 0; \\ \dfrac{\pi}{2} < \arg(z_0) < \dfrac{n-1}{2n}\pi, & \textit{if } n < 0. \end{cases}$$

*Remark 4.5.* Our notation using $n$ is different from that of Hoste–Shanahan in [12]. Our $n$ corresponds to $-n$ in [12, Theorem 1].

*Proof.* By the proof of Theorem 1 and Proposition 1 in [12], the complex number $z_0$ is contained in the compact region bounded by the following curves: the imaginary axis $\mathrm{Re}(z) = 0$, the circle $|z| = 2$, and the two hyperbolas $|z + 2i| - |z - 2i| = 4 \sin\theta_i, i = 1, 2$, where

$$\begin{cases} \theta_1 = \dfrac{(2n-3)\pi}{4n} \text{ and } \theta_2 = \dfrac{(2n-2)\pi}{4n} & \text{if } n > 0; \\ \theta_1 = \dfrac{(2n+2)\pi}{4n} \text{ and } \theta_2 = \dfrac{(2n+1)\pi}{4n} & \text{if } n < 0. \end{cases}$$

If $n > 0$ (resp., $n < 0$), the region is in the first (resp., second) quadrant of the complex plane. Moreover, we can assume that $\arg(z_0) \neq \pi/2$.

First, we consider the case in which $n > 0$. The intersection point of two curves $|z| = 2$ and $|z + 2i| - |z - 2i| = 4 \sin \frac{(2n-3)\pi}{4n}$ $(n > 0)$, which lies in the first quadrant of the complex plane, is

$$z = 2\cos^2 \frac{(2n-3)\pi}{4n} + 2i\sqrt{1 - \cos^4 \frac{(2n-3)\pi}{4n}}.$$

Since $2\cos \frac{(2n-3)\pi}{4n} > 2\cos^2 \frac{(2n-3)\pi}{4n}$, $\frac{(2n-3)\pi}{4n} < \arg(z_0) < \pi/2$.

Next, we consider the case in which $n < 0$. The intersection point of the two curves $|z| = 2$ and $|z + 2i| - |z - 2i| = 4 \sin \frac{(2n+2)\pi}{4n}$ $(n < 0)$, which lies in the second quadrant of the complex plane, is

$$z = -2\cos^2\frac{(n+1)\pi}{2n} + 2i\sqrt{1-\cos^4\frac{(n+1)\pi}{2n}}.$$

Since $n < 0$ and $0 < \frac{n+1}{2n}\pi < \frac{\pi}{2}$, we have

$$2\cos\frac{n-1}{2n}\pi = -2\cos\frac{n+1}{2n}\pi < -2\cos^2\frac{n+1}{2n}\pi.$$

Then, $\frac{\pi}{2} < \arg(z_0) < \frac{n-1}{2\pi}\pi$, completing the proof of Lemma 4.4. □

**Lemma 4.6.** *Let $z_0$ be the complex number in Lemma 4.4. If $n \neq 0, \pm 1$, then ${z_0}^3$ is never real or pure imaginary.*

*Proof.* By Lemma 4.4,

$$\begin{cases} \dfrac{-19}{20}\pi \leq \dfrac{-2n-9}{4n}\pi < \arg(z_0^3) < \dfrac{-\pi}{2}, & \text{if } n \geq 5; \\ -\pi < \arg(z_0^3) < \dfrac{-\pi}{2} \text{ or } \dfrac{3}{8}\pi \leq \dfrac{6n-9}{4n}\pi < \arg(z_0^3) \leq \pi, & \text{if } 2 \leq n \leq 4; \\ \dfrac{-\pi}{2} < \arg(z_0^3) < \dfrac{1}{4}\pi, & \text{if } n = -2; \\ \dfrac{-\pi}{2} < \arg(z_0^3) < \dfrac{-n-3}{2n}\pi \leq 0, & \text{if } n \leq -3. \end{cases}$$

Hence, if $n \leq -3$ or $n \geq 5$, ${z_0}^3$ is never real or pure imaginary.

Next, we consider the exceptional cases ($n = \pm 2, 3, 4$). Here, we provide a proof only for the case in which $n = 4$, following which we can show other cases in a similar manner.

Let $z_0$ be the unique root of the Riley equation $\phi_K(1, y) = \phi_K(1, z+2) = 0$ such that $-\pi < \arg(z_0^3) < \frac{-\pi}{2}$ or $\frac{15}{16}\pi < \arg({z_0}^3) \leq \pi$. We must show that $\arg({z_0}^3) \neq \pi$. Setting $z_0 = re^{\frac{\pi}{3}i}$ $(r > 0)$ and substituting it for $\phi_K(1, z+2)$ yields

$$\phi_K(1, re^{\frac{\pi}{3}i} + 2) = \frac{1}{2}p(r) + \frac{\sqrt{3}}{2}q(r)i,$$

where

$$\begin{aligned} p(r) &= -2 + 4r + 6r^2 - 20r^3 + 5r^4 + 6r^5 - 2r^6 + r^7, \\ q(r) &= r(r+1)(r-2)(r^2+2r-2)(r^2-r+1). \end{aligned}$$

Since $r$ is a positive real number, $\mathrm{Im}(\phi_K(1, re^{\frac{\pi}{3}i} + 2)) = 0$ if and only if $r = 2, -1+\sqrt{3}$. Simple calculations show that $\mathrm{Re}(\phi_K(1, re^{\frac{\pi}{3}i} + 2)) = 71$ if $r = 2$, and $-541 + 312\sqrt{3}$ if $r = -1+\sqrt{3}$, leading to a contradiction. Hence, $\arg({z_0}^3) \neq \pi$, completing the proof of Lemma 4.6. □

## 4.3. **Proof of Theorem 4.2**

In this subsection, we provide a proof of Theorem 4.2. Note that a twist knot $K$ is amphicheiral if and only if $K$ is the figure-eight knot. By Corollary 3.2, condition (1) implies (2) and (3). For the converse direction, we first show the assertions for the case of $\alpha = 0$, a lift of the holonomy representation $\rho_0 = (1, y_0) \in X_K$ of the complete hyperbolic structure of the complement of $K_{2n}$. Then, we discuss local deformation of holonomy representations of hyperbolic 3-cone-manifolds.

Let $K_{2n}$ be a chiral hyperbolic twist knot. We assume that $n \neq 0, \pm 1$, excluding the trivial, trefoil and figure-eight knots.

(3) $\Rightarrow$ (1). Setting $m = 1$ and $x = 2$ in Lemma 4.1 (2), the coefficient of the highest-degree term of $\Delta_{K,\mathrm{Ad}\circ\rho_0}(t)$ is $n/{z_0}^3$, where we have used the relation $y_0 - 2 = z_0$. Since ${z_0}^3$ is never real or pure imaginary by Lemma 4.6, the same is true of $n/{z_0}^3$.

(2) $\Rightarrow$ (1). Setting $m = 1$, $x = 2$, and $t = 1$ in Lemma 4.1 (1) yields the Reidemeister torsion $\tau(K_{2n}, \rho_0) = \Delta_{K_{2n},\rho_0}(1) = 2S_{n-1}(\lambda)$ (Remark 2.3), where $\lambda = {z_0}^2 + 2$. If $\tau(K_{2n}, \rho_0) \notin \mathbb{R}$ for a chiral hyperbolic twist knot, the assertion follows.

For the complete hyperbolic structure of the complement of $K_{2n}$, the restricted Riley equation becomes

$$\phi_K(1, y_0) = S_{n-2}(\lambda) - (3 - y_0)S_{n-1}(\lambda) = 0.$$

Hence, $S_{n-2}(\lambda) = (3 - y_0)S_{n-1}(\lambda) = (1 - z_0)S_{n-1}(\lambda)$. Substituting this for the relation

$$S_{n-1}^2(\lambda) - \lambda S_{n-1}(\lambda)S_{n-2}(\lambda) + S_{n-2}^2(\lambda) = 1$$

(see (4.1)) yields $S_{n-1}^2(\lambda){z_0}^3 = 1$. Thus, $\tau^2(K_{2n}, \rho_0) = 4/{z_0}^3$. Since Lemma 4.6 implies that $\tau^2(K_{2n}, \rho_0)$ is never real, we can conclude $\tau(K_{2n}, \rho_0) \notin \mathbb{R}$ for a chiral hyperbolic twist knot.

Now let us consider local deformation of cone angles. We note that the Riley polynomial $\phi_K(s, y)$ is a regular function on $\mathbb{C}^2 \setminus \{s = 0, y = 2\}$, and $\rho_0 = (1, y_0)$ is a smooth point of $X_K$ (see [27]). Hence, the implicit function theorem implies that there is a neighborhood $U \subset \mathbb{C}$ of $s = 1$ and a regular function $\varphi : U \to \mathbb{C}$ such that $\varphi(1) = y_0$ and $\phi_K(s, \varphi(s)) = 0$ for $s \in U$. Composing $y = \varphi(s)$ with the coefficient of the highest-degree term of $\Delta_{K,\mathrm{Ad}\circ\rho}(t)$ and the Reidemeister torsion $\tau(K_{2n}, \rho_0)$, we can regard them as continuous functions on $U$.

On the other hand, for example, $\Delta_{K,\rho_0}(t) \notin \mathbb{R}[t^{\pm 1}]$ being an open condition, this property is retained under a continuous function. Hence, there is a neighborhood $V \subset U$ of $s = 1$ such that $\Delta_{K,\rho(s)}(t) \notin \mathbb{R}[t^{\pm 1}]$ for any $\rho(s) = (s, \varphi(s)), s \in V$.

Finally, taking the intersection of $V$ and the path connecting the complete hyperbolic structure ($\alpha = 0$) and the Euclidean cone-structure ($\alpha = \alpha_K$, where $\alpha_K \in [2\pi/3, \pi)$) enables us to obtain the desired local deformation of holonomy representations.

This completes the proof of Theorem 4.2.

*Remark 4.7.* Note that conditions (1), (2) and (3) in Theorem 4.2 are equivalent to $\tau(K, \rho_\alpha) = \Delta_{K,\rho_\alpha}(1) \in \mathbb{R}$.

## 4.4. **Concluding remark**

As mentioned at the beginning of this section, Dunfield, Friedl and Jackson propose some open questions on chirality of hyperbolic knots (See [5, Section 1.7]).

(1) Does $\Delta_{K,\rho_0}(t)$ contain information regarding symmetries of the knot other than information on chirality?

(2) If $\Delta_{K,\rho_0}(t)$ is a real polynomial, is $K$ necessarily amphicheiral?

(3) For an amphicheiral knot, is the top coefficient of $\Delta_{K,\rho_0}(t)$ always positive?

As for the question (1), we refer to the papers [9, 11, 14]. Our result (see Subsection 3.2 and Theorem 4.2) answers the question (2) for higher-dimensional twisted Alexander polynomials corresponding to local deformation of holonomy representations of hyperbolic twist knots. Therefore, it is natural to ask the following question.

**Question 4.8.** *Does the statement analogous to Theorem 4.2 hold for the hyperbolic genus one two-bridge knot $J(2m, 2n)$? Moreover, is the top coefficient of $\Delta_{K,\rho_\alpha}(t)$ always positive for an amphicheiral genus one two-bridge knot $K = J(2m, 2n)$?*

Since the point of our proof of Theorem 4.2 is Lemma 4.4, we must first generalize Theorem 1 in [12] to genus one two-bridge knots. Another question is to clarify a relation with global deformation of holonomy representations. Accordingly, we conclude this paper with the following problem.

**Problem 4.9.** *Prove Theorem 4.2 for cone angle $\alpha \in [0, \alpha_K)$, where $\alpha_K \in [2\pi/3, \pi)$.*

## Appendix

Here, we state the definition of the Reidemeister torsion, and summarize the relationship between the twisted Alexander polynomial and the Reidemeister torsion for a knot. For more details, see [8, 15, 27, 35].

Let $\mathbb{F}$ be a field and $C_* = (C_*, \partial)$ a chain complex of finite dimensional $\mathbb{F}$-vector spaces:

$$0 \to C_d \xrightarrow{\partial} C_{d-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \to 0.$$

For each $i$, we denote by $B_i = \operatorname{Im}(C_{i+1} \xrightarrow{\partial} C_i)$, $Z_i = \ker(C_i \xrightarrow{\partial} C_{i-1})$, and the homology is denoted by $H_i = Z_i/B_i$. Then, we have the following exact sequence:

$$0 \to Z_i \to C_i \xrightarrow{\partial} B_{i-1} \to 0,$$
$$0 \to B_i \to Z_i \to H_i \to 0.$$

Let $\widetilde{B}_{i-1}$ be a lift of $B_{i-1}$ to $C_i$, and $\widetilde{H}_i$ a lift of $H_i$ to $Z_i$. Then, we can decompose $C_i$ as follows:

$$\begin{aligned} C_i &= Z_i \oplus \widetilde{B}_{i-1} \\ &= B_i \oplus \widetilde{H}_i \oplus \widetilde{B}_{i-1}. \end{aligned}$$

Let $c^i$ be a basis of $C_i$ and $\mathbf{c}$ the collection $\{c^i\}_{i\geq 0}$, and let $h^i$ be a basis of $H_i$ if nonzero, and $\mathbf{h}$ the collection $\{h^i\}_{i\geq 0}$. We choose $b^i$ a basis of $B_i$. Let $\widetilde{b}^{i-1}$ be a lift of $b^{i-1}$ to $C_i$, and $\widetilde{h}^i$ a lift of $h^i$ to $Z_i$, then we have a new basis $b^i \sqcup \widetilde{b}^{i-1} \sqcup \widetilde{h}^i$ of $C_i$, where $\sqcup$ means a disjoint union. We denote by $[b^i, \widetilde{b}^{i-1}, \widetilde{h}^i/c^i]$ the determinant of the transformation matrix from the basis $c^i$ to $b^i \sqcup \widetilde{b}^{i-1} \sqcup \widetilde{h}^i$.

**Definition A.1.** The torsion of the chain complex $C_*$ with basis $\mathbf{c}$ and $\mathbf{h}$ for $H_i$ is

$$\operatorname{tor}(C_*, \mathbf{c}, \mathbf{h}) = \prod_{i=0}^{d} [b^i, \widetilde{b}^{i-1}, \widetilde{h}^i/c^i]^{(-1)^{i+1}} \quad \in \mathbb{F}^*/\{\pm 1\}.$$

It is known that $\operatorname{tor}(C_*, \mathbf{c}, \mathbf{h})$ is independent of the choice of $b^i$ and the lifts $\widetilde{b}^{i-1}$ and $\widetilde{h}^i$.

Let $W$ be a finite CW-complex, and $\rho : \pi_1(W) \to \mathrm{SL}(k, \mathbb{F})$ a representation of its fundamental group. Consider the chain complex of vector spaces

$$C_*(W, \rho) := \mathbb{F}^k \otimes_\rho C_*(\widetilde{W}; \mathbb{Z})$$

where $C_*(\widetilde{W}; \mathbb{Z})$ denotes the simplicial complex of the universal covering of $W$ and $\otimes_\rho$ means that one takes the quotient of $\mathbb{F}^k \otimes_{\mathbb{Z}} C_*(\widetilde{W}; \mathbb{Z})$ by $\mathbb{Z}$-module generated by

$$\rho(\gamma)^{-1} v \otimes c - v \otimes \gamma \cdot c.$$

Here, $v \in \mathbb{F}^k$, $\gamma \in \pi_1(W)$ and $c \in C_*(\widetilde{W}; \mathbb{Z})$. Namely,

$$v \otimes \gamma \cdot c = \rho(\gamma)^{-1} v \otimes c$$

for any $\gamma \in \pi_1(W)$. The boundary operator is defined by linearity and $\partial(v \otimes c) = (\mathrm{Id} \otimes \partial)(v \otimes c) = v \otimes \partial c$. We denote by $H_*(W, \rho)$ the homology group of this complex.

Let $\{v_1, \dots, v_k\}$ be a basis of $\mathbb{F}^k$ and let $c^i_1, \dots, c^i_{l_i}$ denote the set of $i$-dimensional cells of $W$. We take a lift $\widetilde{c}^i_j$ of the cell $c^i_j$ in $\widetilde{W}$. Then, for each $i$, $\widetilde{c}^i = \{\widetilde{c}^i_1, \dots, \widetilde{c}^i_{l_i}\}$ is a basis of the $\mathbb{Z}[\pi_1(W)]$-module $C_i(\widetilde{W}; \mathbb{Z})$. Thus, we have the following basis of $C_i(W, \rho)$:

$$c^i = \{v_1 \otimes \widetilde{c}^i_1, v_2 \otimes \widetilde{c}^i_1, \dots, v_k \otimes \widetilde{c}^i_{l_i}\}.$$

Suppose $H_i(W, \rho) \neq 0$, and $h^i$ be a basis of $H_i(W, \rho)$. We denote by $\mathbf{h}$ the basis $\{h^0, \dots, h^{\dim W}\}$ of $H_*(W, \rho)$. Then, $\mathrm{tor}(C_*(W, \rho), \mathbf{c}, \mathbf{h})$ $(\in \mathbb{F}^*/\{\pm 1\})$ is well-defined. Note that it does not depend on the lifts of the cells $\widetilde{c}^i$ since $\det \rho = 1$. Further, if the Euler characteristic of $W$ is equal to zero (e.g. the case that $W$ corresponds to the exterior of a knot), it does not depend on the choice of a basis $\{v_1, \dots, v_k\}$.

As in Subsection 2.2, let $\mathfrak{a}$ be a surjective homomorphism from $\pi_1(W)$ to the multiplicative group $\langle t \rangle$. Instead of a representation $\rho : \pi_1(W) \to \mathrm{SL}(k, \mathbb{C})$, consider the tensor representation

$$\mathfrak{a} \otimes \rho : \pi_1(W) \to M(k, \mathbb{C}(t)),$$

where $\mathbb{C}(t)$ is the field of fraction of the polynomial ring $\mathbb{C}[t]$. By the same method as above, we can define $\mathrm{tor}(C_*(W, \mathfrak{a} \otimes \rho), \mathbf{1} \otimes \mathbf{c}, \mathbf{h})$ $(\in \mathbb{C}^*(t)/\{\pm t^{k\mathbb{Z}}\})$.

Let $K$ be a knot in the 3-sphere $S^3$. In this paper, we focus on the knot exterior $E_K = S^3 - \mathrm{int}(N(K))$. In this setting, a result of Waldhausen [34] implies that the torsion $\mathrm{tor}(C_*(E_K, \mathfrak{a} \otimes \rho), \mathbf{1} \otimes \mathbf{c}, \mathbf{h})$ does not depend on the choice of the CW-structure. Thus, we may denote it by $\tau(K, \rho, \mathbf{h})$ if $\mathbf{h} \neq \emptyset$ and by $\tau(K, \rho)$ if $\mathbf{h} = \emptyset$, and call it the Reidemeister torsion of a knot $K$.

**Proposition A.2** ([15, Theorem A]). *Suppose $\rho$ is a non-trivial representation such that $H_*(E_K, \rho) = 0$. Then, $H_*(E_K, \mathfrak{a} \otimes \rho) = 0$ and $\tau(K, \mathfrak{a} \otimes \rho) = \Delta_{K,\rho}(t)$, in particular, $\tau(K, \rho) = \Delta_{K,\rho}(1)$ holds.*

See also [27, Theorem 2.13].

Suppose $K$ is a hyperbolic knot. As in Subsections 2.1 and 2.3, we consider the holonomy representation $\rho_0$, its lift and the composition with $\sigma_k : G(K) \to \mathrm{SL}(k, \mathbb{C})$, denoted by $\rho^{(k)}$. By Corollary 3.7 in [20], we have that $\dim H_i(E_K, \rho^{(k)}) = 0$ $(i = 0, 1, 2)$ if $k$ is even, and that $\dim H_0(E_K, \rho^{(k)}) = 0$, $\dim H_1(E_K, \rho^{(k)}) = \dim H_2(E_K, \rho^{(k)}) = 1$ if $k$ is odd. Further, in [21], Menal-Ferrer and Porti proved the following.

**Proposition A.3** ([21, Proposition 4.6]). *Suppose that $H_*(\partial E_K, \rho^{(k)}) \neq 0$. Let $G < G(K)$ be some fixed realization of the fundamental group of $\partial E_K$ as a subgroup of $G(K)$. Choose a non-trivial cycle $\theta \in H_1(\partial E_K; \mathbb{Z})$, and a nonzero vector $v \in V_k$ fixed by $\rho^{(k)}(G)$. Then, the following holds:*

(1) *A basis for $H_1(E_K, \rho^{(k)})$ is given by $i_*([v \otimes \widetilde{\theta}])$.*

(2) *A basis for* $H_2(E_K, \rho^{(k)})$ *is given by* $i_*([v \otimes \widetilde{\partial E_K}])$.

*Here,* $i : \partial E_K \hookrightarrow E_K$ *denotes the inclusion.*

The homology group $H_1(\partial E_K; \mathbb{Z})$ has the basis $\{[\mu], [\lambda]\}$, where $[\mu]$ is the homology class of the meridian of $K$ and $[\lambda]$ is that of a longitude of $K$. Set $h^1 = i_*([v \otimes \widetilde{\lambda}])$, $h^2 = i_*([v \otimes \widetilde{\partial E_K}])$ and $\mathbf{h} = \{h^1, h^2\}$. Then, we have the following proposition.

**Proposition A.4** ([8, 35])**.** *Under the above notations, the following holds:*

(1) *If* $k$ *is even, then* $\tau(K, \rho^{(k)}) = \Delta_{K,\rho^{(k)}}(1)$.

(2) *If* $k$ *is odd, then* $\tau(K, \rho^{(k)}, \mathbf{h}) = \lim_{t\to 1} \frac{\Delta_{K,\rho^{(k)}}(t)}{t-1}$.

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